# 10. Probability distributions

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# Introduction

These notes consider the Chapter 10 of the handbook on various probability distributions.

# Discrete distributions

For random variables with countable number of possible values.

## Binomial distribution

See page 200 for proper definition.

Useful when a random variable $X$ has exactly two exclusive possible outcomes (e.g. success/fail) with known probabilities $p \in [0, 1]$ (success) and $q = 1 - p$ (fail). For $n \in \mathbb{N}$ trials with $k = 0, 1, 2, \ldots, n$ successes

$$X \sim Bin(n, p)$$

Probability Mass Function (PMF)

$$\mathrm{P}(X = k) = \binom{n}{k} p^k q^{n-k}$$

Cumulative Distribution Function (CDF)

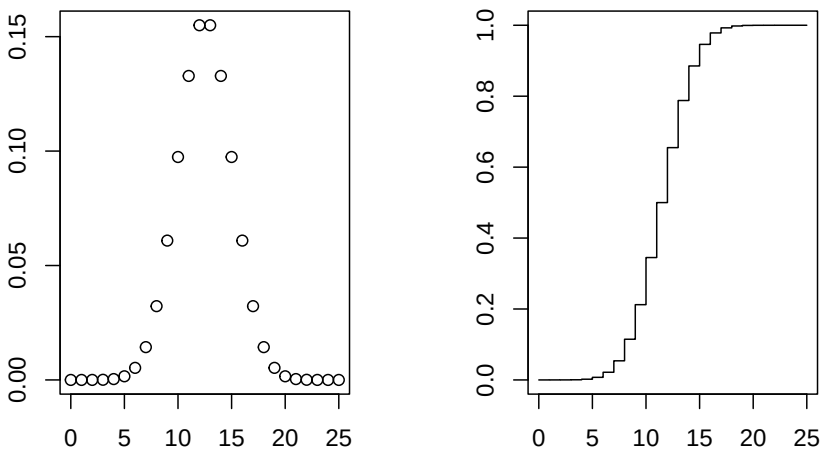
$$\mathrm{F}(k) = \mathrm{P}(X \leqslant k) = \sum_{i=0}^{k} \binom{n}{i} p^i q^{n-i}$$

In the following code examples, equal probabilities are assumed, $n =$ **size**, $p =$ **prob** $= 0.5$.

```r
# Sequence for visualization
binomial_seq <- seq(0, 25, by = 1)

# Functions
binomial_pmf <- dbinom(x = binomial_seq, size = 25, prob = 0.5)
binomial_cdf <- pbinom(q = binomial_seq, size = 25, prob = 0.5)

# Plot
par(mfrow = c(1, 2))
plot(binomial_seq, binomial_pmf, ann = FALSE)
plot(binomial_seq, binomial_cdf, type = "S", ann = FALSE)
```




```r
# Probability for exactly 3 successes out of 10 trials
dbinom(x = 3, size = 10, prob = 0.5)
```

```
## [1] 0.1171875
```

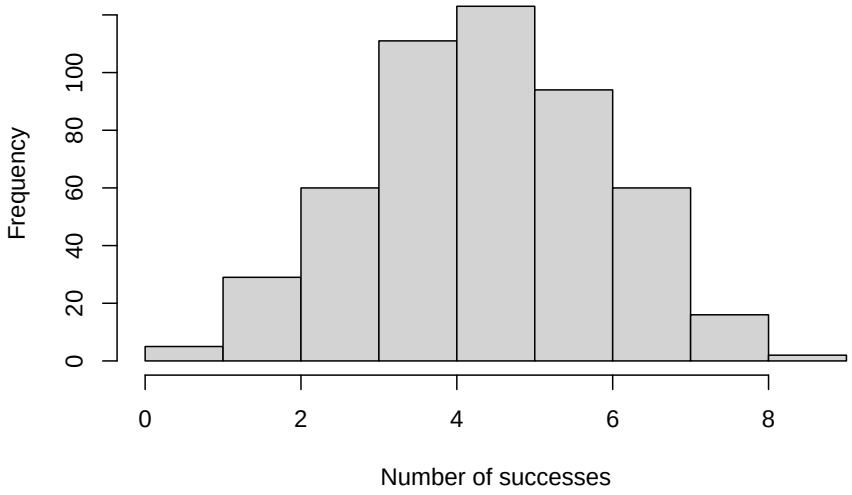
```r
# Probability for up to 3 successes out of 10 trials
pbinom(q = 3, size = 10, prob = 0.5)
```

```
## [1] 0.171875
```

```r
# Simulate 10 times how many successes there is using random numbers
rbinom(n = 10, size = 10, prob = 0.5)
```

```
##  [1] 7 2 7 4 5 8 6 3 5 3
```

```r
# With large enough n, expected value (np = 5) should become visible
rbinom500 <- rbinom(n = 500, size = 10, prob = 0.5)
hist(rbinom500, xlab = "Number of successes", ylab = "Frequency", main = NULL)
```




```r
summary(rbinom500)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   0.000   4.000   5.000   4.862   6.000   9.000
```

## Poisson distribution

See page 204.

