FYS3150 Computational Physics – Project 4

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Abstract

This project simulated phase transitions in magnetic systems using the Ising model and the Metropolis algorithm. It was observed that the system reached an equilibrium after 10^6 Monte Carlo cycles and showed a phase transition at a temperature of $T \approx 2.27$ where it went from begin a system with a finite amount of magnetic moment to a system with zero magnetic moment.

GitHub repository with code can be found at: github.com/harmoh/FYS3150_Project_4

1 Introduction

The objective of this project is to study phase transitions in magnetic systems. This is performed using the Ising model in two dimensions. At a certain temperature these systems goes from having a finite amount of magnetic moment to a phase with zero magnetization. The basis of the Ising model is shown for the energy in this simple equation:

$$E = -J \sum_{<kl>}^{N} s_k s_l \tag{1}$$

where $s_k = \pm 1$ and $s_l = \pm 1$. They represent the spin direction of elements in a lattice. The energy is only affected by the change of nearby spins, indicated with $< kl >$. N represents the number of spins in the lattice and J is a constant expressing the strength of interaction between neighbouring spins. Periodic boundary condition will be used in order to get a better approximation of the numerical calculations.

In this project, we will first go through a simple case with a 2×2 lattice in order to compare analytical and numerical results. Then, a 20×20 lattice will be tested to see how many Monte Carlo cycles it takes before the system reaches an equilibrium. Both temperatures $T = 1.0$ and $T = 2.4$ will be investigated. The probability distribution between the two temperatures will be analyzed. Phase transitions over temperature for different lattice sizes is studied before finally extracting the critical temperature for phase transition.

2 Method

2.1 Ising model

In order to find the analytical results, expressions for expectation values have to be found. First we set up the expression for the energy based on eq. (1). The expectation value of the energy can be written as

$$\langle E \rangle = \frac{1}{Z} \sum_{i=1}^{N} E_i e^{-\beta E_i} \tag{2}$$

where N is the total number of microstates,

$$E_i = -J \sum_{<kl>} S_k S_l \tag{3}$$

and the partition function Z becomes

$$Z = \sum_{i=1}^{N} e^{-\beta E_i} \tag{4}$$

where β becomes a function of time defined as:

$$\beta = \frac{1}{k_B T} \tag{5}$$

We also have that:

$$\langle E^2 \rangle = \frac{1}{Z} \sum_{i=1}^{N} E_i^2 e^{-\beta E_i} \tag{6}$$

Such that we can find the variance in energy:

$$\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2 \tag{7}$$

We can then find the heat capacity C_V which is simply

$$C_V = \frac{\sigma_E^2}{k_B T^2} = \frac{\langle E^2 \rangle - \langle E \rangle^2}{k_B T^2} \tag{8}$$

where k_B is the Boltzmann constant, T is the temperature. Expectation values for the magnetic moment can be found from the same basis:

$$\langle M \rangle = \frac{1}{Z} \sum_{i=1}^{N} M_i e^{-\beta E_i} \tag{9}$$

where

$$M_i = \sum_{i=1}^{N} S_i \tag{10}$$

and the partition function Z and the β function remains the same as above. We have also that:

$$\langle M^2 \rangle = \frac{1}{Z} \sum_{i=1}^{N} M_i^2 e^{-\beta E_i} \tag{11}$$

Such that we can find the variance in magnetic moment:

$$\sigma_M^2 = \langle M^2 \rangle - \langle M \rangle^2 \tag{12}$$

We can then find the susceptibility χ which becomes:

$$\chi = \frac{\langle M^2 \rangle - \langle M \rangle^2}{k_B T} = \frac{\langle M^2 \rangle - \langle M \rangle^2}{k_B T} \tag{13}$$

The number of microstates is given by the lattice size L for two dimensions:

$$N = 2^{L \times L} \tag{14}$$

When solving for a 2×2 lattice, it can be expressed analytically by setting up the energy as:

$$E = -J(S_1 S_2 + S_2 S_1 + S_1 S_3 + S_3 S_1 + S_2 S_3 + S_3 S_2 + S_2 S_4 + S_4 S_2) \tag{15}$$

when using periodic boundary conditions. This means (as seen in eq. (15)) that the spins with each neighbour is added twice, which gives an accurate result of the energy. A lattice with the size $L = 2$ results in 16 microstates as $N = 2^{2 \times 2} = 16$. Spin up equals $S_i = +1$ and spin down equals $S_i = -1$. Table 1 shows the energy and the magnetic moment of the microstates of a 2×2 lattice.

Table 1: The energy and magnetic moment of all the microstates of a 2×2 lattice

Spins up	Energy	Magnetic moment	# Microstates
4	-8J	4	1
3	0	2	4
2	8J	0	2
2	0	0	4
1	0	-2	4
0	-8J	-4	1

With this, we can set up the partition function for the chosen system:

$$Z = \sum_{i=1}^{16} e^{-\beta E_i} = 2e^{-(-8J)\beta} + 2e^{-8J\beta} + 12e^{0\beta}$$

$$Z = 2e^{8J\beta} + 2e^{-8J\beta} + 12 = 2(e^{8J\beta} + e^{-8J\beta}) + 12$$

$$Z = 4\cosh(8J\beta) + 12 \tag{16}$$

The expectation value for the energy becomes:

$$\langle E \rangle = \frac{1}{Z} \sum_{i=1}^{16} E_i e^{-\beta E_i} = \frac{1}{Z} 16J e^{-8J\beta} - 16J e^{8J\beta} = \frac{1}{Z} 16J(e^{-8J\beta} - e^{8J\beta})$$

$$\langle E \rangle = \frac{32J \sinh(8J\beta)}{Z} \tag{17}$$

The expectation value for the magnetic moment becomes:

$$\langle M \rangle = \frac{1}{Z} \sum_{i=1}^{16} M_i e^{-\beta E_i} = 0 \tag{18}$$

But when using the absolute values of the magnetic moment (which we will use from now on), the expectation value becomes:

$$\langle |M| \rangle = \frac{1}{Z} \sum_{i=1}^{16} |M_i| e^{-\beta E_i} = \frac{8e^{8J\beta} + 16}{Z} \tag{19}$$

The specific heat capacity is:

$$C_V = \frac{\langle E^2 \rangle - \langle E \rangle^2}{k_B T^2} = \frac{\frac{1}{Z}\sum_{i=1}^{16} E_i^2 e^{-\beta E_i} - \langle E \rangle^2}{k_B T^2}$$

$$C_V = \frac{\frac{1}{Z}(128Je^{-8J\beta} + 128Je^{8J\beta}) - \langle E \rangle^2}{k_B T^2} = \frac{\frac{1}{Z}(128J(e^{-8J\beta} + e^{8J\beta})) - \langle E \rangle^2}{k_B T^2}$$

$$C_V = \frac{\frac{1}{Z}(256J\cosh(8J\beta)) - (\frac{1}{Z}(32J\sinh(8J\beta)))^2}{k_B T^2} \tag{20}$$

Finally, the susceptibility becomes:

$$\chi = \frac{\langle M^2 \rangle - \langle |M| \rangle^2}{k_B T} = \frac{\frac{1}{Z}\sum_{i=1}^{16} M_i^2 e^{-\beta E_i} - (\frac{1}{Z}\sum_{i=1}^{16} M_i e^{-\beta E_i})^2}{k_B T}$$

$$\chi = \frac{\frac{1}{Z}(32e^{8J\beta} + 32) - (\frac{1}{Z}(8e^{8J\beta} + 16))^2}{k_B T} \tag{21}$$

2.2 Monte Carlo method

In this project, we will use the Monte Carlo method in order to simulate the system. The specific Monte Carlo method that will be used is called the Metropolis algorithm, which is a popular algorithm when it comes to solving the Ising model numerically. This algorithm is based on that a system can go into different states, and will, with a certain probability, move between these states.

