CS 302.1 - Automata Theory

Lecture 03

Shantanav Chakraborty

Center for Quantum Science and Technology (CQST)

Center for Security, Theory and Algorithms (CSTAR)

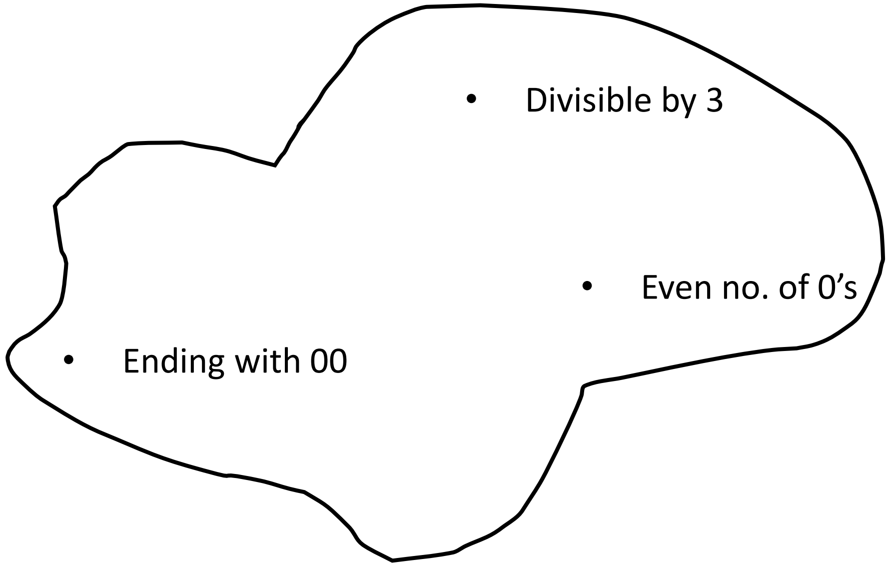
IIIT Hyderabad

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

Quick Recap

- DFAs and NFAs are equivalent
- For every NFA we can obtain a "Remembering DFA" that accepts the same language.
- The language accepted by finite automata are called Regular Languages.



Set of all regular Languages

A language is called a **Regular Language** if there exists some finite automata recognizing it.

If M be a finite automaton (DFA/NFA) and,

$$L(M) = \{\omega | \omega \text{ is accepted by } M\}$$

$\boldsymbol{L(M)}$ **is regular.**

Regular Languages

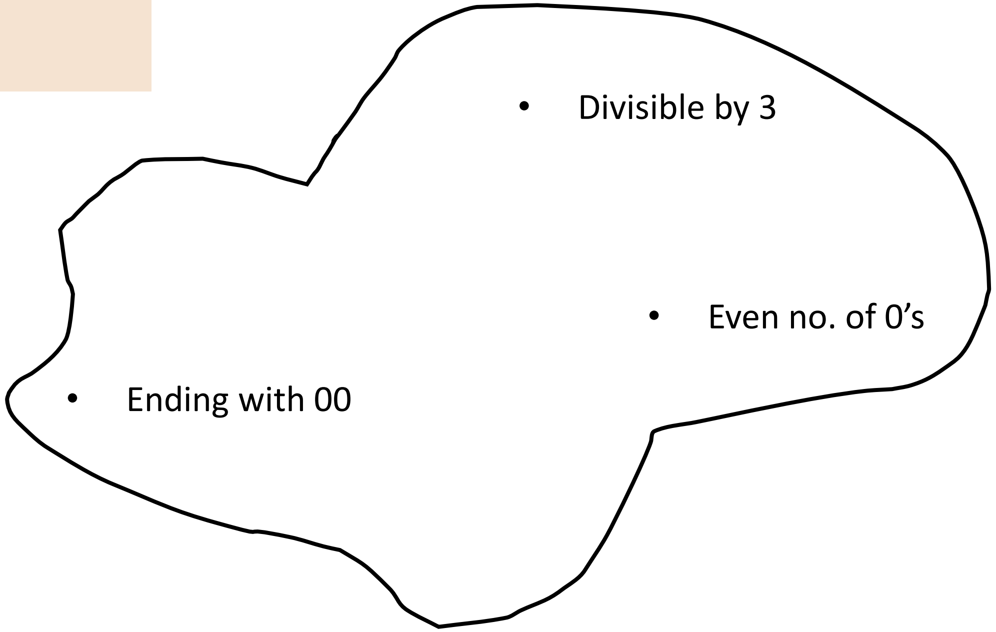
A language is called a **Regular Language** if there exists some finite automata recognizing it.

If M be a finite automaton (DFA/NFA) and,

$$L(M) = \{\omega | \omega \text{ is accepted by } M\}$$

$\boldsymbol{L(M)}$ **is regular.**

- Any language has associated with it, a set of operations that can be performed on it.
- These operations help us to understand the properties of that language, e.g. closure properties
- For regular languages, this will help us prove that certain languages are non-regular and hence we cannot hope to design a finite automaton for them



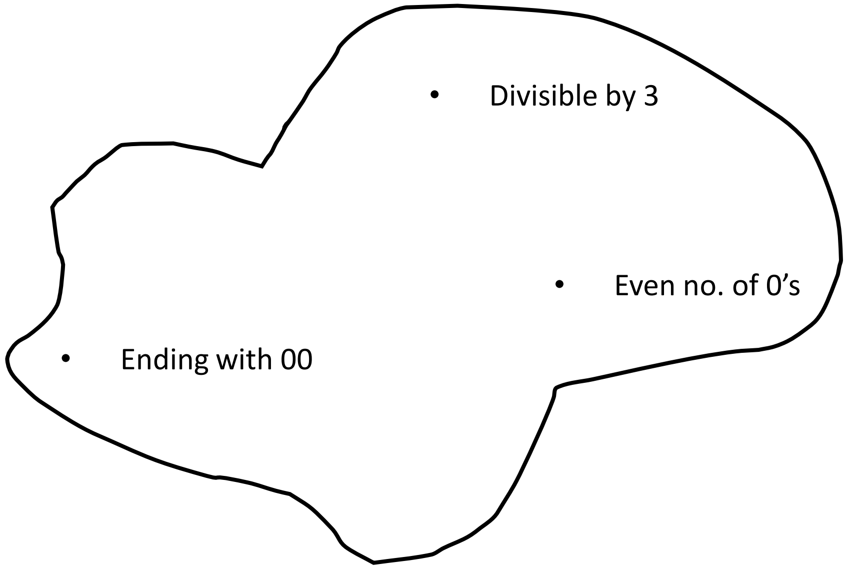
Set of all regular Languages

Regular Languages

Regular Operations:

Let L_1 and L_2 be languages. The following are the *regular operations*:

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k |\ k \geq 0 \text{ and each } x_i \in L_1\}$



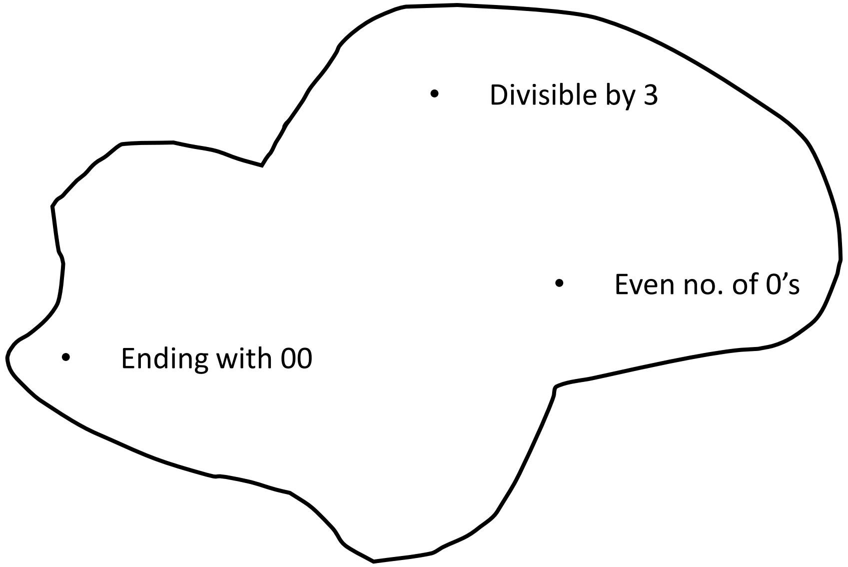
Set of all regular Languages

Regular Languages

Regular Operations:

Let L_1 and L_2 be languages. The following are the *regular operations*:

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k | \; k \geq 0 \text{ and each } x_i \in L_1\}$



Set of all regular Languages

Star operation: It is a unary operation (unlike the other two) and involves putting together *any number of strings in L_1 together to obtain a new string.*

Note: Any number of strings includes "0" as a possibility and so the empty string ϵ is a member of L_1^***.**

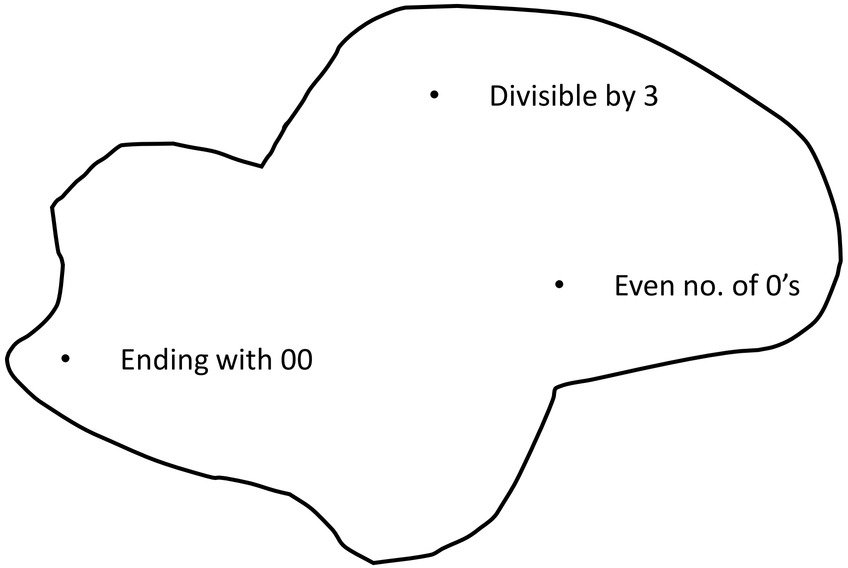
If $\Sigma = \{a\}$, $\Sigma^* = \{\epsilon, a, aa, aaa, \ldots \ldots\}$; If $\Sigma = \{\Phi\}, \Sigma^* = \{\epsilon\}$

Regular Languages

Regular Operations:

Let L_1 and L_2 be languages. The following are the *regular operations*:

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k |\ k \geq 0 \text{ and each } x_i \in L_1\}$



Set of all regular Languages

Star operation: It is a unary operation (unlike the other two) and involves putting together *any number of strings in L_1 together to obtain a new string.*

Note: Any number of strings includes "0" as a possibility and so the empty string ϵ is a member of L_1^***.**

If $\mathrm{L} = \{0,1\}$, we have that $\mathrm{L}^* = \{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \ldots\}$

Regular Languages

Regular Operations: Let L_1 and L_2 be languages.

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k |\ k \geq 0 \text{ and each } x_i \in L_1\}$

Example: Let the alphabet $\Sigma = \{a, b, \cdots, z\}$. If $L_1 = \{$*social, economic*$\}$ and $L_2 = \{$*justice, reform*$\}$, then

- $L_1 \cup L_2 = \{$*social, economic, justice, reform*$\}$

Regular Languages

Regular Operations: Let L_1 and L_2 be languages.

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k |\ k \geq 0 \text{ and each } x_i \in L_1\}$

Example: Let the alphabet $\Sigma = \{a, b, \cdots, z\}$. If $L_1 = \{\textit{social, economic}\}$ and $L_2 = \{\textit{justice, reform}\}$, then

- $L_1 \cup L_2 = \{\textit{social, economic, justice, reform}\}$
- $L_1 . L_2 = \{\textit{socialjustice, socialreform, economicjustice, economicreform}\}$

Regular Languages

Regular Operations: Let L_1 and L_2 be languages.

