CS 302.1 - Automata Theory

Lecture 07

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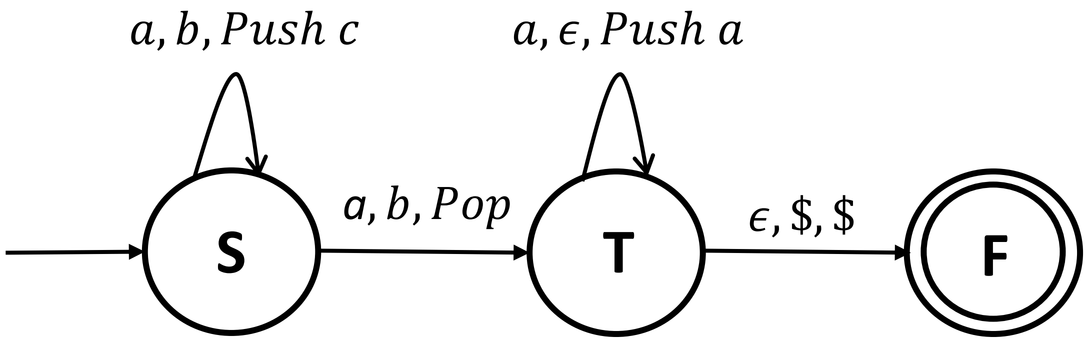
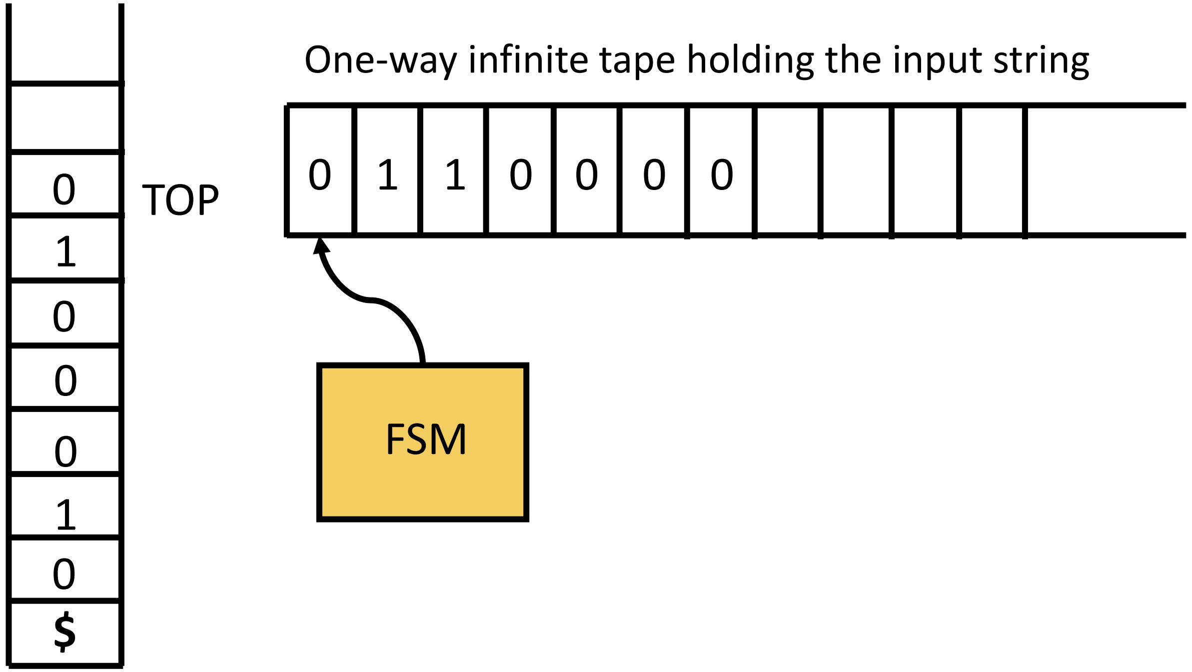
Quick Recap

Pushdown Automata

- Automata that recognizes CFLs
- FSM + stack
- FSM transitions by reading an input symbol and by interacting with the stack

- $\boldsymbol{\delta(q_i, a, b) = (q_j, c)}$: If the input symbol read is a and the stack top = b, then push c onto the stack and transition from q_i to q_j
- $\boldsymbol{\delta(q_i, a, \epsilon) = (q_j, c)}$: If the input symbol read is a, then push c onto the stack and transition from q_i to q_j
- $\boldsymbol{\delta(q_i, a, b) = (q_j, \epsilon)}$: If the input symbol read is a, and the stack top = b, then transition from q_i to q_j
- $\boldsymbol{\delta(q_i, \epsilon, \$) = (q_j, \$)}$: Transition from q_i to q_j if the stack is empty.





Pushdown Automata

Let $\Sigma = \{0,1\}$ consider the language $\boldsymbol{L} = \{\boldsymbol{w} \in \boldsymbol{\Sigma}^* \mid \boldsymbol{w}$ **is a Palindrome**$\}$. Design a PDA P that recognizes L.

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- The above intuition is applicable for even length palindromes of the form ww^R.