Useful when estimating amounts in random processes where the expected value ($\lambda$) is known

$$X \sim Poi(\lambda)$$

PMF

$$\mathrm{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}\ ,\ k = 0, 1, 2, \ldots, n$$

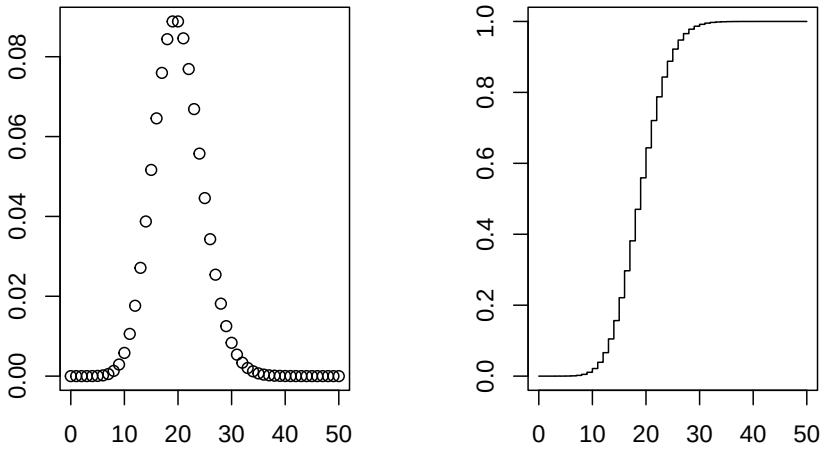
Note that the Poisson distribution can be used to approximate the binomial distribution when $n$ is large and $p$ is small. In this case, $\lambda = np$ and

$$\mathrm{P}(X = k) = \frac{(np)^k e^{-np}}{k!}$$

```r
# Sequence for visualization
poisson_seq <- seq(0, 50, by = 1)

# Functions
poisson_pmf <- dpois(x = poisson_seq, lambda = 20)
poisson_cdf <- ppois(q = poisson_seq, lambda = 20)
```

```r
# Plot
par(mfrow = c(1, 2))
plot(poisson_seq, poisson_pmf, ann = FALSE)
plot(poisson_seq, poisson_cdf, type = "S", ann = FALSE)
```




```r
# A bridge is crossed by 12 people per minute, on average.
# What is the probability that exactly 16 people crosses it in a minute?
dpois(x = 16, lambda = 12)
```

```
## [1] 0.05429334
```

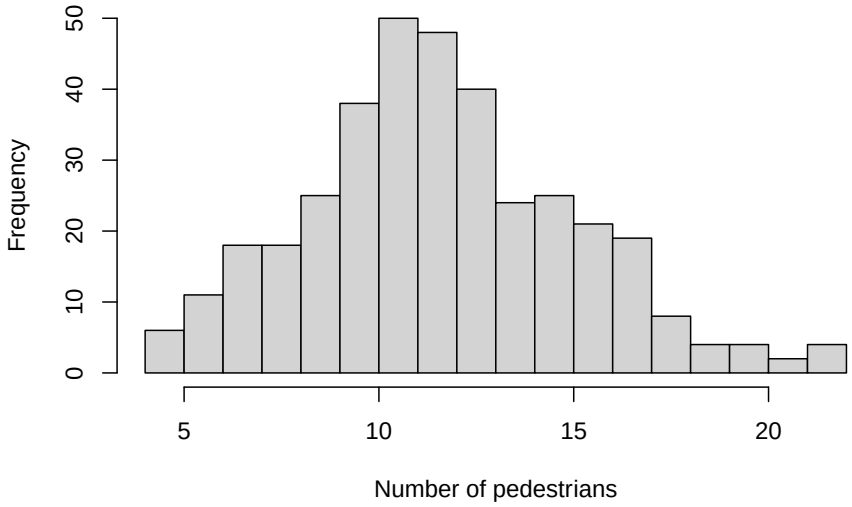
```r
# A bridge is crossed by 12 people per minute, on average.
# What is the probability that up to 16 people crosses it in a minute?
ppois(q = 16, lambda = 12)
```

```
## [1] 0.898709
```

```r
# Simulate amount of people per minute 10 times
rpois(n = 10, lambda = 12)
```

```
##  [1]  7 13  9  9 22 15 11 14 13  8
```

```r
# By repeating the observation once every day for a year,
# the expected value (lambda = 12) should become visible
poisson_pedestrians <- rpois(n = 365, lambda = 12)
hist(poisson_pedestrians, breaks = 20,
     xlab = "Number of pedestrians", ylab = "Frequency", main = NULL)
```




```
summary(poisson_pedestrians)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##     4.0    10.0    12.0    12.1    14.0    22.0
```

