Probability and Statistics: MA6.101

Tutorial 6

Topics Covered: Conditional Probability, Conditional Expectation

Q1: Let X and Y be two independent $N(0,1)$ random variables, and define:

$$Z = 1 + X + XY^2$$
$$W = 1 + X$$

Find $\text{Cov}(Z, W)$.

Q2: The joint density function is given as $f_{X,Y}(x,y) = cx(y-x)e^{-y}$ for $0 \leq x \leq y < \infty$.

(a) Find c.

(b) Show that:

$$f_{X|Y}(x|y) = \frac{6x(y-x)}{y^3}, \quad 0 \leq x \leq y$$
$$f_{Y|X}(y|x) = (y-x)e^{x-y}, \quad 0 \leq x \leq y < \infty$$

(c) Deduce that:

$$\mathbb{E}(X|Y) = \frac{Y}{2}$$

Q3: You throw a fair six-sided die until you get 6. What is the expected number of throws (including the throw giving 6) conditioned on the event that all throws gave even numbers?

Q4: Let X and Y be two independent Uniform$(0,1)$ random variables, and define:

$$Z = \frac{X}{Y}$$

(a) Find CDF of Z .

(b) Find PDF of Z

Q5: Let X, Y, and Z be discrete random variables. Show the following generalizations of the law of iterated expectations.

(a) $\mathbb{E}[Z] = \mathbb{E}\left[\mathbb{E}[Z \mid X, Y]\right]$.

(b) $\mathbb{E}[Z \mid X] = \mathbb{E}\left[\mathbb{E}[Z \mid X, Y] \mid X\right]$.

Q6: If X and Y are arbitrary random variables for which the necessary expectations and variances exist, then prove that $\textbf{Var}(Y) = \mathbb{E}[\text{Var}_X(Y|X)] + \textbf{Var}[\mathbb{E}_X(Y|X)]$.

Q7: Consider a gambler who at each gamble either wins or loses his bet with probabilities p and $1 - p$, independent of earlier gambles. When $p > \frac{1}{2}$, a popular gambling system, known as the Kelly strategy, is to always bet the fraction $2p - 1$ of the current fortune. Compute the expected fortune after n gambles, starting with x units and employing the Kelly strategy.

Q8: There are n letters and n envelopes. You put the letters randomly in the envelopes so that each letter is in one envelope. (Effectively a random permutation of n numbers chosen uniformly). Calculate the expected number of envelopes with the correct letter inside them.