- What about odd length palindromes?
  - Non-determinism to the rescue once again

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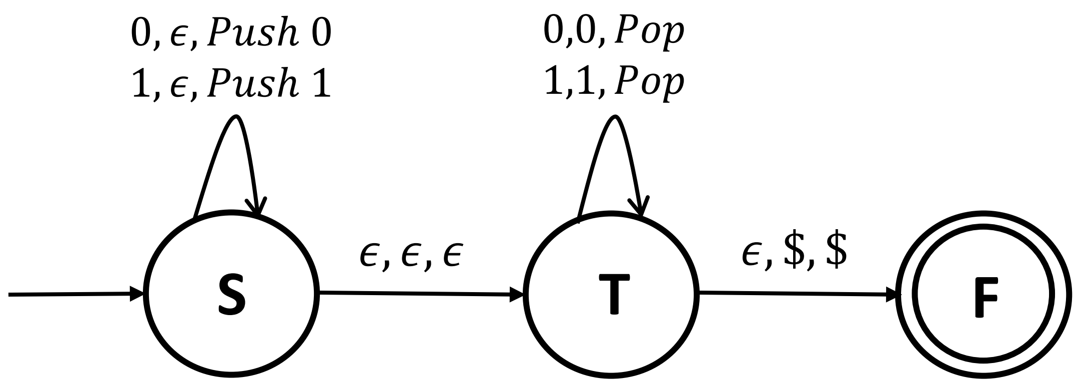

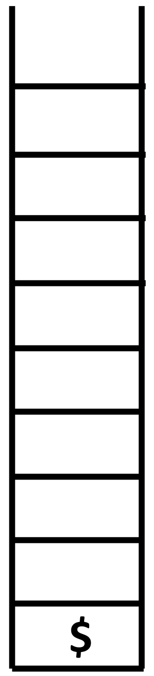

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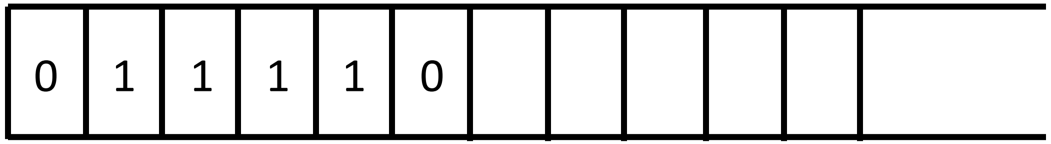

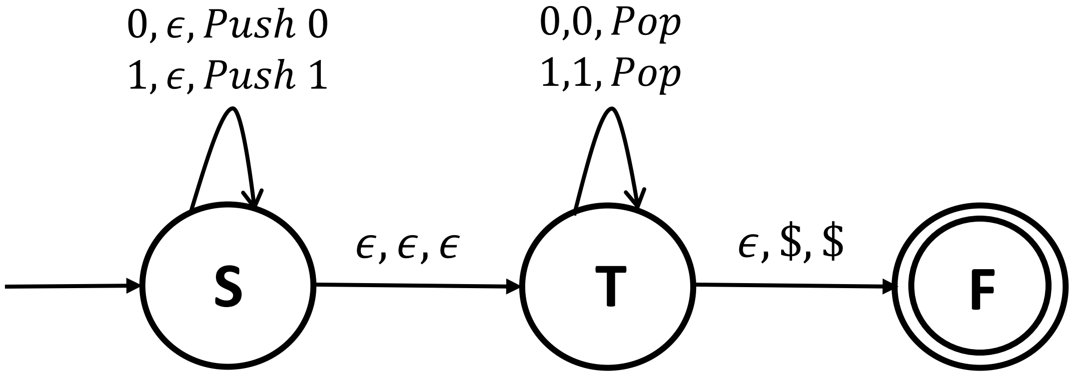

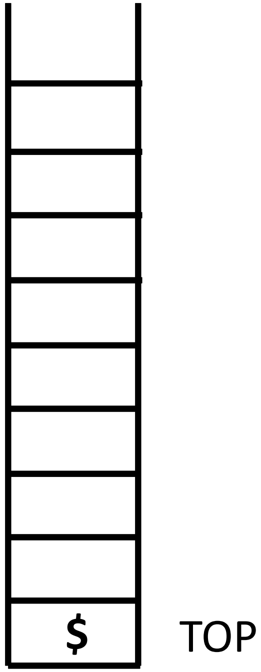

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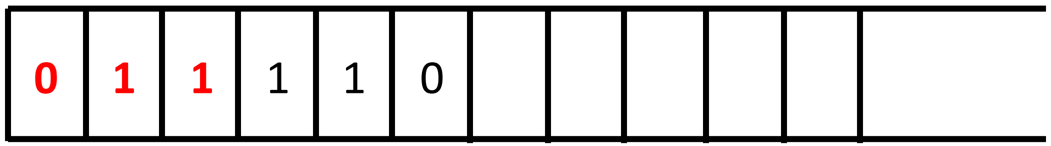