It starts with a microstate of the system and calculate the energy and the magnetic moment of the system. Then a random spin is chosen and flipped. The change in energy and magnetic moment is calculated. If the energy difference leads the system into a state with lower energy ($\Delta E < 0$) or a random number is within a certain range ($r \leq e^{-\beta \Delta E}$), the flip is accepted. It the conditions are not fulfilled, the flip is not accepted and the microstate from before the flip is still the current one. Finally the energy and magnetic moment is added to the total sum. This process will repeat for each Monte Carlo cycle. A large number of Monte Carlo cycles will lead the system into equilibrium.

2.3 Phase transitions

This project will study phase transitions over different temperatures. The Ising model is used to study the system phase transition from a system with a finite amount of magnetic moment at one temperature to a system with zero magnetic moment at another temperature. The critical temperature is where the phase transition happens, and the exact solution after Lars Onsager is found to be:

$$\frac{kT_C}{J} = \frac{1}{\ln(1+\sqrt{2})} \approx 2.269 \tag{22}$$

The critical temperature can also be found as:

$$T_C(L) - T_C(L = \infty) = aL^{-1/v} \tag{23}$$

which we can rewrite to

$$T_C(L_i) - T_C(L_j) = a(L_i^{-1/v} - L_j^{-1/v}) \tag{24}$$

where

$$a = \frac{T_C(L_i) - T_C(L_j)}{L_i^{-1/v} - L_j^{-1/v}} \tag{25}$$

3 Implementation

In this project, the main program `main.cpp` is written in C++ and performs all the calculations and writes the result to a text file. Python is used to run `main.cpp`, read the computed files and plot the corresponding data. Parallelization of `main.cpp` was performed using the library OpenMPI. Several Python scripts were made in order to run different systems and plot different values.

4 Result

4.1 Compare analytical and numerical

For a simple 2×2 lattice, analytical expressions can be found by studying the energy and the magnetic moment is the different microstates, as seen in section 2. When choosing a temperature $T = 1.0$ (in the units kT/J), we can find the analytical results. After normalizing by diving by the total number of spins, we the the expectation value for the heat capacity:

$$C_V = 0.0320823 \tag{26}$$

and the susceptibility:

$$\chi = 0.00401074 \tag{27}$$

Table 2 shows the numerical expectations values and relative error compared with the analytical values over different number of Monte Carlo cycles. Error values are taken from the average of

Table 2: Numerical expectation values, relative error and computation time for a 2×2 lattice with up to 10^{10} Monte Carlo cycles

MC cycles	Error (C_V)	Error (χ)	Computation time
1e03	9.12667e-01	1.01896e-00	0.000626 seconds
1e04	3.03758e-01	3.46814e-01	0.003065 seconds
1e05	1.02483e-01	1.07535e-01	0.026724 seconds
1e06	3.24851e-02	3.71210e-02	0.244516 seconds
1e07	7.75695e-03	1.19753e-02	2.502890 seconds
1e08	3.12559e-03	3.43325e-03	23.965100 seconds
1e09	1.28677e-03	1.86825e-03	241.072000 seconds
1e10	6.98657e-05	1.54865e-04	2419.430000 seconds

up to 1000 calculations depending on the calculation time. In order to achieve a good agreement between the numerical and the analytical results, the number of Monte Carlo cycles needs to be at least 10^8, where of course the higher, the better. When using fewer Monte Carlo cycles, there is a chance of getting a larger error than shown in table 2 due to uncertainty of random number used for determining whether a spin is turned or not.

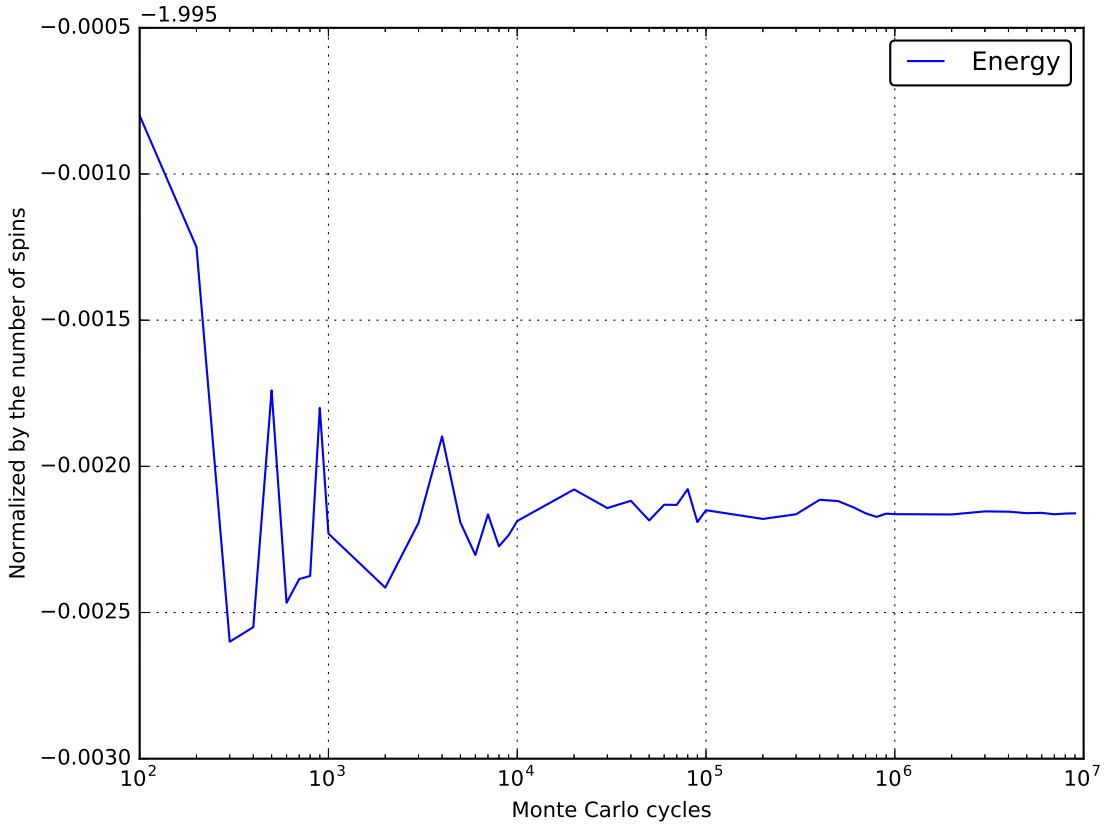

Figure 1: Energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 1.0$ when the initial spin direction is up

4.2 Equilibrium

We can now choose a lattice with the dimensions 20×20. This part studies the period of time it takes before the system reaches an equilibrium, where the number of Monte Carlo cycles is used as a measure of time. The expectation values of the energy and the absolute value of the magnetic moment is plotted to study the evolution over a number of Monte Carlo cycles. Figure 1 shows the energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 1.0$ when the initial spin direction is up. The change in energy decreases over time (as it should) and shows good improvement at every increase of magnitude of Monte Carlo cycles. The expectation value of the energy does not reach an equilibrium until it passes 10^6 Monte Carlo cycles.

