Probability and Statistics: MA6.101

Tutorial 11

Topics Covered: Probability Inequalities, Statistics (Classical and Bayesian)

Probability Inequalities

Q1: A biased coin, which lands heads with probability 1/10 each time it is flipped, is flipped 200 times consecutively. Give an upper bound on the probability that it lands heads at least 120 times.

Q2: Show that convergence in mean square implies convergence in probability.

Q3: Chebyshev Inequality states that:
If X is any random variable, then for any $b > 0$ we have:

$$P\big(|X - EX| \geq b\big) \leq \frac{Var(X)}{b^2}.$$

Let $X \sim Binomial(n, p)$. Using Chebyshev's inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Evaluate the bound for $p = \frac{1}{2}$ and $= \frac{3}{4}$. Calculate using Markov's inequality for the similar parameters, and comment on the betterness of the bounds obtained.

Statistics

Classical/Frequentist Methods

Q1: Prove that maximising the likelihood is the same as maximising the log likelihood. [Hint: Start by taking 2 different values of the parameter; one maximises the likelihood function, and the other maximises the log-likelihood function].

Q2: Let $X_1, X_2, \ldots, X_n$ be a random sample from a $Geometric(p)$ distribution. Suppose we observe the data $\{x_1, x_2, x_3, x_4\} = \{2, 1, 7, 3\}$. The probability mass function of the Geometric distribution is given by:

$$P_X(x; p) = p(1-p)^{x-1}$$

Find the maximum likelihood estimate (MLE) of p

Q3: In the Cilantro experiment, assume 55 out of 100 people said Cilantro tastes like soap. Find the maximum likelihood estimate for p, the true proportion of people who feel that way.

Q4: Let $\mathcal{D} = \{x_1, ..., x_n\}$ denote i.i.d samples from a uniform random variable $U[0, a]$, where a is unknown. Find an MLE estimate for the unknown parameter a.

Q5: Assume our data $\mathbf{Y} = (y_1, y_2, ..., y_n)^T$ given X is independently identically distributed, i.i.d. $Y|X = x \sim Exponential(\lambda = x)$, and we chose the prior to be $X \sim Gamma(\alpha, \beta)$.

a. Find the likelihood of the function, $L(\mathbf{Y}; X) = f_{\mathbf{Y}|X}(y_1, y_2, ..., y_n|x)$.

b. Using the likelihood function of the data, show that the posterior distribution is $Gamma(\alpha + n, \beta + \sum_{i=1}^{n} y_i)$.

c. Write out the PDF for the posterior distribution, $f_{X|\mathbf{Y}}(x|\mathbf{y})$.

Q6: Let $X_1, \ldots, X_n$ be a random sample from a Poisson(λ) distribution.

(a) Find the likelihood function, $L(x_1, \ldots, x_n; \lambda)$

(b) Find the log-likelihood function and use that to obtain the MLE for λ, $\hat{\lambda}_{\text{ML}}$.

Q7: Suppose you are trying to fit a distribution

$$P(X = x|\theta) = (\tfrac{\theta}{2})^{|x|}(1-\theta)^{1-|x|}$$

where the support set is $\{-1, 0, 1\}$ and θ is a real number in the range $[0, 1]$.

(a) Define an estimator as

$$T(X) = 2 \text{ if } x = 1 \text{ otherwise } T(X) = 0.$$

Show that $T(X)$ is an unbiased estimator of θ.

(b) Now define another estimator as $G(X) = |X|$. Show that $G(x)$ is also an unbiased estimator of θ.

(c) Which of the estimators is better for θ? Justify your answer.

Q8: Using a rod of length μ, you lay out a square plot whose length of each side is μ. Thus, the area of the plot will be μ^2 (unknown). Based on n independent measurements $X_1, X_2, \ldots, X_n$ of the length, estimate μ^2. Assume that each X_i has mean μ and variance σ^2.

(a) Show that $\overline{X}^2$ is not an unbiased estimator for μ^2.

(b) For what value of k is the estimator $\overline{X}^2 - kS^2$ unbiased for μ^2?

