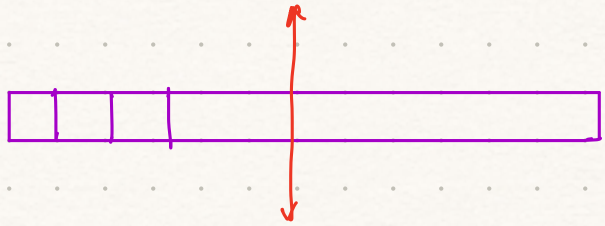
Divide and Conquer

Prev examples: Merge sort.



Paradigm:
- "Break" your problem into disjoint parts and these form a "smaller" instance of the same problem
- Put together solutions of "smaller" problems

Sort (A)

Sort ($A_{\leq 1/2}$)   Sort ($A_{> 1/2}$)

$$T(n) = 2 \cdot T(n/2) + c \cdot n.$$

$$= 2\left[2T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}\right] + c \cdot n$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + c \cdot [n + n]$$

$$= 2^i \cdot T\left(\frac{n}{2^i}\right) + c \cdot n \cdot i$$

$$i \leq \lceil \log_2 n \rceil$$

$$T(n) \leq 2^{\lceil \log_2 n \rceil} \cdot T(1) + c \cdot n \cdot \lceil \log_2 n \rceil$$

Integer Multiplication.

We are given 2 natural numbers a, b.

binary repr.

Want: $a \times b$.

Input size: $\overset{k}{\log_2 a}$, $\overset{l}{\log_2 b}$

Trivial complexity:

$a: 2^0 \times 1 + 2^1 \times 1 + 2^2 \times 0 + 2^3 \times 1 + 2^4 \times 0$

$b: 2^0 \times 0 + 2^1 \times 1 + 2^2 \times 1 + 2^3 \times 0 + 2^4 \times 0$

MSB   LSB

	+1 +1
a:	0 1 0 1 1
b:	0 0 1 1 0
	1 0 0 0 1

$2^0 \times 1 + 2^4 \times 1$

$O(\max\{k, \ell\})$.

$$a = \sum_{i=0}^{k-1} a_i \cdot 2^i \qquad b = \sum_{j=0}^{\ell-1} b_j \cdot 2^j$$

Note that each a_i, b_j are $\underbrace{0, 1}_{\text{bits}}$.

$$a \times b = \underbrace{\left(\sum_{i=0}^{k-1} a_i \cdot 2^i\right)\left(\sum_{j=0}^{\ell-1} b_j \cdot 2^j\right)}_{k\ell \text{ terms}}$$

$k, \ell \approx n$

Complexity $\sim n^2$

Polynomial Mult.

$P(z) = \sum_{i=0}^{d} a_i z^i$	$Q(z) = \sum_{j=0}^{d} b_j \cdot z^j$

$\hookrightarrow P(z) \cdot Q(z)$.

"Algo for polynomial mult. $\Rightarrow$ Algo for Integer mult. w/ some overhead"

Matrix Mult.

$C = A \cdot B$

$$\underbrace{\begin{bmatrix} & | & \\ - & a_{ij} & \\ & & \end{bmatrix}}_{n \times n}^{A} \underbrace{\begin{bmatrix} & | & \\ - & b_{k\ell} & \\ & & \end{bmatrix}}_{n \times n}^{B}$$

$$C_{st} = \sum_u a_{su} \cdot b_{ut}$$

$= \langle$ Row s in A, Col t in $B\rangle$

$O(n^3)$ to compute all of C.
($O(n)$ per entry of C).
(worst case/ brute-force).

(Assume that n is a power of 2.)

Karatsuba's Integer Mult.

$$a = (\overset{\text{LSB}}{a_0}, \ldots, \overset{\text{MSB}}{a_{n-1}}) \qquad b = (\overset{\text{LSB}}{b_0}, \ldots, \overset{\text{MSB}}{b_{n-1}})$$

$$a = \sum_{i=0}^{n-1} a_i \cdot 2^i \qquad b = \sum_{j=0}^{n-1} b_j \cdot 2^j$$

$$= \underbrace{\left(\sum_{i=0}^{n/2-1} a_i \cdot 2^i\right)}_{A_0} + 2^{n/2}\underbrace{\left(\sum_{i'=0}^{\frac{n}{2}-1} a_{i'+\frac{n}{2}} \cdot 2^{i'}\right)}_{A_1} \qquad = \underbrace{\sum_{j=0}^{\frac{n}{2}-1} b_j \cdot 2^j}_{B_0} + 2^{n/2}\underbrace{\left(\sum_{j'=0}^{\frac{n}{2}-1} b_{j'+\frac{n}{2}} \cdot 2^{j'}\right)}_{B_1}$$

