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Highly precise analytic solutions for classic problem of projectile motion in the air

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Abstract. Here is studied a classic problem of the motion of a projectile thrown at an angle to the horizon. The number of publications on this problem is very large. The air drag force is taken into account as the quadratic resistance law. An analytic approach is used for the investigation. Equations of the projectile motion are solved analytically. All the basic functional dependencies of the problem are described by elementary functions. There is no need to study the problem numerically. The found analytical solutions are highly accurate over a wide range of parameters. The motion of a baseball and a badminton shuttlecock is presented as examples.

1. Introduction
The problem of the motion of a projectile in midair arouses interest of authors as before [1–8]. The number of publications on this problem is very large. Together with the investigation of the problem by numerical methods, attempts are still being made to obtain the analytical solutions. Many such solutions of a particular type are obtained. They are valid for limited values of the physical parameters of the problem (for the linear law of the medium resistance at low speeds, for short travel times, for low, high and split angle trajectory regimes and others). For the construction of the analytical solutions various methods are used – both the traditional approaches [1,3,4], and the modern methods [2, 5]. All proposed approximate analytical solutions are rather complicated and inconvenient for educational purposes. In addition, many approximate solutions use special functions, for example, the Lambert W function. This is why the description of the projectile motion by means of simple approximate analytical formulas under the quadratic air resistance is of great methodological and educational importance.

The purpose of the present work is to give simple formulas for the construction of the trajectory of the projectile motion with quadratic air resistance. In this paper, two variants of approximation of the sought functions (the projectile coordinates) is realized. It allows to construct a trajectory of the projectile with the help of elementary functions without using numerical schemes. Following other authors, we call this approach the analytic approach. The conditions of applicability of the quadratic resistance law are deemed to be fulfilled, i.e. Reynolds number Re lies within $1\times10^3 < \mathrm{Re} < 2\times10^5$.

2. Equations of projectile motion
We now state the formulation of the problem and the equations of the motion according to [8]. Suppose that the force of gravity affects the projectile together with the force of air resistance $\boldsymbol{R}$ (see figure 1). Air resistance force is proportional to the square of the velocity V of the projectile and is directed opposite the velocity vector. For the convenience of further calculations, the drag force will

be written as $R = mgkV^2$. Here m is the mass of the projectile, g is the acceleration due to gravity, k is the proportionality factor. Vector equation of the motion of the projectile has the form

$$m\boldsymbol{w} = m\boldsymbol{g} + \boldsymbol{R},$$

where $\boldsymbol{w}$ – acceleration vector of the projectile. Differential equations of the motion, commonly used in ballistics, are as follows [9]

$$\frac{dV}{dt} = -g\sin\theta - gkV^2, \quad \frac{d\theta}{dt} = -\frac{g\cos\theta}{V}, \quad \frac{dx}{dt} = V\cos\theta, \quad \frac{dy}{dt} = V\sin\theta. \qquad (1)$$

Here V is the velocity of the projectile, θ is the angle between the tangent to the trajectory of the projectile and the horizontal, x, y are the Cartesian coordinates of the projectile,

$$k = \frac{\rho_a c_d S}{2mg} = \frac{1}{V_{term}^2} = const,$$

ρ_a is the air density, c_d is the drag factor for a sphere, S is the cross-section area of the object, and V_{term} is the terminal velocity. The first two equations of the system (1) represent the projections of the vector equation of motion on the tangent and principal normal to the trajectory, the other two are kinematic relations connecting the projections of the velocity vector projectile on the axis x, y with derivatives of the coordinates.