# Continuous distributions

For random variables with infinite number of possible values.

## Generic definitions

See page 208.

The probability that an event happens between an interval $[a, b]$ can be calculated from the Probability Density Function (PDF) $f(x)$ as an integral

$$\mathrm{P}(a \leqslant X \leqslant b) = \int_a^b f(x)\ \mathrm{d}x$$

Note that the point probabilities for continuous random variables are intrinsically zero

$$\int_a^a f(x)\ \mathrm{d}x = 0$$

CDF

$$\mathrm{P}(X \leqslant t) = \int_{-\infty}^t f(x)\ \mathrm{d}x$$

## Normal distribution

See page 210.

Many phenomena in nature follow the normal distribution. Additionally, so called central limit theorem states that the averages of idenpendet random samples drawn from a population approximately follow the normal distribution - even if the population is better desrcibed by some other distribution!

Normal distribution is defined by expected value (and median and mode due to symmetry) $\mu$ and (standard) deviation $\sigma$.

$$X \sim N(\mu, \sigma^2)$$

PDF

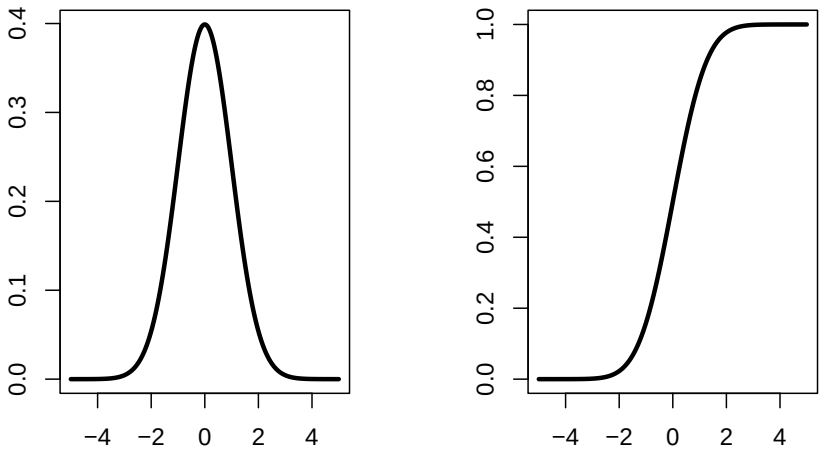
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

In the following code examples, $\mu =$ **mean**, $\sigma =$ **sd**.

```r
# Sequence for visualization
normal_seq <- seq(-5, 5, by = 0.01)

# Functions
normal_pdf <- dnorm(x = normal_seq, mean = 0, sd = 1)
normal_cdf <- pnorm(q = normal_seq, mean = 0, sd = 1)

# Plot
par(mfrow = c(1, 2))
plot(normal_seq, normal_pdf, type = "l", lwd = 3, ann = FALSE)
plot(normal_seq, normal_cdf, type = "l", lwd = 3, ann = FALSE)
```



```r
# The average length of a zebrafish is 22 mm with a standard diviation of 1 mm.
# How likely is a fish 23 mm long?
dnorm(x = 23, mean = 22, sd = 1)
```

```
## [1] 0.2419707
```