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k |\ k \geq 0 \text{ and each } x_i \in L_1\}$

Example: Let the alphabet $\Sigma = \{a, b, \cdots, z\}$. If $L_1 = \{$*social, economic*$\}$ and $L_2 = \{$*justice, reform*$\}$, then

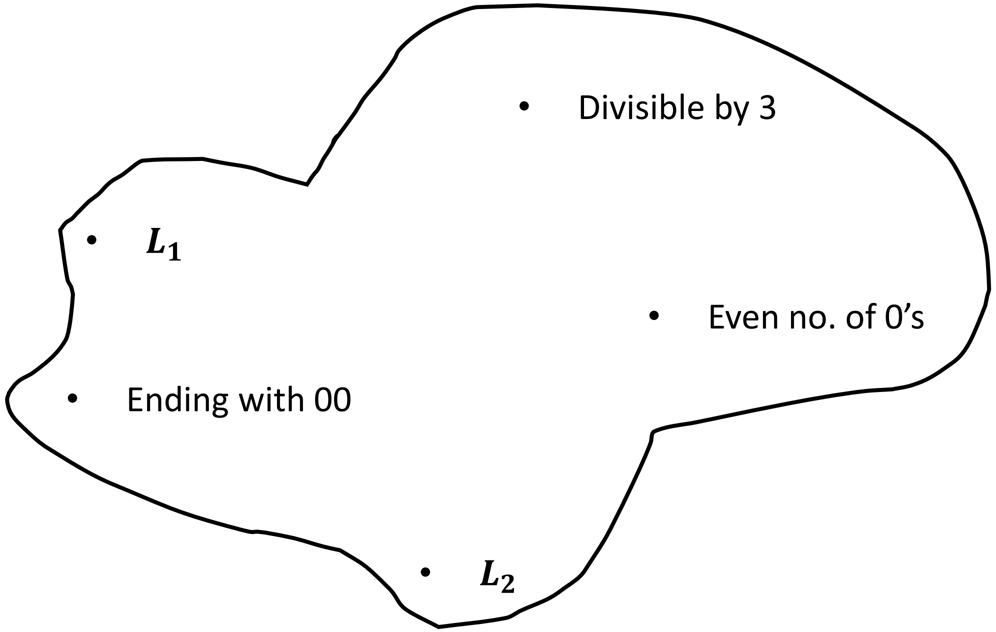
- $L_1 \cup L_2 = \{$*social, economic, justice, reform*$\}$
- $L_1 . L_2 = \{$*socialjustice, socialreform, economicjustice, economicreform*$\}$
- $L_1^* = \{\epsilon$, *social, economic, socialsocial, socialeconomic, economicsocial, economiceconomic, socialsocialsocial, socialsocialeconomic, socialeconomiceconomic,……*$\}$
- $L_2^* = \{\epsilon$, *justice, reform, justicejustice, justicereform, reformjustice, reformreform, justicejusticejustice, ……*$\}$

Closure of Regular Languages

We want to check whether the set of regular languages are **closed** under some operations.

What does this mean?

- We pick up points within the set of all regular languages (say L_1 and L_2)
- Perform *set operations* such as Union, concatenation, Star, intersection, reversal, complement etc on them.
- Observe whether the resulting language still belongs to the set of all regular languages.
- If so, we say, regular languages are **closed** under that operation.



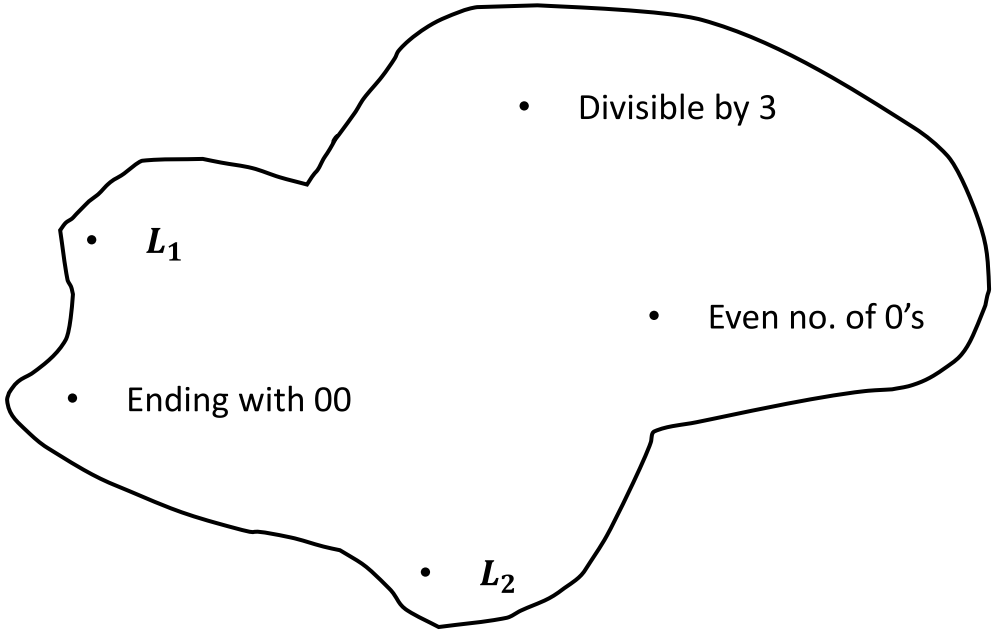
Set of all regular Languages

Closure of Regular Languages

We want to check whether the set of regular languages are **closed** under some operations.

What does this mean?

- We pick up points within the set of all regular languages (say L_1 and L_2)
- Perform *set operations* such as Union, concatenation, Star, intersection, reversal, complement etc on them.
- Observe whether the resulting language still belongs to the set of all regular languages.
- If so, we say, regular languages are **closed** under that operation.



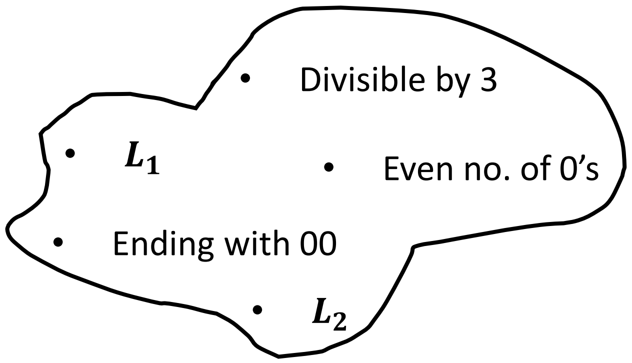
Set of all regular Languages

For example, the **natural numbers are closed under addition/multiplication** and **not under subtraction/division**.

Closure of Regular Languages

Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?



Set of all regular Languages

Closure of Regular Languages

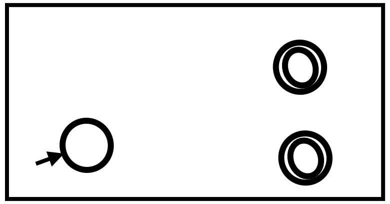
Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?

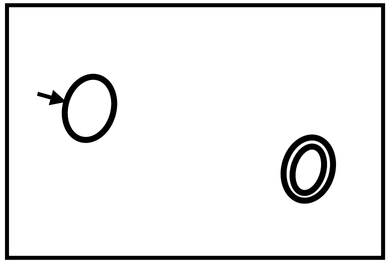
Proof: Since L_1 and L_2 are regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$ and a DFA M_2 that accepts L_2, i.e. $L(M_2) = L_2$.

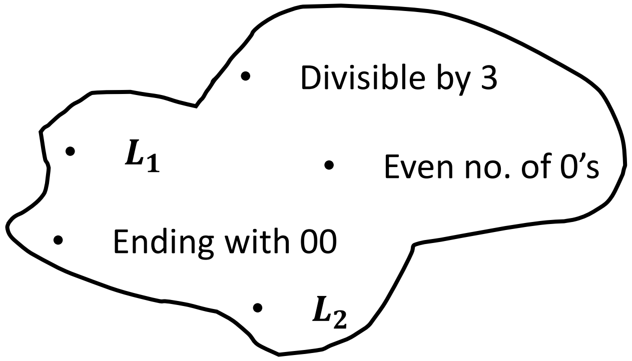
Using M_1 and M_2, we will show how to construct an NFA M that accepts $L = L_1 \cup L_2$, i.e. $L(M) = L_1 \cup L_2$.

Suppose the DFA M_1 is



And the DFA M_2 is





Set of all regular Languages

Closure of Regular Languages

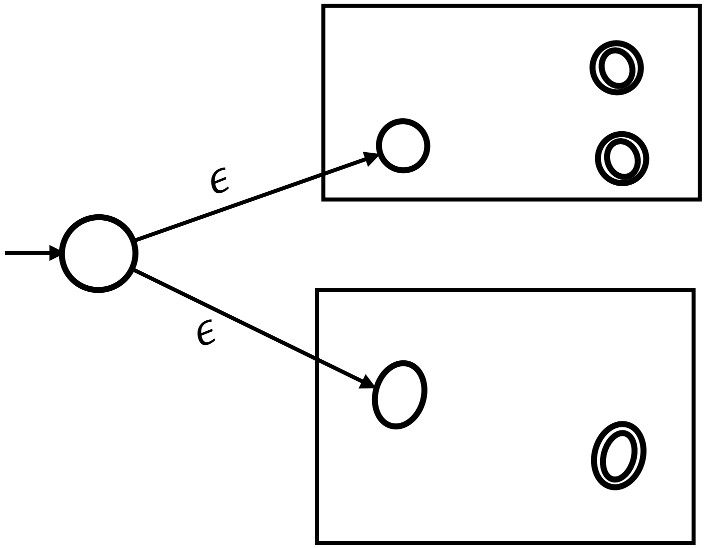
Q: Is the set of all regular languages **closed under union**?

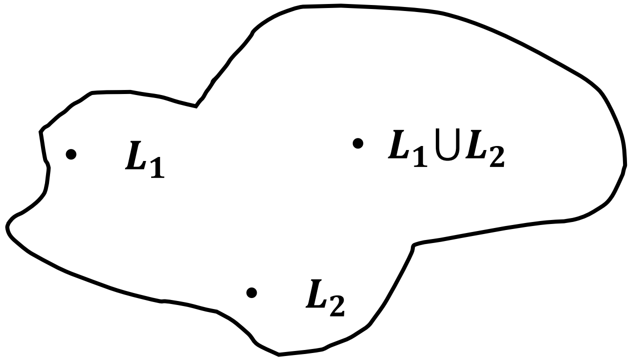
Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?

Proof: Since L_1 and L_2 are regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$ and a DFA M_2 that accepts L_2, i.e. $L(M_2) = L_2$.

Using M_1 and M_2, we will show how to construct an NFA M that accepts $L = L_1 \cup L_2$, i.e. $L(M) = L_1 \cup L_2$.

NFA M accepting $\boldsymbol{L = L_1 \cup L_2}$





Set of all regular Languages

Closure of Regular Languages

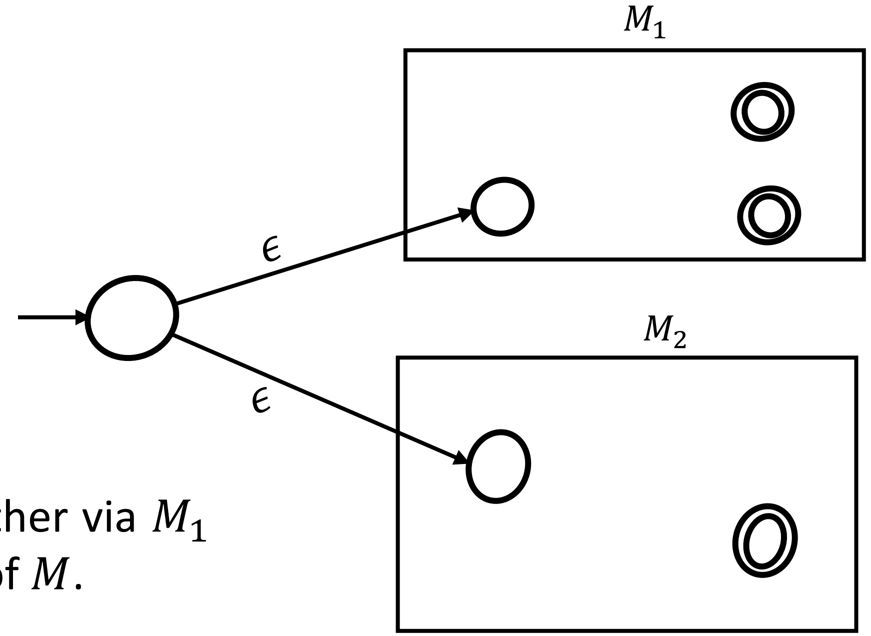
Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?

Proof: In order to prove that $L(M) = L_1 \cup L_2$, we show two things:

(i) $L \subseteq L_1 \cup L_2$

Let $\omega \in L$, i.e. ω is accepted by M. The final state for L can be reached either via M_1 or M_2. Thus ω must be accepted by either of them to reach the final state of M.