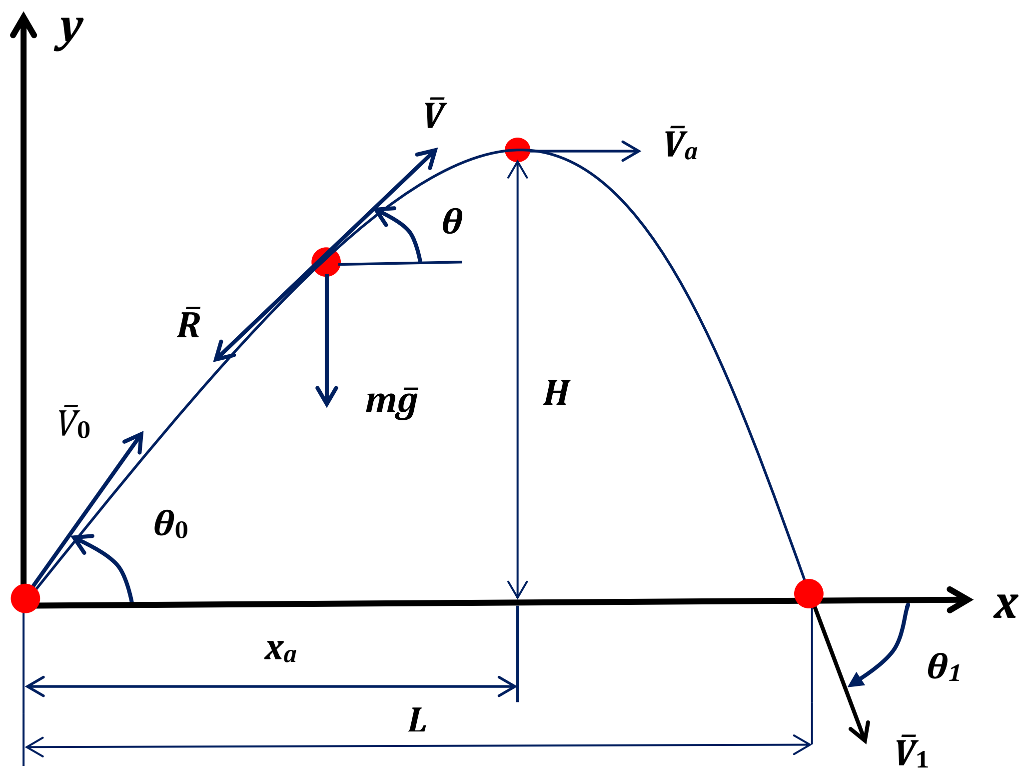

Figure 1. Basic motion parameters.

The well-known solution of system (1) consists of an explicit analytical dependence of the velocity on the slope angle of the trajectory and three quadratures

$$V(\theta)=\frac{V_0\cos\theta_0}{\cos\theta\sqrt{1+kV_0^2\cos^2\theta_0\left(f(\theta_0)-f(\theta)\right)}},\quad f(\theta)=\frac{\sin\theta}{\cos^2\theta}+\ln tg\left(\frac{\theta}{2}+\frac{\pi}{4}\right),\tag{2}$$

$$x=x_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2d\theta,\quad y=y_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2tg\theta d\theta,\qquad t=t_0-\frac{1}{g}\int_{\theta_0}^{\theta}\frac{V}{cos\theta}d\theta\ .\tag{3}$$

Here V_0 and θ_0 are the initial values of the velocity and of the slope of the trajectory respectively, t_0 is the initial value of the time, x_0, y_0 are the initial values of the coordinates of the projectile (usually accepted $t_0=x_0=y_0=0$). The derivation of the formulae (2) is shown in the well-known monograph [10]. The integrals on the right-hand sides of formulas (3) cannot be expressed in terms of elementary functions. Hence, to determine the variables *t, x* and *y* we must either integrate system (1) numerically or evaluate the definite integrals (3).