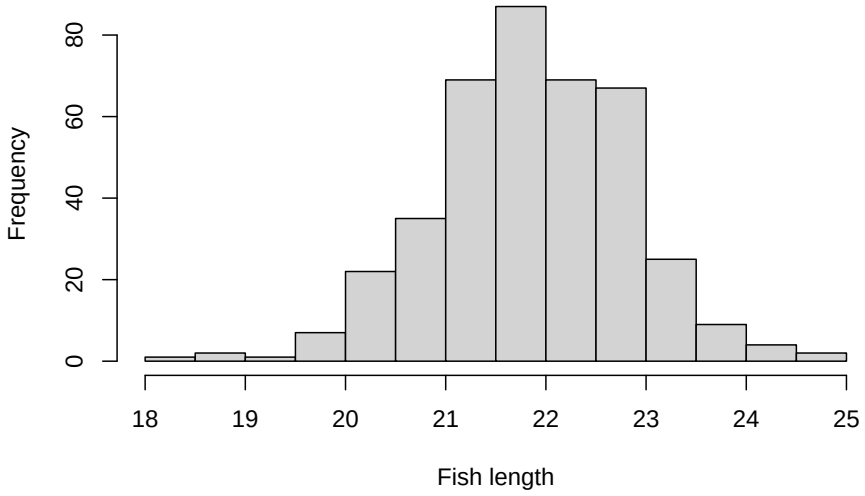
```r
# What fraction of the fish are up to 23 mm long?
pnorm(q = 23, mean = 22, sd = 1)
```

```
## [1] 0.8413447
```

```r
# Simulate length of five random fish
rnorm(n = 5, mean = 22, sd = 1)
```

```
## [1] 20.35990 23.21866 22.63573 22.38878 22.39099
```

```r
# With large enough n, expected value should become visible
normal_zebrafish <- rnorm(n = 400, mean = 22, sd = 1)
hist(normal_zebrafish, breaks = 20,
     xlab = "Fish length", ylab = "Frequency", main = NULL)
```




```r
summary(normal_zebrafish)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   18.48   21.22   21.88   21.87   22.53   24.90
```

## Student's t-distribution

See page 215.

Similar to the normal distribution, Student's $t$ distribution can be used to estimate the expected value of a population using large enough amout of sample averages.

The Student's $t$ distribution is defined by just one parameter, namely the degrees of freedom (DoF) $\nu$. Compared to the normal distribution, the advantage here is that prior knowledge of parameters $\mu$ and $\sigma$ is not needed - which oftentimes is the case. Additionally, with a large $\nu$, the Student's $t$ distribution approaches the standard normal distribution N(0,1).

PDF is defined using the Gamma function $\Gamma(n) = (n-1)!$.

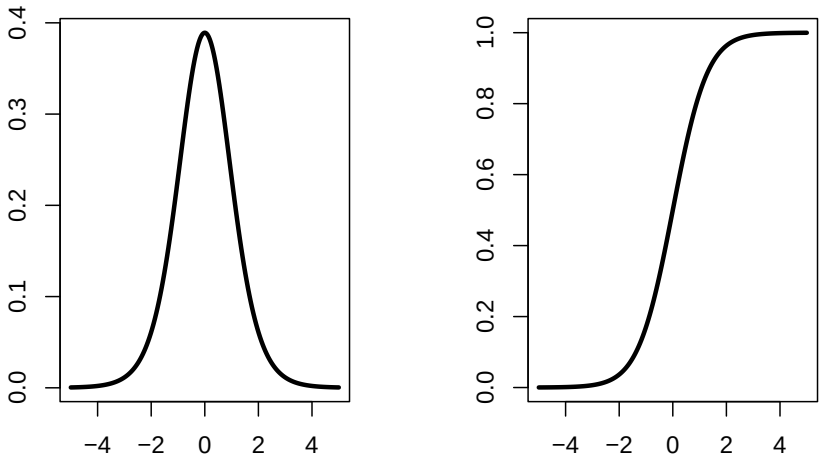
$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

In the following code examples, $\nu = \mathbf{df}$.

```r
# Sequence for visualization
students_t_seq <- seq(-5, 5, by = 0.01)

# Functions
students_t_pdf <- dt(x = students_t_seq, df = 10)
students_t_cdf <- pt(q = students_t_seq, df = 10)

# Plot
par(mfrow = c(1, 2))
plot(normal_seq, students_t_pdf, type = "l", lwd = 3, ann = FALSE)
plot(normal_seq, students_t_cdf, type = "l", lwd = 3, ann = FALSE)
```



```r
# Same as in the earlier examples
dt(x = 2, df = 20)
```

```
## [1] 0.05808722
```

