# Probability and Statistics: MA6.101

# Tutorial 8

Topics Covered: Convergence

Q1: Consider a sequence $\{X_n, n = 1, 2, 3, \dots\}$ such that

$$X_n = \begin{cases} -\frac{1}{n^2}, & \text{with probability } 0.6, \\ \frac{1}{n^2}, & \text{with probability } 0.4. \end{cases}$$

Show that $X_n \xrightarrow{a.s.} 0$.

Q2: Consider the sequence of random variables $X_n(\omega) = \omega + \omega^n$, $\omega \in [0, 1]$. Show that $X_n$ converges almost surely.

Q3: Consider a sequence of discrete random variables $Y_n$ with the following distribution:

$$P(Y_n = y) = \begin{cases} 1 - e^{-n}, & \text{for } y = 0, \\ e^{-n}, & \text{for } y = n^2, \\ 0, & \text{elsewhere.} \end{cases}$$

Prove that $Y_n$ converges in probability to 0. Does it also converge in distribution to 0?

Q4: Let $X_1, X_2, X_3, \dots$ be a sequence of random variables with density

$$f_{X_n}(x) = \frac{n}{2} e^{-n|x|}$$

Show that $X_n$ converges to 0 in probability and in distribution

Q5: Let $X_1, X_2, \dots$ be a sequence of random variables, defined by

$$X_n = \begin{cases} 0 & \text{with probability } 1 - \frac{1}{n} \\ 1 & \text{with probability } \frac{1}{n} \end{cases}$$

Show that $X_n$ converges to 0 in mean square

Q6: Let $X_1, X_2, X_3, \cdots$ be a sequence of random variables such that

$$X_n \sim \text{Poisson}(n\lambda), \quad \text{for } n = 1, 2, 3, \cdots,$$

where $\lambda > 0$ is a constant. Define a new sequence $Y_n$ as

$$Y_n = \frac{X_n}{n}, \quad \text{for } n = 1, 2, 3, \cdots.$$

Show that $Y_n$ converges in mean square to $\lambda$, i.e.,

$$Y_n \xrightarrow{\text{m.s.}} \lambda.$$

Q7: Let $X_1, X_2, X_3, \cdots$ be a sequence of random variables such that

$$X_n \sim Binomial\left(n, \frac{\lambda}{n}\right), \qquad \text{for } n \in \mathbb{N}, n > \lambda,$$

where $\lambda > 0$ is a constant. Show that $X_n$ converges in distribution to $Poisson(\lambda)$

Q8: Let $X_2, X_3, X_4, \ldots$ be a sequence of random variables such that

$$F_{X_n}(x) = \begin{cases} \frac{e^{nx} + xe^n}{e^{nx} + \left(\frac{n+1}{n}\right)e^n}, & 0 \le x \le 1, \\ \frac{e^{nx}}{e^{nx} + \left(\frac{n+1}{n}\right)e^n}, & x > 1 \end{cases}$$

Show that $X_n$ converges in distribution to Uniform$(0, 1)$.

Q9: Suppose $X_n$, $n = 1, 2, 3, \ldots$ are i.i.d. uniform $U[0, 1]$ and let $Y_n = \min(X_1, \ldots, X_n)$. Show that $Y_n$ converges to 0 in probability.