Figure 2 shows the energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 1.0$ when the initial spin direction is random. The equilibrium of the system is much clearer when the initial spin direction is random due to the high initial value of the expectation value of the energy. On the other hand, the large variance makes the plotting much less precise than with ordered spin.

Figure 3 shows the absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for temperature $T = 1.0$ when the initial spin direction is up. The expectation value for the magnetic moment correlates with the energy and does not reach equilibrium until it passes 10^6 Monte Carlo cycles. Figure 4 shows the

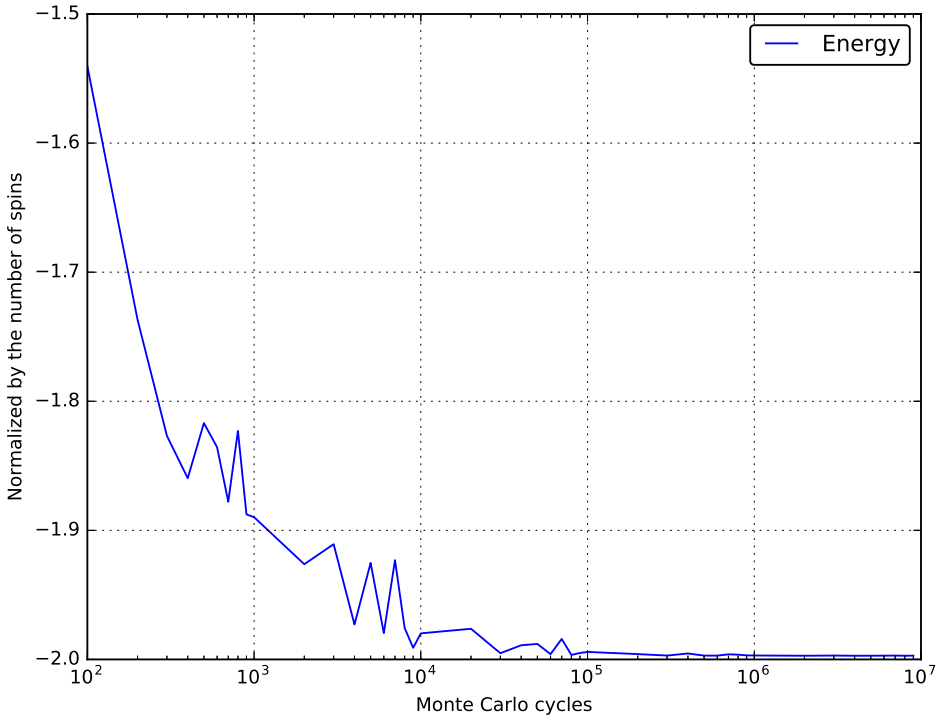

Figure 2: Energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 1.0$ when the initial spin direction is random

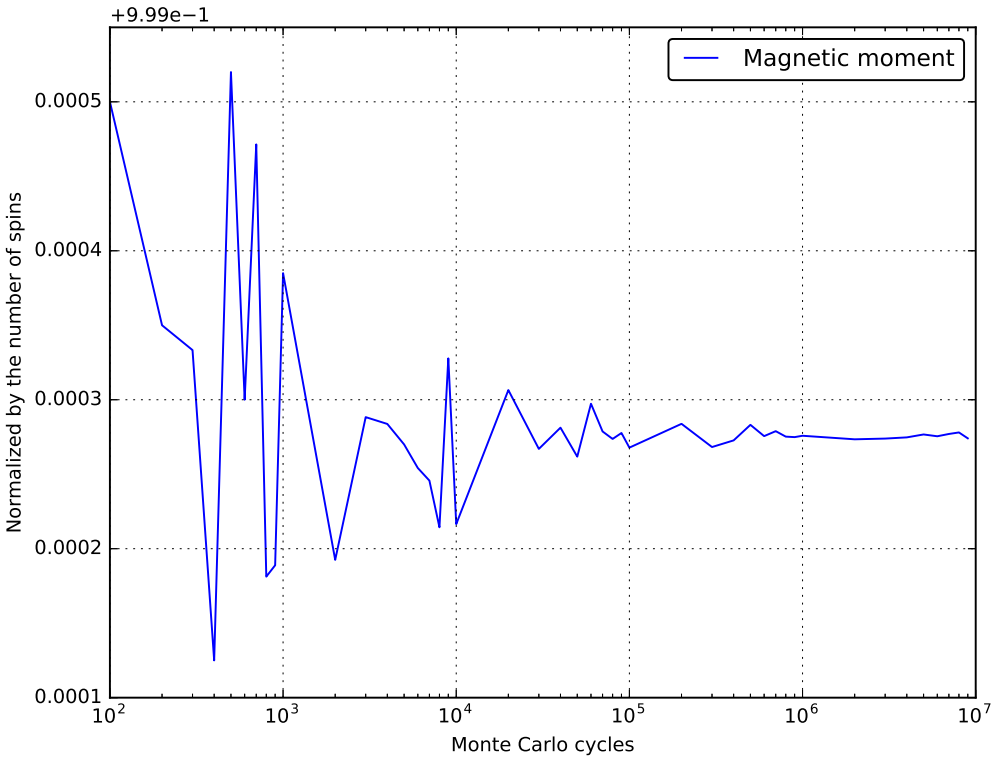

Figure 3: Absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 1.0$ when the initial spin direction is up

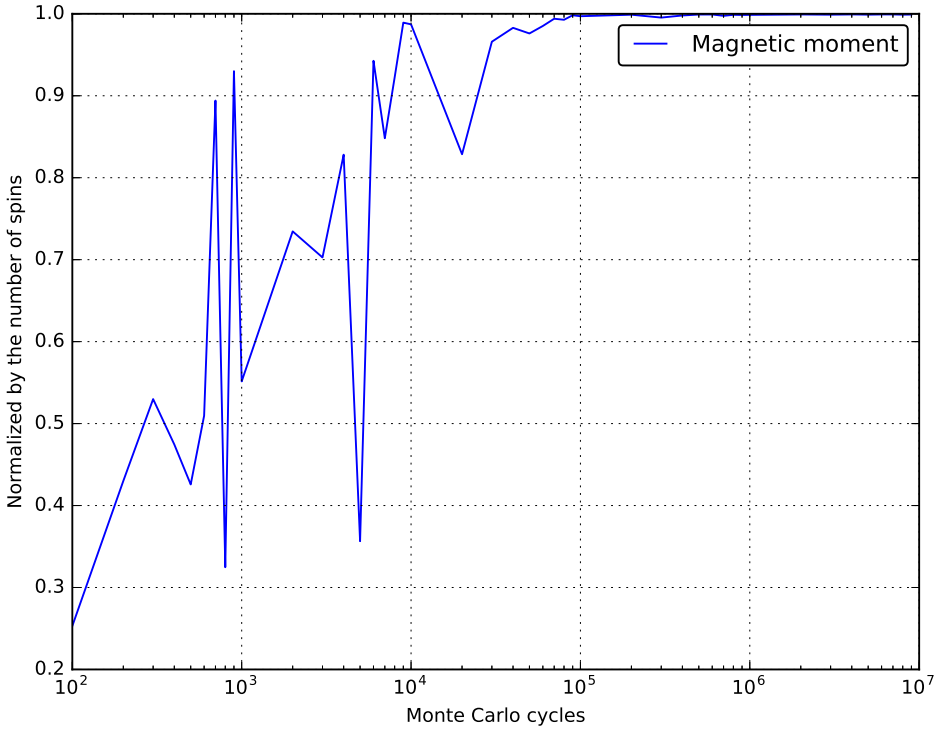

Figure 4: Absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 1.0$ when the initial spin direction is random