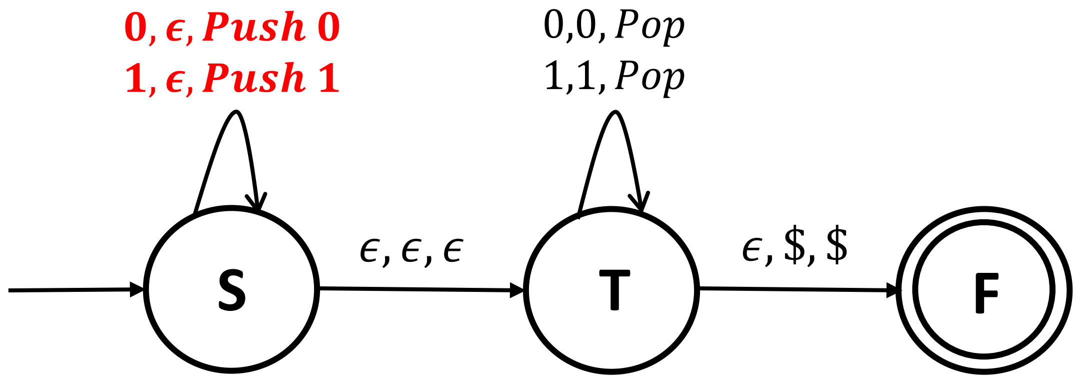

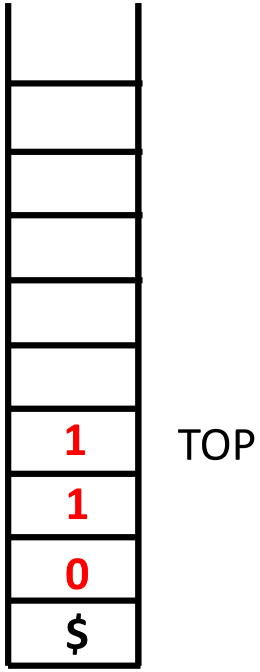

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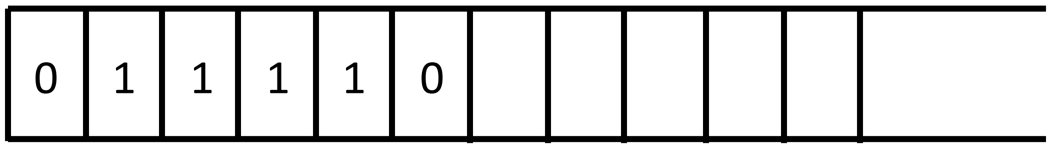

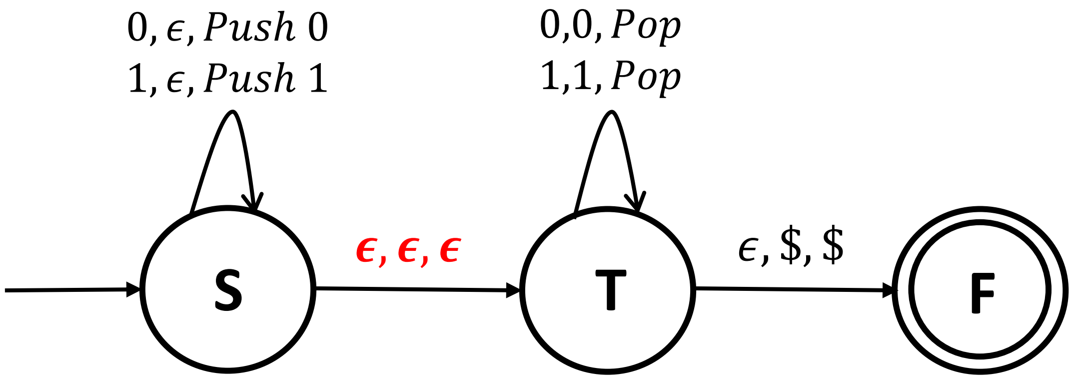

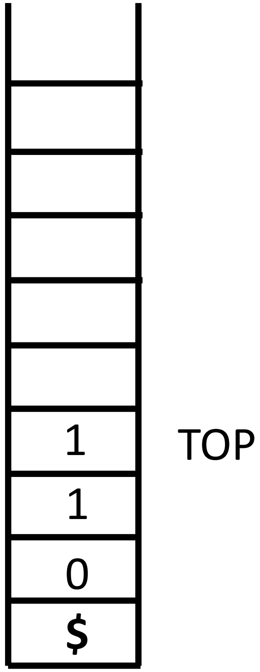

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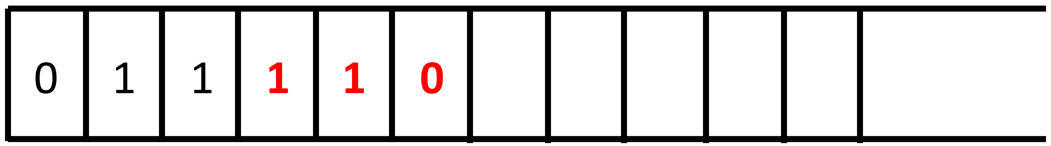

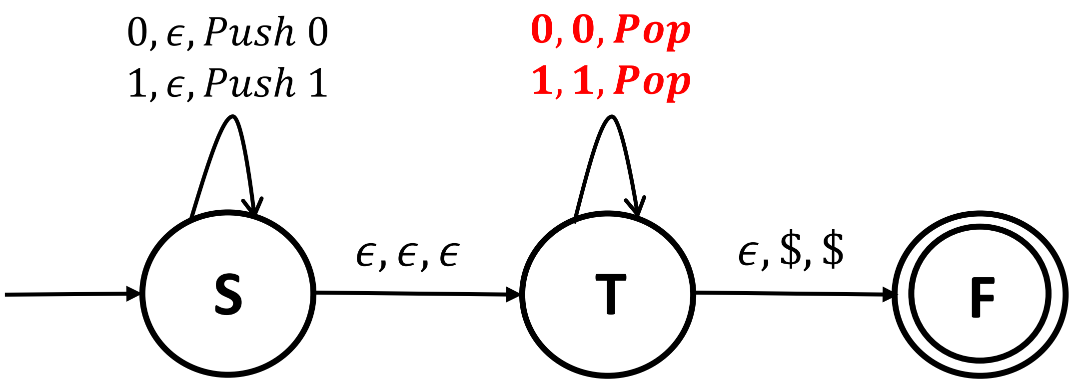

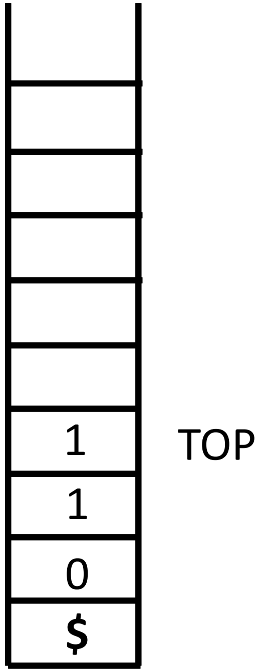

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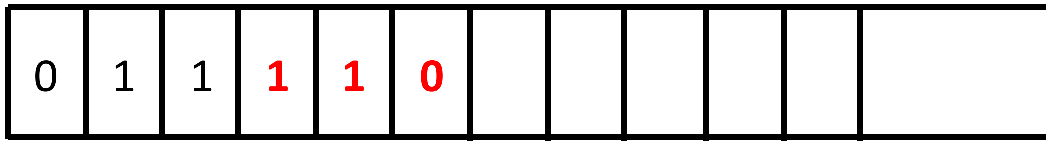