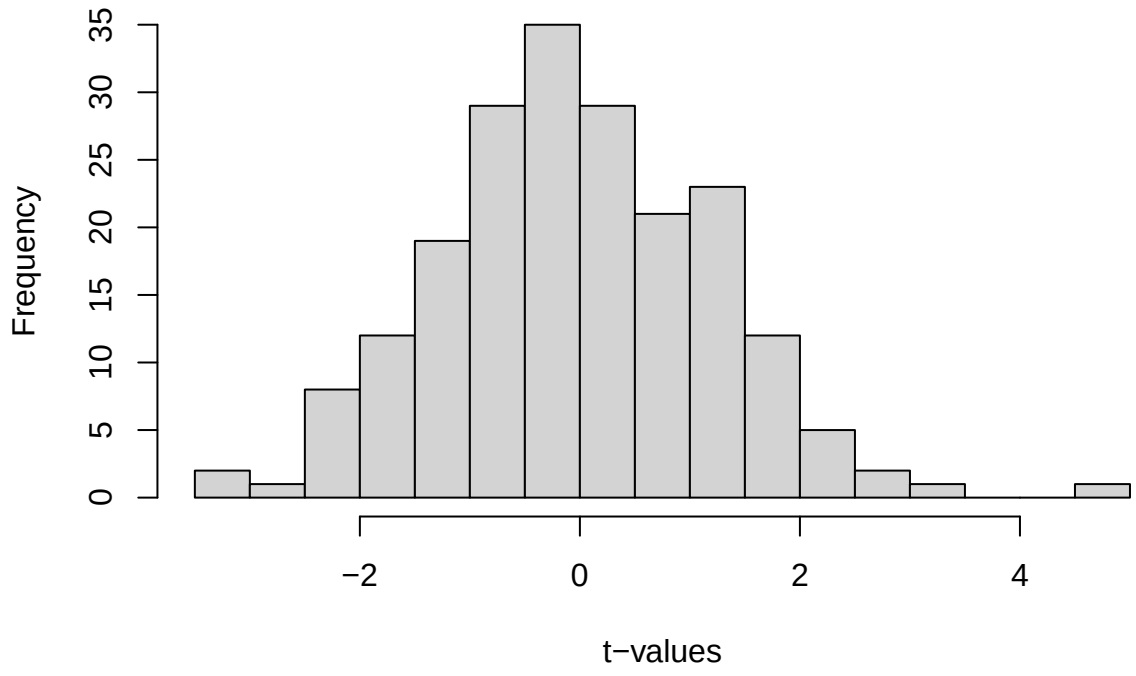
```r
pt(q = 2, df = 20)
```

```
## [1] 0.9703672
```

```r
rt(n = 5, df = 20)
```

```
## [1]  1.254897  1.088066  1.412907  1.559818 -0.703659
```

```r
# The mean should be around 0 regardless of DoF
student_values <- rt(n = 200, df = 5)
hist(student_values, breaks = 20,
     xlab = "t-values", ylab = "Frequency", main = NULL)
```




```r
summary(student_values)
```

```
##     Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
## -3.49870 -0.83202 -0.05487 -0.02296  0.77763  4.85570
```

## Chi-squared distribution

See page 217.

The $\chi^2$ distribution can be used to estimate the distribution of population variances. Suppose we have independent standard normal ($N(0,1)$) random variables $X_1, X_2, \ldots, X_k$, then the random variable

$$Q = \sum_{i=1}^{k} \chi_i^2$$

follows the $\chi^2$ distribution ($Q \sim \chi^2(k)$) with $k-1$ DoF.

PDF

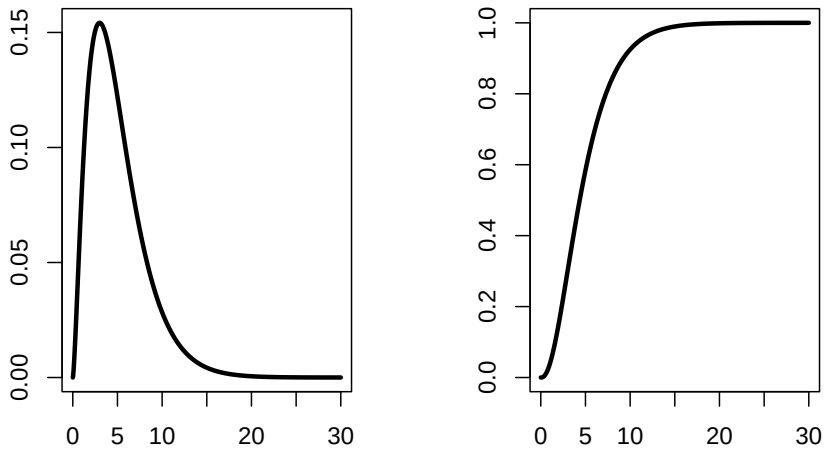
$$f(x;\, k) = \begin{cases} \dfrac{x^{k/2-1} e^{-x/2}}{2^{k/2}\Gamma\left(\frac{k}{2}\right)}, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

