Ballistic Trajectory with Drag

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No Drag

Firing Artillery is an interesting proposition. Hitting the target with as few rounds as possible is very important. Each round fired comes at a cost. Besides the monetary cost, each round draws unwanted attention from the enemy. Therefore, each round needs to be carefully considered. Each target needs to provide enough payoff to outweigh the cost and the calculations for each round needs to be meticulous and timely in order to have as little adjustment as possible and requiring subsequent rounds.

The initial physics behind firing artillery seems pretty straight forward. Initial velocity at some angle: the initial velocity vector V

We analyze the components of that initial vector separately Vx and Vy.

$$V_x = V cos\theta$$
$$V_y = V sin\theta$$

If there were no drag on the round, Vx would remain constant and Vy would be affected only by gravity. For this paper,

$$x_0 = y_0 = t_0 = x_f = y_f = 0$$

$$\frac{\text{dx}}{\text{dt}} = \ V cos\theta, \quad x = V t cos\theta, \quad \frac{\text{dy}}{\text{dt}} = \ V sin\theta - gt, \quad y = V t sin\theta - \frac{1}{2} g t^2$$

So, for a given initial velocity, in order to hit the target, we need x to equal the range (r) to the target at the same time y equals the height of the target (zero in this case).

$$\frac{r}{\text{Vcos}} = t = \frac{2V sin\theta}{g} \rightarrow r = \frac{2V^2 \text{sin cos}}{g} = \frac{V^2 sin2\theta}{g}$$

So, for a target at a given range and a round fired at a set velocity, we know the angle that must be shot.

$$\theta = \frac{1}{2} arcsin\left(\frac{\text{rg}}{V^2}\right)$$

We also know the maximum height (H) of the round happens at half the range and half the time.

$$If\ halftime = \frac{\text{Vsin}}{g}$$

, then

$$H = \frac{V^2 \sin^2\theta}{2g}$$

For example, to hit a target at 15km with an initial velocity of 671 m/s, the initial angle needs to be .1664 radians, 9.54 degrees, or 169.7 mils. Time of flight is 22.668. Max height is 629.87 m at 11.3339 seconds. Impact angle is -169.7 mils. Impact velocity is 671 m/s.

This is all very predictable, but there is a great deal of drag so things change.

Drag

If there is drag on the round, there are many variables that affect the velocity of the round. Velocity in the x direction is no longer constant. I get these formulas from Peter Chudinov from the journal of Physics[1].

The drag constant

$$k = \frac{\rho_a c_d S}{2mg} = \frac{1}{V_t^2}$$

;

a is the air density, *cd* is the drag factor for a sphere, *S* is the cross-section area of the object, and *Vt* is the terminal velocity.

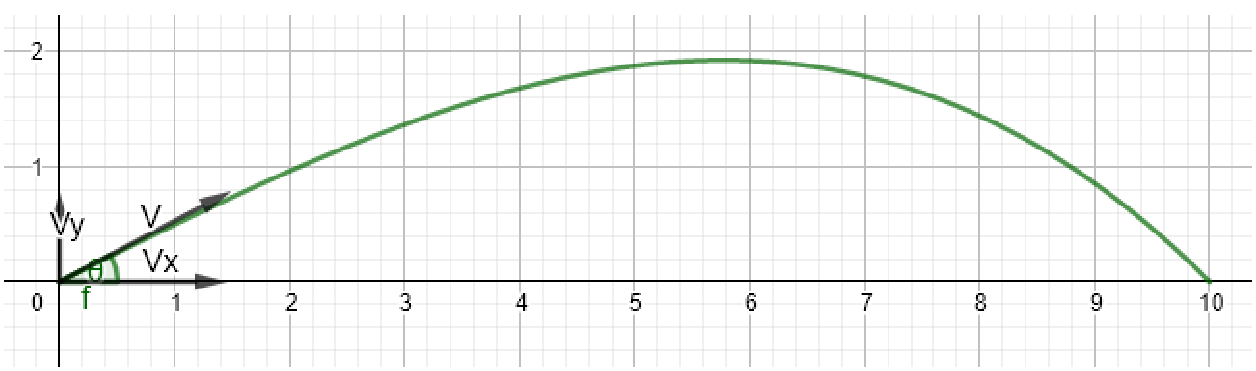
I will treat k as a constant even though changes with altitude which may be significant with artillery shots; in this case k could become a function. All variables are functions of angle. Theta is the angle of trajectory that varies throughout the flight.

