Probability and Statistics: MA6.101

Tutorial 10

Topics Covered: Markov Chains

Q1: Suppose a machine can be either operational or in maintenance, and the machine's status on successive days follows a Markov chain with stationary transition probabilities. Suppose the transition matrix is as follows:

	Operational	Maintenance
Operational	0.7	0.3
Maintenance	0.6	0.4

(a) If the machine is in maintenance on a given day, what is the probability that it will also be in maintenance the next day?

(b) If the machine is operational on a given day, what is the probability that it will remain operational for the next two days?

(c) If the machine is in maintenance on a given day, what is the probability that it will be operational on at least one of the next three days?

A:

(a) Probability of Maintenance the Next Day Given Maintenance Today
Let s_n denote the machine's status on a given day. From the transition matrix, the probability $P(s_{n+1} = \text{Maintenance} \mid s_n = \text{Maintenance})$ is given by the bottom-right element of the transition matrix:

$$P(s_{n+1} = \text{Maintenance} \mid s_n = \text{Maintenance}) = 0.4$$

(b) Probability of Operational Status for the Next Two Days Given Operational Today
We need to find $P(s_{n+2} = \text{Operational} \cap s_{n+1} = \text{Operational} \mid s_n = \text{Operational})$. This can be written as:

$$\begin{aligned}P(s_{n+2} = \text{Operational} \cap s_{n+1} = \text{Operational} \cap s_n = \text{Operational}) = \\ P(s_n = \text{Operational}) \cdot P(s_{n+1} = \text{Operational} \mid s_n = \text{Operational}) \\ \cdot P(s_{n+2} = \text{Operational} \mid s_{n+1} = \text{Operational})\end{aligned}$$

Since $P(s_n = \text{Operational}) = 1$ (we start with an operational day), this simplifies to:

$$\begin{aligned}&P(s_{n+2} = \text{Operational} \mid s_{n+1} = \text{Operational}) \cdot P(s_{n+1} = \text{Operational} \mid s_n = \text{Operational}) \\ &\quad = (0.7)(0.7) \\ &\quad = 0.49\end{aligned}$$

(c) Probability of Operational Status on At Least One of the Next Three Days Given Maintenance Today
We are asked to find the probability that the machine is operational on at least one of the next three days, given that today it is in maintenance. This probability is given by:

$$1 - P(s_{n+3} = \text{Maintenance} \cap s_{n+2} = \text{Maintenance} \cap s_{n+1} = \text{Maintenance} \mid s_n = \text{Maint})$$

Using the transition matrix, we have:

$$
\begin{aligned}
&P(s_{n+3} = \text{Maintenance}, s_{n+2} = \text{Maintenance}, s_{n+1} = \text{Maintenance} \mid s_n = \text{Maintenance}) \\
&\quad = (0.4)(0.4)(0.4) \\
&\quad = 0.064
\end{aligned}
$$

Thus, the required probability that the machine will be operational on at least one of the next three days is:

$$1 - 0.064 = 0.936$$

Q2: A person walks along a straight line and, at each time period, takes a step to the right with probability b and a step to the left with probability $1 - b$. The person starts in one of the positions $1, 2, \ldots, m$, but if they reach position 0 (or position $m + 1$), their step is instantly reflected back to position 1 (or position m, respectively). Equivalently, we may assume that when the person is in positions 1 or m, they will stay in that position with probability $1 - b$ and b, respectively.

(a) Find the transition probability matrix P.

(b) Find the stationary distribution using the formula $\pi = \pi P$.