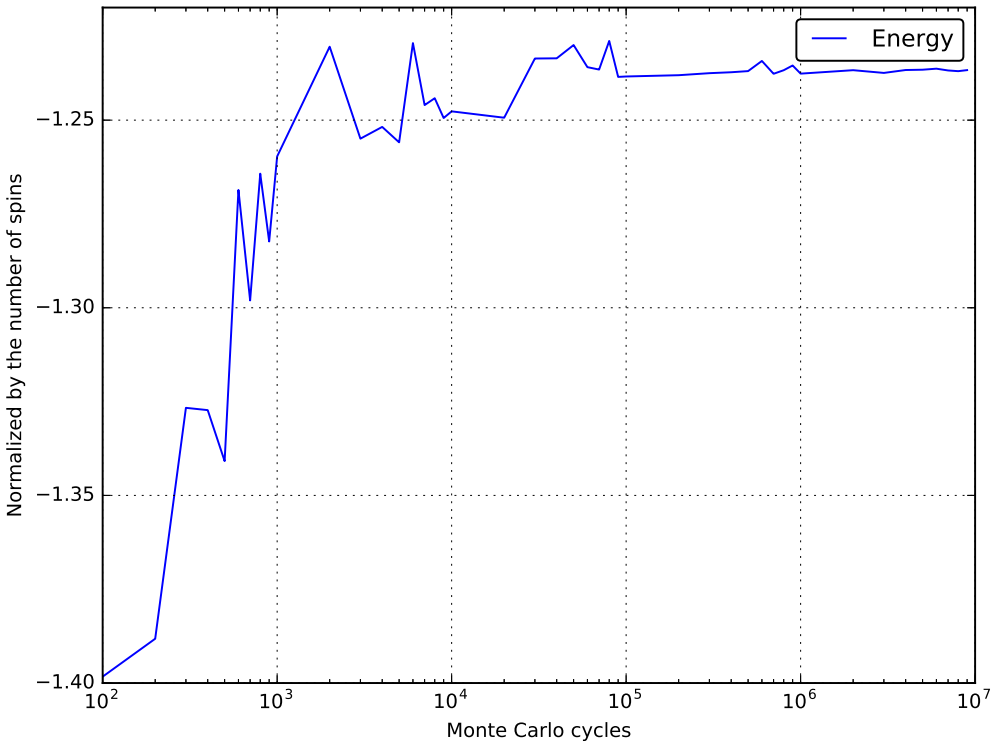

Figure 5: Energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is up

absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for temperature $T = 1.0$ when the initial spin direction is random. This figures shows the same pattern as with the energy when the initial spin direction is random.

When plotting the expectation values for the energy and the magnetic moment with a temperature $T = 2.4$, the result is noticeably different. Figure 5 shows the energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is up. Now the expectation value of the energy ends up somewhere between -1.2 and -1.25 instead of around -2. Figure 6 shows the energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is random. This starts high and stabilizes around -1.24. Figure 7 shows the absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is up. This stabilizes at around 0.45 instead of around 1. The increased temperature also shows a larger variation throughout the plot which, again, makes it less precise and more difficult to establish when it has reached an equilibrium.

Figure 8 shows the absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is random. The same pattern as with $T = 1.0$ for random spins is visible here. Figure 9 shows

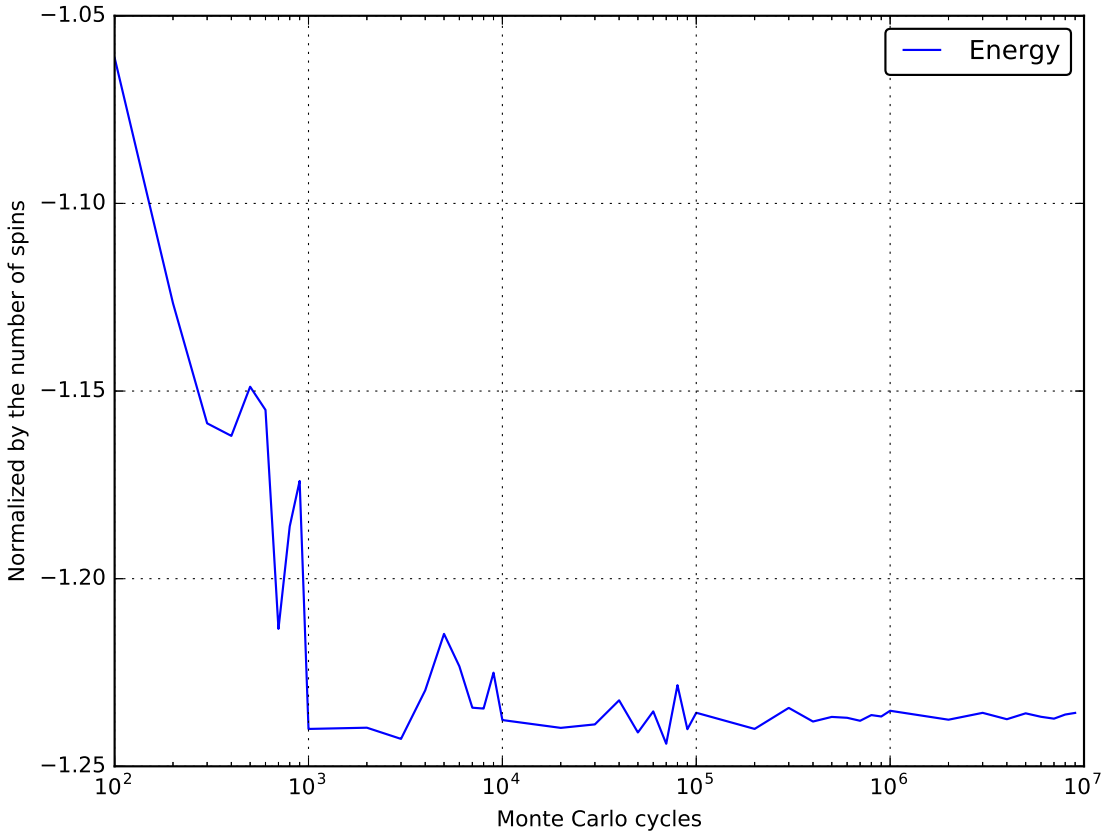

Figure 6: Energy as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is random