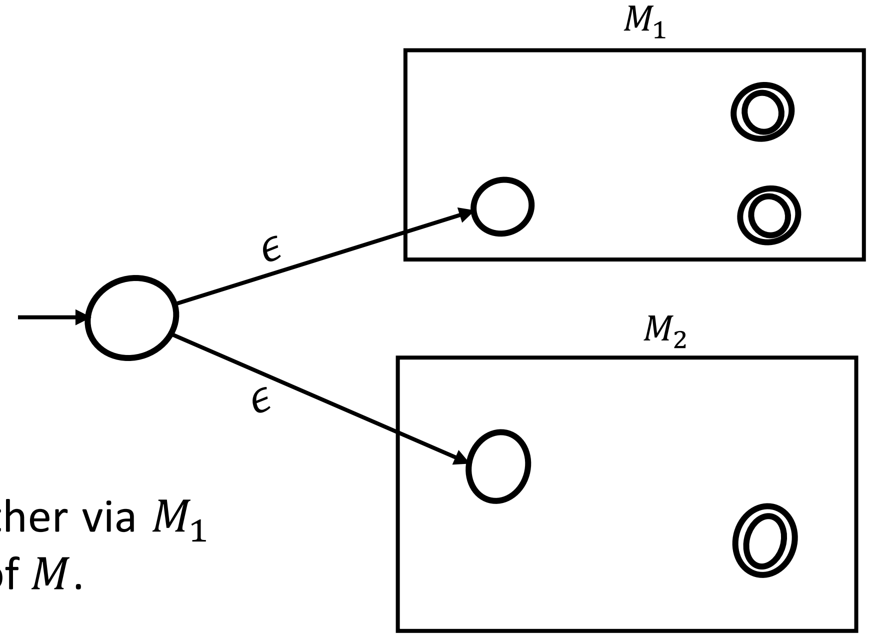


Closure of Regular Languages

Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?

Proof: In order to prove that $L(M) = L_1 \cup L_2$, we show two things:



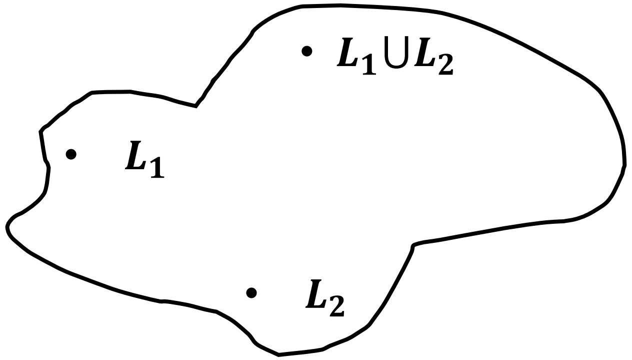
(i) $L \subseteq L_1 \cup L_2$

Let $\omega \in L$, i.e. ω is accepted by M. The final state for L can be reached either via M_1 or M_2. Thus ω must be accepted by either of them to reach the final state of M.

(ii) $L_1 \cup L_2 \subseteq L$

Let $\omega \in L_1 \cup L_2$. Then, $\omega \in L_1$ or $\omega \in L_2$.

Thus, ω must reach the final state of M_1 or M_2. But since the start state of M_1 or M_2 can be reached from the start state of M by taking an ϵ-transition, $\omega \in L$.



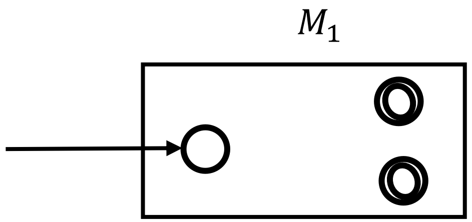
Set of all regular Languages

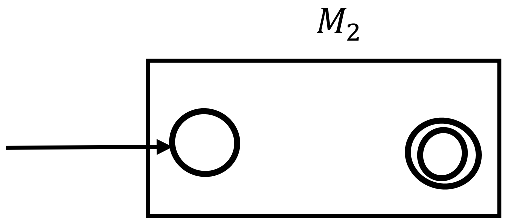
Closure of Regular Languages

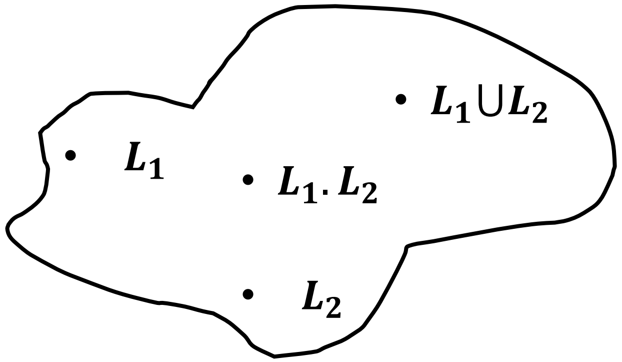
Q: Is the set of all regular languages **closed under concatenation**? Suppose L_1 and L_2 are regular languages. Is $L = L_1 . L_2$ also regular?

Proof: Since L_1 and L_2 are regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$ and a DFA M_2 that accepts L_2, i.e. $L(M_2) = L_2$.

Using M_1 and M_2, we will show how to construct an NFA M that accepts $L = L_1 . L_2$.

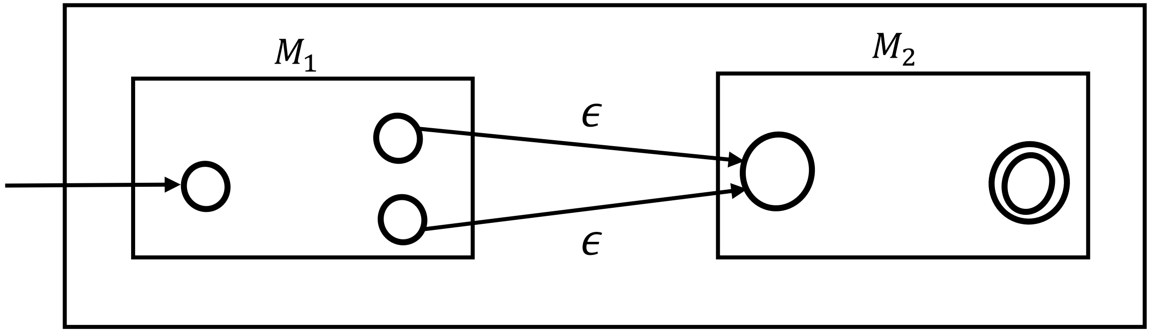






Set of all regular Languages

$L_1 . L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$

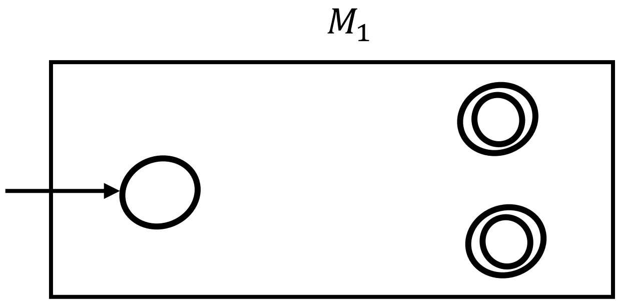


NFA M accepting $\boldsymbol{L = L_1 . L_2}$

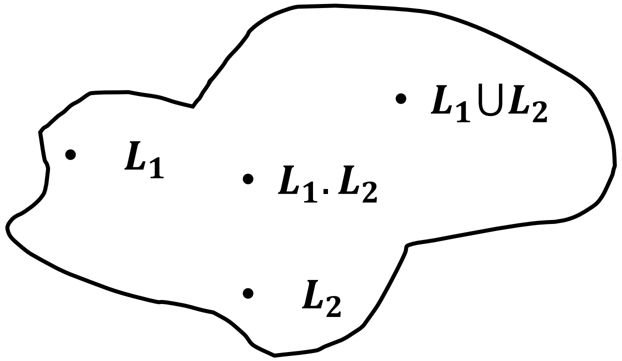
Closure of Regular Languages

Q: Is the set of all regular languages **closed under star**? Suppose L_1 is a regular language. Is L_1^* also regular?

Proof: Since L_1 is regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$**.** Using M_1, we will show how to construct an NFA M that accepts $L = L_1^*$.



$L_1^* = \{x_1 x_2 \cdots x_k \mid k \geq 0$ and each $x_i \in L_1\}$

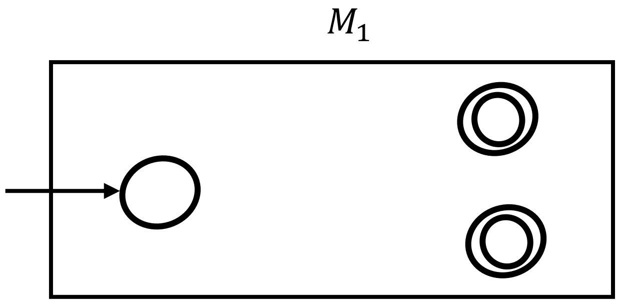


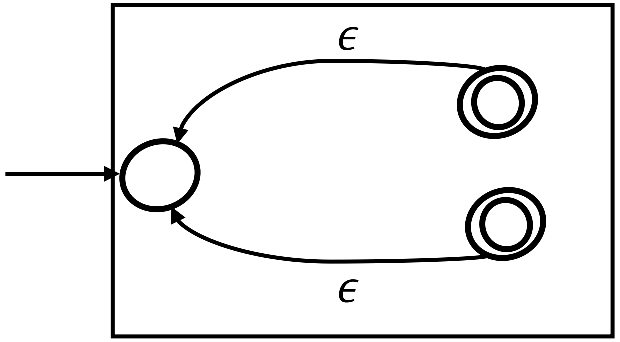
Set of all regular Languages

Closure of Regular Languages

Q: Is the set of all regular languages **closed under star**? Suppose L_1 is a regular language. Is L_1^* also regular?

Proof: Since L_1 is regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$**.** Using M_1, we will show how to construct an NFA M that accepts $L = L_1^*$.

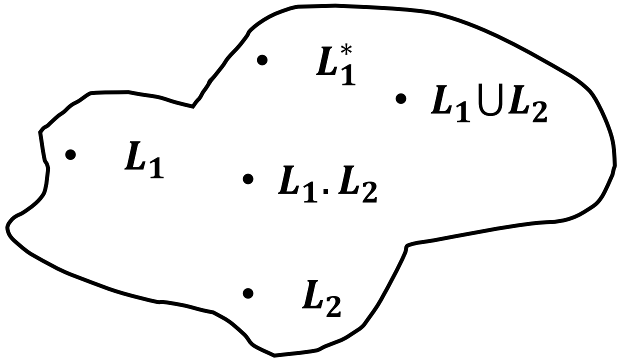




$L_1^* = \{x_1 x_2 \cdots x_k |\ k \geq 0 \text{ and each } x_i \in L_1\}$

Steps:

- Make ϵ-transitions from the final states of L_1 to the initial state of L_1.



Set of all regular Languages

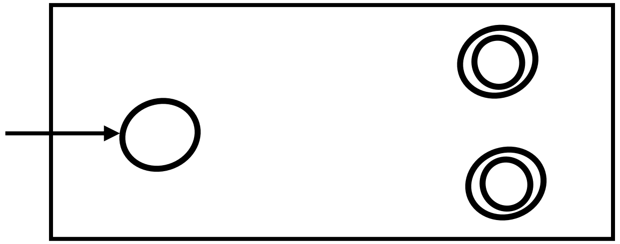
Closure of Regular Languages

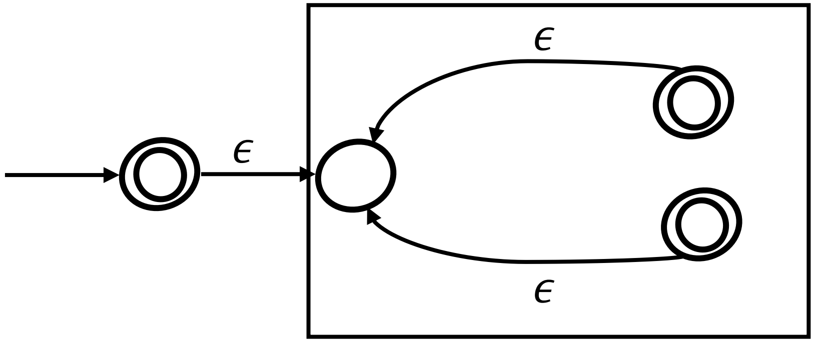
Q: Is the set of all regular languages **closed under star**? Suppose L_1 is a regular language. Is L_1^* also regular?

