Probability and Statistics: MA6.101

Homework 7

Topics Covered: Moment Generating Functions and Stochastic Simulation

Q1: Let $X_1, X_2, \ldots, X_n$ be n i.i.d exponential random variables with parameter λ. Let $Z_{\min} = \min(X_1, X_2, \ldots, X_n)$ and $Z_{\max} = \max(X_1, X_2, \ldots, X_n)$.

Generate $Z_{\min}$ and $Z_{\max}$.

Answer: The CDF of $Z_{\min}$ is:

$$G_{\min}(z) = P(Z_{\min} \leq z) = 1 - P(Z_{\min} > z)$$

Since $Z_{\min} = \min(X_1, X_2, \ldots, X_n)$:

$$P(Z_{\min} > z) = P(X_1 > z, X_2 > z, \ldots, X_n > z) = \prod_{i=1}^{n} P(X_i > z) = (P(X_1 > z))^n$$

For $X_i \sim \text{Exp}(\lambda)$:

$$P(X_i > z) = e^{-\lambda z}$$
$$P(Z_{\min} > z) = e^{-n\lambda z}$$

Therefore, the CDF of $Z_{\min}$ is:

$$G_{\min}(z) = 1 - e^{-n\lambda z}$$

To generate $Z_{\min}$:

$$G_{\min}(z) = U$$
$$1 - e^{-n\lambda z} = U$$
$$z = -\frac{1}{n\lambda} \ln(1 - U)$$

Thus,

$$Z_{\min} = -\frac{1}{n\lambda} \ln(U)$$

The CDF of $Z_{\max}$ is:

$$G_{\max}(z) = P(Z_{\max} \leq z) = P(X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z)$$
$$G_{\max}(z) = (P(X_1 \leq z))^n$$

For $X_i \sim \text{Exp}(\lambda)$:

$$P(X_i \leq z) = 1 - e^{-\lambda z}$$

Therefore:

$$G_{\max}(z) = (1 - e^{-\lambda z})^n$$

To generate $Z_{\max}$:

$$G_{\max}(z) = U$$

$$(1-e^{-\lambda z})^n = U$$
$$z = -\frac{1}{\lambda}\ln(1-U^{1/n})$$

Thus, to generate from $Z_{\max}$:

$$Z_{\max} = -\frac{1}{\lambda}\ln(1-U^{1/n})$$

Q2: Let X be a discrete random variable with the following moment-generating function:

$$M(t) = \frac{1}{10}e^{t} + \frac{2}{10}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t}$$

for all t.

Determine the p.m.f of X.

Answer:

The moment-generating function is defined as:

$$M(t) = E[e^{tX}] = \sum_x p_X(x)e^{tx}$$

where $p_X(x)$ is the probability mass function.

By comparing the given mgf with the general form, we can identify the probabilities associated with the discrete values of X:

$$p_X(1) = \frac{1}{10},$$
$$p_X(2) = \frac{2}{10},$$
$$p_X(3) = \frac{3}{10},$$
$$p_X(4) = \frac{4}{10}.$$

The probability mass function (p.m.f.) $p_X(x)$ of X can be expressed as:

$$p_X(x) = \begin{cases} \frac{1}{10} & \text{if } x = 1 \\ \frac{2}{10} & \text{if } x = 2 \\ \frac{3}{10} & \text{if } x = 3 \\ \frac{4}{10} & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$

To verify that this is a valid p.m.f., we check that the probabilities sum to 1:

$$p_X(1) + p_X(2) + p_X(3) + p_X(4) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = \frac{10}{10} = 1.$$

Q3: Let X and Y be two independent random variables with respective moment generating functions

$$m_X(t) = \frac{1}{1-5t}, \quad \text{if } t < \frac{1}{5}, \quad m_Y(t) = \frac{1}{(1-5t)^2}, \quad \text{if } t < \frac{1}{5}.$$

Find $\mathbb{E}(X+Y)^2$.

Answer:
First recall that if we let $W = X + Y$, and using that X, Y are independent, then we see that

$$m_W(t) = m_{X+Y}(t) = m_X(t)m_Y(t) = \frac{1}{(1-5t)^3},$$

recall that $\mathbb{E}[W^2] = m''_W(0)$, which we can find from

$$m'_W(t) = \frac{15}{(1-5t)^4},$$

$$m''_W(t) = \frac{300}{(1-5t)^5},$$

thus

$$\mathbb{E}[W^2] = m''_W(0) = \frac{300}{(1-0)^5} = 300.$$

Q4: True or False? If $X \sim \text{Exp}(\lambda_x)$ and $Y \sim \text{Exp}(\lambda_y)$, then $X + Y \sim \text{Exp}(\lambda_x + \lambda_y)$. Justify your answer.