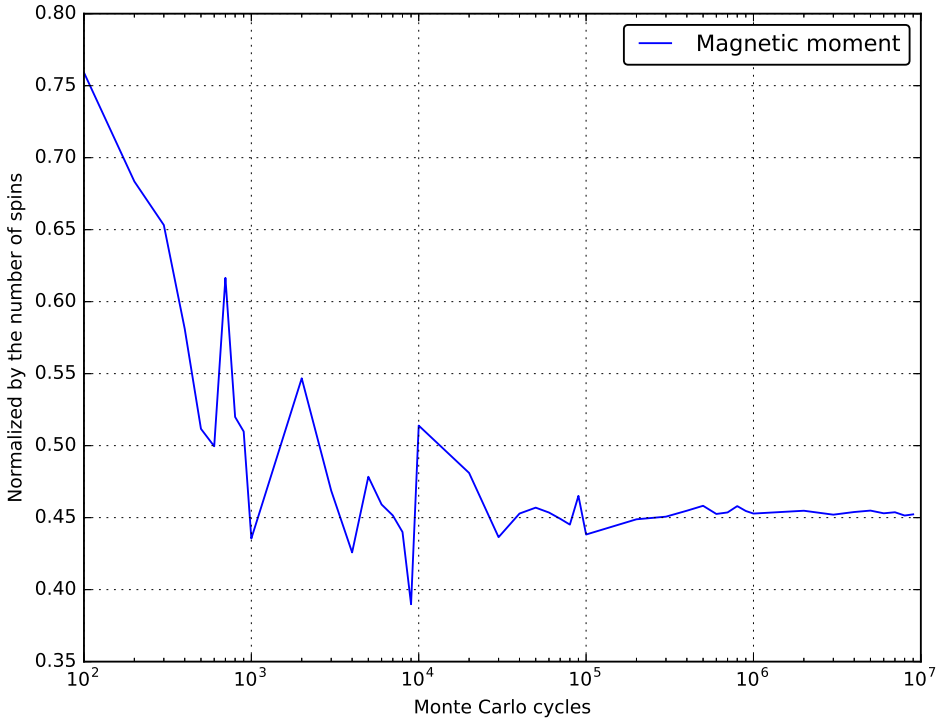

Figure 7: Absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is up

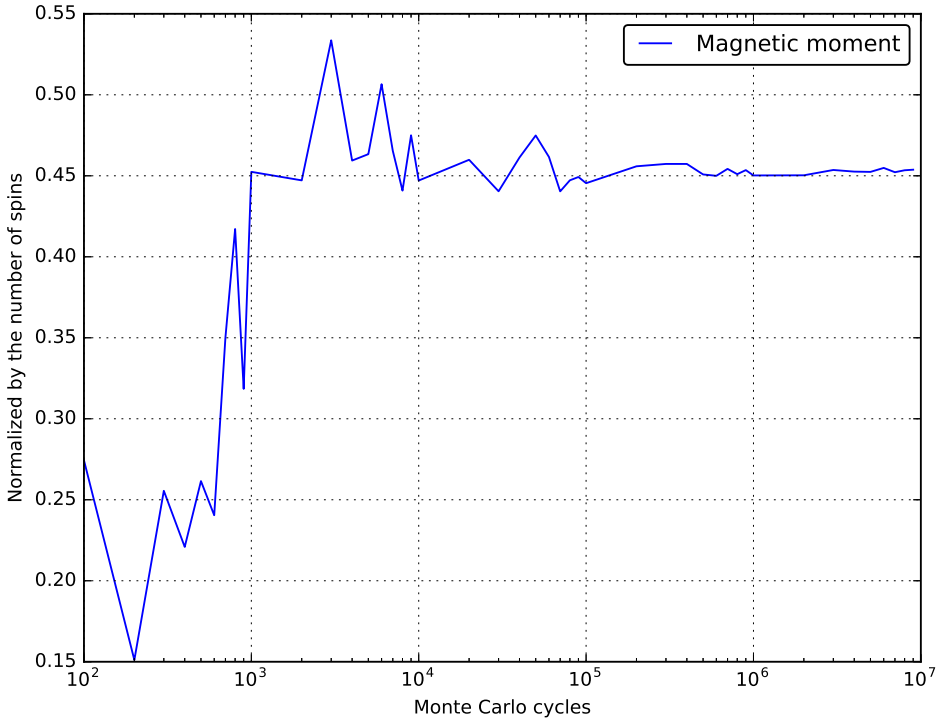

Figure 8: Absolute values of the magnetic moment as a function of Monte Carlo cycles for a 20×20 lattice with up to $9 \cdot 10^6$ Monte Carlo cycles for $T = 2.4$ when the initial spin direction is random

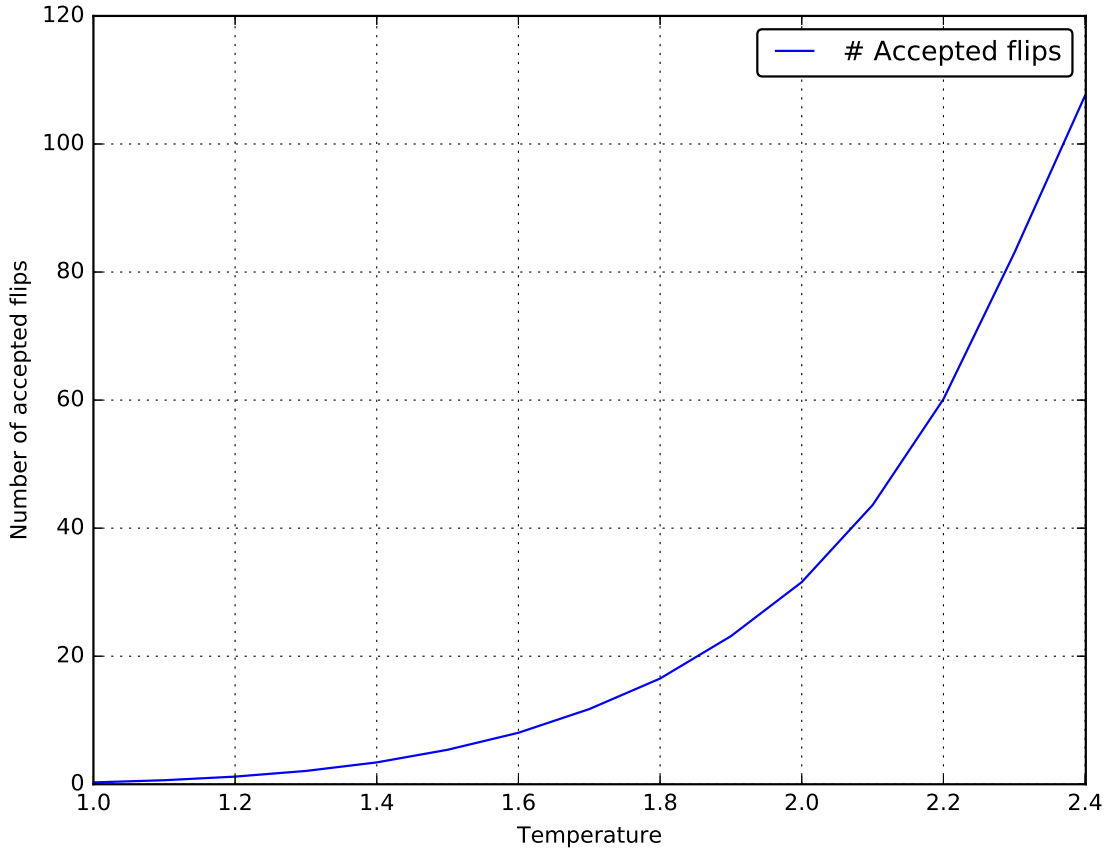

Figure 9: The number of accepted flips as a function of temperature normalized by the number of Monte Carlo cycles of a 20×20 lattice with 10^5 Monte Carlo cycles when initial spin direction is up

the number of accepted flips as a function of temperature of a 20×20 lattice with 10^5 Monte Carlo cycles when initial spin direction is up. The number of flips is normalized by the number of Monte Carlo cycles. The number of accepted configurations (flips) increase exponentially as the temperature increase.