Proof: Since L_1 is regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$**.** Using M_1, we will show how to construct an NFA M that accepts $L = L_1^*$.

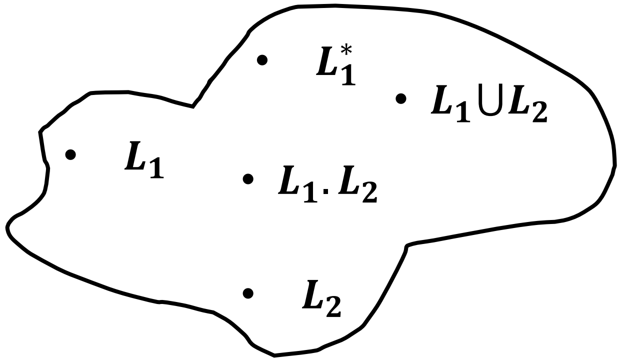
$L_1^* = \{x_1 x_2 \cdots x_k \mid k \geq 0$ and each $x_i \in L_1\}$

M_1





NFA accepting $\boldsymbol{L = L_1^*}$



Set of all regular Languages

Steps:

- Make ϵ-transitions from the final states of L_1 to the initial state of L_1.
- Make a new final state as the start state and make an ϵ-transition from this state to the previous start state of L_1.

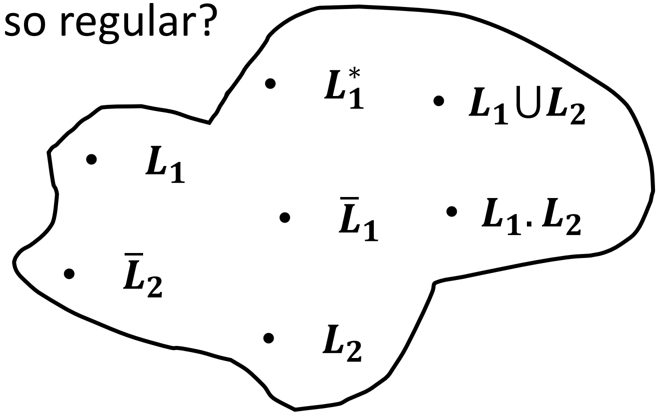
Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

(i) changing all the non-final states of M to be the final states of M' and
(ii) changing all the final states M to be the non-final states of M'.

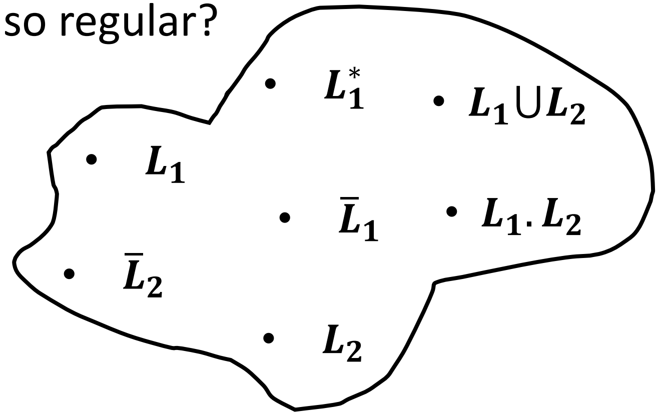
$$\boldsymbol{L(M') = \overline{L}}$$



Set of all regular Languages

Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?



Set of all regular Languages

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

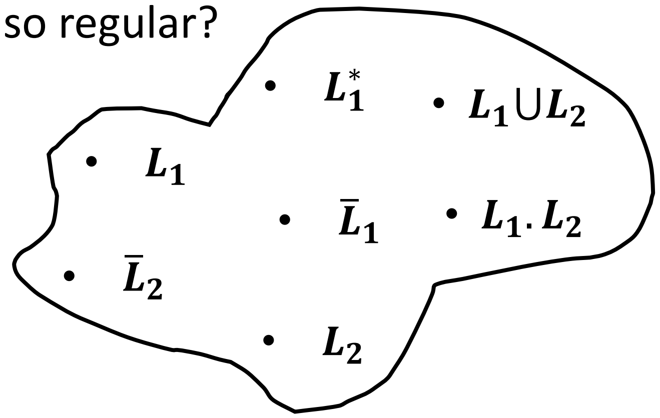
(i) changing all the non-final states of M to be the final states of M' and
(ii) changing all the final states M to be the non-final states of M'.

$$\boldsymbol{L(M') = \overline{L}}$$

Q: If L is the language accepted by an NFA, does "toggling" its states result in an NFA that accepts $\overline{L}$?

Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?



Set of all regular Languages

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

(i) changing all the non-final states of M to be the final states of M' and
(ii) changing all the final states M to be the non-final states of M'.

$$\boldsymbol{L(M') = \overline{L}}$$

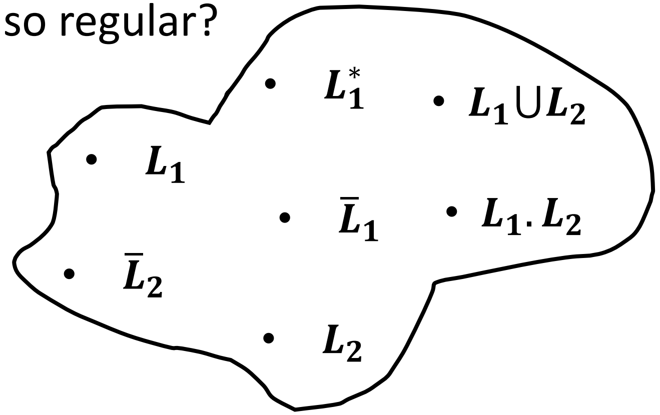
Q: If L is the language accepted by an NFA, does "toggling" its states result in an NFA that accepts $\overline{L}$?

Proof: Consider that for an input string $x \in L$**,** such that N accepts it. Suppose there is a rejecting run and an accepting run for input x. (See Table)

	NFA N	**Toggled NFA N'**
Run 1	Rejecting	
Run 2	Accepting	

Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?



Set of all regular Languages

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

(i) changing all the non-final states of M to be the final states of M' and
(ii) changing all the final states M to be the non-final states of M'.

$$\boldsymbol{L(M') = \overline{L}}$$

Q: If L is the language accepted by an NFA, does "toggling" its states result in an NFA that accepts $\overline{L}$?

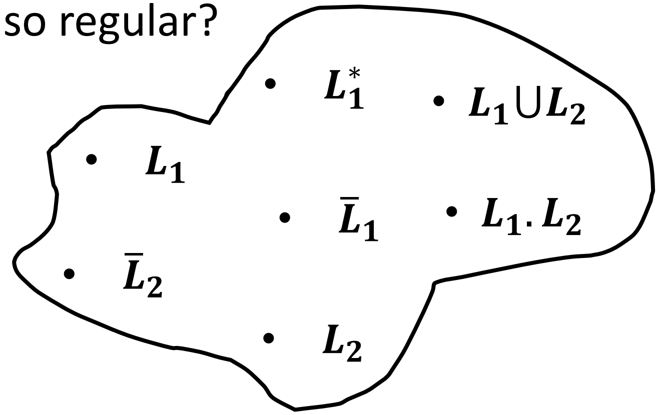
Proof: Consider that for an input string $x \in L$**,** such that N accepts it. Suppose there is an rejecting run and an accepting run for input x. (See Table)

For toggled NFA N' too, there are two runs for x. However, the rejecting run for N is an accepting run for N'. Thus x is accepted by both N and N'.

	NFA N	Toggled NFA N'
Run 1	Rejecting	Accepting
Run 2	Accepting	Rejecting

Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?



Set of all regular Languages

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

(i) changing all the non-final states of M to be the final states of M' and
(ii) changing all the final states M to be the non-final states of M'.

$$\boldsymbol{L(M') = \overline{L}}$$

Q: If L is the language accepted by an NFA, does "toggling" its states result in an NFA that accepts $\overline{L}$?

Proof: Consider that for an input string $x \in L$, such that N accepts it. Suppose there is an rejecting run and an accepting run for input x. (See Table)

For toggled NFA N' too, there are two runs for x. However, the rejecting run for N is an accepting run for N'. Thus x is accepted by both N and N'.

	NFA N	**Toggled NFA N'**
Run 1	Rejecting	Accepting
Run 2	Accepting	Rejecting

Contradiction! So No, the **toggled NFA does not accept $\overline{L}$**.

Closure of Regular Languages

Q: Is the set of all regular languages **closed under intersection**? If L_1 and L_2 are regular, then is $L = L_1 \cap \mathrm{L}_2$ also regular?

Proof: We shall use the fact that regular languages are **closed** under union and complement.

Closure of Regular Languages

Q: Is the set of all regular languages **closed under intersection**? If L_1 and L_2 are regular, then is $L = L_1 \cap \mathrm{L}_2$ also regular?

Proof: We shall use the fact that regular languages are **closed** under union and complement.

Note that using De Morgan's laws:

$$\boldsymbol{L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}}$$

Closure of Regular Languages

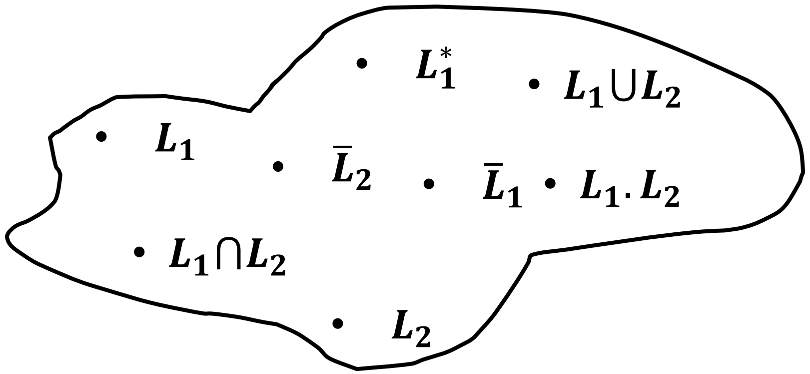
Q: Is the set of all regular languages **closed under intersection**? If L_1 and L_2 are regular, then is $L = L_1 \cap \mathrm{L}_2$ also regular?

Proof: We shall use the fact that regular languages are **closed** under union and complement.

Note that using De Morgan's laws:

$$\boldsymbol{L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}}$$

Given a DFA for L_1 and a DFA for L_2, we know how to construct an NFA for $\overline{L_1}, \overline{L_2}$ as well as for $L_1 \cup L_2$. Using these constructions and the aforementioned relationship, we can construct an NFA for $L = L_1 \cap \mathrm{L}_2$



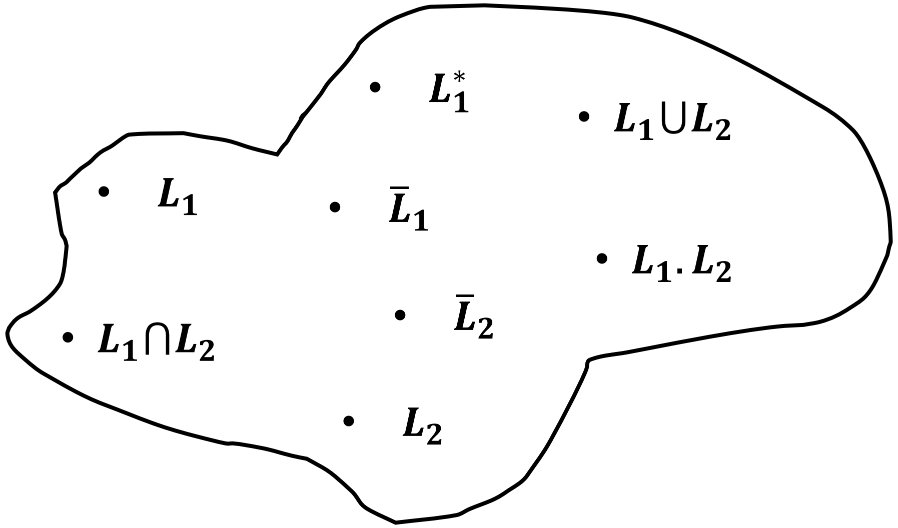
Set of all regular Languages

Closure of Regular Languages

Summary:

Regular Languages are closed under:

- **Union**
- **Intersection**
- **Star**
- **Complement**
- **Concatenation**



Set of all regular Languages

Regular Languages

If Σ is an alphabet, then

- $\Sigma^0 = \{\epsilon\}$
- $\Sigma^2 = \{a_1 a_2 | a_1 \in \Sigma,\ a_2 \in \Sigma\}$
- $\Sigma^k = \{a_1 a_2 \cdots a_k | a_i \in \Sigma \,|1 \leq i \leq k\}$
- $\Sigma^* = \{\bigcup_{i \geq 0} \Sigma^i\} = \{\Sigma^0 \cup \Sigma^1 \cup \Sigma^2\ \cdots\} = \{a_1 a_2 \cdots a_k | k \in \{0,1,\cdots\}\ \&\ a_j \in \Sigma, \forall j \in \{1,2,\cdots,k\}\}$

A Language $L \subset \Sigma^*$ and $L^* = \{\bigcup_{i \geq 0} \mathrm{L}^i\}$

Regular Languages

If Σ is an alphabet, then

- $\Sigma^0 = \{\epsilon\}$
- $\Sigma^2 = \{a_1 a_2 | a_1 \in \Sigma,\ a_2 \in \Sigma\}$
- $\Sigma^k = \{a_1 a_2 \cdots a_k | a_i \in \Sigma\ | 1 \leq i \leq k\}$
- $\Sigma^* = \{\bigcup_{i \geq 0} \Sigma^i\} = \{\Sigma^0 \cup \Sigma^1 \cup \Sigma^2\ \cdots\} = \{a_1 a_2 \cdots a_k | k \in \{0,1,\cdots\}\ \&\ a_j \in \Sigma, \forall j \in \{1,2,\cdots,k\}\}$

A Language $L \subset \Sigma^*$ and $L^* = \{\bigcup_{i \geq 0} \mathrm{L}^i\}$

Regular Language (alternate definition): Let Σ be an alphabet. Then the following are the regular languages over Σ:

- The empty language Φ is regular
- For each $a \in \Sigma, \{a\}$ is regular.
- Let L_1, L_2 be regular languages. Then $L_1 \cup L_2$, $L_1 . L_2$, L_1^* are regular languages.

