Probability and Statistics: MA6.101

Tutorial 1

Topics Covered: Sigma Algebra, Set Theory, Probability Axioms, Conditional Probability, Permutations and Combinations

Q1: You purchase a certain product. The manual states that the lifetime T of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = e^{-t/5}, \quad \text{for all } t \geq 0.$$

For example, the probability that the product lasts more than (or equal to) 2 years is $P(T \geq 2) = e^{-2/5} = 0.6703$. I purchase the product and use it for two years without any problems. What is the probability that it breaks down in the third year?

Q2: There are 30 people in a room. What is the chance that any two of them celebrate their birthday on the same day? Assume 365 days in a year.

Q3: Prove the following inequality without the use of induction:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Q4: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

Q5: A permutation σ is called a *derangement* if $\forall i, \sigma(i) \neq i$. Consider a uniform random permutation σ of $\{1, \ldots, n\}$, and let D_n be the event that σ is a derangement. Use the inclusion-exclusion principle to find a formula for the number of derangements, and show that $\mathbb{P}(D_n) \xrightarrow{n\to\infty} e^{-1}$.

Q6: A 6-sided die is rolled n times. What is the probability all faces have appeared? (Hint: Use Principle of Inclusion and Exclusion)

Q7: Let $E_1, E_2, \ldots, E_n$ be n events, each with positive probability. Prove that

$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = \mathbb{P}\{E_1\} \cdot \mathbb{P}\{E_2 \mid E_1\} \cdot \mathbb{P}\{E_3 \mid E_1 \cap E_2\} \cdots \mathbb{P}\left\{E_n \mid \bigcap_{i=1}^{n-1} E_i\right\}.$$

Q8: Queueville Airlines knows that on average 5% of the people making flight reservations do not show up. (They model this information by assuming that each person independently does not show up with probability of 5%.) Consequently, their policy is to sell 52 tickets for a flight that can only hold 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?