4.3 Probability distribution

In the Metropolis algorithm, random walks are used in order to achieve a more natural simulation of the problem. The random number generator used in this project follows a normal distribution. The objective of this section is to show that the system indeed shows a normal distribution. When finding the probability, each value of the energy is simply counted and shown using a histogram. The calculation of the probability starts after 10^6 Monte Carlo cycles and runs until a total of 10^7 Monte Carlo cycles.

Figure 10 shows the probability distribution of the energy calculated between 10^6 and 10^7 Monte Carlo cycles of a 20×20 lattice for the temperature $T = 1.0$. There is a slight sign of the normal distribution, but since there is no energies registered lower than -800, it looks like the distribution is cut off. Figure 11 shows the probability distribution of the energy calculated between 10^6 and 10^7 Monte Carlo cycles of a 20×20 lattice for the temperature $T = 2.4$. This figure shows a clearer

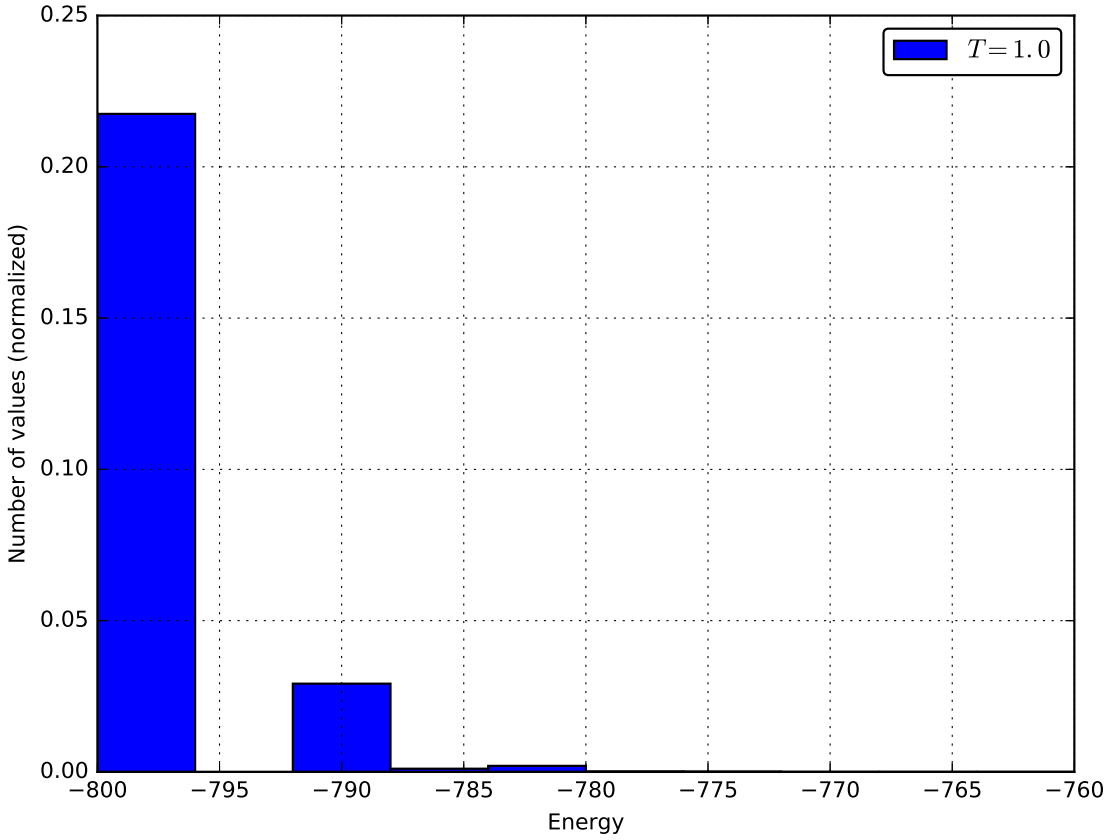

Figure 10: Probability distribution of the energy calculated between 10^6 and 10^7 Monte Carlo cycles of a 20×20 lattice for the temperature $T = 1.0$

normal distribution with the center at -500. The variance for the for the energy was calculated for both temperatures and found to be:

$$\sigma_E^2 = 9.34081 \quad \text{for} \quad T = 1.0 \tag{28}$$

and

$$\sigma_E^2 = 3237.96 \quad \text{for} \quad T = 2.4 \tag{29}$$

It can be seen in the figures that there are fewer energies at $T = 1.0$ than at $T = 2.4$. This seems to be the reason for the large difference in variance between the two temperatures.

4.4 Phase transitions

A part of this project is to study the phase transition in it self. Here we will plot $\langle E \rangle$, $\langle |M| \rangle$, C_V and χ as functions of the temperature, where $T \in [2.18, 2.32]$ and with a temperature step of $\Delta T = 0.01$. The number of Monte Carlo cycles will set to 10^6.

Figure 12 shows the energy as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles. The energies for the different lattice sizes have approximately the same value until the temperature reaches around $T = 2.26$, where they start to slightly differ. The largest lattice shows the highest energy. Figure 13 shows the absolute value of the magnetic moment as a function of temperature for the lattice sizes $L = 40$,

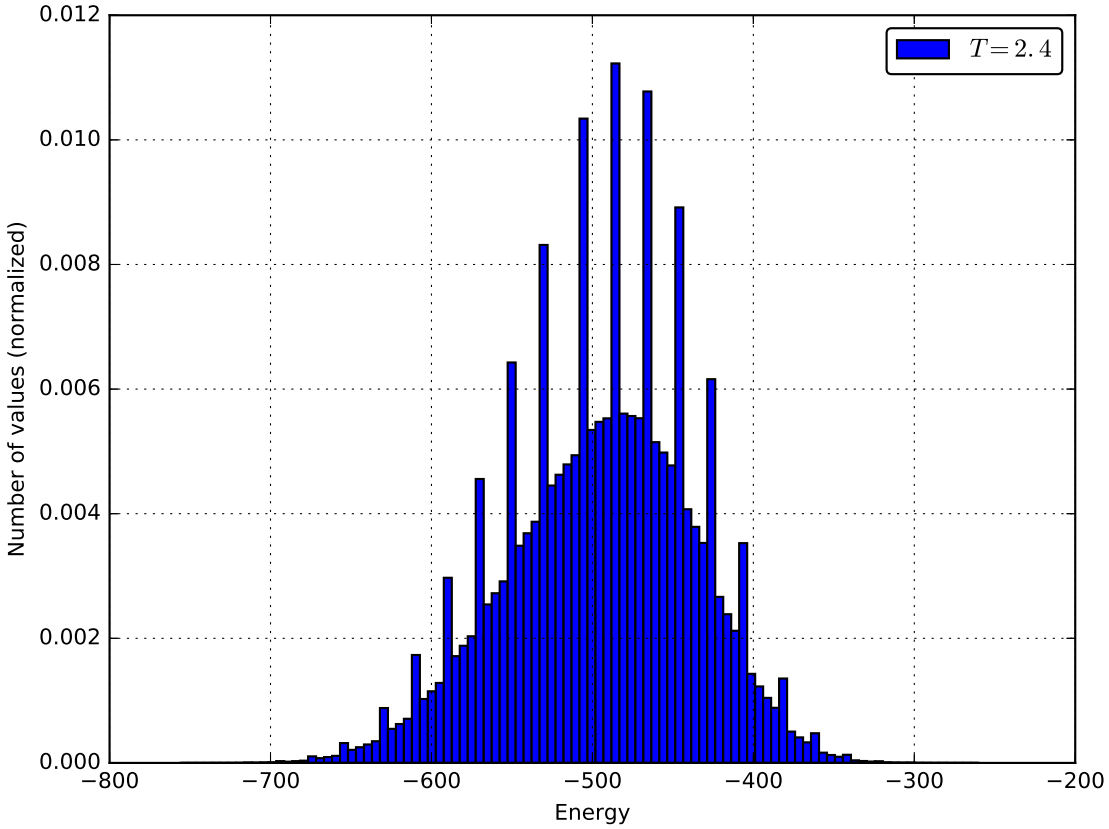

Figure 11: Probability distribution of the energy calculated between 10^6 and 10^7 Monte Carlo cycles of a 20×20 lattice for the temperature $T = 2.4$

$L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles. The absolute value of the magnetic moment decrease with increasing temperature where the phase transition starts somewhere after $T = 2.24$. The larger the lattice, the lower the absolute value of the magnetic moment. This is easily explained by the fact that larger sizes gives a more accurate representation of a natural system, an will therefore give a more correct value than the smaller lattices.