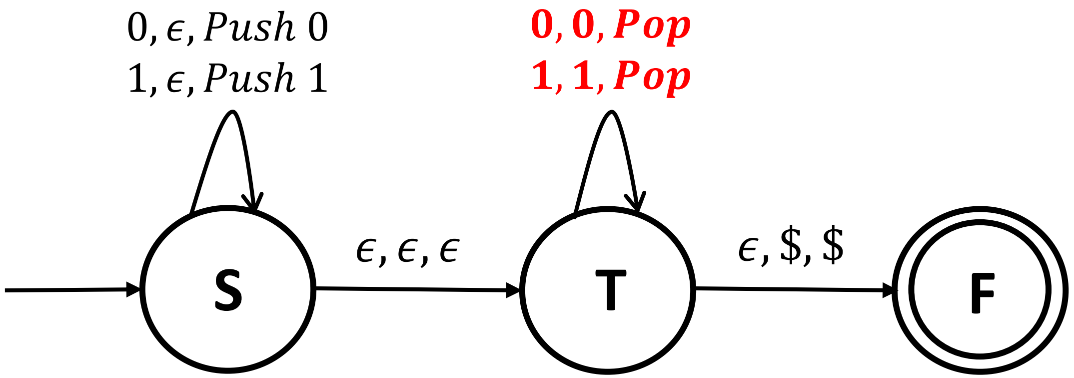

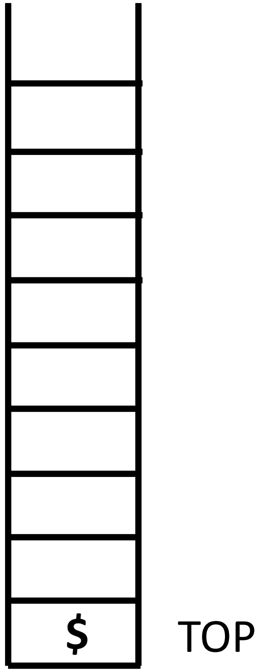

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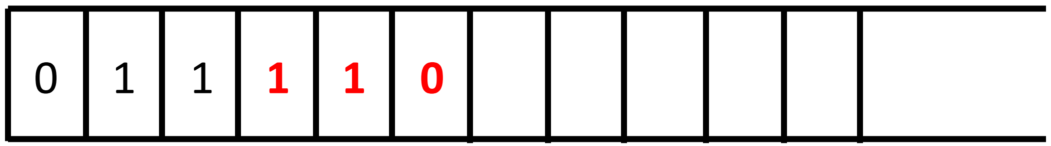

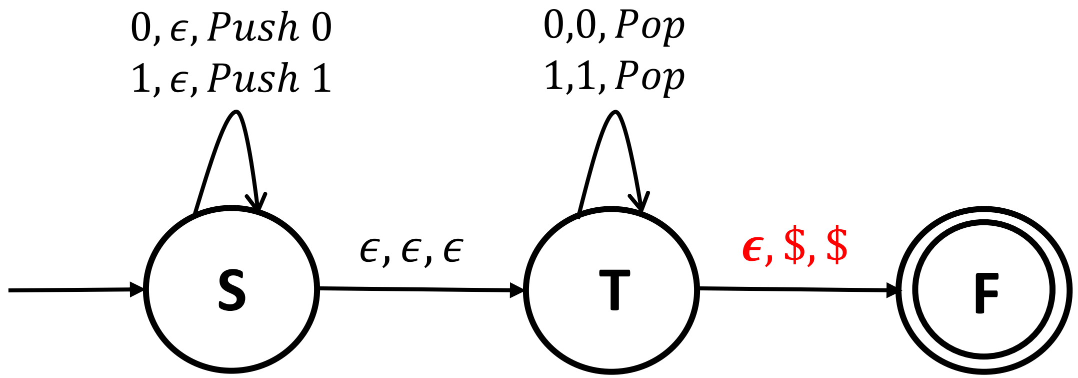

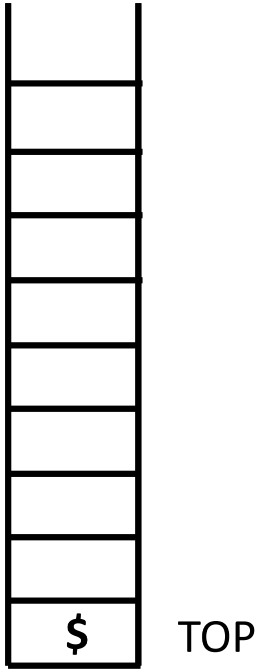

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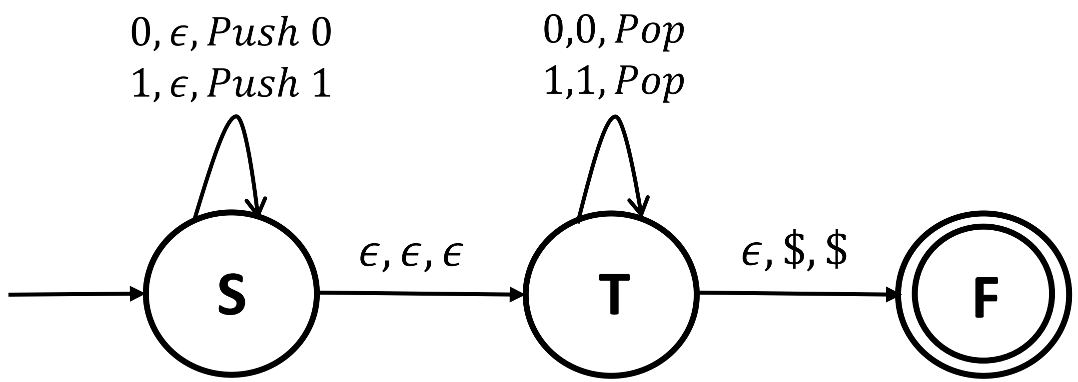

