End of Semester Review-1

Given a directed graph $G=(V,E)$

↳ Find if the graph is strongly connected.

↳ Start from an arb. vertex and DFS/BFS.

If all other vertices are reachable from s

(if some vertices are not reached then return false)

↳ G_{rev}: Graph obtained from G by reversing dir (s)

(BFS/DFS).

"

Is s reachable from all other vertices.

Obs1: If u and v are mutually reachable
v and w "
then u and w are mutually reachable.

Obs2: For any pair of vert. u and w, their str. conn comp are either identical or disjoint.

Case-1: If u and w are mut. rec.

$v \leftarrow \underline{\underline{C_u}} \equiv \underline{\underline{C_w}} \leftarrow v'$

Case-2: They are not.

For the sake of contradiction, $\underline{v} \in \underline{\underline{C_u}} \cap \underline{\underline{C_w}}$

↳ From the defns of C_u and C_w: All vertices in C_u are mut. rec. from u. ⇒ v is mut. rec. from u — (i)

→ All vert. in C_w are mut. rec. from w ⇒ v is mut. rec.

from w —②

Putting ① and ② together with Obs 1: we get that u and w are mut. rec. This contradicts our assumption that they are not.

Interference free run

→ $G=(V,E)$

R_1 $\quad$ R_2

(s_1, d_1) $\quad$ (s_2, d_2)

Start $\quad$ Dest

1. ~~Paths of robots cannot intersect~~ (at an instant)
2. Robots at any instant cannot be $\leq r$ dist from each other. ✓
3. Robots move in alt. time steps

Qn: Given G, param r, nodes s_1, s_2, d_1, d_2, check if an interference free run is possible.

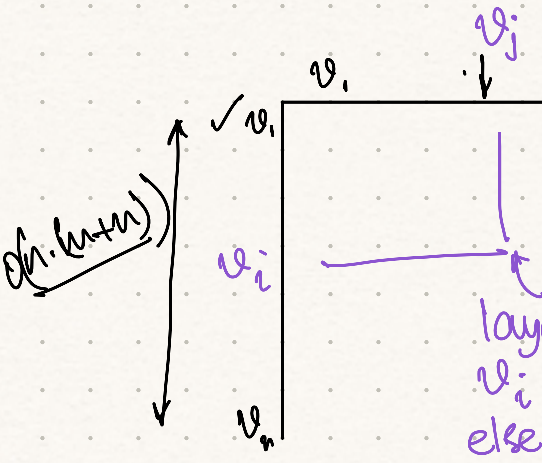
$$G' = (V', E')$$

$$V' = \{ (\underbrace{u}_{\text{position of } R_1}, \overbrace{v}^{\text{position of } R_2}) \mid (u,v) \in V \times V \text{ s.t. } dist(u,v) > r \}$$

$$E' : \quad (u, v) \longrightarrow (u', v')$$

if $((u,u') \in E \wedge v = v')$
$\vee\ ((v, v') \in E \wedge u = u')$
$\cancel{\vee\ ((u,u') \in E \wedge (v,v') \in E)}$

$(s_1, s_2) \rightsquigarrow (d_1, d_2)$



entry is 1 if layer no. of v_j in BFS from $v_i > r$ else 0.

$\leq n^2$ vertices in G'.

$\hookrightarrow \leq O(n^3)$ edges.

$O(mn + n^2) + O(n^3 + n^2) \leq O(n^3)$

(Solved example 2 in Chapter 3).
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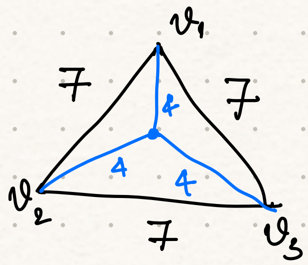
Chapter 4
Q 30

Qn: Steiner tree of min wt.

$G=(V,E) \leftarrow K_n$

$\underbrace{G_X=(V_X,E_X)}$



V

X $|X|=k$.

$\underbrace{X \subseteq Z \subseteq V}$

distances follow triangle inequality.
a, b, c

$w_{ab} \leq w_{ac} + w_{bc}$
$w_{ac} \leq w_{bc} + w_{ab}$

We are given X (a set of terminals) which must be "spanned".

We say that a Steiner tree on X is a set Z s.t $X \subseteq Z \subseteq V$ together w/ a tree T of $G[Z]$.
We want to find a tree T with min wt.