Figure 14 shows the heat capacity as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles. Here, the critical temperature is visible where the heat capacity diverges. This happens around $T = 2.28$. Again, the largest lattice shows the most accurate values with the highest peak. Figure 15 shows the susceptibility as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles. This shows the same pattern as the heat capacity regarding the critical temperature.

When studying the critical temperature manually by investigating the plots for the heat capacity and the susceptibility, a trend is appearing where a larger lattice size give a slightly lower critical temperature. For the heat capacity, at $L = 40$, the critical temperature lies somewhere between $T = 2.28$ and $T = 2.3$. For $L = 60$ and $L = 100$, the critical temperature is approximately $T = 2.28$ and for $L = 140$, the critical temperature is between $T = 2.27$ and $T = 2.28$. We can set the following:

$$T_C(L = 40) = 2.29$$

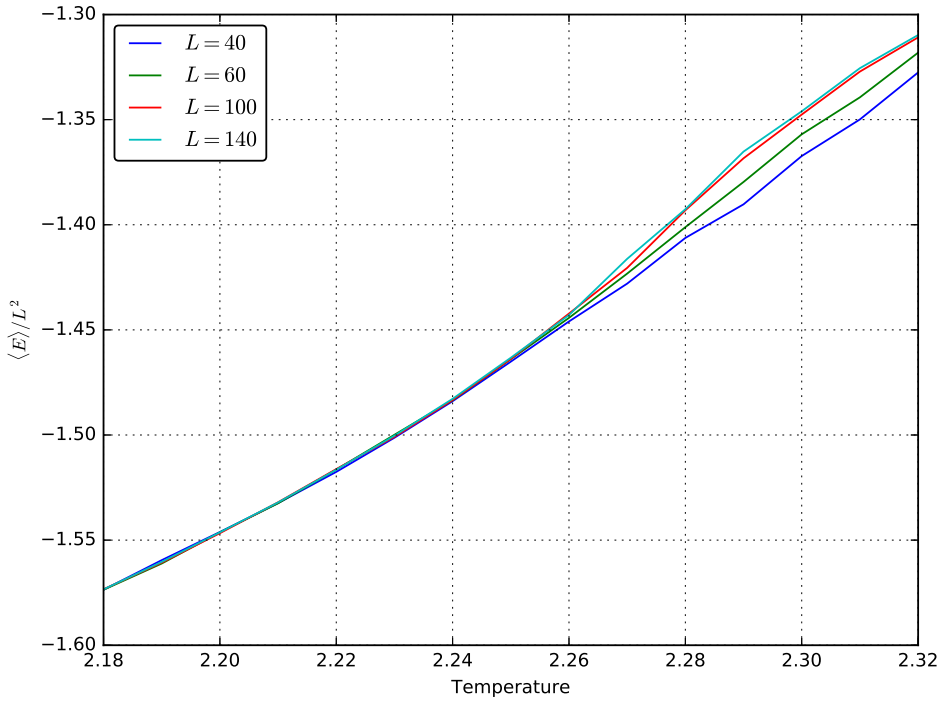

Figure 12: Energy as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles

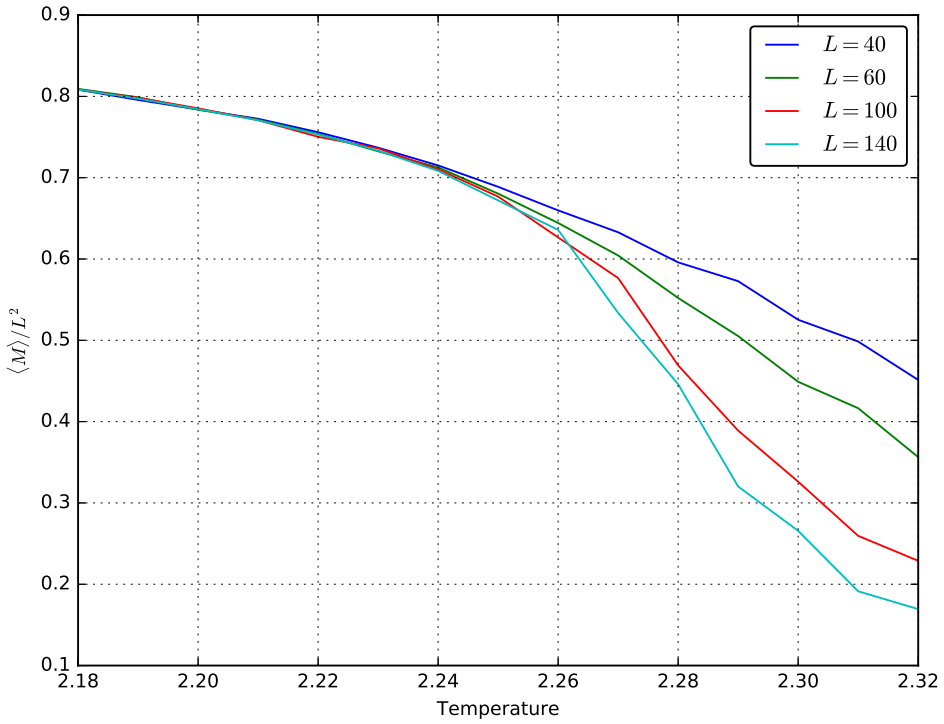

Figure 13: Absolute value of the magnetic moment as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles

