Ballistic Trajectory with Drag

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No Drag

Firing Artillery is an interesting proposition. Hitting the target with as few rounds as possible is very important. Each round fired comes at a cost. Besides the monetary cost, each round draws unwanted attention from the enemy. Therefore, each round needs to be carefully considered. Each target needs to provide enough payoff to outweigh the cost and the calculations for each round needs to be meticulous and timely in order to have as little adjustment as possible and requiring subsequent rounds.

The initial physics behind firing artillery seems pretty straight forward. Initial velocity at some angle: the initial velocity vector V

We analyze the components of that initial vector separately Vx and Vy.

$$V_x = Vcos\theta$$

$$V_y = Vsin\theta$$

If there were no drag on the round, Vx would remain constant and Vy would be affected only by gravity. For this paper,

$$x_0 = y_0 = t_0 = x_f = y_f = 0$$

$$\frac{\mathrm{dx}}{\mathrm{dt}} = Vcos\theta, \ \ x = Vtcos\theta, \ \ \frac{\mathrm{dy}}{\mathrm{dt}} = Vsin\theta - gt, \ \ y = Vtsin\theta - \frac{1}{2}gt^2$$

So, for a given initial velocity, in order to hit the target, we need x to equal the range (r) to the target at the same time y equals the height of the target (zero in this case).

$$\frac{r}{\mathrm{V}\cos\theta} = t = \frac{2V sin\theta}{g} \rightarrow r = \frac{2V^2 \sin\theta\cos\theta}{g} = \frac{V^2 sin2\theta}{g}$$

So, for a target at a given range and a round fired at a set velocity, we know the angle that must be shot.

$$\theta = \frac{1}{2} arcsin\left(\frac{\mathrm{rg}}{V^2}\right)$$

We also know the maximum height (H) of the round happens at half the range and half the time.

$$If\ halftime = \frac{\mathrm{V}\sin\theta}{g}$$

, then

$$H = \frac{V^2 \sin^2\theta}{2g}$$

For example, to hit a target at 15km with an initial velocity of 671 m/s, the initial angle needs to be .1664 radians, 9.54 degrees, or 169.7 mils. Time of flight is 22.668. Max height is 629.87 m at 11.3339 seconds. Impact angle is -169.7 mils. Impact velocity is 671 m/s.

This is all very predictable, but there is a great deal of drag so things change.

Drag

If there is drag on the round, there are many variables that affect the velocity of the round. Velocity in the x direction is no longer constant. I get these formulas from Peter Chudinov from the journal of Physics[1].

The drag constant

$$k = \frac{\rho_a c_d S}{2mg} = \frac{1}{V_t^2}$$

;

ρ*a* is the air density, *cd* is the drag factor for a sphere, *S* is the cross-section area of the object, and *Vt* is the terminal velocity.

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[1] https://iopscience.iop.org/article/10.1088/1742-6596/1287/1/012032

I will treat k as a constant even though ρ changes with altitude which may be significant with artillery shots; in this case k could become a function. All variables are functions of θ. Theta is the angle of trajectory that varies throughout the flight.

Launch angle:

$$\theta \epsilon \left[0, \frac{\pi}{2}\right]$$

Impact angle:

$$-\theta \epsilon \left[0, \frac{\pi}{2}\right]$$

Here are the functions I am going to use to predict artillery shots given and initial velocity and launch angle.

These first two equations I can compute directly from θ.

$$f(\theta) = \frac{\sin\theta}{\cos^2\theta} + ln\left(\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right)$$

$$V(\theta) = \frac{V_0 \cos\theta_0}{\cos\theta\sqrt{1 + kV_0^2\cos^2\theta_0\big(f(\theta_0) - f(\theta)\big)}}$$

The following formulas I cannot compute directly but I can get a numerical solution for each θ.

The velocity in the x and y directions is now based off of instantaneous velocity and trajectory angle; not just initial velocity and launch angle.

$$\frac{\mathrm{dx}}{\mathrm{dt}} = Vcos\theta, \frac{\mathrm{dy}}{\mathrm{dt}} = Vsin\theta, \frac{\mathrm{d\theta}}{\mathrm{dt}} = \frac{-gcos\theta}{V}, \frac{\mathrm{dV}}{\mathrm{dt}} = -gsin\theta - gkV^2$$

Now convert all these so they are in terms of θ:

$$\frac{\mathrm{dV}}{\mathrm{d\theta}} = Vtan\theta + \frac{kV^3}{\cos\theta}, \quad \frac{\mathrm{dx}}{\mathrm{d\theta}} = \frac{V^2}{-g}, \quad \frac{\mathrm{dy}}{\mathrm{d\theta}} = \frac{V^2}{-g}tan\theta, \quad \frac{\mathrm{dt}}{\mathrm{d\theta}} = \frac{V}{-gcos\theta}$$

$$x = \int_{\theta_0}^{\theta} \frac{V^2}{-g} \mathrm{d}\theta\,, \quad y = \int_{\theta_0}^{\theta} \frac{V^2}{-g} \tan\theta \,\mathrm{d}\theta\,, \quad t = \int_{\theta_0}^{\theta} \frac{V}{-g cos\theta} \mathrm{d}\theta$$

Once I have and instantaneous altitude, I can alter k for appropriate air pressure.