Pf: $\underbrace{V \supseteq Z \supseteq X}$ 2^{n-k} many choices for Z. } Brute force on each
$\hookrightarrow$ poly (n) . 2^{n-k}.

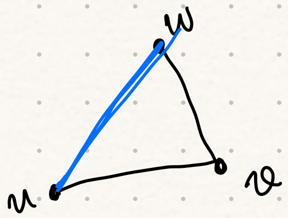
Obs: It is only useful to add vertices from $V \setminus X$ with at least 3 neighbours in X

X Y $X \sqcup Y = Z$

new vertices we need to add.

$v \in V \setminus X$ s.t $\#(N(v) \cap X) = 2$.
u, w

u, w ∈ X
v ∈ Y.



Obs: Adding v is only useful if $wt_{(w,v)} + wt_{(u,v)}$ is lower than wt of (u,w) ← This cannot happen because of Δ^{le} ineq.

$\Rightarrow$ We <u>may</u> gain by adding vertices of degree ≥ 3.

<u>Obs</u>: At least 3 edges incident on v must show up in min wt Steiner tree.

$$\underbrace{X \sqcup Y}_{\boxed{\leq P}} \Big\} \; Z$$

$$|X \sqcup Y| = t$$

$$\underbrace{|Y| < \frac{t}{2}} \;\Big|\; k = |X| \geq \frac{t}{2}$$

$$|Y| < k$$

$$\sum_{i=0}^{k} \binom{n-k}{i} \, \mathrm{poly}(n)$$

$$\underline{\sum_{i=1}^{P} \binom{n-k}{i}} \cdot \mathrm{poly}(n)$$

$$\leq \left(e \, \frac{(n-k)}{P} \right)^{P} \cdot P$$

2^{n-k}

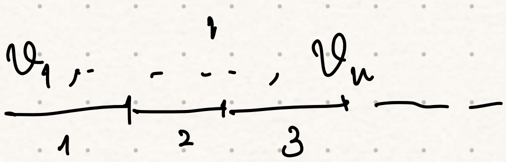
$2^{P \log \left(\frac{n-k}{P}\right)}$

$\underline{P \sim \sqrt{n}}$ $\quad$ $\underline{k \sim \sqrt{n}}$

$2^{\sqrt{n} \log n}$ $\quad$ $2^{n - \sqrt{n}}$

<u>Claim</u>: # of vertices in a spanning tree on n-vertices, with degree at least 3, is at most $\frac{n}{2} - 1$.

$v_1, \ldots \ldots, v_n$



$t_1 \leftarrow$ # of vert. of deg 1

$t_2 \leftarrow$ # of vert w/ deg 2

$t_{\geq 3} \leftarrow$ # of vert of deg ≥ 3.

- # of edges in a spanning tree = $n-1$.

$$t_1 + t_2 + t_{\geq 3} = n$$

- Sum of degrees over all vertices = $2 \cdot$ # edges.

$$1 \cdot t_1 + 2 \cdot t_2 + 3 \cdot t_{\geq 3} \leq 2 \cdot (n-1).$$

$(t_1 + t_2 + t_{\geq 3}) + (t_2 + 2t_{\geq 3}) \leq 2n-2 \Rightarrow t_2 + 2t_{\geq 3} \leq n-2$

$\Rightarrow t_{\geq 3}$ cannot be larger than $\frac{n-2}{2}$.

- For every set $Y \subseteq V \setminus X$, of size $< k$, compute the wt of MST on $\underbrace{X \sqcup Y}$ $\longleftarrow \sum_{i=0}^{k} \binom{n-k}{i}$; for each set computing MST $\leftarrow O(\text{poly}(n))$.
- Return the min wt amongst all of these
  - $\hookrightarrow$ This is min wt Steiner tree wrt X and $G=(V,E)$

$\hookrightarrow \left(\sum_{i=0}^{k} \binom{n-k}{i} \right) \cdot \text{poly}(n)$.

$\hookrightarrow \leq k \binom{n-k}{k} \cdot \text{poly}(n)$

$\hookrightarrow n^{O(k)} \cdot \text{poly}(n)$

$e\left(\frac{n-k}{k}\right)^k \leq n^k$

n

$k < \cancel{\frac{n}{4}} \cdot \frac{n}{3}$.

$k < \frac{n-k}{2}$

$\frac{3k}{2} < \frac{n}{2}$

$k < \frac{n}{3}$