Recognizes even length palindromes of the form: $\boldsymbol{ww^R}$

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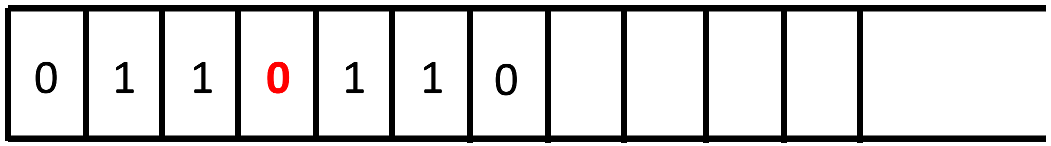

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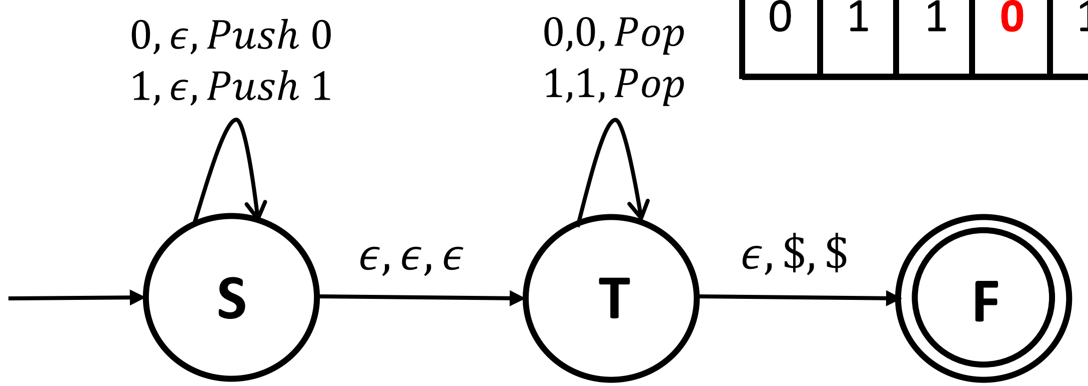

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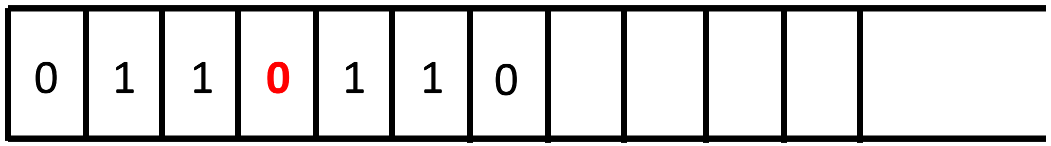

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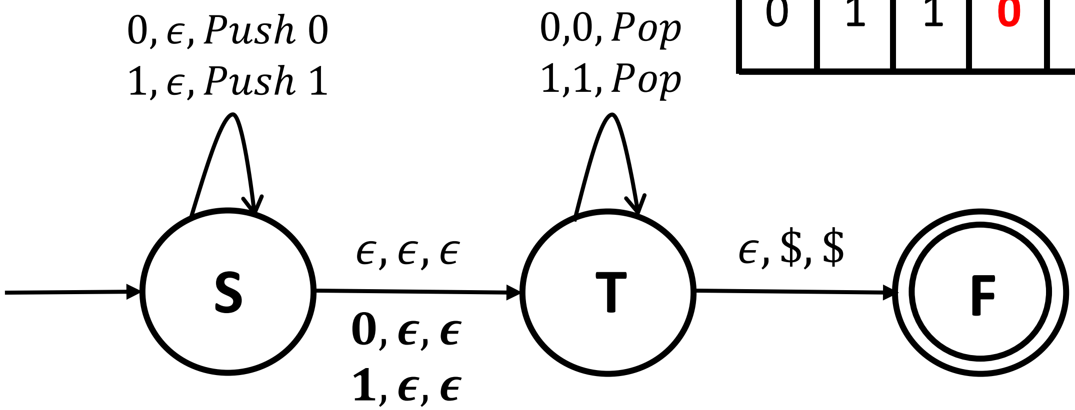

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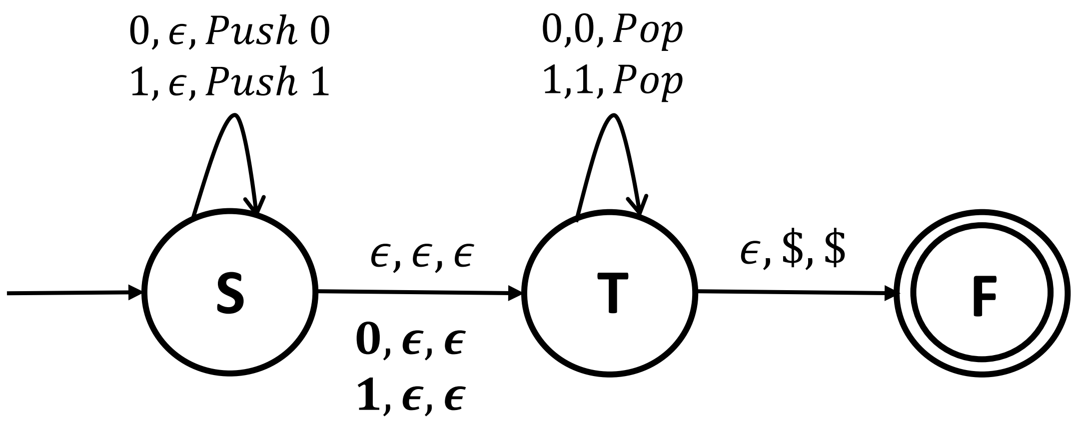