Regular Expressions

A regular expression describes regular languages algebraically. The algebraic formulation also provides a powerful set of tools which will be leveraged to prove

- languages are regular
- derive properties of regular languages

Regular Expressions

A regular expression describes regular languages algebraically. The algebraic formulation also provides a powerful set of tools which will be leveraged to prove

- languages are regular
- derive properties of regular languages

Syntax for regular expressions (Recursive definition): R is said to be a regular expression if it has one of the following forms:

- Φ is a regular expression, $L(\Phi) = \Phi$
- ϵ is a regular expression, $L(\epsilon) = \{\epsilon\}$
- Any $a \in \Sigma$ is a regular expression, $L(a) = \{a\}$

Regular Expressions

A regular expression describes regular languages algebraically. The algebraic formulation also provides a powerful set of tools which will be leveraged to prove

- languages are regular
- derive properties of regular languages

Syntax for regular expressions (Recursive definition): R is said to be a regular expression if it has one of the following forms:

- Φ is a regular expression, $L(\Phi) = \Phi$
- ϵ is a regular expression, $L(\epsilon) = \{\epsilon\}$
- Any $a \in \Sigma$ is a regular expression, $L(a) = \{a\}$
- $R_1 + R_2$ is a regular expression if R_1 and R_2 are regular expressions, $L(R_1 + R_2) = L(R_1) \cup L(R_2)$
- R^* is a regular expression if R is a regular expression, $L(R^*) = \left(L(R)\right)^*$

Regular Expressions

A regular expression describes regular languages algebraically. The algebraic formulation also provides a powerful set of tools which will be leveraged to prove

- languages are regular
- derive properties of regular languages

Syntax for regular expressions (Recursive definition): R is said to be a regular expression if it has one of the following forms:

- Φ is a regular expression, $L(\Phi) = \Phi$
- ϵ is a regular expression, $L(\epsilon) = \{\epsilon\}$
- Any $a \in \Sigma$ is a regular expression, $L(a) = \{a\}$
- $R_1 + R_2$ is a regular expression if R_1 and R_2 are regular expressions, $L(R_1 + R_2) = L(R_1) \cup L(R_2)$
- R^* is a regular expression if R is a regular expression, $L(R^*) = \left(L(R)\right)^*$
- R_1R_2 is a regular expression if R_1 and R_2 are regular expressions, $L(R_1R_2) = L(R_1).L(R_2)$
- (R) is a regular expression if R is a regular expression, $L\left((R)\right) = R$

Regular Expressions

Syntax for regular expressions:

Regular Expression	Regular Language	Comment
Φ	$\{\}$	The empty set
ϵ	$\{\epsilon\}$	The set containing ϵ only
a	$\{a\}$	Any $a \in \Sigma$
$R_1 + R_2$	$L(R_1) \cup L(R_2)$	For regular expressions R_1 and R_2
$R_1 R_2$	$L(R_1).L(R_2)$	For regular expressions R_1 and R_2
R^*	$\left(L(R)\right)^*$	For regular expressions R
(R)	$L(R)$	For regular expressions R

Order of precedence: (), *, . , +

A language L is regular if and only if for some regular expression R, $L(R) = L$.

RE's are equivalent in power to NFAs/DFAs

Regular Expressions

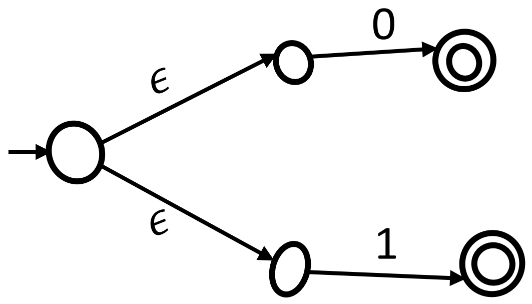
Syntax for regular expressions:

Regular Expression R	$\boldsymbol{L(R)}$
01	$\{01\}$
$01 + 1$	$\{01,1\}$
$(0+1)^*$	$\{\epsilon, 0, 1, 00,01, \cdots\}$
$(01+\epsilon)1$	$\{011,1\}$
$(0+1)^*01$	$\{01,001,101,0001, \cdots\}$
$(0+10)^*(\epsilon+1)$	$\{\epsilon, 0, 10, 00, 001,010, 0101, \cdots\}$

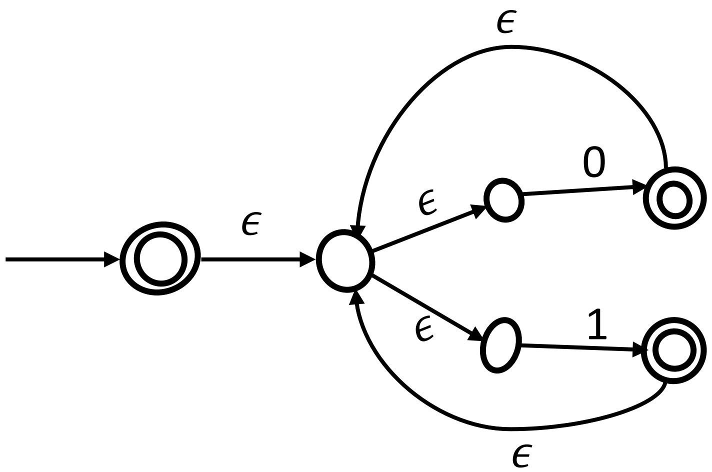
Regular Expressions

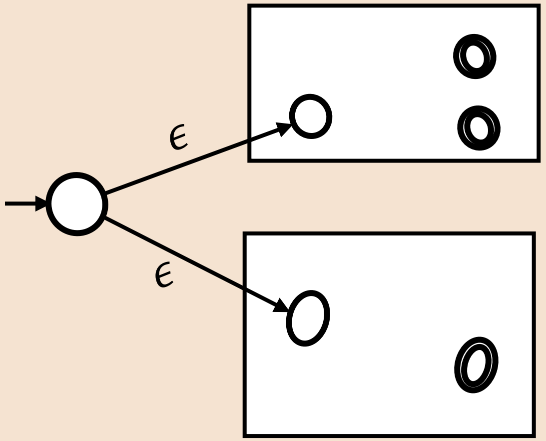
NFA for RE: $(\mathbf{0}+\mathbf{1})^{*}\mathbf{01}$

(i) NFA for $(0+1)$

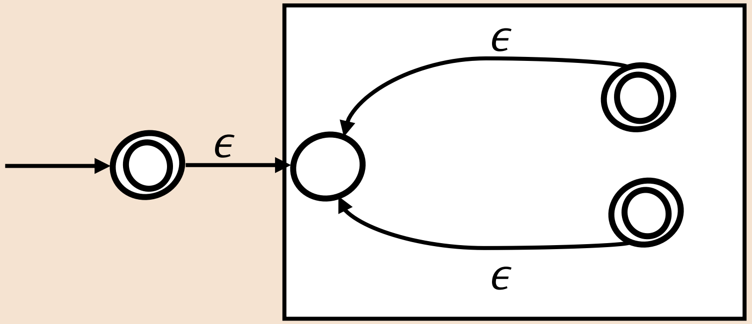


(ii) NFA for $(0+1)^{*}$





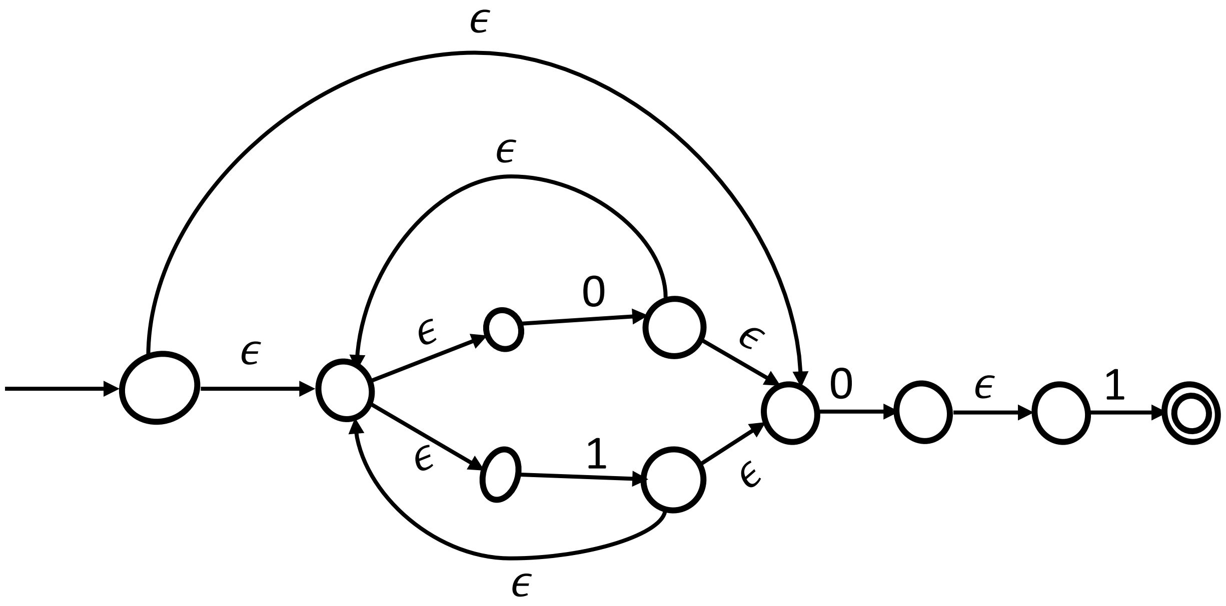
$\boldsymbol{L_1} \cup \boldsymbol{L_2}$

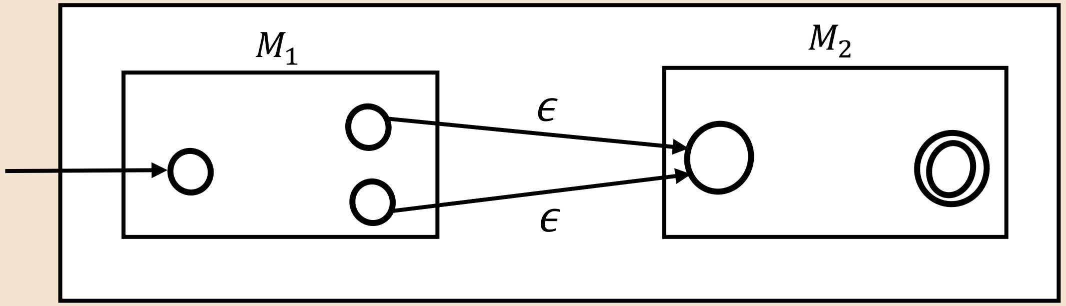


$\boldsymbol{L_1^{*}}$

Regular Expressions

NFA for $(\mathbf{0} + \mathbf{1})^{*}\mathbf{01}$





$\boldsymbol{L_1.L_2}$

Regular Expressions

Let $\Sigma = \{a, b\}$.

