Probability and Statistics: MA6.101

Homework 10

Topics Covered: Markov Chains

Q1: Consider the Markov chain with three states $S = \{1, 2, 3\}$ and the state transition diagram given below: Suppose $P(X_1 = 1) = \frac{1}{2}$ and $P(X_1 = 2) = \frac{1}{4}$.

(a) Find the state transition matrix for this chain.

(b) Find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

(c) Find $P(X_1 = 3, X_3 = 1)$.

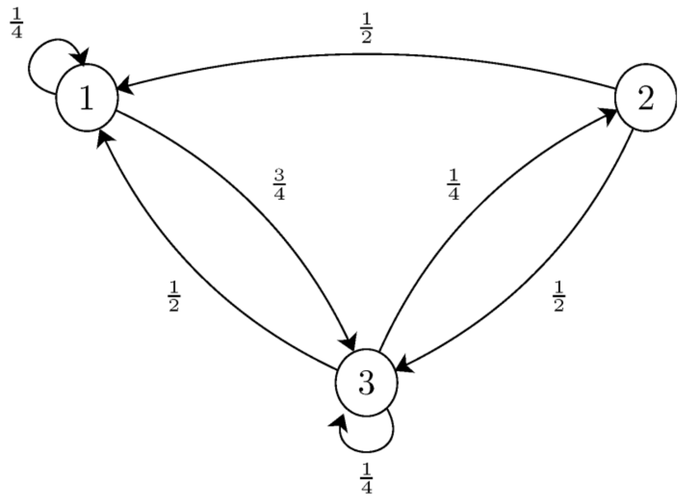

Figure 1: Transition Diagram Q1

Q2: A transition matrix is said to be double stochastic if $\sum_{i=0}^{M} P_{ij} = 1$ for all states $j = 0, 1 \ldots M$ (i.e. every column sums to 1). Show that such a DTMC has the stationary distribution $\pi_j = \frac{1}{M+1} \forall j$

Q3: Write the transition matrix of the following Markov chains

a) n black balls and n white balls are placed in two urns so that each urn contains n balls. At each stage one ball is selected at random from each urn and the two balls interchange. The state of the system is the number of white balls in the first urn.

b) Consider two urns A and B containing a total of n balls. An experiment is performed in which a ball is selected at random at time $t(t = 1, \ldots)$ from among the totality of n balls. Then an urn is selected at random (probability of selecting A is p) and the ball previously drawn is placed in this urn. The state of the system at each trial is the number of balls in A.

Q4: Consider a spinner with numbers 1 through 4 that is spun repeatedly. Define the following processes:

(a) Let S_n represent the highest number observed on the spinner up to the n-th spin.

(b) At the n-th spin, let T_n denote the number of spins required to observe the next "4."

Prove that both S_n and T_n follow the Markov property, and determine the transition probabilities for each process.

Q5: A digital signal processing device operates using only two signals, represented by the values -1 and 1. The device is designed to transmit one of these signals through multiple stages. However, at each stage, there is a probability r that the signal entering the stage will be flipped when it exits, and a probability $s = 1 - r$ that it will remain unchanged.

(a) Construct a Markov chain to model the transmission process.

(b) Determine the transition probability matrix.

(c) Assuming that both signals are equally likely at the initial stage, calculate the probability that the device outputs the signal -1 after passing through two stages.

Q6: Consider the Markov chain in figure below.

(a) Find the recurrent classes R_1 and R_2.

(b) Assuming $X_0 = 3$, find the probability that the chain gets absorbed in R_1.

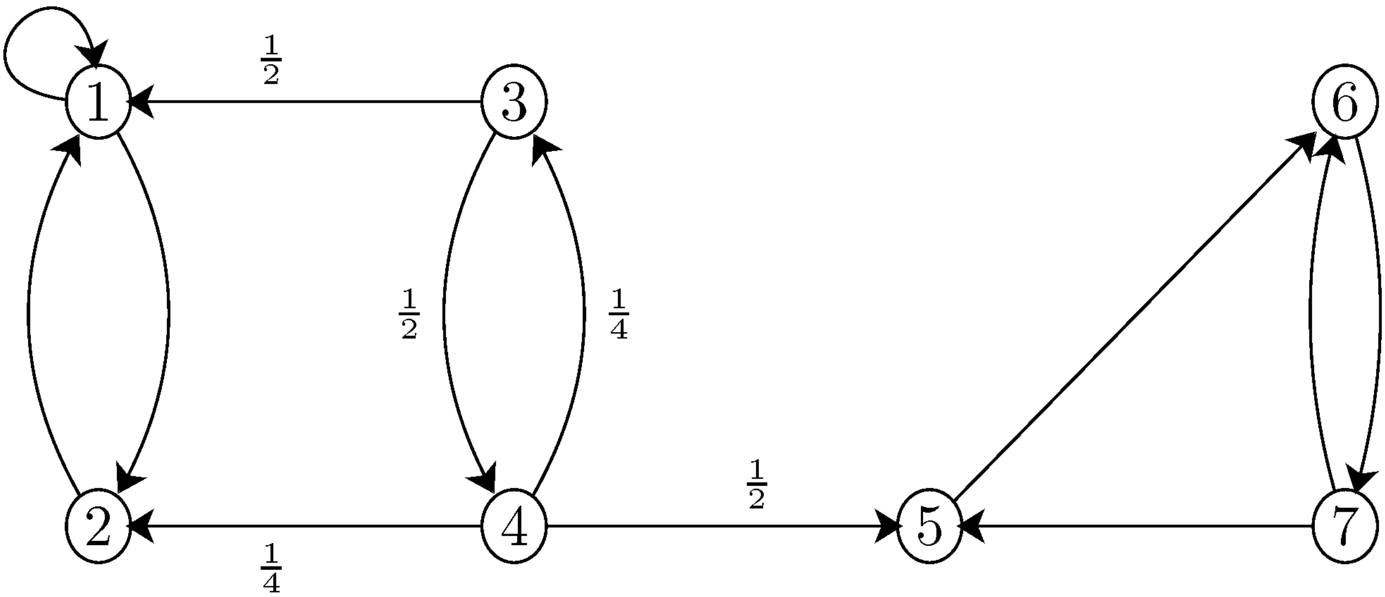

Figure 2: Markov Chain

Q7: Let $X = [X_0, X_1, \dots]$ be a Markov chain having transition matrix P. Recall that for any non-negative integer n, we have $P_{ij}^{(n)} = \Pr(X_n = j \mid X_0 = i)$. Then for any $m \geq 0$ and $n \geq 0$, we get

$$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)}.$$

This equation is called the *Chapman-Kolmogorov equation*. Prove this.

If X has a finite state space, we can write $P^{(n)}$ as a matrix, called the n-step transition matrix. Then $P^{(n)} = P^n$ for all $n \geq 0$.