3. Proposed approximations for the obtaining an analytical solutions of the problem

The task analysis shows, that equations (3) are not exactly integrable owing to the complicated nature of function (2)

$$f(\theta)=\frac{\sin\theta}{\cos^2\theta}+\ln tg\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\ .$$

The odd function $f(\theta)$ is defined in the interval $-\pi/2<\theta<\pi/2$. Therefore, it can be assumed that a successful approximation of this function will make it possible to calculate analytically the definite integrals (3) with the required accuracy. An analysis of the problem shows that it is convenient to approximate the function $f(\theta)$ only by polynomials of the second or third degree. The first-order polynomial does not provide the required accuracy of the approximation. Polynomials of higher orders do not allow us to calculate the integrals (3) in elementary functions. The Ref. [1] presents a simple approximation in the mathematical sense of a function $f(\theta)$ by a second-order polynomial of the following form (polynomial is with respect to a function $\tan\theta$)

$$f_a(\theta)=a_1\tan\theta+b_1\tan^2\theta.$$

The function $f_a(\theta)$ well approximates the function $f(\theta)$ only on the specified interval $[0,\theta_0]$, since the function $f_a(\theta)$ contains an even term. Under the condition $\theta<0$, another approximation is required because the function $f(\theta)$ is odd.

In the present paper we propose two approximations of the function $f(\theta)$ on the whole interval $-\pi/2<\theta<\pi/2$. The first approximation uses a second-order polynomial, the second approximation uses a third-order polynomial. Approximation of the function $f(\theta)$ by a second order polynomial $f_2(\theta)$ has the following form

$$f_2(\theta)=\begin{cases}a_1\tan\theta+b_1\tan^2\theta, & \text{on}\quad \text{condition}\quad \theta\ge 0,\\ a_1\tan\theta-b_1\tan^2\theta, & \text{on}\quad \text{condition}\quad \theta\le 0.\end{cases}$$

The coefficients a_1 and b_1 can be chosen in such a way as to smoothly connect the functions $f(\theta)$ and $f_2(\theta)$ to each other with the help of conditions

$$f_2(\theta_0) = f(\theta_0), \quad f_2'(\theta_0) = f'(\theta_0). \tag{4}$$

$$a_1 = \frac{2\ln\tan(\theta_0/2+\pi/4)}{\tan\theta_0}, \quad b_1 = \frac{1}{\sin\theta_0} - \frac{\ln\tan(\theta_0/2+\pi/4)}{\tan^2\theta_0}.$$

Approximation of the function $f(\theta)$ by a third-order polynomial $f_3(\theta)$ has the following form

$$f_3(\theta) = a_1\tan\theta + b_1\tan^3\theta.$$

The function $f_3(\theta)$ is formed by two odd functions, and therefore it is applicable over the whole interval $-\pi/2 < \theta < \pi/2$. As well as for the function $f_2(\theta)$, we find coefficient values a_1 и b_1 from the conditions (4)

$$a_1 = \frac{1}{2\cos\theta_0} + \frac{3\ln\tan(\theta_0/2+\pi/4)}{2\tan\theta_0}, \quad b_1 = \frac{1}{2\tan^2\theta_0}\left(\frac{1}{\cos\theta_0} - \frac{\ln\tan(\theta_0/2+\pi/4)}{\tan\theta_0}\right).$$

These functions $f_2(\theta)$, $f_3(\theta)$ well approximate the function $f(\theta)$ throughout the whole interval of its definition for any values θ_0. As an example, we give graphs of functions $f(\theta)$, $f_2(\theta)$, $f_3(\theta)$ in the interval $-80° \le \theta \le 80°$. Coefficients a_1, b_1 are calculated at a value $\theta_0 = 60°$.

