Probability and Statistics: Endsem Q3 (8 Marks)

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1. Question

Consider a sequence $\{X_n, n = 1, 2, 3, \ldots\}$ such that:

$$X_n = \begin{cases} n, & \text{with probability } \frac{1}{n^2}, \\ 0, & \text{with probability } 1 - \frac{1}{n^2}. \end{cases}$$

(a) Show that $X_n \xrightarrow{p} 0$ (Convergence in probability to 0).

(b) Show that $X_n \xrightarrow{\text{a.s.}} 0$ (Almost sure convergence to 0).

2. Convergence in Probability

($X_n \xrightarrow{p} 0$): This step is missing in the solution, but it's crucial for completeness. Here's how you can add it:

For convergence in probability, we need:

$$\forall \epsilon > 0, \lim_{n \to \infty} P(|X_n - 0| > \epsilon) = 0.$$

In this case:

$$P(|X_n - 0| > \epsilon) = P(X_n \neq 0) = P(X_n = n) = \frac{1}{n^2}.$$

As $n \to \infty$, $P(|X_n - 0| > \epsilon) = \frac{1}{n^2} \to 0$. Hence, $X_n \xrightarrow{p} 0$.

3. Almost sure convergence: Solution Using the Borel-Cantelli Lemma

To show $X_n \xrightarrow{\text{a.s.}} 0$, we use the Borel-Cantelli lemma. Recall the lemma: - If $\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty$, then $X_n \xrightarrow{\text{a.s.}} X$.

For the given sequence, we have:

$$P(|X_n| > \epsilon) = P(X_n = n) = \frac{1}{n^2}.$$

Thus:

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a p-series with $p = 2 > 1$, so the series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By the Borel-Cantelli lemma, $X_n \xrightarrow{\text{a.s.}} 0$. Therefore, the sequence almost surely converges to 0.

4. Solution Without Using the Borel-Cantelli Lemma

To prove $X_n \xrightarrow{\text{a.s.}} 0$, we directly analyze the probabilities.

From the definition of X_n, we know:

$$P(X_n = n) = \frac{1}{n^2}, \quad P(X_n = 0) = 1 - \frac{1}{n^2}.$$

1. As $n \to \infty$,

$$P(X_n = n) = \frac{1}{n^2} \to 0, \quad P(X_n = 0) = 1 - \frac{1}{n^2} \to 1.$$

2. The probability $P(X_n = n)$ becomes negligibly small, and $X_n = 0$ dominates for sufficiently large n.

3. Since $P(X_n = 0) \to 1$, the probability that $X_n \neq 0$ happens infinitely often is 0. Thus, with probability 1, $X_n = 0$ for all sufficiently large n.

Hence, we conclude that $P(\lim_{n\to\infty} X_n = 0) = 1$, and $X_n \xrightarrow{\text{a.s.}} 0$.

5. Why the Misinterpreted Solution is Incorrect

An incorrect argument for almost sure convergence is as follows:

- $P(X_n = 0) \to 1$, so the sequence X_n "should" almost surely converge to 0. - This conclusion is based on the idea that the probability of $X_n = 0$ being dominant implies almost sure convergence.

Why This is Wrong:

1. Conflating Probability with Almost Sure Convergence: - $P(X_n = 0) \to 1$ shows that the value 0 is more probable as $n \to \infty$, but this does not ensure almost sure convergence. Almost sure convergence requires that $X_n = 0$ eventually (for all sufficiently large n) in almost every realization of the sequence.

2. Ignoring Infinite Occurrences: - The argument does not address whether $X_n \neq 0$ happens infinitely often. Even small probabilities can accumulate over an infinite sequence, so this must be explicitly ruled out to establish almost sure convergence.

3. Missing Analysis of the Event: - The correct approach requires analyzing the sequence of probabilities $P(X_n \neq 0)$ across all n, either via tools like the Borel-Cantelli lemma or direct reasoning. Simply stating that $P(X_n = 0) \to 1$ is insufficient.

6. Correct proof without using Borel-Cantelli Lemma

To prove $X_n \xrightarrow{\text{a.s.}} 0$ **without using the Borel-Cantelli lemma**, we directly analyze the probability that $X_n \neq 0$ infinitely often and show it is zero. Here's the detailed proof:

We aim to show:

$$P\left(\lim_{n\to\infty} X_n = 0\right) = 1,$$

which is equivalent to showing that the probability of $X_n \neq 0$ infinitely often is zero. Denote this event as:

$$A_{\text{inf}} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{X_n \neq 0\}.$$

Complement of A_{inf} The complement of A_{inf} is the event where $X_n = 0$ for all n sufficiently large:

$$A_{\text{inf}}^c = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{X_n = 0\}.$$

If we can show that $P(A_{\text{inf}}) = 0$, then $P(A_{\text{inf}}^c) = 1$, which implies $X_n \xrightarrow{\text{a.s.}} 0$.

Probability of $X_n \neq 0$ By the definition of X_n, we know:

$$P(X_n \neq 0) = P(X_n = n) = \frac{1}{n^2}.$$

The probability that $X_n \neq 0$ for infinitely many n can be analyzed using the following reasoning.

Bound the probability of infinitely many $X_n \neq 0$ The event $X_n \neq 0$ for infinitely many n can be written as:

$$P(A_{\text{inf}}) = P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{X_n \neq 0\}\right).$$

Using the union bound, the probability of $\bigcup_{n=m}^{\infty}\{X_n \neq 0\}$ is:

$$P\left(\bigcup_{n=m}^{\infty} \{X_n \neq 0\}\right) \leq \sum_{n=m}^{\infty} P(X_n \neq 0).$$

Now, substitute $P(X_n \neq 0) = \frac{1}{n^2}$:

$$\sum_{n=m}^{\infty} P(X_n \neq 0) = \sum_{n=m}^{\infty} \frac{1}{n^2}.$$

This is a tail sum of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because $p = 2 > 1$. Let the total sum of the series be S, then:

$$\sum_{n=m}^{\infty} \frac{1}{n^2} \to 0 \quad \text{as } m \to \infty.$$

Thus, for any $\epsilon > 0$, there exists an m such that:

$$P\left(\bigcup_{n=m}^{\infty} \{X_n \neq 0\}\right) < \epsilon.$$

Conclude that $P(A_{\text{inf}}) = 0$ As $m \to \infty$, the probability that $X_n \neq 0$ for infinitely many n approaches 0:

$$P(A_{\text{inf}}) = 0.$$

Therefore, with probability 1, $X_n = 0$ for all sufficiently large n, which implies:

$$P\left(\lim_{n\to\infty} X_n = 0\right) = 1.$$

We have shown directly that $P(A_{\text{inf}}) = 0$ without using the Borel-Cantelli lemma. Hence, $X_n \xrightarrow{\text{a.s.}} 0$.

7. Conclusion

The problem illustrates the distinction between convergence in probability and almost sure convergence. Using the Borel-Cantelli lemma provides a rigorous way to establish almost sure convergence. Without it, the proof requires carefully analyzing the cumulative probability of $X_n \neq 0$ and ensuring it vanishes over all n. Misinterpreting the dominance of $P(X_n = 0)$ can lead to incorrect conclusions about almost sure convergence.

8. Marking Scheme

- Mathematical expressions are a must. Long written text has not been awarded any marks.
- Solution similar to Section 4 is given 0 marks.
- Correct mathematical condition for convergence in probability and almost sure convergence has been given 1 mark each.