Language	**Regular Expression**
$\{\omega \mid \omega$ ends in "ab"$\}$	$(a+b)^*ab$
$\{\omega \mid \omega$ has a single a $\}$	b^*ab^*
$\{\omega \mid \omega$ has at most one $a\}$	$b^* + b^*ab^*$
$\{\omega \mid \ \lvert\omega\rvert$ is even$\}$	$((a+b)(a+b))^* = (aa+bb+ab+ba)^*$
$\{\omega \mid \omega$ has "ab" as a substring$\}$	$(a+b)^*ab(a+b)^*$
$\{\omega \mid \lvert\omega\rvert$ is a multiple of 3$\}$	$\big((a+b)(a+b)(a+b)\big)^*$

Regular Expressions

Let $\Sigma = \{a, b\}$.

Language	Regular Expression
$\{\omega \mid \omega$ ends in "ab"$\}$	$(a+b)^*ab$
$\{\omega \mid \omega$ has a single a $\}$	b^*ab^*
$\{\omega \mid \omega$ has at most one $a\}$	$b^* + b^*ab^*$
$\{\omega \mid \ \lvert\omega\rvert$ is even$\}$	$((a+b)(a+b))^* = (aa+bb+ab+ba)^*$
$\{\omega \mid \omega$ has "ab" as a substring$\}$	$(a+b)^*ab(a+b)^*$
$\{\omega \mid \lvert\omega\rvert$ is a multiple of 3$\}$	$\big((a+b)(a+b)(a+b)\big)^*$

Some algebraic properties of Regular Expressions:

- $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$
- $R_1(R_2R_3) = (R_1R_2)R_3$
- $R_1(R_2 + R_3) = R_1R_2 + R_1R_3$
- $(R_1 + R_2)R_3 = \mathrm{R_1R_3 + R_2R_3}$
- $R_1 + R_2 = R_2 + R_1$
- $R_1^*R_1^* = R_1^*$
- $(R_1^*)^* = R_1^*$
- $R\epsilon = \epsilon R = R$
- $R\Phi = \Phi R = \Phi$
- $R + \Phi = R$
- $\epsilon + RR^* = \epsilon + R^*R = R^*$
- $(R_1 + R_2)^* = (R_1^*R_2^*)^* = (R_1^* + R_2^*)^*$

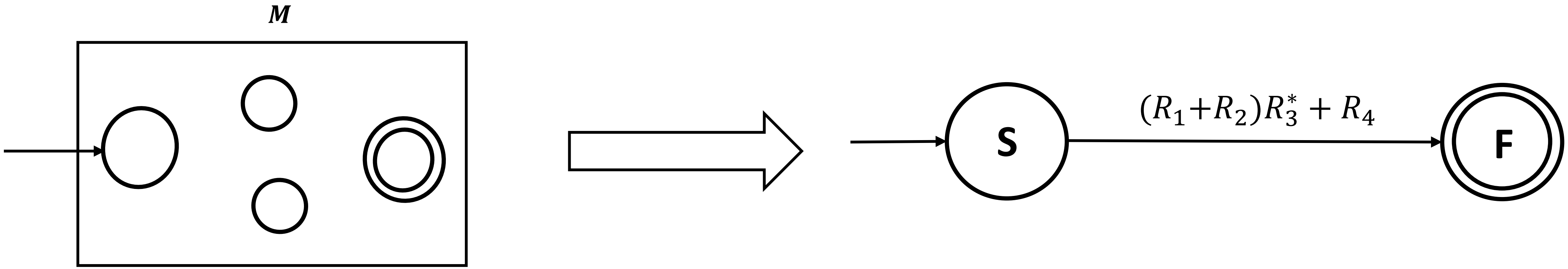
DFA to Regular Expressions

If a language is regular then it accepts a regular expression. We could draw equivalent NFAs for Regular Expressions.

How can we obtain Regular expressions given a DFA?

Given a DFA M, we **recursively** construct a two-state **Generalized NFA** (GNFA) with

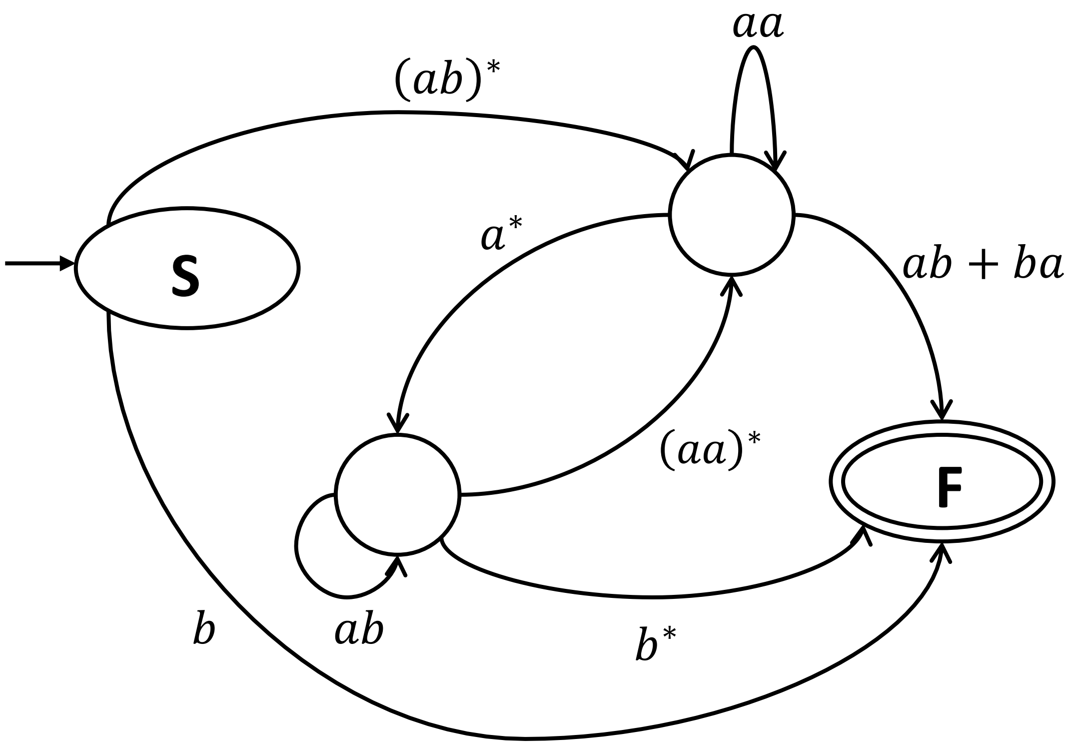
- A start state and a final state
- A single arrow goes from the start state to the final state
- The label of this arrow is the regular expression corresponding to the language accepted by the DFA M.



DFA to Regular Expressions: GNFA

What are GNFAs? They are simply NFAs such that

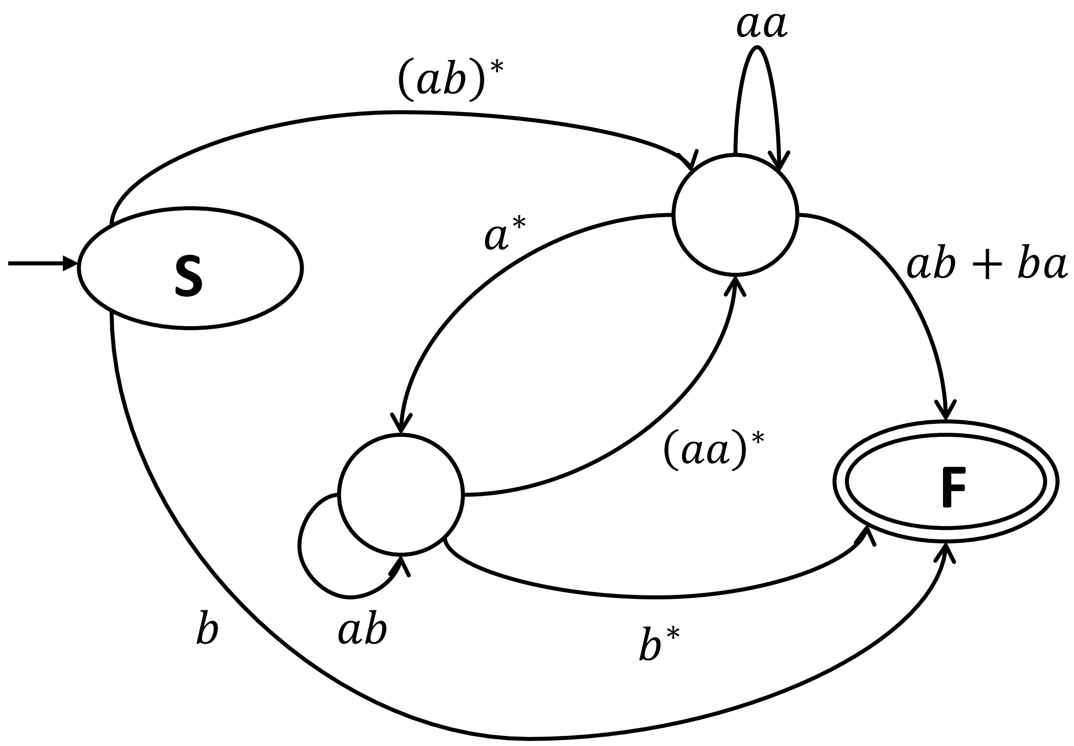
- The transitions may have regular expressions
- A unique start state that has arrows going to other states, but has no incoming arrows
- A unique final state that has arrows incoming from other states, but has no outgoing arrows
- For an input string, **runs** on a GNFA are similar to that of an NFA, except now a block of symbols are read corresponding to the Regular Expressions on the transitions.
- $b, abababab, aaabba$ are some input strings that have accepting runs for the GNFA on the right



DFA to Regular Expressions: GNFA

What are GNFAs? They are simply NFAs such that

- The transitions may have regular expressions
- A unique start state that has arrows going to other states, but has no incoming arrows
- A unique final state that has arrows incoming from other states, but has no outgoing arrows
- For an input string, **runs** on a GNFA are similar to that of an NFA, except now a block of symbols are read corresponding to the Regular Expressions on the transitions.
- $b, abababab, aaabba$ are some input strings that have accepting runs for the GNFA on the right

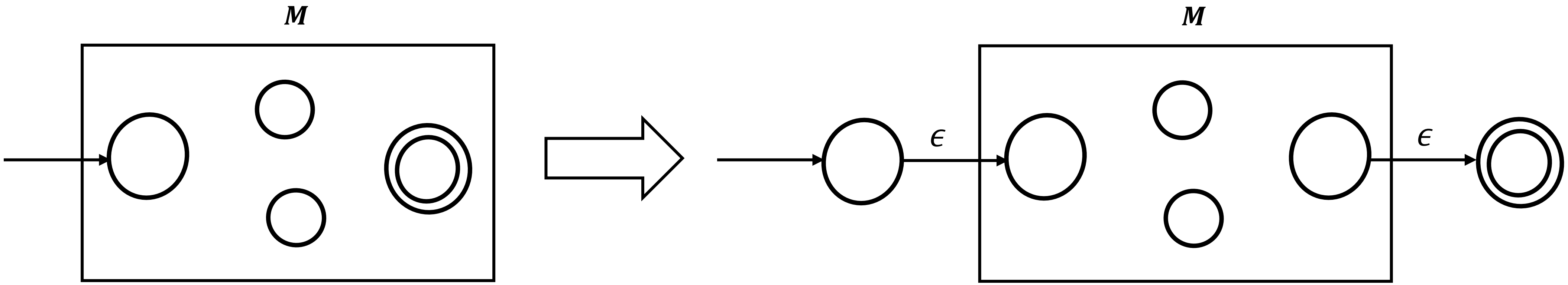


Starting from a DFA we will begin by constructing a GNFA with k states. We then outline a recursive procedure by which at each step, we will construct a GNFA with one less state. This step will be repeated until we obtain the **2-state GNFA**.

DFA to Regular Expressions: GNFA

Starting from the DFA M,

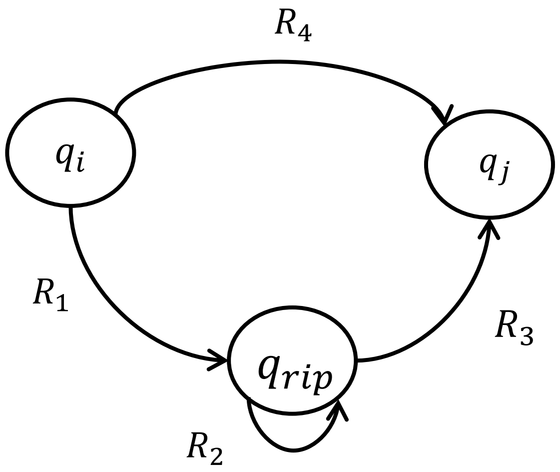
- Add a new start state with an ϵ arrow to the old start state.
- Add a new final state by with an ϵ arrow to the old final state.



