Probability and Statistics: MA6.101

Tutorial 7

Topics Covered: Moment Generating Functions and Stochastic Simulation

Q1: Let X be an exponential random variable with parameter λ and let Y be a random variable with the Gamma distribution $Y \sim \text{Gamma}(k, \theta)$.

a) Show how to generate X using a uniform random variable U drawn from the interval $[0, 1]$.

b) Show how to generate Y using k uniform random variables drawn from $[0, 1]$.

Note: The Gamma distribution $Y \sim \text{Gamma}(k, \theta)$ can be expressed as the sum of k independent exponential random variables $X_1, X_2, \ldots, X_k$, where each $X_i \sim \text{Exp}\left(\frac{1}{\theta}\right)$. That is:

$$Y = \sum_{i=1}^{k} X_i$$

where X_i are independent and identically distributed.

Answer:

a) To generate X:

The cumulative distribution function (CDF) of X is given by:

$$F_X(x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

- Let $U \sim \text{Uniform}(0, 1)$. By the inverse transform method:

$$F_X(X) = U$$
$$1 - e^{-\lambda X} = U$$
$$X = -\frac{1}{\lambda} \ln(1 - U)$$

- Since $U \sim \text{Uniform}(0, 1)$, the distribution of $1 - U$ is also uniform, so $-\ln(1 - U)$ has the same distribution as $-\ln(U)$. Therefore, we can simplify this to:

$$X = -\frac{1}{\lambda} \ln(U)$$

b) To generate $Y \sim \text{Gamma}(k, \theta)$:

- Generate k independent uniform random variables:

$$U_1, U_2, \ldots, U_k \sim \text{Uniform}(0, 1)$$

- Using the inverse transform method for each $X_i \sim \text{Exp}\left(\frac{1}{\theta}\right)$:

$$X_i = -\theta \ln(U_i), \quad i = 1, 2, \ldots, k$$

- Sum the X_i's to obtain Y:

$$Y = \sum_{i=1}^{k} X_i = -\theta \sum_{i=1}^{k} \ln(U_i)$$

Q2: Prove that $x \sim f(x) = xe^{-x}; x \geq 0$ has a moment generating function of $\frac{1}{(1-t)^2}$. Hint: Use the change of variable technique to integrate with respect to $w = x(1-t)$ instead of x.

Answer:
The moment generating function is

$$M(x,t) = E(e^{xt}) = \int_0^\infty e^{xt} xe^{-x} dx = \int_0^\infty xe^{-x(1-t)} dx.$$

Define $w = x(1-t)$. Then

$$x = \frac{w}{1-t} \quad \text{and} \quad \frac{dx}{dw} = \frac{1}{1-t}.$$

The change of variable technique indicates that

$$\int g(x) dx = \int g(x(w)) \frac{dx}{dw} dw,$$

where $g(x) = xe^{-x(1-t)}$. Thus we find that

$$M(x,t) = \int_0^\infty \frac{w}{1-t} e^{-w} \frac{1}{1-t} dw = \frac{1}{(1-t)^2} \int_0^\infty we^{-w} dw = \frac{1}{(1-t)^2}.$$

Here the value of the final integral is unity, since the expression we^{-w}, which is to be found under the integral sign, has the same form as the p.d.f. of x.

To demonstrate directly that the value is unity, we can use the technique of integrating by parts. The formula is

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

Within the expression we^{-w}, we take $u = w$ and $e^{-w} = dv/dw$. Then we get

$$\int_0^\infty we^{-w} dw = \left[-we^{-w}\right]_0^\infty + \int_0^\infty e^{-w} dw = \left[-e^{-w}\right]_0^\infty = 1.$$

Q3: Use the rejection method to generate a random variable having the $Gamma(\frac{5}{2}, 1)$ density function.
Note: The pdf of $Gamma(k, \theta)$ is given by $f(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}$ and $\Gamma(\frac{5}{2}) = \frac{3}{4}\pi$.
Hint: You need to figure out an appropriate distribution you can already sample from to use in the rejection method.

Answer: We pick $exp(\lambda)$ as the distribution we'll be sampling from.

$$f(x) = \frac{4}{3\sqrt{\pi}} x^{\frac{3}{2}} e^{-x}, x > 0$$
$$g(x) = \lambda e^{-\lambda x}, x > 0$$
$$\implies \frac{f(x)}{g(x)} = \frac{4}{3\lambda\sqrt{\pi}} x^{\frac{3}{2}} e^{(\lambda-1)x}$$

We wish to find a c such that $\frac{f(x)}{g(x)} \leq c$ for all x

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 0$$

$$\text{Hence, } x = \frac{3}{2(1-\lambda)}$$

We need to pick an appropriate λ such that $x > 0$. We pick $\lambda = \frac{2}{5}$.