In the following code examples, **df** is the DoF.

```r
# Sequence for visualization
chisq_seq <- seq(0, 30, by = 0.02)

# Functions
chisq_pdf <- dchisq(x = chisq_seq, df = 5)
chisq_cdf <- pchisq(q = chisq_seq, df = 5)

# Plot
par(mfrow = c(1, 2))
plot(chisq_seq, chisq_pdf, type = "l", lwd = 3, ann = FALSE)
plot(chisq_seq, chisq_cdf, type = "l", lwd = 3, ann = FALSE)
```



```r
# Same as in the earlier examples
dchisq(x = 2, df = 4)
```

```
## [1] 0.1839397
```

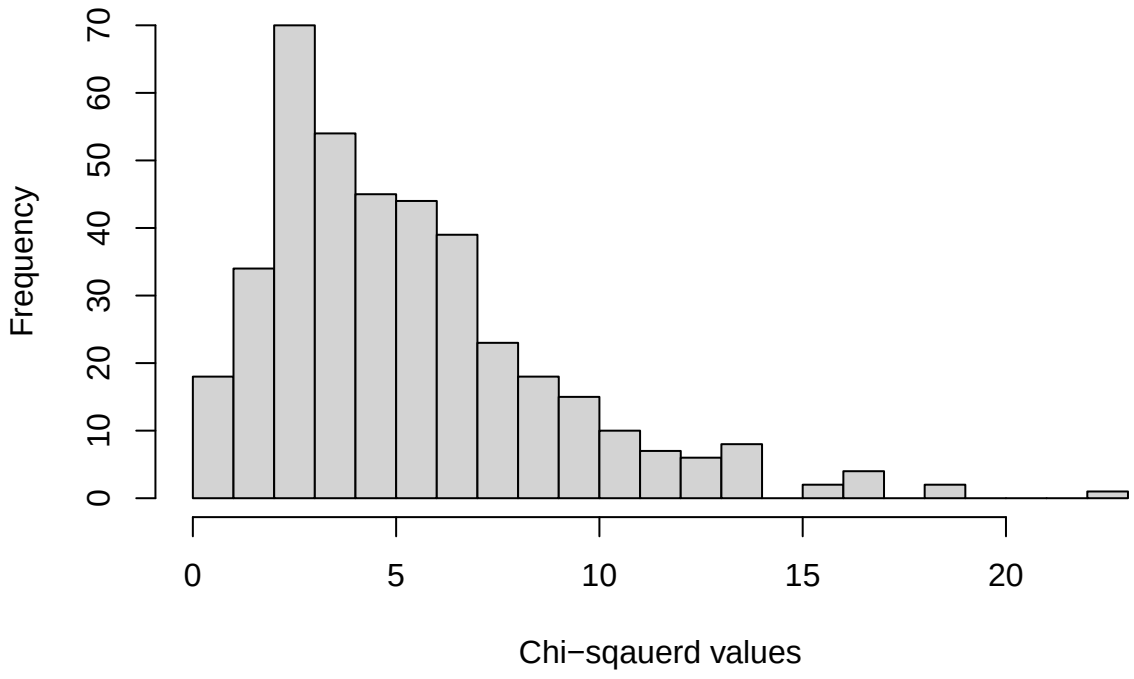
```r
pchisq(q = 2, df = 4)
```

```
## [1] 0.2642411
```

```r
rchisq(n = 5, df = 4)
```

```
## [1] 3.7035521 1.4447092 0.6703702 3.0761773 4.3526684
```

```r
# The mean should be around the DoF
chisq_values <- rchisq(n = 400, df = 5)
hist(chisq_values, breaks = 20,
     xlab = "Chi-sqauerd values", ylab = "Frequency", main = NULL)
```




```r
summary(chisq_values)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.4012  2.7217  4.5814  5.2903  6.9210 22.2281
```