A:

(a) Transition Probability Matrix
Let the states of the system be $\{1, 2, \ldots, m\}$. Then, the transition probability matrix P is an $m \times m$ matrix where:
- For the boundary positions: - $P_{1,1} = 1 - b$: The probability of staying in position 1 is $1 - b$. - $P_{1,2} = b$: The probability of moving from position 1 to position 2 is b.
- $P_{m,m-1} = 1 - b$: The probability of moving from position m to position $m - 1$ is $1 - b$. - $P_{m,m} = b$: The probability of staying in position m is b.
- For the inner positions $2 \leq i \leq m - 1$: - $P_{i,i-1} = 1 - b$: The probability of moving from position i to $i - 1$ is $1 - b$. - $P_{i,i+1} = b$: The probability of moving from position i to $i + 1$ is b.
Thus, the transition probability matrix P can be written as:

$$
P = \begin{bmatrix}
1-b & b & 0 & \ldots & 0 & 0 \\
1-b & 0 & b & \ldots & 0 & 0 \\
0 & 1-b & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1-b & 0 & b \\
0 & 0 & \ldots & 0 & 1-b & b
\end{bmatrix}
$$

Q3: A gambler begins with an initial fortune of i dollars. Each time he plays, he has the possibility of winning 1 dollar with a probability p or losing 1 dollar with a probability $1 - p$. The gambler will only stop playing if he either accumulates N dollars or loses all of his money. What is the probability that he will end up with N dollars ?

A:

Let X_n be the amount of money after playing n times. Then $X_n = i + \Delta_1 + \cdots + \Delta_n$, where $\{\Delta_n\}$ is a random walk with step 1 and probability of going up p.
Now let

$$\tau_i = \min_{n \geq 0} \{X_n \in \{0, N\} | X_0 = i\}$$

(ii)

$$\pi = \pi P$$

$$\pi = [\pi_1 \; \text{———} \; \pi_m]$$

$$[\pi_0 \; \text{———} \; \pi_m] = [\pi_0 \; \text{———} \; \pi_m] \begin{bmatrix} 1-b & b & 0 & \cdots & \cdots \\ 1-b & 0 & b & & \\ \cdots & & \cdots & & \\ & \cdots & \cdots & & \cdots \\ & & \cdots & & 1-b \;\; 0 \;\; b \end{bmatrix}$$

$$\pi_1(1-b) + \pi_2(1-b) = \pi_1 \quad \text{— (i)}$$

$$\pi_1(b) + \pi_3(1-b) = \pi_2 \quad \text{— (ii)}$$

$$\pi_2(b) + \pi_4(1-b) = \pi_3$$

$$\boxed{\pi_1 b = \pi_2(1-b)}$$

using this in eq (ii)

$$\pi_2(1-b) + \pi_3(1-b) = \pi_2$$

$$\pi_2(b) = \pi_3(1-b)$$

<u>in general</u>

$$\pi_i(b) = \pi_{i+1}(1-b)$$

$$\Rightarrow \pi_1 b = \pi_2(1-b)$$

$$\Rightarrow \pi_2 b = \pi_3(1-b)$$

$$\Rightarrow \boxed{\pi_1 \left(\frac{b}{1-b}\right)^2 = \pi_3}$$

$$\Rightarrow \pi_i = \pi_1 \left(\frac{b}{1-b}\right)^{i-1}$$

Now
$$\sum_{i=1}^{m} \pi_i = 1$$

$$\pi_1 + \pi_2 + \text{———} = 1$$

$$\pi_1 + \pi_1\left(\frac{b}{1-b}\right)^1 \text{———} \pi_1\left(\frac{b}{1-b}\right)^{m-1} = 1$$

$$r = \frac{b}{1-b}.$$

$$\pi_1\left(1 + r + \text{———} r^{m-1}\right) = 1$$

$$\Rightarrow \pi_1 \frac{r^m - 1}{r - 1} = 1 \Rightarrow \boxed{\pi_1 = \frac{r-1}{r^m - 1}}$$

Now rest all can be calculated

Figure 1: Question 3 Part (ii)

be the time at which the gambler stops playing. We want to calculate

$$P_i(N) = \mathbb{P}(X_{\tau_i} = N)$$

that is, the probability that the gambler accumulates N dollars starting with i dollars Now if after the first play, if $\Delta_1 = 1$, then the gambler will finally win with a probability $P_{i+1}(N)$ by the markov property of the chain. Similar reasoning holds for $\Delta_1 = -1$. Thus we obtain the equation

$$P_i(N) = pP_{i+1}(N) + (1-p)P_{i-1}(N)$$

We also have the boundary probabilites $P_0(N) = 0, P_N(N) = 1$
Rearranging the terms of the equation, we have

$$P_{i+1}(N) - P_i(N) = \frac{q}{p}(P_i(N) - P_{i-1}(N))$$

where $q = 1 - p$. Recursively backtracking using this equation, we get

$$\begin{aligned} P_{i+1}(N) - P_i(N) &= \left(\frac{q}{p}\right)^i (P_1(N) - P_0(N)) \\ &= \left(\frac{q}{p}\right)^i P_1(N) \end{aligned}$$