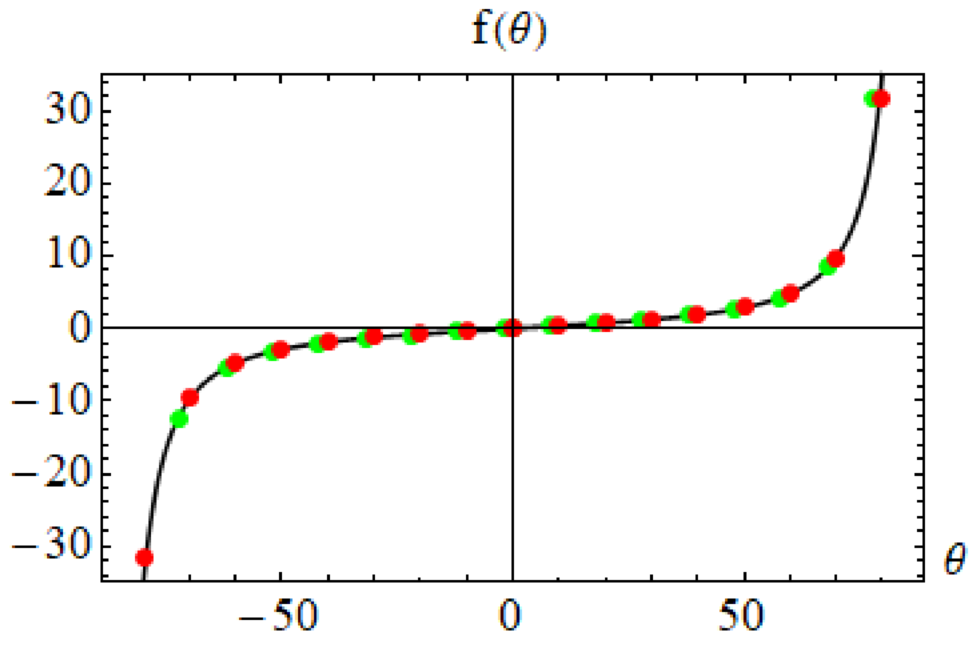

Figure 2. Approximations of the function $f(\theta)$.

The solid curve in figure 2 is a graph of the function $f(\theta)$, the red dots curve is a graph of the function $f_2(\theta)$. The green dots curve is a graph of the function $f_3(\theta)$. All three graphics practically coincide. Hence, the functions $f_2(\theta)$, $f_3(\theta)$ can be used instead of the function $f(\theta)$ in calculating the integrals (3).

4. Analytical solutions for the approximation $f_2(\theta)$

Now the quadratures (3) are integrated in elementary functions. Since the function $f_2(\theta)$ has a different form on the gaps $\theta \geq 0,\ \theta \leq 0$, the integrals (3) also have a different form at these intervals. For the ascending branch of the trajectory it is $\theta \geq 0,$ for the descending branch of the trajectory it is $\theta \leq 0$. In calculating the integrals we take $t_0 = x_0 = y_0 = 0$. We integrate the first of the integrals (3). For the coordinate x we obtain:

$$x_1(\theta) = -\frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta = A_1 \arctan\left(\frac{2b_2 \tan\theta + 1}{b_3}\right)\Bigg|_{\theta_0}^{\theta} \qquad \text{in case of} \quad \theta \geq 0\,,$$

$$x_2(\theta) = -A_2 \arctan\left(\frac{2b_2 \tan\theta - 1}{b_4}\right)\Bigg|_{\theta_0}^{\theta} \qquad \text{in case of} \quad \theta \leq 0.$$

Here we introduce the following notation:

$$A_1 = \frac{2}{gka_1b_3},\ A_2 = \frac{2}{gka_1b_4},\ a_2 = \frac{a}{a_1},\ a = \frac{1}{kV_0^2\cos^2\theta_0} + f(\theta_0),$$

$$b_2 = \frac{b_1}{a_1},\ b_3 = \sqrt{-1 - 4a_2b_2},\ b_4 = \sqrt{-1 + 4a_2b_2}.$$

Thus, the dependence $x(\theta)$ has the following form:

$$x(\theta) = x_1(\theta) - x_1(\theta_0) \qquad \text{in case of} \qquad \theta \geq 0,$$

$$x(\theta) = x_1(0) - x_1(\theta_0) + x_2(\theta) - x_2(0) \qquad \text{in case of} \qquad \theta \leq 0. \qquad (5)$$