Answer:
We first find the MGF of $X + Y$ and compare it to the MGF of a random variable $V \sim \text{Exp}(\lambda_x + \lambda_y)$. The MGF of V is

$$m_V(t) = \frac{\lambda_x + \lambda_y}{\lambda_x + \lambda_y - t}, \quad \text{for } t < \lambda_x + \lambda_y.$$

By independence of X and Y,

$$m_{X+Y}(t) = m_X(t)m_Y(t) = \frac{\lambda_x}{\lambda_x - t} \cdot \frac{\lambda_y}{\lambda_y - t}.$$

but

$$\frac{\lambda_x + \lambda_y}{\lambda_x + \lambda_y - t} \neq \frac{\lambda_x}{\lambda_x - t} \cdot \frac{\lambda_y}{\lambda_y - t},$$

and hence the statement is false.

Q5: Given an integral $I = \int_0^{2\pi} f(x)dx$, solve the following:

(a) Write the Monte Carlo Estimate, assuming X_i's are sampled uniformly over the domain.

(b) Write the Monte Carlo Estimate, assuming X_i's are sampled according to some PDF $g(X_i)$.

(c) Prove that the Monte Carlo Estimates from the previous questions compute the right answer on average.

Answer:

(a) We are given samples from $U \sim [0, 2\pi]$. Using SLLN,

$$I = 2\pi\mathbb{E}[f(U)] \approx \frac{2\pi}{N}\sum_{i=0}^{N} f(U_i)$$

(b) We are given samples of Y with PDF $g(\cdot)$.

$$I = \int_0^{2\pi} \frac{f(y)}{g(y)} g(y)dy = \mathbb{E}\left[\frac{f(y)}{g(y)}\right] \approx \frac{1}{N}\sum_{i=0}^{N} \frac{f(Y_i)}{g(Y_i)}$$

(c) We want to show that the expected value of the estimate of I is equal to I. Using linearity of expectation,

$$\mathbb{E}\left[\frac{2\pi}{N}\sum_{i=0}^{N} f(U_i)\right] = \frac{2\pi}{N}\sum_{i=0}^{N} \mathbb{E}[U_i] = 2\pi\mathbb{E}[U] = I$$

and

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=0}^{N} \frac{f(Y_i)}{g(Y_i)}\right] = \frac{1}{N}\sum_{i=0}^{N} \mathbb{E}\left[\frac{f(Y_i)}{g(Y_i)}\right] = \mathbb{E}\left[\frac{f(Y)}{g(Y)}\right] = I$$

Q6: Let $X \sim Gamma(\alpha, \beta)$. Find the MGF of X and its region of convergence.

Note: The pdf of $Gamma(\alpha, \beta)$ is given by $f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x \geq 0$ and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

Answer:

$$\begin{aligned}
M_X(t) &= E[e^{tX}] \\
\implies M_X(t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} e^{-\beta x} x^{\alpha-1} dx \\
\implies M_X(t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-x(\beta - t)} x^{\alpha-1} dx
\end{aligned}$$

Let $x' = x(\beta - t)$ then $dx = \frac{dx'}{\beta - t}$

$$\begin{aligned}
\implies M_X(t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-x'} \frac{x'^{\alpha-1}}{(\beta - t)^{\alpha-1}} \frac{dx'}{\beta - t} \\
\implies M_X(t) &= \frac{\beta^\alpha}{\Gamma(\alpha)(\beta - t)^\alpha} \int_0^\infty e^{-x'} x'^{\alpha-1} dx' \\
\implies M_X(t) &= \left(\frac{\beta}{\beta - t}\right)^\alpha
\end{aligned}$$

Now looking at the initial expression

$$M_X(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} e^{-\beta x} x^{\alpha-1} dx$$

If we take $\beta = t$, then our integral $\int_0^\infty x^{\alpha-1} dx$ diverges. If $\beta < t$, then our integral $\int_0^\infty e^{kx} x^{k'} dx$ diverges again due to $k > 0$ and exponential dominating polynomials. So our region of convergence is $t < \beta$