Probability and Statistics: MA6.101

Tutorial 8

Topics Covered: Convergence

A1: Theorem: Consider the sequence $X_1, X_2, X_3, \ldots$. If for all $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty,$$

then $X_n \xrightarrow{a.s.} X$.

Note that $|X_n| = \frac{1}{n^2}$. Thus, $|X_n| > \epsilon$ if and only if $n^2 < \frac{1}{\epsilon}$. Therefore, we conclude that

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) \leq \sum_{n=1}^{\lfloor \frac{1}{\sqrt{\epsilon}} \rfloor} P(|X_n| > \epsilon) = \lfloor \frac{1}{\sqrt{\epsilon}} \rfloor < \infty.$$

Since $P(|X_n| > \epsilon) = 0.6 + 0.4 = 1$

This implies that $\sum_{n=1}^{\infty} P(|X_n| > \epsilon)$ is finite, and by the theorem, we conclude that

$$X_n \xrightarrow{a.s.} 0.$$

A2: Let $X(\omega) = \omega$. We want to show that $X_n \xrightarrow{a.s.} X$.

For $\omega \in [0, 1)$, consider the limit of $X_n(\omega) = \omega + \omega^n$:

$$\lim_{n\to\infty} \omega^n = 0 \quad \text{as} \quad \omega \in [0, 1).$$

Thus,

$$\lim_{n\to\infty} (\omega + \omega^n) = \omega.$$

When $\omega = 1$, we have:

$$X_n(1) = 1 + 1^n = 2 \quad \text{for all} \quad n.$$

Note that $X_n(1) = 2$ is not equal to 1, but since $P(\omega = 1) = 0$, this does not affect almost sure convergence.

Therefore, we conclude that:

$$X_n \xrightarrow{a.s.} X.$$

A3: To prove that Y_n converges in probability to 0, we need to show that for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|Y_n| \geq \epsilon) = 0.$$

From the given probability distribution of Y_n, we know that $y \geq \epsilon$ is only possible when $y \neq 0$, i.e., $y = n^2$, since $\epsilon > 0$.

For sufficiently large n ($n \to \infty$), n^2 will always be greater than any fixed $\epsilon > 0$. So, for any $\epsilon > 0$, we have

$$P(|Y_n| \geq \epsilon) = P(Y_n = n^2) = e^{-n}.$$

Thus, for large n, the condition $|Y_n| \geq \epsilon$ corresponds to $Y_n = n^2$, and the probability of this happening is $P(Y_n = n^2) = e^{-n}$. Since $e^{-n} \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} P(|Y_n| \geq \epsilon) = 0.$$

This shows that Y_n converges in probability to 0.

The fact that Y_n also converges in distribution to 0 can be shown by considering the CDF of Y_n:

$$F_{Y_n}(y) = \begin{cases} 0, & x < 0, \\ 1 - e^{-n}, & 0 \leq x < n^2, \\ 1, & x \geq n^2. \end{cases}$$

As $n \to \infty$, the CDF of Y_n converges pointwise to (since, $\lim_{n\to\infty} 1 - e^{-n} = 1$):

$$F(y) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

This is the CDF of the constant random variable 0, meaning Y_n converges in distribution to 0.

Note that, since Y_n converges in probability to 0, i.e.,

$$Y_n \xrightarrow{P} 0,$$

therefore, it must also converges in distribution to 0, i.e.,

$$Y_n \xrightarrow{d} 0.$$

A4: For any $\epsilon > 0$,

$$\begin{aligned} \lim_{n\to\infty} P(|X_n - 0| \geq \epsilon) &= \lim_{n\to\infty} P(|X_n| \geq \epsilon) \\ &= \lim_{n\to\infty} 2P(X_n \geq \epsilon) \quad \text{(by symmetry)} \\ &= \lim_{n\to\infty} \int_{\epsilon}^{\infty} ne^{-nx} dx \\ &= \lim_{n\to\infty} \frac{1}{e^{n\epsilon}} \\ &= 0 \end{aligned}$$

Therefore, X_n converges to 0 in probability.