$$c = \frac{10}{3\sqrt{\pi}}\left(\frac{5}{2}\right)^{\frac{3}{2}} e^{-\frac{3}{2}}$$

$$\frac{f(x)}{cg(x)} = \frac{x^{\frac{3}{2}} e^{-\frac{3x}{5}}}{\left(\frac{5}{2}\right)^{\frac{3}{2}} e^{-\frac{3}{2}}}$$

Now for to finally generate the required random number (i.e using the rejection method algorithm)

(a) Generate a random number U_1 and use that to generate a random number from $exp(\frac{2}{5})$ ($Y = -\frac{5}{2}\log U_1$)

(b) Generate a random number U_2

(c) If $U_2 < \frac{Y^{\frac{3}{2}} e^{-\frac{3Y}{5}}}{\left(\frac{5}{2}\right)^{\frac{3}{2}} e^{-\frac{3}{2}}}$, set $X = Y$. Otherwise, execute the step (a).

Q4: What is the expected number of iterations to generate k random numbers from a distribution using the rejection method?

Answer: Let f be the pdf of the distribution we wish to sample from and g be the pdf of the distribution we sample from such that $support(f) \subseteq support(g)$.

Then for rejection sampling we have a c such that $\frac{f(y)}{g(y)} \leq c$ for all y.

We claim that $\mathbb{P}\left(U \leq \frac{f(Y)}{Mg(Y)}\right) = \frac{1}{c}$

$$\begin{aligned}
\mathbb{P}\left(U \leq \frac{f(Y)}{cg(Y)}\right) &= E_{g(Y)}[P(U \leq \frac{f(y)}{cg(y)}|Y = y)] \\
&= E_{g(Y)}\left[\frac{f(y)}{cg(y)}\right] \\
&= E_{g(Y)}\left[\frac{f(y)}{cg(y)}\right] \\
&= \int_{y:g(y)>0} \frac{f(y)}{cg(y)} g(y)dy \\
&= \frac{1}{c}
\end{aligned}$$

The last step comes from the fact that $support(f) \subseteq support(g)$.

Now notice that the number of iteration required to successfully generate one number is a geometric random variable with parameter $\frac{1}{c}$. So the expected number of iterations for generating one sample is c and for k samples is kc

Q5: (a) Let $M_X(s)$ be finite for $s \in [-c, c]$, where $c > 0$. Show that the MGF of Y$= aX + b$ is given by

$$M_Y(s) = e^{sb} M_X(as)$$

and it is finite in $\left[-\frac{c}{|a|}, \frac{c}{|a|}\right]$.

(b) If $X_1, X_2, \ldots, X_n$ are n independent random variables with respective moment-generating functions $M_{X_i}(t) = \mathbb{E}[e^{tX_i}]$ for $i = 1, 2, \ldots, n$, then prove the moment-generating function of the linear combination: $Y = \sum_{i=1}^{n} a_i X_i$ is:

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

Answer:

(a)

$$\begin{aligned} M_Y(s) &= E[e^{sY}] \\ &= E[e^{s(aX+b)}] \\ &= e^{sb} E[e^{(sa)X}] \\ &= e^{sb} M_X(as). \end{aligned}$$

Where, $as \in [-c, c]$. So, $M_X(as)$ is finite for $s \in \left[-\frac{c}{|a|}, \frac{c}{|a|}\right]$.

(b)

$$\begin{aligned} M_Y(t) &= \mathbb{E}[e^{tY}] \\ &= \mathbb{E}[e^{t(a_1X_1 + a_2X_2 + \cdots + a_nX_n)}] \\ &= \mathbb{E}[e^{a_1tX_1}]\mathbb{E}[e^{a_2tX_2}] \ldots \mathbb{E}[e^{a_ntX_n}] \\ &= M_{X_1}(a_1t)M_{X_2}(a_2t) \ldots M_{X_n}(a_nt) \\ &= \prod_{i=1}^{n} M_{X_i}(a_it) \end{aligned}$$

Q6: Let $X \sim$ Normal$(Y, 1)$ where $Y \sim$ Exponential(λ). Find the MGF of X.

Answer:

$$\mathbb{E}[e^{tX}] = \int_0^\infty \mathbb{E}[e^{sX}|Y = y] f_Y(y) dy$$

using the MGF of Gaussian

$$\begin{aligned} &= \int_0^\infty e^{yt} e^{\frac{1}{2}t^2} \lambda e^{-\lambda y} dy \\ &= \lambda e^{\frac{1}{2}t^2} \int_0^\infty e^{y(t-\lambda)} dy \\ &= \frac{\lambda}{t - \lambda} e^{\frac{1}{2}t^2} \left[e^{y(t-\lambda)}\right]_0^\infty \end{aligned}$$

when $t < \lambda$

$$= \frac{\lambda}{t - \lambda} e^{\frac{1}{2}t^2} [-1] = \frac{\lambda}{\lambda - t} e^{\frac{1}{2}t^2}$$