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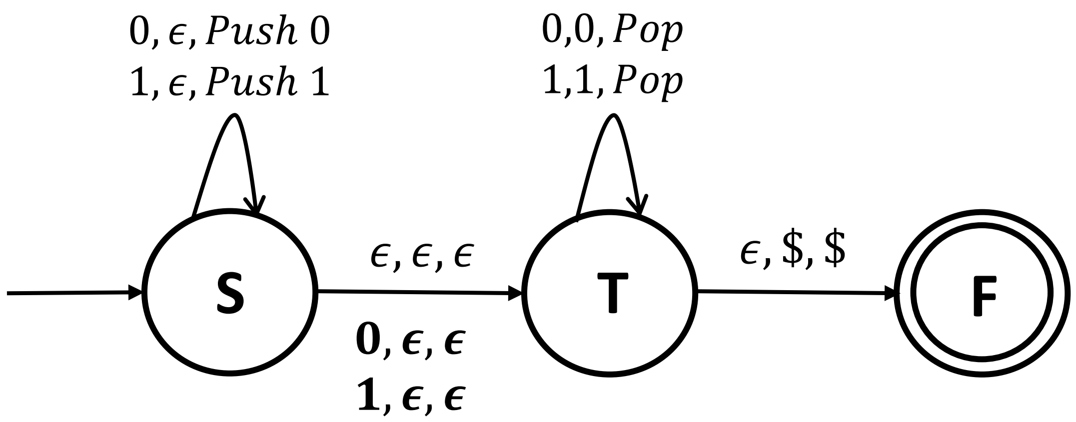

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This allows the PDA to **recognize strings of the form: $\boldsymbol{\omega c w^R}$**, where the aforementioned transitions non-deterministically guessed $c \in \{0{,}1\}$

Equivalence between PDA and CFL

- We already know that a language is Context-Free if and only if there exists a CFG that generates all the strings belonging to the CFL.

- It can be shown that a language is context free if and only if a PDA recognizes it.

    - If L is context free then there exists a PDA that recognizes L. (We'll prove this next)

    - If there exists a PDA for L, then L is context-free. (Won't prove this in class. Look up a standard text book)

Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

- Before formally proving this, we will use some examples in order to provide some intuition.
- For any L, we can write a context free grammar that can generate all strings that are in L.
- Any string w is generated by the CFG if there exists a derivation $S \overset{*}{\Rightarrow} w$.

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- For any L, we can write a context free grammar that can generate all strings that are in L.
- Any string w is generated by the CFG if there exists a derivation $S \overset{*}{\Rightarrow} w$.
- The proof consists of using the rules of the CFG to build a PDA so that it can simulate any derivation $S \overset{*}{\Rightarrow} w$.
    - The PDA accepts an input w if the CFG G generates w
    - It determines whether $\exists$ a derivation for w.
    - Takes advantage of non determinism

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Intuitions

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Example: Consider the grammar G with the rules: $S \rightarrow aTb|b$

$T \rightarrow Ta|\epsilon$

The string $\boldsymbol{w = aaab}$ can be generated by G. Derivation:

$$\boldsymbol{S \rightarrow aTb \rightarrow aTab \rightarrow aTaab \rightarrow aaab}$$

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Example: $S \to aTb|b$
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Input to PDA: $\mathbf{w} = \boldsymbol{aaab}$

Derivation for input string $w = aaab$ can be generated by G:

$$S \to aTb \to aTab \to aTaab \to aaab$$

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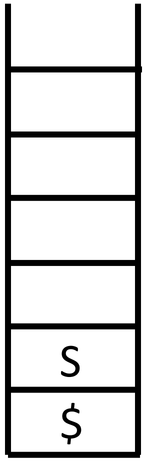

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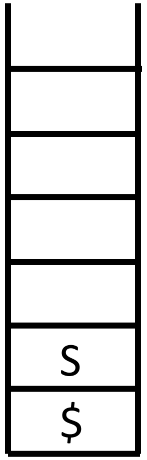

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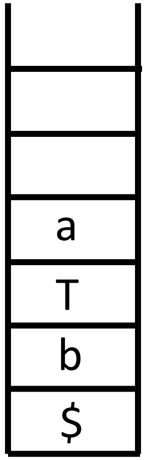

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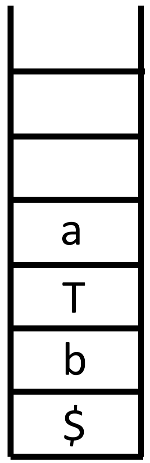

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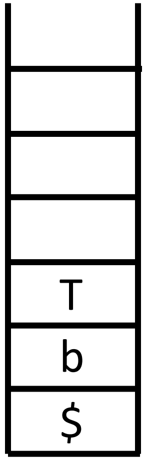

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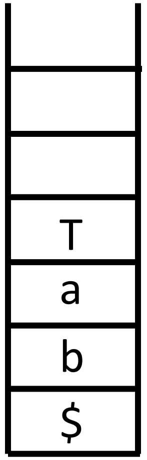