We integrate the second of the integrals (3). For the coordinate y we obtain:

$$y_1(\theta) = -\frac{1}{g}\int_{\theta_0}^{\theta} V^2 \tan\theta d\theta = \left(-B_1 \arctan\left(\frac{1 + 2b_2\tan\theta}{b_3}\right) + B_2 \ln\left|-a_2 + \tan\theta + b_2\tan^2\theta\right|\right)\Bigg|_{\theta_0}^{\theta}$$

in case of $\theta \geq 0$,

$$y_2(\theta) = \left(-B_3 \arctan\left(\frac{-1 + 2b_2\tan\theta}{b_4}\right) - B_2 \ln\left|a_2 - \tan\theta + b_2\tan^2\theta\right|\right)\Bigg|_{\theta_0}^{\theta} \qquad \text{in case of} \quad \theta \leq 0.$$

Here we introduce the following notation:

$$B_1 = \frac{1}{kga_1b_2b_3},\ B_2 = \frac{1}{2kga_1b_2},\ B_3 = \frac{1}{kga_1b_2b_4}.$$

Thus, the dependence $y(\theta)$ has the following form:

$$y(\theta) = y_1(\theta) - y_1(\theta_0) \qquad \text{in case of} \qquad \theta \geq 0,$$

$$y(\theta) = y_1(0) - y_1(\theta_0) + y_2(\theta) - y_2(0) \qquad \text{in case of} \qquad \theta \leq 0. \qquad (6)$$

For the variable t we get:

$$t_1(\theta) = B_4 \arctan\left[\frac{(1 + 2b_2\tan\theta)\sqrt{a_2 - \tan\theta - b_2\tan^2\theta}}{2\sqrt{b_2}\left(-a_2 + \tan\theta + b_2\tan^2\theta\right)}\right]\Bigg|_{\theta_0}^{\theta} \qquad \text{in case of} \quad \theta \geq 0,$$

$$t_2(\theta) = -B_4 \ln\left|-1 + 2b_2\tan\theta + 2\sqrt{b_2}\sqrt{a_2 - \tan\theta + b_2\tan^2\theta}\right|\Bigg|_{\theta_0}^{\theta} \qquad \text{in case of} \quad \theta \leq 0.$$

Here we introduce the notation:

$$B_4 = \frac{1}{g\sqrt{ka_1 b_2}}.$$

Thus, the dependence $t(\theta)$ has the following form:

$$t(\theta) = t_1(\theta) - t_1(\theta_0) \qquad \text{in case of } \theta \geq 0,$$
$$t(\theta) = t_1(0) - t_1(\theta_0) + t_2(\theta) - t_2(0) \qquad \text{in case of } \theta \leq 0. \qquad (7)$$

Consequently, the basic functional dependencies of the problem $x(\theta), y(\theta), t(\theta)$ are written in terms of elementary functions.

The main characteristics of the projectile's motion are (figure 1):

- H – the maximum height of ascent of the projectile ,
- T – motion time,
- L – flight range ,
- x_a – the abscissa of the trajectory apex,
- t_a – the time of ascent,
- θ_1 – impact angle with respect to the horizontal .

Using formulas (5) – (7), we find:

$$x_a = x_1(0) - x_1(\theta_0), \;\; H = y_1(0) - y_1(\theta_0), \;\; t_a = t_1(0) - t_1(\theta_0). \qquad (8)$$

Then formulas (5) – (7) can be rewritten as:

$$x(\theta) = x_1(\theta) - x_1(\theta_0) \qquad \text{in case of } \theta \geq 0,$$
$$x(\theta) = x_a + x_2(\theta) - x_2(0) \qquad \text{in case of } \theta \leq 0.$$
$$y(\theta) = y_1(\theta) - y_1(\theta_0) \qquad \text{in case of } \theta \geq 0,$$
$$y(\theta) = H + y_2(\theta) - y_2(0) \qquad \text{in case of } \theta \leq 0.$$
$$t(\theta) = t_1(\theta) - t_1(\theta_0) \qquad \text{in case of } \theta \geq 0,$$
$$t(\theta) = t_a + t_2(\theta) - t_2(0) \qquad \text{in case of } \theta \leq 0.$$