$$= A_0 + (A_1 \cdot 2^{n/2}) \qquad = B_0 + (B_1 \cdot 2^{n/2})$$

$$a \cdot b = (A_0 + A_1 \cdot 2^{n/2})(B_0 + B_1 \cdot 2^{n/2}) = \overbrace{A_0B_0}^{C_1} + \overbrace{(A_1B_0 + A_0B_1)}^{C_2} \cdot 2^{n/2} + \underbrace{A_1 \cdot B_1}_{C_3} \cdot 2^n$$

Obs: A_0, A_1, B_0, B_1 are integers of bit complexity $\leq \frac{n}{2}$.

$$T(n) = 4 \cdot T\left(\frac{n}{2}\right) + O(n) = n^{\log_2 4} \simeq n^2$$

Can we cut down on the no. of $\frac{n}{2}$-sized mult.

$$\underbrace{(A_0 + A_1)}\underbrace{(B_0 + B_1)} \qquad \underbrace{A_0B_0}_{C_1} \qquad \underbrace{A_1B_1}_{C_3}.$$

$$= \underbrace{A_0B_0 + A_1B_1 + (A_0B_1 + A_1B_0)}_{C^*}$$

$\hookrightarrow A_0B_1 + A_1B_0 = C^* - C_1 - C_3$

$A_0 + A_1$, $B_0 + B_1$ could have $\frac{n}{2}+1$ bits. $\frac{n}{2}+1 \leftarrow$

$$T(n) = 3 \cdot T\left(\frac{n}{2}+1\right) + O(n). \quad \sim n^{\log_2 3}$$

Increasing no. of additions/sub to optimize # of mult.

mult of $\frac{n}{2}$ bit nos. $\left\{ \begin{array}{l} A_0 \cdot B_0 \leftarrow C_1 \leftarrow (1) \text{ mult} \\ A_1 \cdot B_1 \leftarrow C_3 \leftarrow (2) \text{ mult} \end{array} \right.$

$A_0 + A_1 \leftarrow \tilde{A} \leftarrow (1)$ add

$B_0 + B_1 \leftarrow \tilde{B} \leftarrow (2)$ add

$\{(\tilde{A} \cdot \tilde{B}) \leftarrow \tilde{A} \cdot \tilde{B} \leftarrow (3)$ mult

mult. of $\leq \frac{n}{2}+1$ bit numbers.

$$C_2 = \underbrace{\tilde{A} \cdot \tilde{B} - C_1 - C_3}_{(3) \text{ sub/add}}$$

$$C_1, \; C_2, \; C_3 \longrightarrow \left.\underbrace{C_1 + \underbrace{C_2 \cdot 2^{n/2}}_{\text{Bit shifts}} + \underbrace{C_3 \cdot 2^{n}}}_{\text{Add}}\right\} O(n)$$

Nearest power of 2 is at most

$$n_0 \longrightarrow 2n_0$$

$$\left.\begin{aligned} a &= \tilde{A}_0 + \tilde{A}_1 \cdot 2^{n/3} + \tilde{A}_2 \cdot 2^{2n/3} \\ b &= \tilde{B}_0 + \tilde{B}_1 \cdot 2^{n/3} + \tilde{B}_2 \cdot 2^{2n/3} \end{aligned}\right\}$$

$\tilde{A}_i, \tilde{B}_j$ are $\frac{n}{3}$ bit no.s.

Brute force $\rightarrow n^{\log_3 9}$

$$T(n) = 9 \cdot T\left(\frac{n}{3}\right) + O(n).$$

Brief look into Strassen's matrix mult:

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{n\times n} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}_{n\times n} = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

$A_i, B_j \leftarrow$ are of size $\frac{n}{2} \times \frac{n}{2}$.

$$T(n) = 8 \cdot T\left(\frac{n}{2}\right) + O(n^2) \sim O\left(n^{\log_2 8}\right) \underset{\sim}{= cn^3}$$

Strassen gave a procedure with 7 matrix mult of $\frac{n}{2} \times \frac{n}{2}$ mat. at the expense of matrix additions. $\longrightarrow n^{\log_2 7} \sim n^{2.81}$

$\hookrightarrow$ "Laser method" + "tensor alg" w/ Coppersmith-Winograd tensor.

$\hookrightarrow \omega = 2.3$ ---

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Deepmind