For convergence in distribution:

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} P(X_n \le x)$$

$$= \lim_{n\to\infty} \int_{-\infty}^{x} \frac{n}{2} e^{-n|x|}\, dx$$

when $x > 0$:

$$= \lim_{n\to\infty} \left(\int_{-\infty}^{0} \frac{n}{2} e^{nx}\, dx + \int_{0}^{x} \frac{n}{2} e^{-nx}\, dx \right)$$

$$= \lim_{n\to\infty} (1 + 1 - e^{-nx})/2 = 1$$

when $x < 0$:

$$= \lim_{n\to\infty} \left(\int_{-\infty}^{x} \frac{n}{2} e^{nx}\, dx \right)$$

since, $x < 0$

$$= \lim_{n\to\infty} -e^{nx}/2 = 0$$

$$= \begin{cases} 1, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$= F_X(x)$$

where X is a constant random variable 0. Therefore, X_n converges to 0 in distribution.

Note: At x=0, the cdf is 1/2 (symmetric). Since, x = 0 is a point of discontinuity, the CDF need not match at that point.

A5: We first show that X_n converges to 0 in mean square. Convergence in mean square means that:

$$\lim_{n\to\infty} \mathbb{E}\left[(X_n - 0)^2\right] = 0.$$

Since X_n takes only the values 0 and 1, we have:

$$(X_n - 0)^2 = X_n^2 = X_n,$$

because $X_n^2 = X_n$ (since X_n is either 0 or 1). Therefore, we need to compute $\mathbb{E}[X_n]$. By the definition of X_n, we have:

$$\mathbb{E}[X_n] = 0 \cdot \left(1 - \frac{1}{n}\right) + 1 \cdot \frac{1}{n} = \frac{1}{n}.$$

Thus:

$$\mathbb{E}[X_n^2] = \mathbb{E}[X_n] = \frac{1}{n}.$$

As $n \to \infty$, $\mathbb{E}[X_n^2] \to 0$. Therefore, we conclude that X_n converges to 0 in mean square.

A6:

$$\begin{aligned}
E[|Y_n - \lambda|^2] &= E[(Y_n - \lambda)^2] \\
&= E\left[\left(\frac{1}{n}X_n - \lambda\right)^2\right] \\
&= E\left[\frac{1}{n^2}(X_n - \lambda n)^2\right] \\
&= \frac{1}{n^2}E\left[(X_n - E[X_n])^2\right], \text{ since, for } X_n \sim \text{Poisson}(n\lambda),\ E[X_n] = n\lambda \\
&= \frac{1}{n^2}Var[X_n] \\
&= \frac{1}{n^2}n\lambda, \text{ since, for } X_n \sim \text{Poisson}(n\lambda),\ Var[X_n] = n\lambda \\
&= \frac{\lambda}{n}.
\end{aligned}$$

Thus, $\lim_{n\to\infty} E[|Y_n - \lambda|^2] = 0$, or $Y_n \xrightarrow{m.s.} \lambda$.

A7: We will use the following theorem to prove this question:

Consider the sequence $X_1, X_2, X_n, \cdots$ and the random variable X. Assume that X and X_n (for all n) are non-negative and inter-valued, i.e.

$$\begin{aligned}
&R_X \subset \{0,1,2,\cdots\}, \\
&R_{X_n} \subset \{0,1,2,\cdots\}, \qquad \text{for } n = 1,2,3,\cdots.
\end{aligned}$$

Then $X_n \xrightarrow{d} X$ (convergence in probabilty) if and only if

$$\lim_{n\to\infty} P_{X_n}(k) = P_X(k), \qquad \text{for } k = 0,1,2,\cdots.$$

Now, We have

$$\begin{aligned}
\lim_{n\to\infty} P_{X_n}(k) &= \lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \lambda^k \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\left(\frac{1}{n^k}\right)\left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!}.\lim_{n\to\infty}\left(\left[\frac{n(n-1)(n-2)...(n-k+1)}{n^k}\right]\left[\left(1-\frac{\lambda}{n}\right)^n\right]\left[\left(1-\frac{\lambda}{n}\right)^{-k}\right]\right).
\end{aligned}$$

Note that for a fixed k, we have

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k+1)}{n^k} = 1,$$
$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-k} = 1,$$
$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}.$$

Thus, we conclude

$$\lim_{n\to\infty} P_{X_n}(k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