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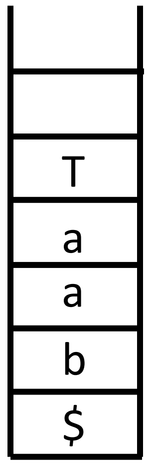

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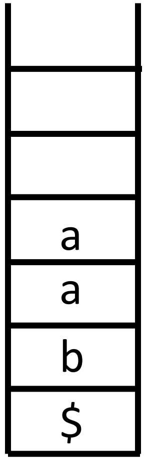

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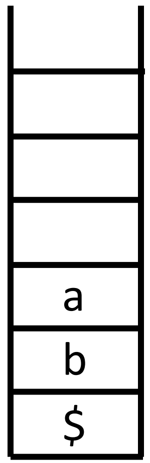

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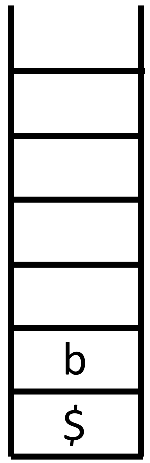

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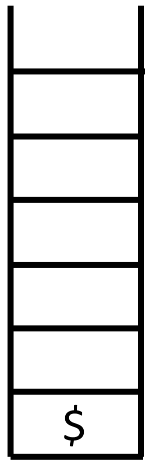
10. Since the stack is empty exactly when the input has been read, accept w.



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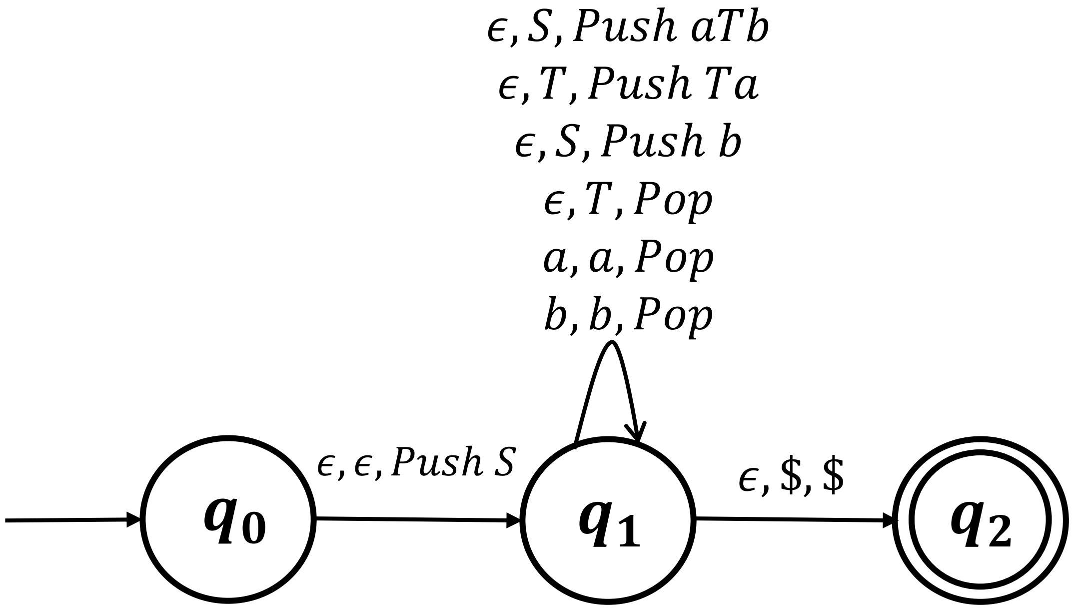

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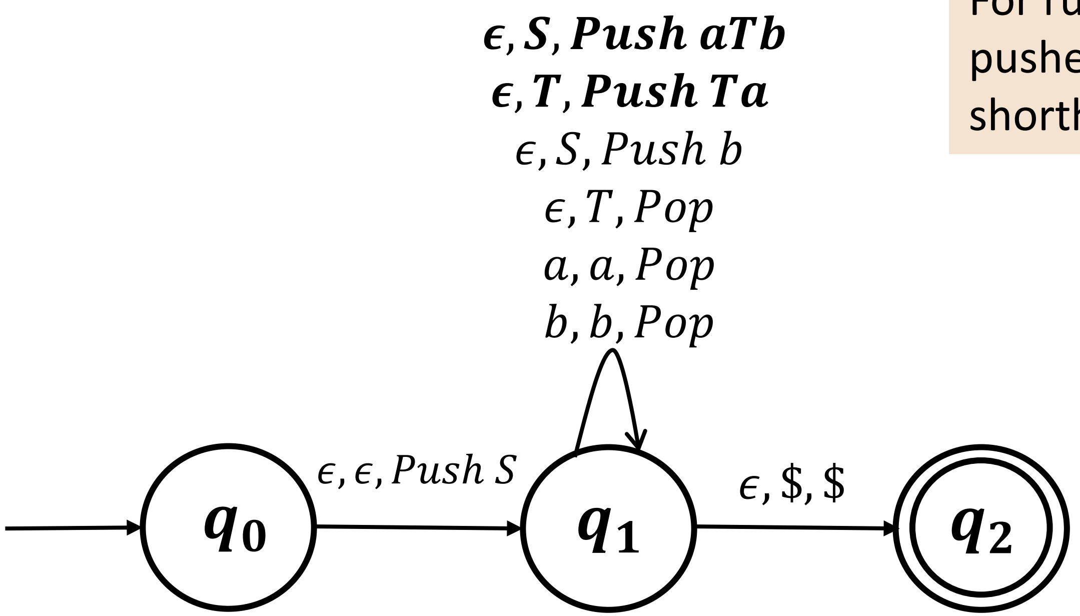
For rules where several elements need to be pushed, new states are introduced. This is only a shorthand for that.