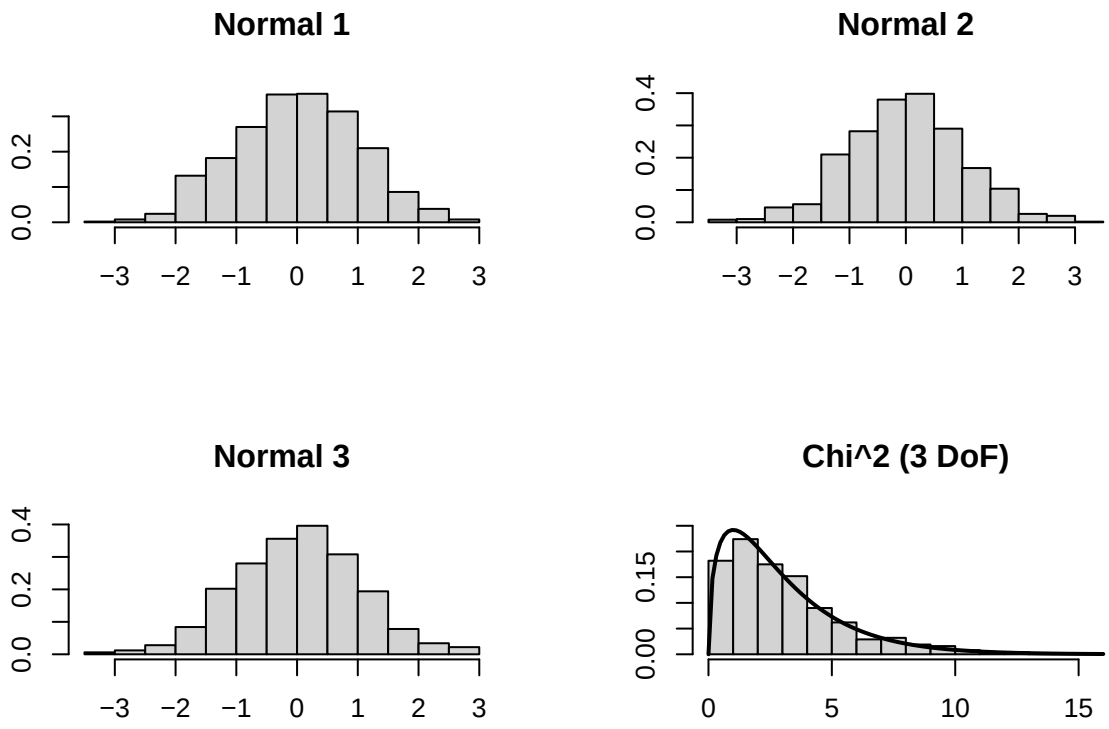
```r
# Let's also visualize the connection to the standard normal distribution
# First, generate some distributions
standard_normal_1 <- rnorm(n = 1000, mean = 0, sd = 1)
standard_normal_2 <- rnorm(n = 1000, mean = 0, sd = 1)
standard_normal_3 <- rnorm(n = 1000, mean = 0, sd = 1)

# Calculate the sum of squares (see random variable Q in the definition)
sum_of_squares <- standard_normal_1^2 + standard_normal_2^2 + standard_normal_3^2

# Plots
par(mfrow = c(2, 2))
hist(standard_normal_1, breaks = 20,
     xlab = "", ylab = "", main = "Normal 1", prob = TRUE)
hist(standard_normal_2, breaks = 20,
     xlab = "", ylab = "", main = "Normal 2", prob = TRUE)
hist(standard_normal_3, breaks = 20,
     xlab = "", ylab = "", main = "Normal 3", prob = TRUE)

hist(sum_of_squares, breaks = 20, ylim = c(0, 0.25),
     xlab = "", ylab = "", main = "Chi^2 (3 DoF)", prob = TRUE)

# Overlap an ideal distribution with 3 DoF
curve(dchisq(x, df = 3), lwd = 2, add = TRUE)
```




Intuitively this should make sense: square of a standard normal distribution "flips" the negative values over the mean creating a skewed distribution and summing these enhances the effect.

## F-distribution

See page 220.

The $F$-distribution is defined as the ratio of two independent random variables that follow the $\chi^2$ distribution:

$$F = \frac{S_1/d_1}{S_2/d_2},$$

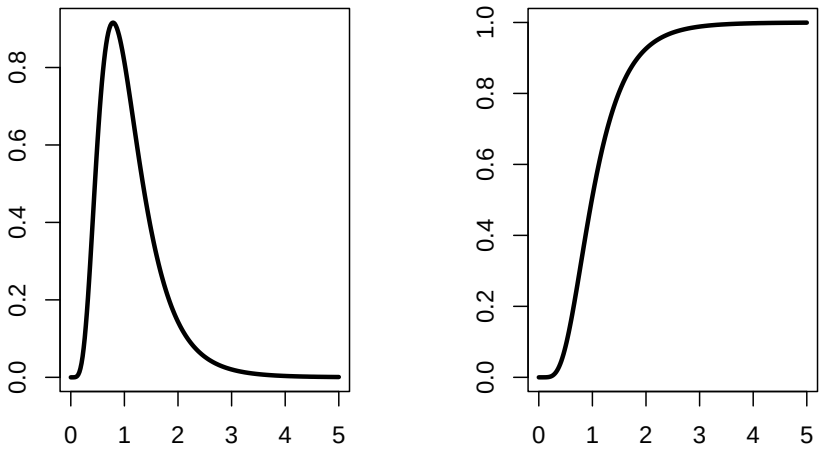
where $S_1 \sim \chi^2(i)$ with $d_1 = i - 1$ DoF and $S_2 \sim \chi^2(j)$ with $d_2 = j - 1$ DoF. The $F$ distribution, therefore, estimates the distribution of ratios of two variances in a population. The expected value is always 1.