Launch angle:

$$\left[0, \frac{\pi}{2}\right]$$

Impact angle:

$$- \quad \left[0, \frac{\pi}{2}\right]$$



[1] https://iopscience.iop.org/article/10.1088/1742-6596/1287/1/012032

Here are the functions I am going to use to predict artillery shots given and initial velocity and launch angle.

These first two equations I can compute directly from angle.

1. $$f(\theta) = \frac{\sin}{\cos^2\theta} + ln\left(\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right)$$

2. $$V(\theta) = \frac{V_0 \cos\theta_0}{\cos\sqrt{1 + kV_0^2\cos^2\theta_0\left(f(\theta_0) - f(\theta)\right)}}$$

The following formulas I cannot compute directly but I can get a numerical solution for each angle.

The velocity in the x and y directions is now based off of instantaneous velocity and trajectory angle; not just initial velocity and launch angle.

1. $$\frac{\mathrm{dx}}{\mathrm{dt}} = Vcos\theta, \frac{\mathrm{dy}}{\mathrm{dt}} = Vsin\theta, \frac{\mathrm{d}}{\mathrm{dt}} = \frac{-gcos\theta}{V}, \frac{\mathrm{dV}}{\mathrm{dt}} = -gsin\theta - gkV^2$$

Now convert all these so they are in terms of angle:

1. $$\frac{\mathrm{dV}}{\mathrm{d}} = Vtan\theta + \frac{kV^3}{\cos}, \quad \frac{\mathrm{dx}}{\mathrm{d}} = \frac{V^2}{-g}, \quad \frac{\mathrm{dy}}{\mathrm{d}} = \frac{V^2}{-g}tan\theta, \quad \frac{\mathrm{dt}}{\mathrm{d}} = \frac{V}{-gcos\theta}$$

2. $$x = \int_{\theta_0}^{\theta}\frac{V^2}{-g}\mathrm{d}\,, \quad y = \int_{\theta_0}^{\theta}\frac{V^2}{-g}\tan\ \mathrm{d}\,, \quad t = \int_{\theta_0}^{\theta}\frac{V}{-gcos\theta}\mathrm{d}$$

Once I have and instantaneous altitude, I can alter k for appropriate air pressure.