Q9: Consider an experiment with two possible outcomes:

- **{Success}**, with probability p
- **{Failure}**, with probability $1 - p$

The probability of success, p, is known to lie within the interval $\left[\frac{1}{10}, \frac{1}{5}\right]$. Suppose the experiment is repeated independently n times. The estimator for p is defined as:

$$\widehat{p} = \frac{\text{Successes obtained}}{\text{Total experiments performed}}.$$

Our objective is to determine the minimum value of n such that the standard deviation of the estimator $\widehat{p}$ is guaranteed to be less than $\frac{1}{100}$ for all $p \in \left[\frac{1}{10}, \frac{1}{5}\right]$.

Q10: Let $X_1, X_2, \ldots, X_n$ be a random sample from the following distribution:

$$f_X(x) = \begin{cases} \theta\left(x - \frac{1}{2}\right) + 1 & \text{for } 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta \in [-2, 2]$ is an unknown parameter. We define the estimator $\hat{\Theta}_n$ as:

$$\hat{\Theta}_n = 12\overline{X} - 6$$

to estimate θ.

(a) Is $\hat{\Theta}_n$ an unbiased estimator of θ?

(b) Is $\hat{\Theta}_n$ a consistent estimator of θ?

(c) Find the mean squared error (MSE) of $\hat{\Theta}_n$.

Bayesian Inference

Q11: Let $X_1, X_2, X_3, ..., X_n$ be a random sample from a distribution with mean $E[X_i] = \theta$ and variance $Var(X_i) = \sigma^2$. Consider the following two estimators for θ:

1. $\hat{\Theta}_1 = X_1$ 2. $\hat{\Theta}_2 = \overline{X} = \frac{X_1+X_2+...+X_n}{n}$

Find $MSE(\hat{\Theta}_1)$ and $MSE(\hat{\Theta}_2)$ and show that for $n > 1$ we have $MSE(\hat{\Theta}_1) > MSE(\hat{\Theta}_2)$.

Q12: Let $X \sim Uniform(0, 1)$. Suppose that we know

$$Y|X = x \sim Geometric(x).$$

Find the posterior density of X given $Y = 2$, $f_{X|Y}(x|2)$.

Q13: Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, suppose that $Y|X = x \sim Geometric(x)$. Find the MAP estimate of X given $Y = 5$.

Q14: Suppose $D = \{x_1, ..., x_n\}$ is a data set consisting of independent samples of a Bernoulli random variable with unknown parameter θ, i.e., $f(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$ for $x_i \in \{0, 1\}$. Obtain an expression for the posterior distribution on θ. Using this, obtain $\hat{\theta}_{MAP}$ and the conditional expectation estimator $\hat{\theta}_{CE}$.

(Hint: $\int_0^1 \theta^m(1-\theta)^r d\theta = \frac{m!r!}{(m+r+1)!}$)

Q15: Suppose we have a prior $\theta \sim \mathcal{N}(4, 8)$, and the likelihood function is $\phi(x_1|\theta) \sim \mathcal{N}(\theta, 5)$. Suppose also that we have one measurement $x_1 = 3$. Show that the posterior distribution is normal.

Q16: Suppose that the signal $X \sim N(0, \sigma_X^2)$, is transmitted over a communication channel. Assume that the received signal is given by:

$$Y = aX + bW \quad \text{where} \quad W \sim N(0, \sigma_W^2)$$

and W is independent of X.

(a) Find the ML estimate of X, given Y=y is observed.

(b) Find the MAP estimate of X, given Y=y is observed.

Q17: Let Θ be a continuous random variable with pdf as $\frac{1}{6}$ for $\theta \in [4, 10]$ and 0 elsewhere. And we know that $X = \Theta + U[-1, 1]$. Find the conditional expectation estimator.

Q18: Show that the conditional expectation estimator is the best guess, in terms of minimizing mean square error, i.e. show the following -

$$E[(\theta - E[\Theta|X])^2|X] \leq E[(\theta - g(x))^2|X]$$