Attempt at prediction

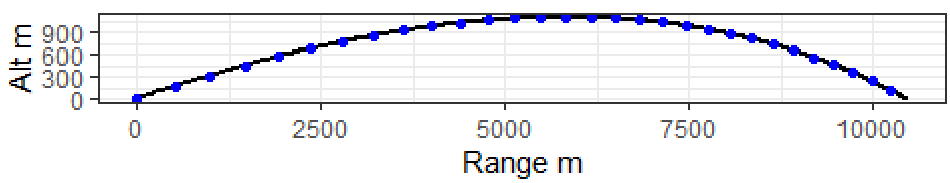
Actual shot data that I am trying to match: Initial velocity 547 m/s at an initial angle of (330.2 mils). My prediction needs to hit a level target at 10000 m in 28.8 seconds while reaching a maximum altitude of 1070 m. Also, impact angle of (-483 mils) with and impact velocity of 298 m/s.

```
options(scipen=999)
v0 <- 547 # initial velocity in m/s for M795 with M232A1 4H
am0 <- 330.2 # QE in mils for a level 15000 m shot

th0 <- am0 * pi / 3200 # initial angle in radians
x0 <- 0 #Initial x
y0 <- 0 # initial y
t0 <- 0 # initial time
g <- 9.80665 # gravitational force in m/s/s
press <- data.frame(cbind(
  alt = c(0,200,500,1000,1500,2000,2500,3000,3500,
          4000,4500,5000,6000,7000,8000,9000),
  rho = c(1.2250,1.2133,1.1844,1.1392,1.0846,1.0320,
          .9569,.8632,.7768,.6971,.5895,.4664,.3612,.2655,.1937,.1413)))
c.press <- glm(rho~poly(alt,6,raw=TRUE), data = press)
pk <- .000006
rho <- as.numeric(predict(c.press, data.frame(alt = y0), type = "response"))
k <- rho*pk # is the drag constant at 0 alt
#k <- .0000075 # Drag
m <- 46.94681 #mass in kg

traj <- data.frame(matrix(ncol = 7))
colnames(traj)=c("Time s","k","Vel m/s","Angle r","Angle mils","Range m","Alt
m")
t <- t0
v <- v0
th <- th0
x <- x0
y <- y0
firstrow <- c(t,k,v,th,th*3200/pi,x,y)
traj[1,] <- firstrow
dt <- .1
while (y > -1){
t <- t + dt
rho <- as.numeric(predict(c.press, data.frame(alt = y), type = "response"))
k <- rho*pk # is the drag constant at y alt
v <- v - (sin(th) + k*v^2) * g * dt
th <- th - (g*cos(th)/v)*dt
```

```
x <- x + v*dt*cos(th)
y <- y + v*dt*sin(th)
nextrow <- c(t,k,v,th,th*3200/pi,x,y)
traj <- rbind(traj,nextrow)
}
trajw <- traj[seq(1, nrow(traj), 10), ]
traj %>% ggplot(aes(`Range m`,`Alt m`))+geom_point(size=.1) +
  geom_point(data=trajw,aes(`Range m`,`Alt m`),color="blue") +
    coord_fixed(ratio = 1)
```




```
pander(trajw)
```

	Time s	k	Vel m/s	Angle r	Angle mils	Range m	Alt m
1	0	0.000007383	547	0.3242	330.2	0	0
11	1	0.000007281	523.3	0.3067	312.4	507.5	165.2
21	2	0.000007191	501.7	0.2884	293.7	996.4	314.7
31	3	0.000007117	482	0.2692	274.2	1468	449.4
41	4	0.000007054	463.9	0.2491	253.7	1925	569.9
51	5	0.000006999	447.2	0.2281	232.4	2366	676.9
61	6	0.000006952	431.8	0.2063	210.1	2795	771
71	7	0.000006911	417.6	0.1836	187	3211	852.7
81	8	0.000006875	404.5	0.1601	163	3615	922.4
91	9	0.000006845	392.3	0.1357	138.2	4008	980.6

101	10	0.00000682	381.1	0.1105	112.5	4392	1028
111	11	0.0000068	370.6	0.08446	86.03	4765	1064
121	12	0.000006786	361	0.05768	58.75	5130	1089
131	13	0.000006778	352.1	0.03016	30.72	5485	1104
141	14	0.000006774	343.8	0.001943	1.979	5833	1109
151	15	0.000006776	336.3	-0.02693	-27.43	6172	1105
161	16	0.000006783	329.3	-0.05641	-57.46	6504	1090
171	17	0.000006796	322.9	-0.08645	-88.06	6829	1067
181	18	0.000006812	317	-0.117	-119.2	7147	1034
191	19	0.000006834	311.7	-0.1479	-150.7	7458	991.9
201	20	0.000006859	306.8	-0.1793	-182.6	7763	941.1
211	21	0.000006889	302.4	-0.2109	-214.8	8061	881.7
221	22	0.000006923	298.5	-0.2427	-247.2	8354	813.7
231	23	0.00000696	295	-0.2747	-279.8	8640	737.4
241	24	0.000007002	291.9	-0.3068	-312.5	8921	652.9
251	25	0.000007048	289.2	-0.3388	-345.1	9196	560.4
261	26	0.000007099	286.8	-0.3708	-377.7	9466	460
271	27	0.000007156	284.8	-0.4026	-410.1	9730	351.8
281	28	0.000007221	283.1	-0.4342	-442.2	9990	236.1
291	29	0.000007297	281.6	-0.4655	-474.1	10244	113