Using this equation, we can successively evaluate $P_i(N)$ as an expression of $P_1(N)$

$$\begin{aligned} P_2 &= \left(1 + \frac{q}{p}\right) P_1 \\ P_3 &= \left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2\right) P_1 \\ &\vdots \\ P_i &= \left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right) P_1 \end{aligned}$$

Setting $P_N(N) = 1$ we get

$$P_N(N) = 1 = \left(1 + \frac{q}{p} + \ldots \left(\frac{q}{p}\right)^{N-1}\right) P_1(N)$$

$$P_N(N) = \begin{cases} \frac{1-(q/p)^N}{1-(q/p)} P_1(N) & q \neq p \\ NP_1(N) & q = p \end{cases}$$

Solving for $P_1(N)$ and substituting, we get

$$P_i(N) = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & q \neq p \\ \frac{i}{N} & q = p \end{cases}$$

Q4: Simulate a markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.9 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \end{bmatrix}$$

and find its limiting distribution.

A:

```
import numpy as np
import matplotlib.pyplot as plt

P = np.array([[0.2, 0.7, 0.1],
              [0.9, 0.0, 0.1],
              [0.2, 0.8, 0.0]])


def simulate(P, num_iterations=1000, tol=1e-8):
    state = np.random.rand(P.shape[0])
    state = state / state.sum()
    state_history = [state]

    for i in range(num_iterations):
        new_state = np.dot(state, P)
        state_history.append(new_state)

        if np.allclose(new_state, state, atol=tol):
            print(f"Converged after {i+1} iterations")
            break
        state = new_state

    return np.array(state_history)

num_trials = 5
num_iterations = 1000

plt.figure(figsize=(12, 8))

for trial in range(num_trials):
    print(f"\nTrial {trial + 1}")
    state_history = simulate(P, num_iterations=num_iterations)

    stationary_distribution = state_history[-1]
    print(f"Stationary Distribution for Trial {trial + 1}: {stationary_c

    for state_index in range(P.shape[0]):
        plt.plot(state_history[:, state_index], label=f"State {state_ind

        final_value = state_history[-1, state_index]
        plt.text(len(state_history), final_value -0.008, f"{final_value:.
                 verticalalignment='center', horizontalalignment='right'
                 color=f"C{state_index}", fontsize=10)

plt.xlabel("Iterations")
plt.ylabel("Probability")
plt.title("Convergence to Stationary Distribution of Markov Chain")
plt.legend(loc="best")
plt.show()
```

From the results, we can see that the probability distribution of the states converges to a fixed value irrespective of the starting distribution. This is the limiting distribution