The angle of incidence of the projectile θ_1 is determined from the condition $y(\theta_1) = 0$. Then we have

$$L = x_a + x_2(\theta_1) - x_2(0), \qquad T = t_a + t_2(\theta_1) - t_2(0). \qquad (9)$$

We note that formulas (5) – (7) also define the dependences $y = y(x),\; y = y(t),\; x = x(t)$ in a parametric way.

5. Analytical solutions for the approximation $f_3(\theta)$

Now let the function $f(q)$ in formula (2) be approximated by a function $f_3(q)$. We integrate the first of the integrals (3). For the coordinate x we obtain:

$$x = x_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta = \left.\left(A_1 \ln\left|\frac{(\tan\theta - b)^2}{\tan^2\theta + b\tan\theta + c}\right| - A_2 \arctan\left(\frac{2\tan\theta + b}{\Delta}\right)\right)\right|_{\theta_0}^{\theta}.$$

Here we introduce the following notation:

$$a = \frac{1}{kV_0^2 \cos^2 \theta_0} + f(\theta_0), \quad d_0 = -\frac{a}{b_1}, \quad d_1 = \frac{a_1}{b_1}, \quad p_1 = \sqrt[3]{-\frac{d_0}{2} + \sqrt{\frac{d_0^2}{4} + \frac{d_1^3}{27}}}, \quad p_2 = -\frac{d_1}{3p_1},$$

$$b = p_1 + p_2, \quad c = d_1 + b^2, \quad \Delta = \sqrt{\left|b^2 - 4c\right|}, \quad A_1 = \frac{1}{2gkb_1\left(c + 2b^2\right)}, \quad A_2 = \frac{6bA_1}{\Delta},$$

$$F_1(\theta) = A_1 \ln\left|\frac{(\tan\theta - b)^2}{\tan^2\theta + b\tan\theta + c}\right| - A_2 \arctan\left(\frac{2\tan\theta + b}{\Delta}\right).$$

Thus, the dependence $x(\theta)$ has the following form:

$$x(\theta) = F_1(\theta) - F_1(\theta_0). \tag{10}$$

We integrate the second of the integrals (3). For the coordinate y we obtain:

$$y = y_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 \tan\theta d\theta = \left(bF_1(\theta) + A_3 \arctan\left(\frac{2\tan\theta + b}{\Delta}\right)\right)\Bigg|_{\theta_0}^{\theta}.$$

Here we introduce the following notation:

$$F_2(\theta) = bF_1(\theta) + A_3 \arctan\left(\frac{2\tan\theta + b}{\Delta}\right), \quad A_3 = \frac{2}{gkb_1\Delta}.$$

Thus, the dependence $y(\theta)$ has the following form:

$$y(\theta) = F_2(\theta) - F_2(\theta_0). \tag{11}$$

Consequently, the basic functional dependencies of the problem $x(\theta), y(\theta)$ are written in terms of elementary functions.

The third integral (3) cannot be taken in elementary functions. However, estimates for the parameters T and t_a can be made using the formulas of [6]. The angle of incidence of the projectile θ_1 is determined from the condition $y(\theta_1) = 0$. Using formulas (10) – (11), we find:

$$x_a = x(0), \quad H = y(0).$$

Then we have

$$L = x(\theta_1), \quad T = 2\sqrt{\frac{2H}{g}}, \quad V_a = \frac{V_0 \cos\theta_0}{\sqrt{1 + kV_0^2 \cos^2\theta_0 f(\theta_0)}}, \quad t_a = \frac{T - kHV_a}{2}.$$

We note that formulas (10) – (11) also define the dependence $y = y(x)$ in a parametric way.