DFA to Regular Expressions: GNFA

The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states. This is what we shall show next.

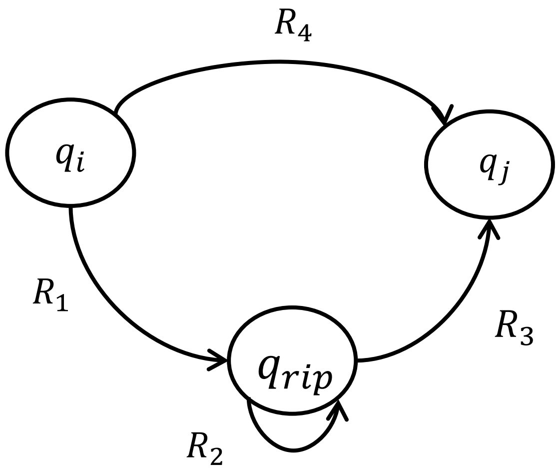
- Start by picking any state of the GNFA (except the new start and final states)
- Let us call this state q_{rip}. We "rip" q_{rip} out of the machine and create a GNFA with $k-1$ states.
- Of course, we need to "repair" the machine by altering the regular expressions that label each of the remaining arrows.
- The new labels compensate for the loss of q_{rip}.

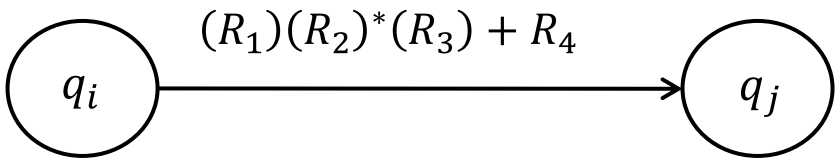


DFA to Regular Expressions: GNFA

The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states. This is what we shall show next.

- Start by picking any state of the GNFA (except the new start and final states)
- Let us call this state q_{rip}. We “rip” q_{rip} out of the machine and create a GNFA with $k-1$ states.
- Of course, we need to “repair” the machine by altering the regular expressions that label each of the remaining arrows.
- The new labels compensate for the loss of q_{rip}.





DFA to Regular Expressions: GNFA

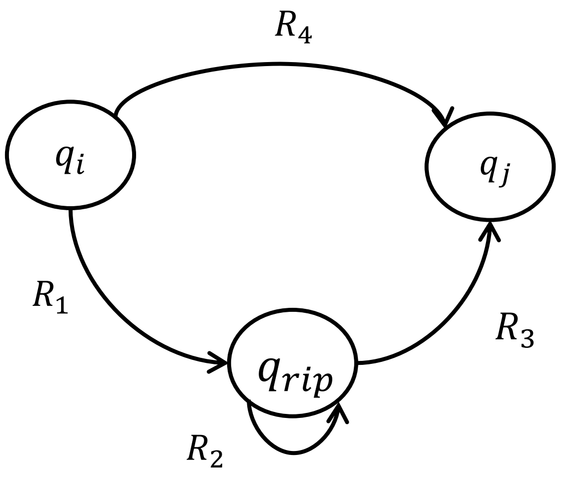
The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states.

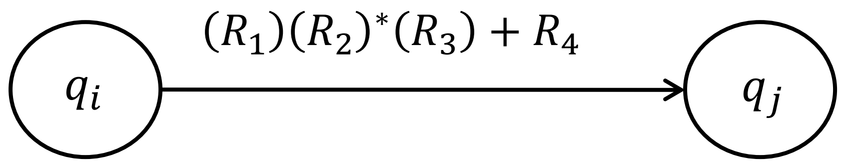
How do we remove q_{rip}? In the old machine if

- q_i goes to q_{rip} with an arrow labelled R_1
- q_{rip} goes to itself with an arrow labelled R_2
- q_{rip} goes to q_j with an arrow labelled R_3
- q_i goes to q_j with an arrow labelled R_4

Repeat this until $\boldsymbol{k = 2}$

then in the new machine, the arrow from q_i to q_j has the label $(R_1)(R_2)^*(R_3) + R_4$

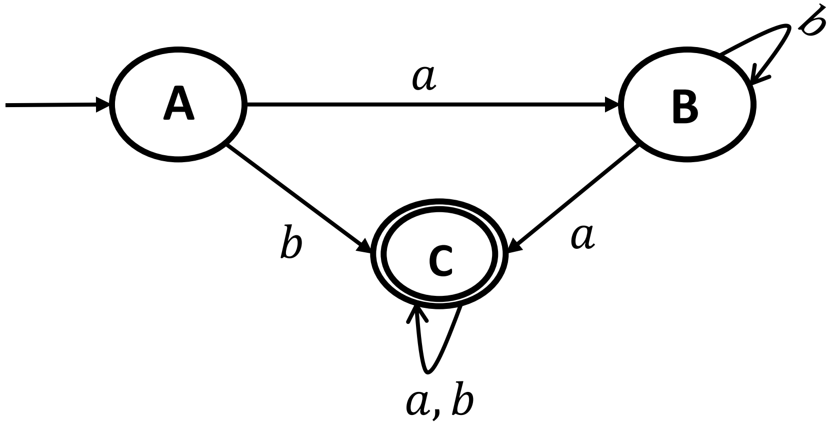




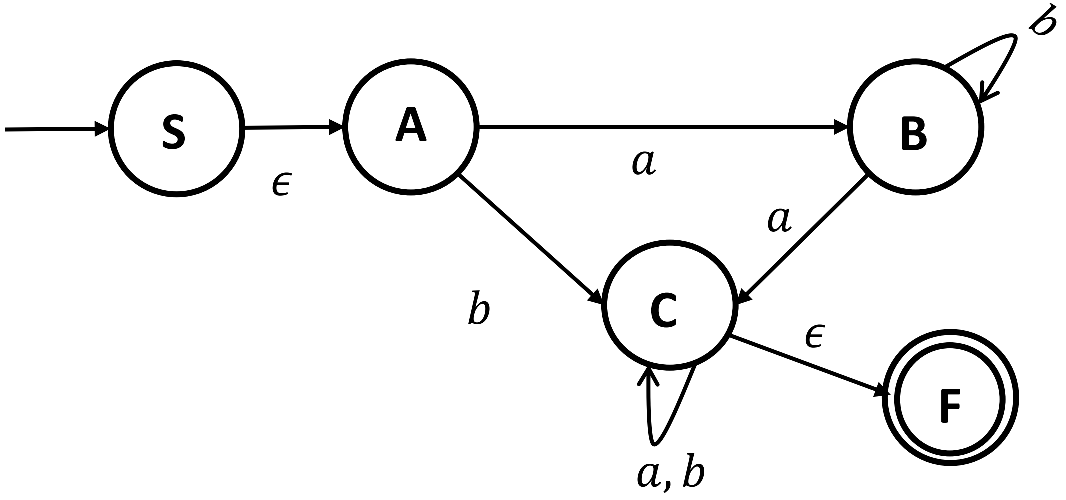
This should be done for **every pair** of arrows outgoing and incoming q_{rip}

DFA to Regular Expressions: GNFA

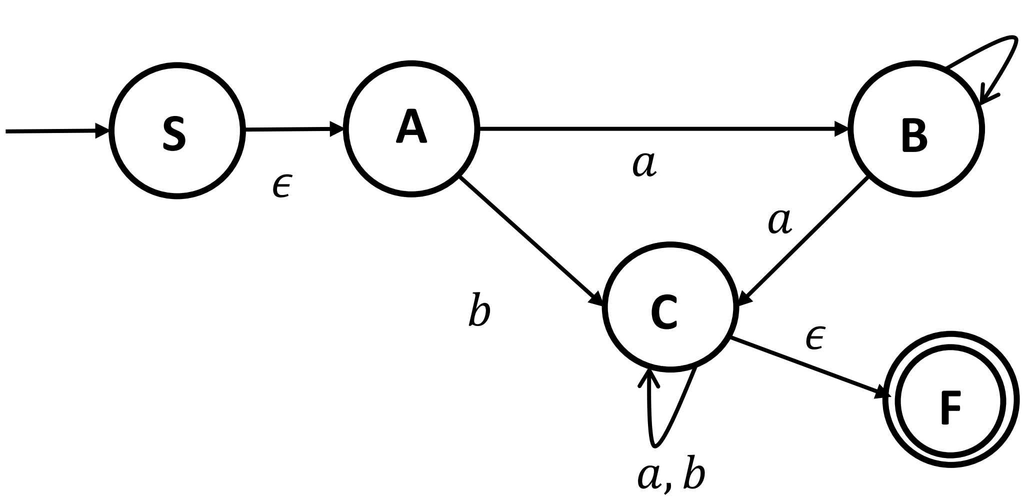
Let us look at an example. Consider the original DFA M below and find the regular expression corresponding to $L(M)$.