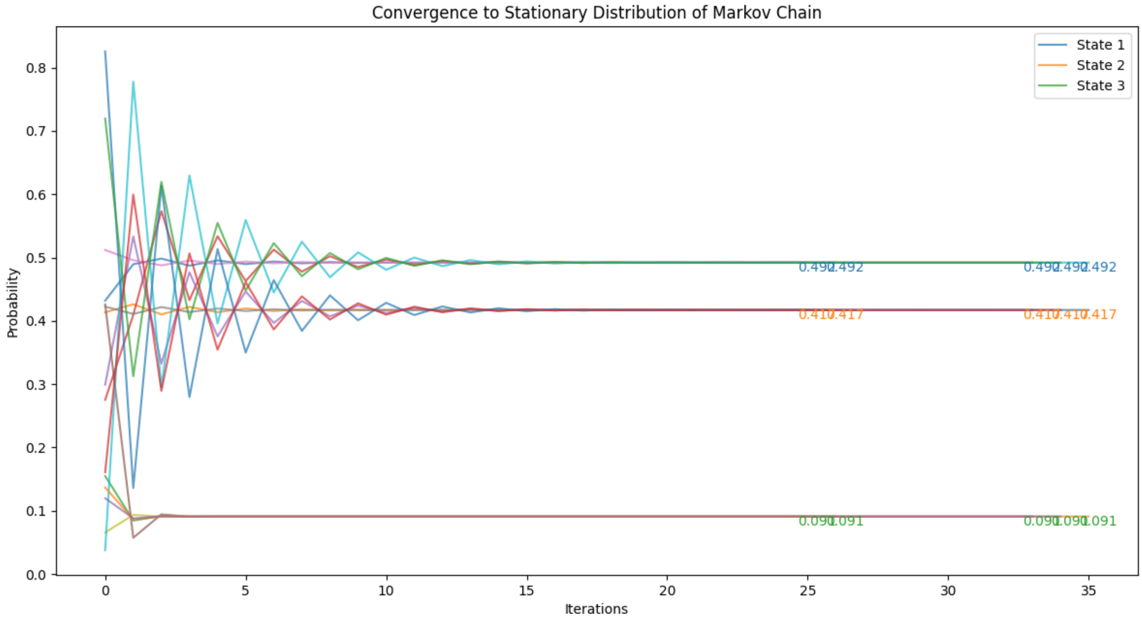

Figure 2: Simulation Results

of the markov chain.

The stationary/ limiting distribution obtained is

$$\alpha = \begin{bmatrix} 0.492 & 0.417 & 0.091 \end{bmatrix}$$

We can verify this by finding the stationary distribution of the markov chain

$$\pi(P - I) = 0$$

$$\pi \begin{bmatrix} -0.8 & 0.7 & 0.1 \\ 0.9 & -1 & 0.1 \\ 0.2 & 0.8 & -1 \end{bmatrix} = 0$$

We get the following sytem of equations

$$\begin{aligned} -8\pi_1 + 9\pi_2 - 2\pi_3 &= 0 \\ 9\pi_1 - 10\pi_2 + \pi_3 &= 0 \\ 2\pi_1 + 8\pi_2 - 10\pi_3 &= 0 \\ \pi_1 + \pi_2 + \pi_3 &= 0 \end{aligned}$$

Solving these, we get

$$\pi = \begin{bmatrix} \frac{92}{187} & \frac{78}{187} & \frac{1}{11} \end{bmatrix} = \begin{bmatrix} 0.491 & 0.417 & 0.091 \end{bmatrix}$$

Which is the same as the distribution obtained from the simulation

Q5: Purpose-flea zooms around the vertices of the transition diagram shown below. Let X_t represent Purpose-flea's state at time t (where $t = 0, 1, \ldots$).

(a) Find the transition matrix P.

(b) Find $P(X_2 = 3 \mid X_0 = 1)$.

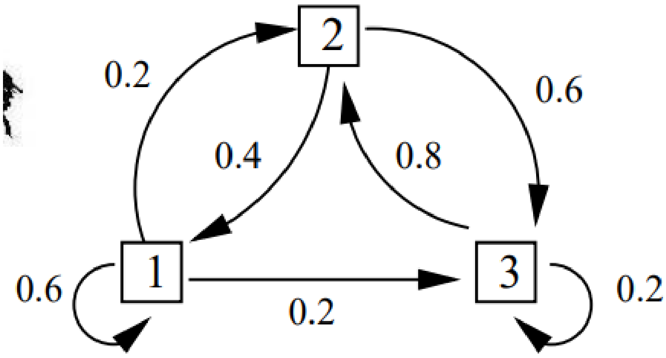

Figure 3: Transition Diagram of Purpose-flea's Movement

(c) Suppose that Purpose-flea is equally likely to start on any vertex at time 0. Find the probability distribution of X_1.

(d) Suppose that Purpose-flea begins at vertex 1 at time 0. Find the probability distribution of X_2.

A:

(a) The transition matrix P is constructed based on the probabilities from the diagram:

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

(b) To find $P(X_2 = 3 \mid X_0 = 1)$, we calculate P^2 and find the entry in the first row and third column.

$$P^2 = P \cdot P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

Calculate P^2 , and use the value in position $(1, 3)$ for $P(X_2 = 3 \mid X_0 = 1)$.