6. The results of the calculations. Field of application of the obtained solutions

Proposed formulas have a wide region of application. We introduce the notation $p = kV_0^2$. The dimensionless parameter p has the following physical meaning – it is the ratio of air resistance to the weight of the projectile at the beginning of the movement. As calculations show, trajectory of the projectile $y = y(x)$ and the main characteristics of the motion L, H, T, x_a, t_a have accuracy to within 1% for values of the launch angle and for the parameter p within ranges

$$0° < \theta_0 < 90°, \quad 0 < p \leq 60.$$

Figure 3 presents the results of plotting the projectile trajectories with the aid of formulas (5) – (6) and (10) – (11) over a wide range of the change of the initial angle θ_0 with the following values of the parameters

$$V_0 = 80 \text{ m/s}, \quad k = 0.000625 \text{ s}^2/\text{m}^2, \quad g = 9.81 \text{ m/s}^2, \quad p = 4.$$

The used value of the parameter k is the typical value of the baseball drag coefficient.

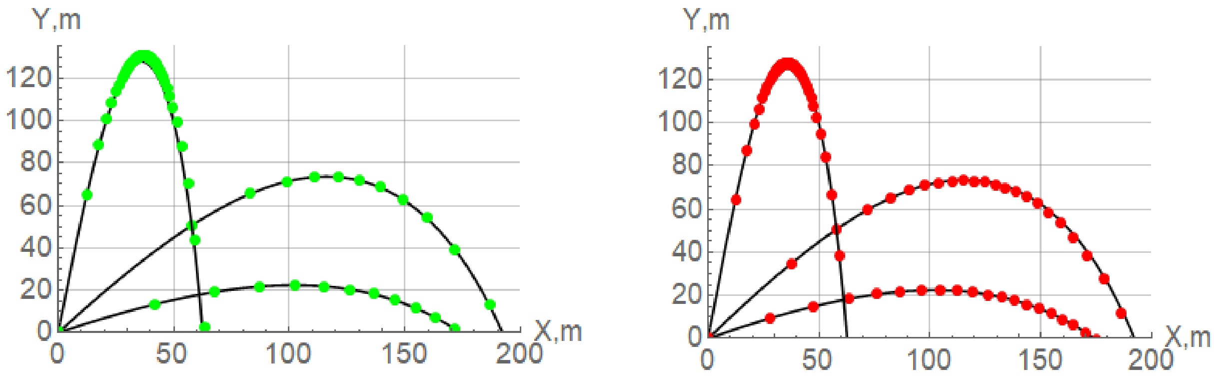

Figure 3. The graphs of the trajectory $y = y(x)$ at launching angles $\theta_0 = 20°, 45°, 80°$.

Figure 4 represents the results of plotting the projectile trajectories with the aid of formulas (5) – (6) and (10) – (11) over a wide range of the change of the initial velocity V_0 with the following values of the parameters

$$\theta_0 = 40°, \quad k = 0.000625 \text{ s}^2/\text{m}^2, \quad g = 9.81 \text{ m/s}^2, \quad p = 1, 4, 9.$$

In this case the values of the parameter p vary from 1 to 9.