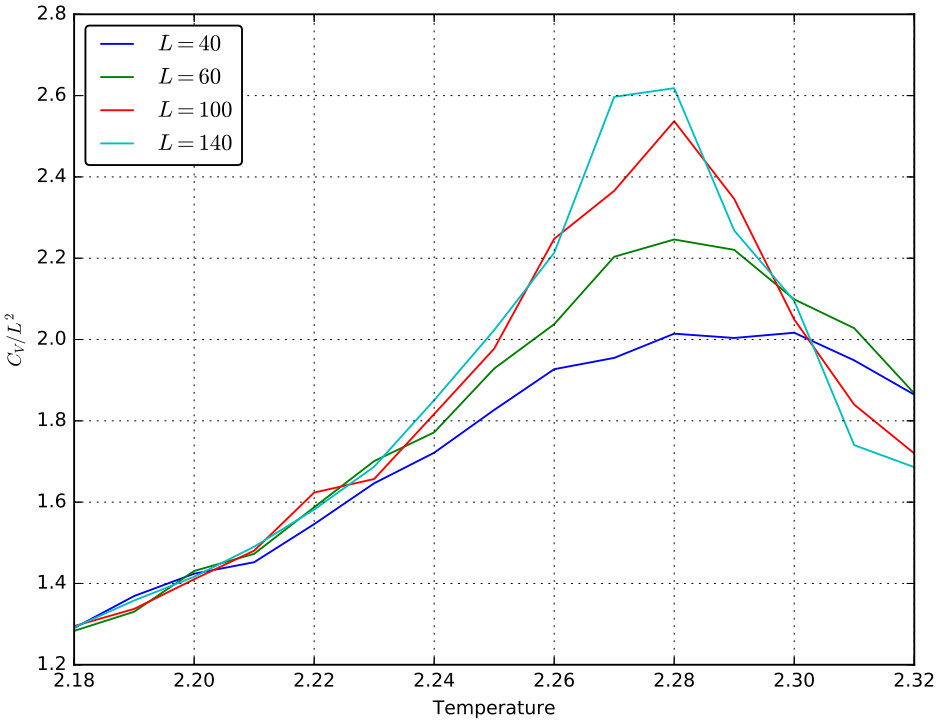

Figure 14: Heat capacity as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles

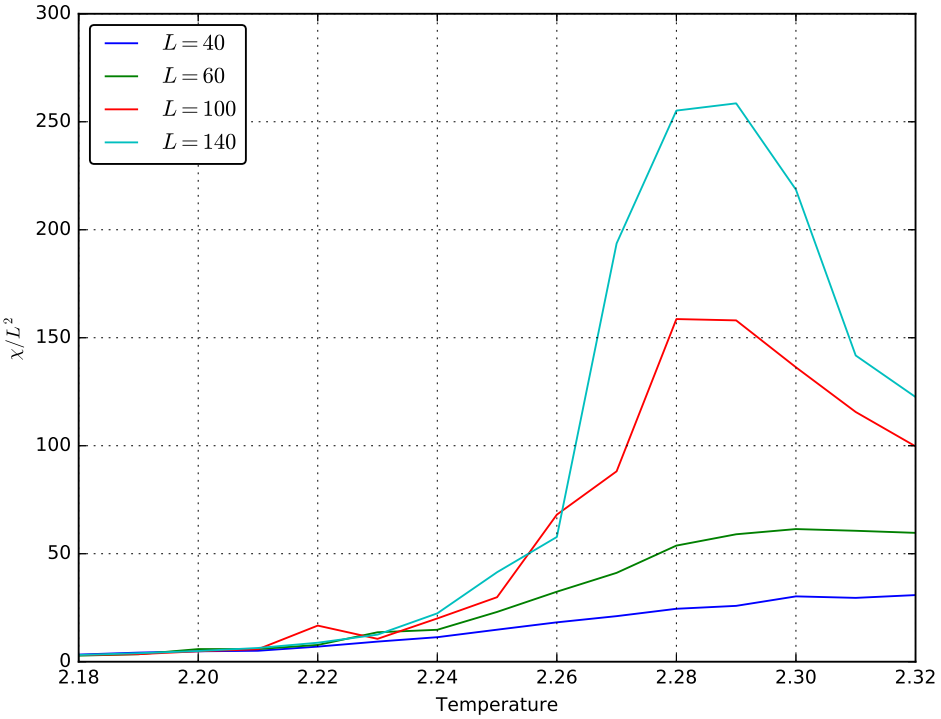

Figure 15: Susceptibility as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ calculated over 10^6 Monte Carlo cycles

$$T_C(L = 60) = 2.28$$

$$T_C(L = 100) = 2.28$$

$$T_C(L = 140) = 2.275$$

By inserting them into eqs. (24) and (25), we get:

$$a = \frac{T_C(L_i) - T_C(L_j)}{L_i^{-1} - L_j^{-1}} \tag{30}$$

$$a = \frac{T_C(100) - T_C(140)}{\frac{1}{100} - \frac{1}{140}} = \frac{2.28 - 2.275}{\frac{1}{100} - \frac{1}{140}} = 1.75 \tag{31}$$

We can then find:

$$T_{C\infty} = T_C(L) - \frac{a}{L} = 2.275 - \frac{1.75}{140} = 2.2625 \tag{32}$$

This indicates that when the lattice size goes to infinity, the critical temperature becomes

$$T_C(L \to \infty) = 2.2625 \tag{33}$$

which seems accurate compared to the plots and is pretty close to the exact solution given in eq. (22).

For plotting $\langle E \rangle$, $\langle |M| \rangle$, C_V and χ as functions of the temperature, both a low time step and a high number of Monte Carlo cycles were necessary in order to observe the phase transition and the critical temperature. However, the number of Monte Carlo cycles used for these plots where just barely enough to reach the equilibrium shown previously (notice that the critical temperature was not visible when using 10^5 Monte Carlo cycles). To get optimal plots, an even smaller time step with more Monte Carlo cycles could be used, but this could easily result in a total computation time of several days. The computation time for these generated plots were slightly more than 4 hours.

A simple comparison of computation time for the parallelization was performed for a lattice with size $L = 140$ for the temperatures $T \in [2.18, 2.32]$ and with a temperature step of $\Delta T = 0.01$. The number of Monte Carlo cycles was set to 10^3. When only using one processor (without parallelization) without optimization flags the computation time was:

```
Time = 84.1645 seconds on number of processors: 1
```

The same conditions was performed with the optimization flag `-O3`, which resulted in the following computation time:

```
Time = 21.2101 seconds on number of processors: 1
```

Finally, a test was run with the optimization flag `-O3` and parallelization on four processors:

```
Time = 8.98085 seconds on number of processors: 4
```

As seen, both optimization flags and parallelization can reduce the computation time by a factor of almost 10.

5 Conclusion

This project has studied phase transitions in magnetic systems using the Ising model in two dimensions. Simulations were performed using Monte Carlo cycles and the Metropolis algorithm. To start with, a lattice with the size $L = 2$ was used to find both numerical and analytical result in order to find the accuracy for a different number of Monte Carlo cycles. It was found that the system gets more accurate with higher number of Monte Carlo cycles (not surprisingly), but this is of course on the expense of the computation time. More than 10^6 cycles is recommended for a good result.

A lattice with the size $L = 20$ was used to find out when the system reaches equilibrium by plotting the energy and the absolute value of the magnetic moment. Both $T = 1.0$ and $T = 2.4$ were used. The equilibrium was found to be reached after around 10^6 Monte Carlo cycles. The distribution of energy values were plotted and showed a normal distribution (especially for $T = 2.4$) as expected due to a normal distribution for the random number generator used.

Most of the calculations in this project used parallelization with OpenMPI and optimization flags in order to achieve a lower computation time. These simple measures reduced the computation time in our case by a factor of almost 10.

Finally, the phase transitions were studied. $\langle E \rangle$, $\langle |M| \rangle$, C_V and χ were plotted as a function of temperature for the lattice sizes $L = 40$, $L = 60$, $L = 100$ and $L = 140$ when using 10^6 Monte Carlo cycles. A phase transition was visible at around $T = 2.28$ where a higher lattice size showed a more accurate result. The critical temperature was visible for heat capacity and susceptibility, and was calculated (from manual reading of the plots) to be $T_C(L \to \infty) = 2.2625$, which is close to the exact value by Lars Onsager to be $\frac{kT_C}{J} = \frac{1}{\ln(1+\sqrt{2})} \approx 2.269$.