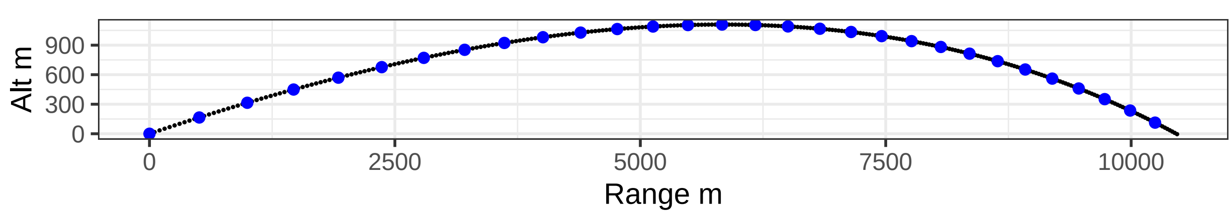
Attempt at prediction

Actual shot data that I am trying to match: Initial velocity 547 m/s at an initial angle of (330.2 mils). My prediction needs to hit a level target at 10000 m in 28.8 seconds while reaching a maximum altitude of 1070 m. Also, impact angle of (-483 mils) with and impact velocity of 298 m/s.

```
v0 <- 547 # initial velocity in m/s for M795 with M232A1 4H
am0 <- 330.2 # QE in mils for a level 15000 m shot

th0 <- am0 * pi / 3200 # initial angle in radians
x0 <- 0 #Initial x
y0 <- 0 # initial y
t0 <- 0 # initial time
g <- 9.80665 # gravitational force in m/s/s
press <- data.frame(cbind(
  alt = c(0,200,500,1000,1500,2000,2500,3000,3500,
          4000,4500,5000,6000,7000,8000,9000),
  rho = c(1.2250,1.2133,1.1844,1.1392,1.0846,1.0320,
          .9569,.8632,.7768,.6971,.5895,.4664,.3612,.2655,.1937,.1413)))
c.press <- glm(rho~poly(alt,6,raw=TRUE), data = press)
pk <- .000006 * 46.94681
m <- 46.94681 #mass in kg
rho <- as.numeric(predict(c.press, data.frame(alt = y0), type = "response"))
k <- rho*pk/m # is the drag constant at 0 alt
#k <- .0000075 # Drag
```

```r
traj <- data.frame(matrix(ncol = 7))
colnames(traj)=c("Time s","k","Vel m/s","Angle r","Angle mils","Range m","Alt m")
t <- t0
v <- v0
th <- th0
x <- x0
y <- y0
firstrow <- c(t,k,v,th,th*3200/pi,x,y)
traj[1,] <- firstrow
dt <- .1
while (y > -1){
t <- t + dt
rho <- as.numeric(predict(c.press, data.frame(alt = y), type = "response"))
k <- rho*pk/m # is the drag constant at y alt
v <- v - (sin(th) + k*v^2) * g * dt
th <- th - (g*cos(th)/v)*dt
x <- x + v*dt*cos(th)
y <- y + v*dt*sin(th)
nextrow <- c(t,k,v,th,th*3200/pi,x,y)
traj <- rbind(traj,nextrow)
}
trajw <- traj[seq(1, nrow(traj), 10), ]
traj %>% ggplot(aes(`Range m`,`Alt m`))+geom_point(size=.1) +
  geom_point(data=trajw,aes(`Range m`,`Alt m`),color="blue") +
    coord_fixed(ratio = 1)
```



```r
pander(trajw)
```

	Time s	k	Vel m/s	Angle r	Angle mils	Range m	Alt m
1	0	7.383e-06	547	0.3242	330.2	0	0
11	1	7.281e-06	523.3	0.3067	312.4	507.5	165.2
21	2	7.191e-06	501.7	0.2884	293.7	996.4	314.7
31	3	7.117e-06	482	0.2692	274.2	1468	449.4
41	4	7.054e-06	463.9	0.2491	253.7	1925	569.9
51	5	6.999e-06	447.2	0.2281	232.4	2366	676.9
61	6	6.952e-06	431.8	0.2063	210.1	2795	771
71	7	6.911e-06	417.6	0.1836	187	3211	852.7
81	8	6.875e-06	404.5	0.1601	163	3615	922.4
91	9	6.845e-06	392.3	0.1357	138.2	4008	980.6
101	10	6.82e-06	381.1	0.1105	112.5	4392	1028
111	11	6.8e-06	370.6	0.08446	86.03	4765	1064
121	12	6.786e-06	361	0.05768	58.75	5130	1089
131	13	6.778e-06	352.1	0.03016	30.72	5485	1104
141	14	6.774e-06	343.8	0.001943	1.979	5833	1109
151	15	6.776e-06	336.3	-0.02693	-27.43	6172	1105
161	16	6.783e-06	329.3	-0.05641	-57.46	6504	1090
171	17	6.796e-06	322.9	-0.08645	-88.06	6829	1067
181	18	6.812e-06	317	-0.117	-119.2	7147	1034
191	19	6.834e-06	311.7	-0.1479	-150.7	7458	991.9
201	20	6.859e-06	306.8	-0.1793	-182.6	7763	941.1
211	21	6.889e-06	302.4	-0.2109	-214.8	8061	881.7
221	22	6.923e-06	298.5	-0.2427	-247.2	8354	813.7
231	23	6.96e-06	295	-0.2747	-279.8	8640	737.4
241	24	7.002e-06	291.9	-0.3068	-312.5	8921	652.9
251	25	7.048e-06	289.2	-0.3388	-345.1	9196	560.4
261	26	7.099e-06	286.8	-0.3708	-377.7	9466	460
271	27	7.156e-06	284.8	-0.4026	-410.1	9730	351.8
281	28	7.221e-06	283.1	-0.4342	-442.2	9990	236.1
291	29	7.297e-06	281.6	-0.4655	-474.1	10244	113