A8: Given the CDF:

$$F_{X_n}(x) = \begin{cases} \frac{e^{nx}+xe^n}{e^{nx}+\left(\frac{n+1}{n}\right)e^n}, & 0 \le x \le 1, \\ \frac{e^{nx}}{e^{nx}+\left(\frac{n+1}{n}\right)e^n}, & x > 1 \end{cases}$$

We need to show that X_n converges in distribution to Uniform$(0,1)$. CDF of uniform is:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \le x \le 1, \\ 1, & x > 1 \end{cases}$$

For $0 \le x \le 1$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{e^{nx}+xe^n}{e^{nx}+\left(\frac{n+1}{n}\right)e^n}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{e^n}{e^n} \frac{e^{nx-n}+x}{e^{nx-n}+\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{e^n}{e^n} \frac{e^{n(x-1)}+x}{e^{n(x-1)}+\left(1+\frac{1}{n}\right)}$$

since $0 \le x \le 1, -1 \le x-1 \le 0$,

$$\lim_{n\to\infty} F_{X_n}(x) = x$$

For $x > 1$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{e^{nx}}{e^{nx}+\left(\frac{n+1}{n}\right)e^n}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{1}{1+\left(1+\frac{1}{n}\right)e^{n-nx}}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{1}{1+\left(1+\frac{1}{n}\right)e^{n(1-x)}}$$

$$\lim_{n\to\infty} F_{X_n}(x) = 1$$

For $x < 0$

There is no explicit expression for $F_{X_n}(x)$ when $x < 0$, we conclude that:

$$\lim_{n\to\infty} F_{X_n}(x) = 0 \quad \text{for} \quad x < 0$$

For all $x \in \mathbb{R}$, we have shown that $F_{X_n}(x)$ converges for all continuity points to the CDF of the Uniform(0,1) distribution:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \le x \le 1, \\ 1, & x > 1 \end{cases}$$

Therefore, X_n converges in distribution to Uniform$(0,1)$.

A9:

$$F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

Also, $R_{Y_n} = [0,1]$.
For $0 \le y \le 1$,

$$\begin{aligned} F_{Y_n}(y) &= P(Y_n \le y) \\ &= 1 - P(Y_n > y) \\ &= 1 - P(X_1 > y, X_2 > y, \ldots, X_n > y) \\ &= 1 - P(X_1 > y)P(X_2 > y)\ldots P(X_n > y) \quad \text{(since } X_i\text{'s are independent)} \\ &= 1 - (1 - F_{X_1}(y))(1 - F_{X_2}(y))\ldots(1 - F_{X_n}(y)) \\ &= 1 - (1-y)^n \end{aligned}$$

Therefore,

$$F_{Y_n}(y) = \begin{cases} 0, & y < 0 \\ 1-(1-y)^n, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

In particular, note that Y_n is continuous. To show $Y_n \xrightarrow{p} 0$, we need to show that

$$\lim_{n\to\infty} P(|Y_n| \ge \epsilon) = 0 \quad \text{for all } \epsilon > 0$$

Since $Y_n \ge 0$, it suffices to show that

$$\lim_{n\to\infty} P(Y_n \ge \epsilon) = 0 \quad \text{for all } \epsilon > 0$$

For $\epsilon \in (0, 1)$, we have

$$\begin{aligned} P(Y_n \geq \epsilon) &= 1 - P(Y_n < \epsilon) \\ &= 1 - P(Y_n \leq \epsilon) \quad \text{(since } Y_n \text{ is a continuous random variable)} \\ &= 1 - F_{Y_n}(\epsilon) \\ &= (1-\epsilon)^n \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} P(Y_n \geq \epsilon) = \lim_{n\to\infty} (1-\epsilon)^n = 0 \quad \text{for all}\, \epsilon \in (0, 1]$$

And for $\epsilon \in [1, \infty)$, we have

$$\lim_{n\to\infty} P(Y_n \geq \epsilon) = 1 - 1 \quad \text{for all}\, \epsilon \in [1, \infty)$$

Since $F_{Y_n} = 1$ for all $\epsilon \in [1, \infty)$

Thus,

$$\lim_{n\to\infty} P(Y_n \geq \epsilon) = 0 \quad \text{for all}\, \epsilon > 0$$

Thus, Y_n converges to 0 in probability. Hence Proved.