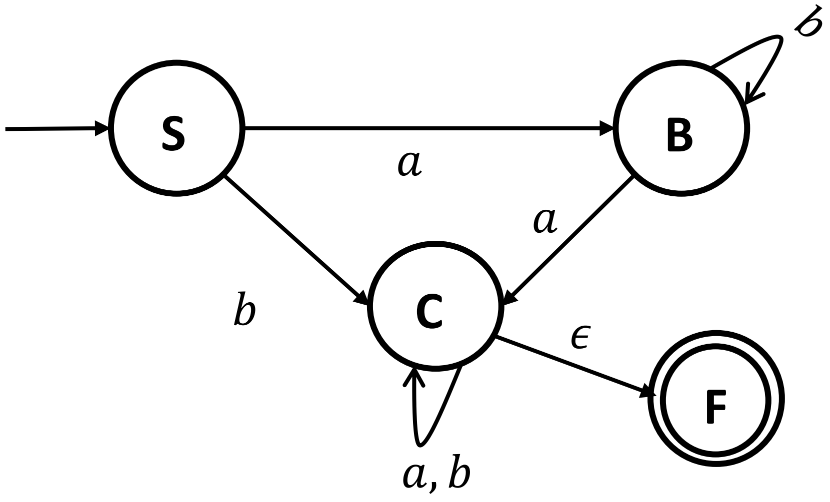
Step 1: Add new start and final states



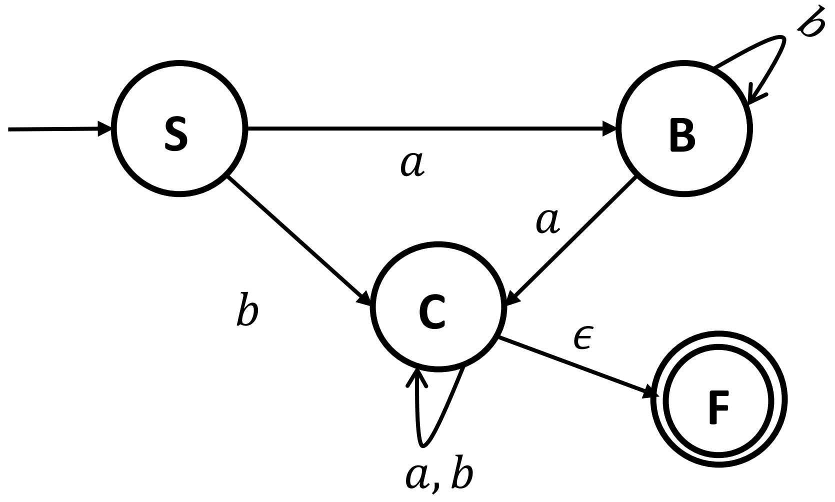
DFA to Regular Expressions: GNFA



Step 2: Eliminate A

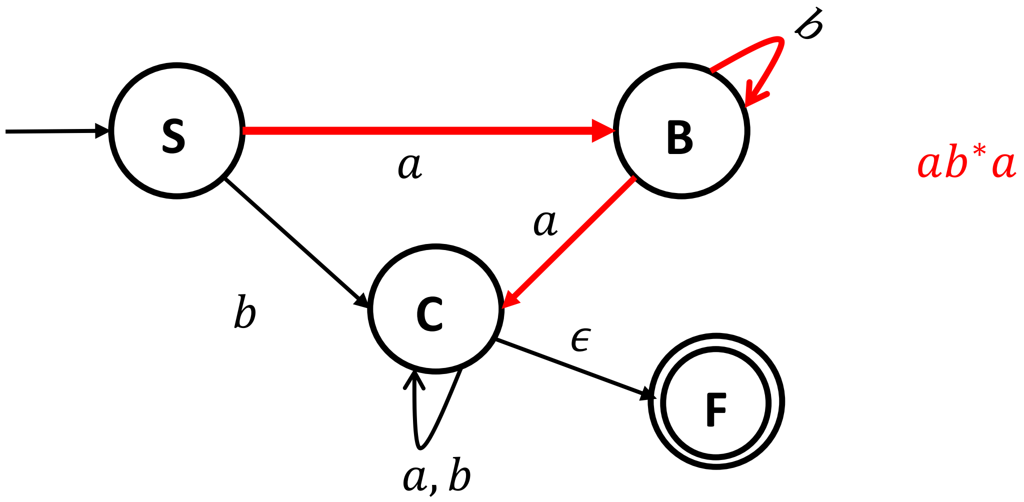


DFA to Regular Expressions: GNFA

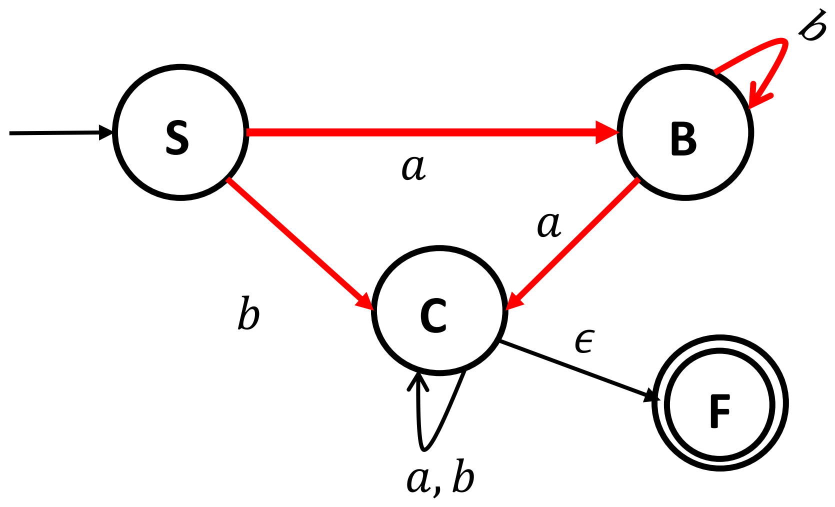


Step 2: Eliminate B

$S \rightarrow C$ via B, RE: ab^*a

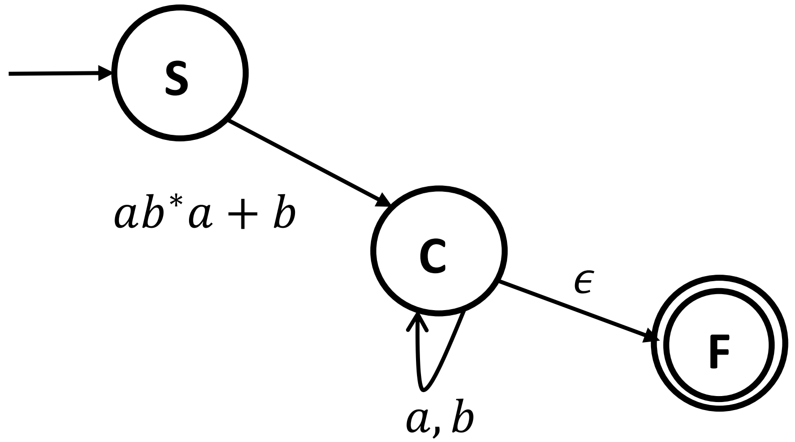


DFA to Regular Expressions: GNFA

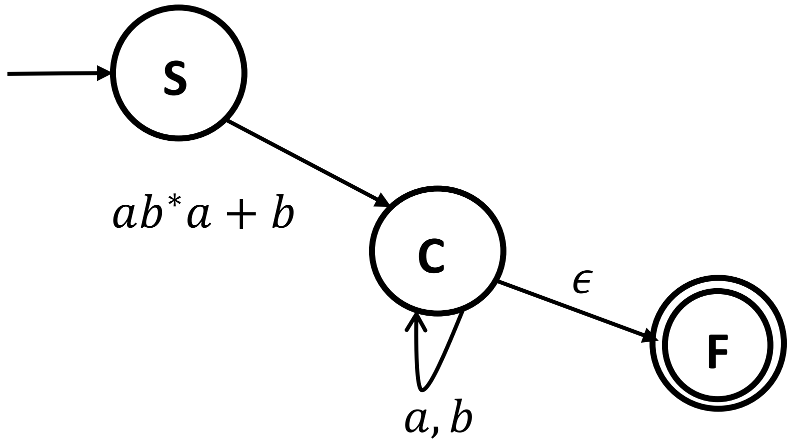


Step 2: Eliminate $\boldsymbol{B}$

$S \to C$ via B, RE: ab^*a
Overall RE for $S \to C$: $\boldsymbol{ab^*a + b}$

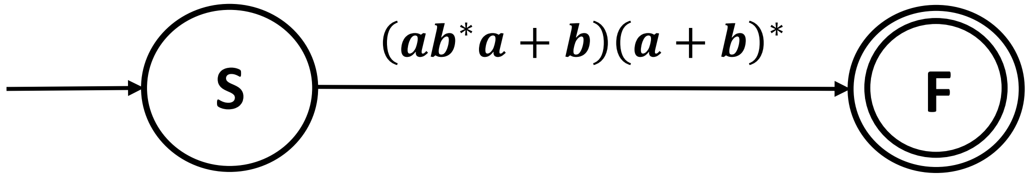


DFA to Regular Expressions: GNFA

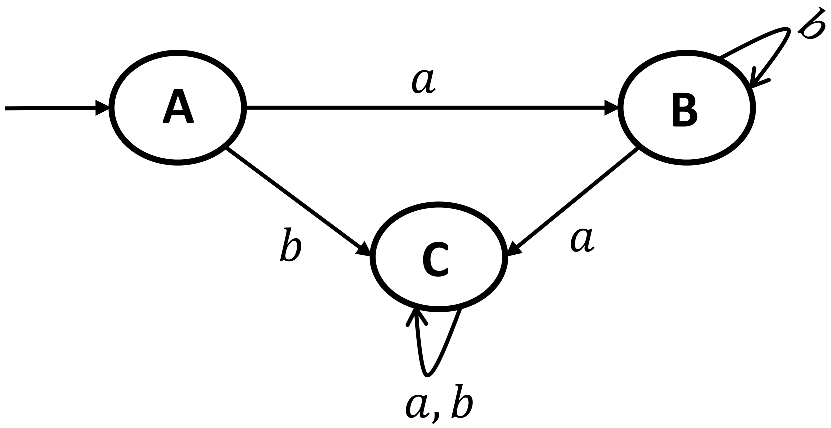


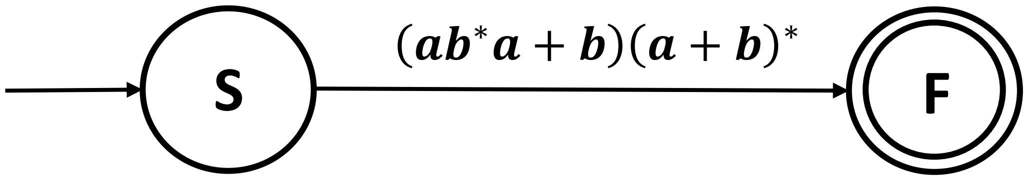
Step 2: Eliminate $\boldsymbol{C}$

$S \rightarrow F$ via C, RE: $(\boldsymbol{ab^*a+b})(\boldsymbol{a+b})^*$



DFA to Regular Expressions: GNFA





Recursively, we managed to convert the DFA M to a 2-state GNFA such that the label from of the arrow from the start state to the final state of the GNFA is the Regular Expression corresponding to $L(M)$.

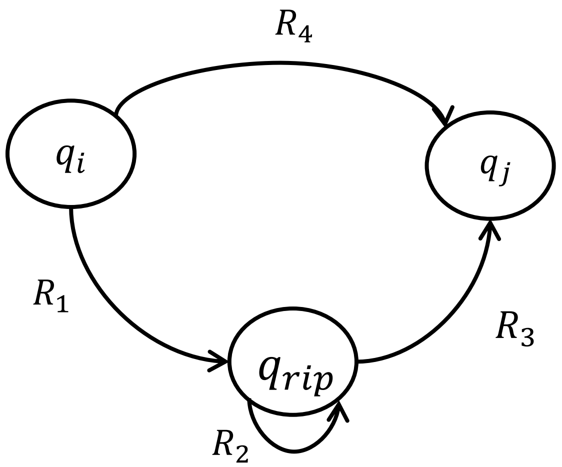
DFA to Regular Expressions: GNFA

Formally, a GNFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states.
- Σ is the input alphabet.
- $\delta: Q - \{q_0\} \times Q - \{F\} \mapsto \mathcal{R}$ is the transition function.
- q_0 is the start state.
- F is the final state.

Convert k-state GNFA to a 2-state GNFA:

We provide a recursive algorithm CONVERT(G) for this.



CONVERT(G):

1. Let k be the number of states of G.
2. If $k = 2$, then return the label R of the arrow between the start and the final state.
3. If $k > 2$, select any state $q_{rip} \in Q$ different from q_0 and F and let G' be the GNFA$(Q', \Sigma, \delta', q_0, F)$, where

$$Q' = Q - \{q_{rip}\},$$

and for any $q_i \in Q' - \{q_0\}$ and any $q_j \in Q' - \{q_0\}$, let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) + R_4,$$

for $R_1 = \delta(q_i, q_{rip})$, $R_2 = \delta(q_{rip}, q_{rip})$, $R_3 = \delta(q_{rip}, q_j)$ and $R_4 = \delta(q_i, q_j)$

4. Compute CONVERT(G') and return its value.

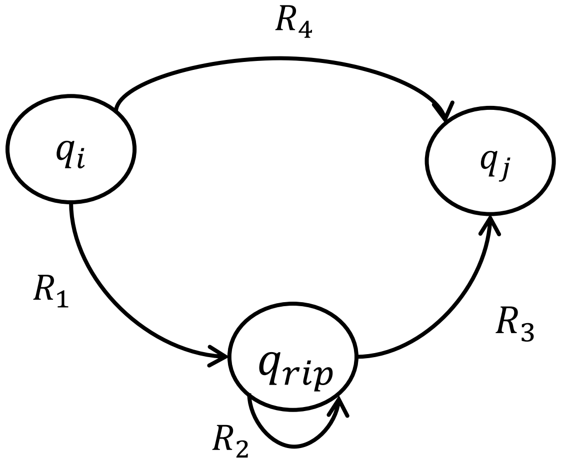
DFA to Regular Expressions: GNFA

Formally, a GNFA is a 5-tuple $(\mathrm{Q}, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states.
- Σ is the input alphabet.
- $\delta: Q - \{q_0\} \times Q - \{F\} \mapsto \mathcal{R}$ is the transition function.
- q_0 is the start state.
- F is the final state.

Convert k-state GNFA to a 2-state GNFA:

We provide a recursive algorithm CONVERT(G) for this.



DFA, NFA, Regular Expressions have equal power and all of them correspond to Regular Languages

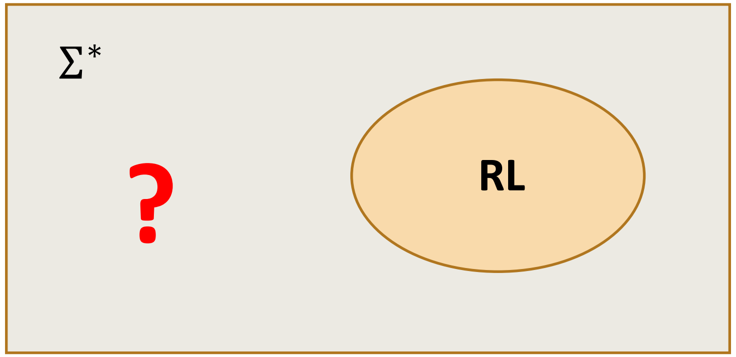
How do Non-regular languages look like?
How can we prove that certain languages are not regular?

Pumping Lemma

Recall that so far, we have proven that the following statements are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathcal{L}(D) = L$.
- There is an NFA N such that $\mathcal{L}(N) = L$.
- There is a regular expression R such that $\mathcal{L}(R) = L$.

- Not all languages are regular.



Thank You!