(c) Let the initial probability distribution vector be $\pi_0 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. The distribution at X_1 is then:

$$\pi_1 = \pi_0 \cdot P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

This calculation yields the probability distribution of X_1.

(d) If Purpose-flea begins at vertex 1 at time 0, the initial distribution vector is $\pi_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. The distribution at X_2 is given by:

$$\pi_2 = \pi_0 \cdot P^2$$

where P^2 is the squared transition matrix. Use the result from part (b) to find the distribution at X_2.

Q6: Consider a Markovian Coin, $S = \{0, 1\}$. Where 0 denotes Head and 1 denotes Tails. Suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e.,

$$\pi^{(0)} = [P(X_0 = 0) \quad P(X_0 = 1)] = [\alpha \quad 1 - \alpha],$$

where $\alpha \in [0, 1]$.

(a) How does transition matrix define the nature of the coin.

(b) Using induction (or any other method), show that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

(c) Show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

(d) Show that

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

A:

(a) Nature of the Markovian coin: The probability that there is a Heads to Tails transition is a, and the probability that there is a Tails to Heads transition is b. The probability that it retains its memory is $1 - \alpha$ where α is the probability that chain changes whatever state it is in.

(b) For $n = 1$, we have

$$P^1 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{1-a-b}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

Assuming that the statement of the problem is true for n, we can write P^{n+1} as

$$\begin{aligned} P^{n+1} = P^n P &= \frac{1}{a+b}\left(\begin{bmatrix} b & a \\ b & a \end{bmatrix} + (1-a-b)^n \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}\right) \cdot \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \\ &= \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^{n+1}}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}, \end{aligned}$$

which completes the proof.

(c) By assumption $0 < a + b < 2$, which implies $-1 < 1 - a - b < 1$. Thus,

$$\lim_{n\to\infty} (1-a-b)^n = 0.$$

Therefore,

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

(d)

$$\begin{aligned} \lim_{n\to\infty} \pi^{(n)} &= \lim_{n\to\infty} [\pi^{(0)} P^n] \\ &= \pi^{(0)} \lim_{n\to\infty} P^n \\ &= [\alpha \quad 1-\alpha] \cdot \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}. \end{aligned}$$

Q7: For the Markovian coin described above:

(a) Calculate the stationary distribution. What do you observe?

(b) Find the mean return times, r_0 and r_1, for this Markov chain. Do you observe anything?

(c) Can you intuitively explain the result above?

A:

(a) The stationary distribution is the same as the limiting distribution.

(b) To calculate r_0:

$$\begin{aligned} r_0 &= E[R \mid X_1 = 0, X_0 = 0] \cdot P(X_1 = 0 \mid X_0 = 0) + E[R \mid X_1 = 1, X_0 = 0] \cdot P(X_1 = 1 \mid X_0 = 0) \\ &= E[R \mid X_1 = 0] \cdot (1 - a) + E[R \mid X_1 = 1] \cdot a. \end{aligned}$$

If $X_1 = 0$, then $R = 1$, so $E[R \mid X_1 = 0] = 1$.
If $X_1 = 1$, then $R \sim 1 + \text{Geometric}(b)$, so

$$E[R \mid X_1 = 1] = 1 + E[\text{Geometric}(b)] = 1 + \frac{1}{b}.$$

Therefore,

$$r_0 = 1 \cdot (1 - a) + \left(1 + \frac{1}{b}\right) \cdot a = \frac{a + b}{b}.$$

Similarly, we can obtain the mean return time to state 1:

$$r_1 = \frac{a + b}{a}.$$

We can notice that:

$$r_0 = \frac{1}{\pi_o} \quad r_1 = \frac{1}{\pi_1}$$

(c) The larger the π_i, the smaller the r_i will be. For example, if $\pi_i = \frac{1}{4}$, we conclude that the chain is in state i one-fourth of the time. In this case, $r_i = 4$, which means that on average it takes the chain four time units to return to state i.