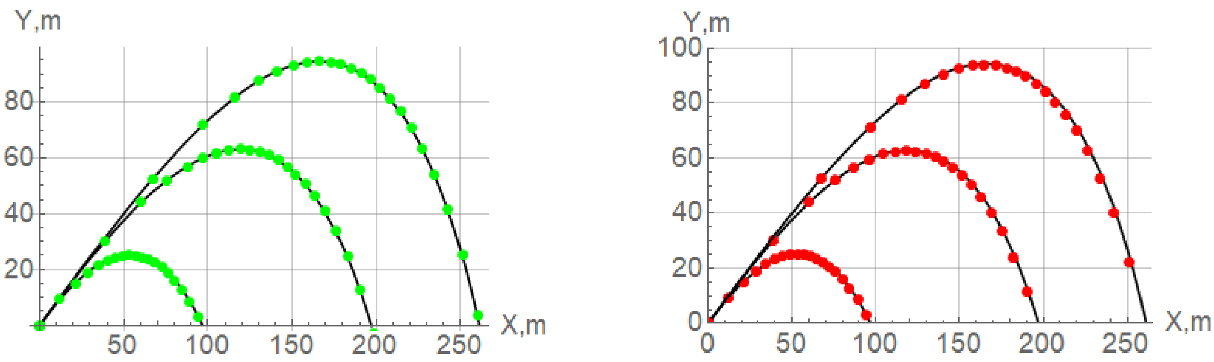

Figure 4. The graphs of the trajectory $y = y(x)$ under the initial conditions $\theta_0 = 40°$, $V_0 = 40$ m/s, 80 m/s, 120 m/s.

Analytical solutions are shown in figure 3, figure 4 by dotted lines. The thick solid lines in figure 3, figure 4 are obtained by numerical integration of system (1) with the aid of the 4-th order Runge-Kutta method. The red dots lines are obtained with using analytical formulas (5) – (6). The green dots lines are obtained with using analytical formulas (10) – (11). As it can be seen from figure 3 and figure 4, the analytical solutions (dotted lines) and numerical solutions are the same. It is interesting that identical trajectories are described with various analytical formulas (5) – (6) and (10) – (11).

As an example of a specific calculation using formulas (5) – (7), we give the trajectory and the values of the basic parameters of the motion $L, H, T, x_a, t_a, \theta_1$ for shuttlecock in badminton. Of all the trajectories of sport projectiles, the trajectory of the shuttlecock has the greatest asymmetry. This is explained by the relatively large value of the drag coefficient k and, accordingly, by the large values of the parameter p. Initial conditions of calculation are

$$k = 0.022 \text{ s}^2/\text{m}^2 ; \quad V_0 = 50 \text{ m/s} ; \ \theta_0 = 40° ; \ p = 55.$$

Table 1. Basic parameters of the shuttlecock movement.

Parameter	Analytical value	Numerical value	Error (%)
L, (m)	11.33	11.34	-0.1
H, (m)	5.09	5.06	0.4
T, (s)	1.97	1.93	2.0
x_a, (m)	7.89	7.84	0.6
t_a, (s)	0.72	0.71	0.4
θ_1	-79.3°	-78°	1.7

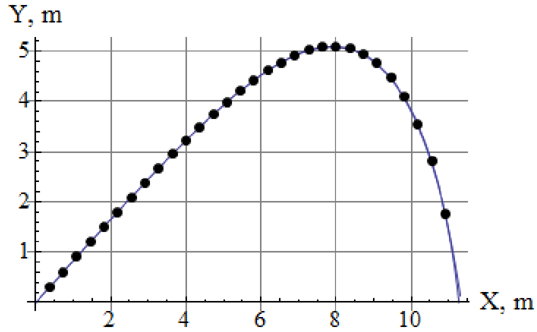

Figure 5. The trajectory of the shuttlecock**.**

The second column of Table 1 contains range values calculated analytically with formulae (8) – (9). The third column of Table 1 contains range values from the integration of the equations of system (1). The fourth column presents the error of the calculation of the parameter in the percentage. The error does not exceed 2 %. Thus, a successful approximation of the function $f(\theta)$ made it possible to calculate the integrals (3) in elementary functions and to obtain a highly accurate analytical solution of the problem of the motion of the projectile in the air.

7. Conclusion

The proposed approach based on the use of analytic formulae makes it possible to simplify significantly a qualitative analysis of the motion of a projectile with the air drag taken into account. All basic variables of the motion are described by analytical formulae containing elementary functions. Moreover, numerical values of the sought variables are determined with high accuracy. It can be implemented even on a standard calculator. Thus, proposed formulas make it possible to study projectile motion with quadratic drag force even for first-year undergraduates.

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