In the following code examples, $d_1 =$ **df1** and $d_2 =$ **df2**.

```
# Sequence for visualization
f_seq <- seq(0, 5, by = 0.02)

# Functions
f_pdf <- df(x = f_seq, df1 = 15, df2 = 20)
f_cdf <- pf(q = f_seq, df1 = 15, df2 = 20)

# Plot
par(mfrow = c(1, 2))
plot(f_seq, f_pdf, type = "l", lwd = 3, ann = FALSE)
plot(f_seq, f_cdf, type = "l", lwd = 3, ann = FALSE)
```




```
# Same as in the earlier examples
df(x = 2, df1 = 10, df2 = 30)
```

```
## [1] 0.1399632
```

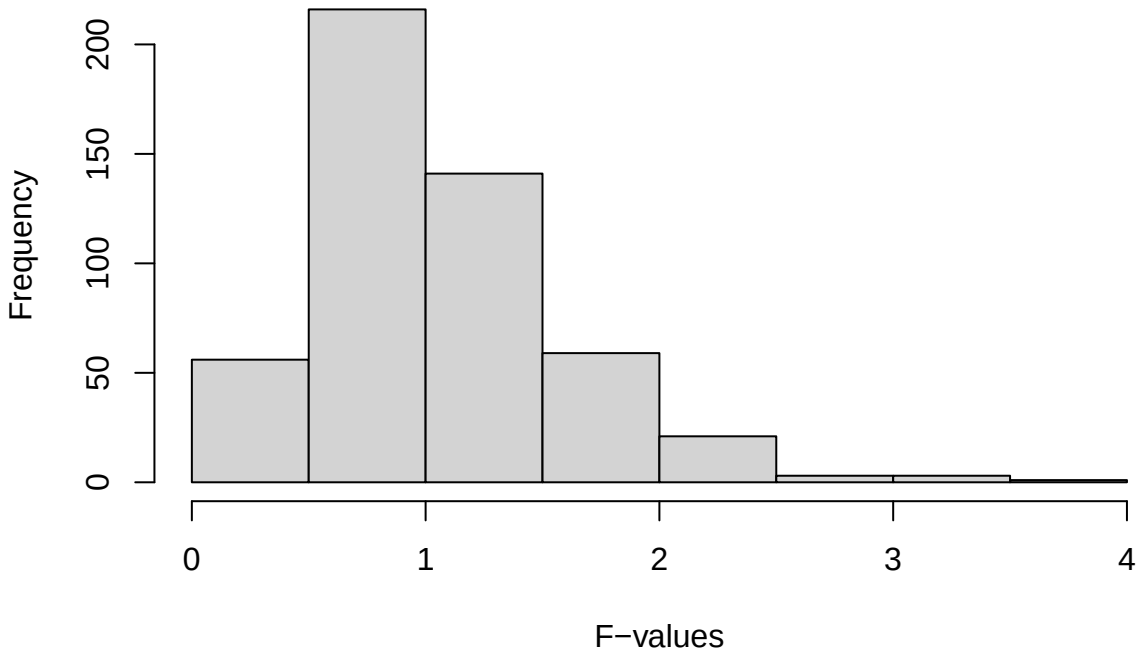
```r
pf(q = 2, df1 = 10, df2 = 30)
```

```
## [1] 0.9303863
```

```r
rf(n = 5, df1 = 10, df2 = 30)
```

```
## [1] 0.9843608 0.5072291 2.1535789 0.4591626 0.9630048
```

```r
# The mean should be around 1
f_values <- rf(n = 500, df1 = 10, df2 = 50)
hist(f_values,
     xlab = "F-values", ylab = "Frequency", main = NULL)
```




```r
summary(f_values)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.1971  0.6975  0.9437  1.0518  1.3182  3.7447
```