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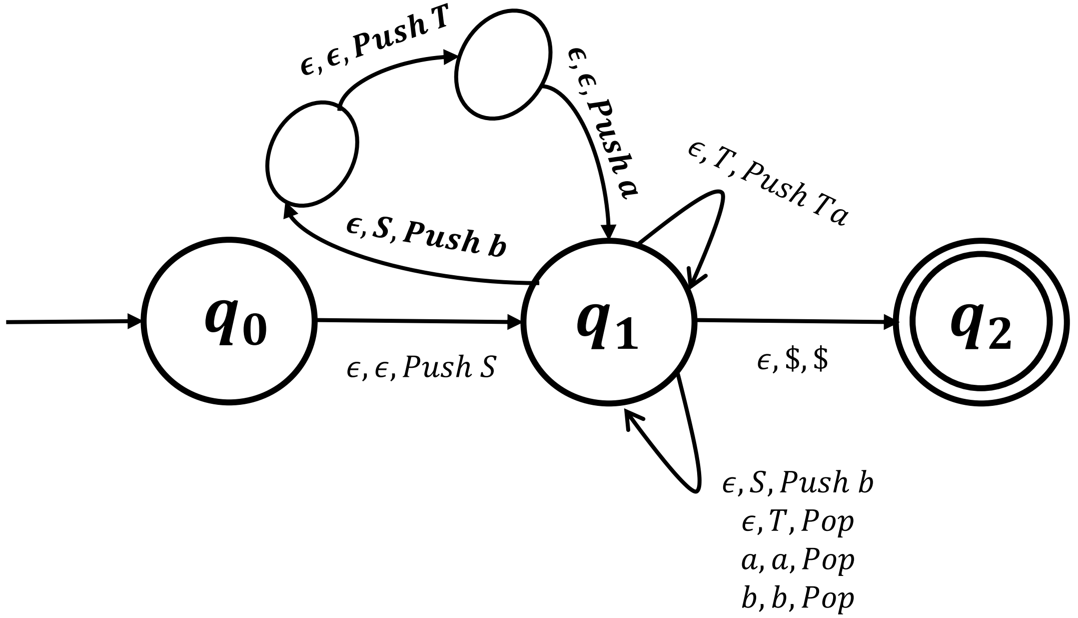

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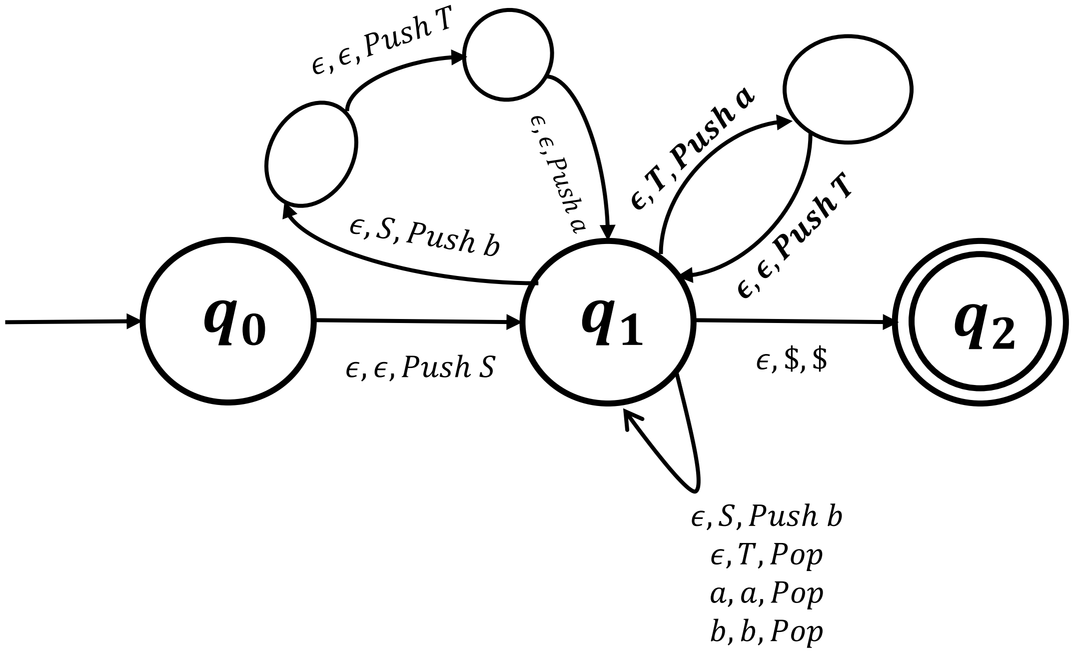


Summary

Given the rules of a CFG G, the equivalent PDA either non deterministically chooses which rule to use or matches part of the input symbol.

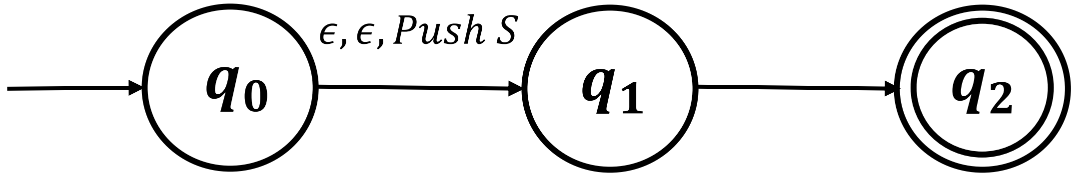
Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Proof: For convenience, we shall be using the shorthand notation.

Let G be a CFG with a set of rules R, then the equivalent PDA P will have three states $\{q_0, q_1, q_2\}$.

The PDA P first pushes the start symbol S into the stack, irrespective of the input symbol and transitions from the initial state q_0 to q_1, i.e. $\delta(q_0, \epsilon, \epsilon) = (q_1, S)$.



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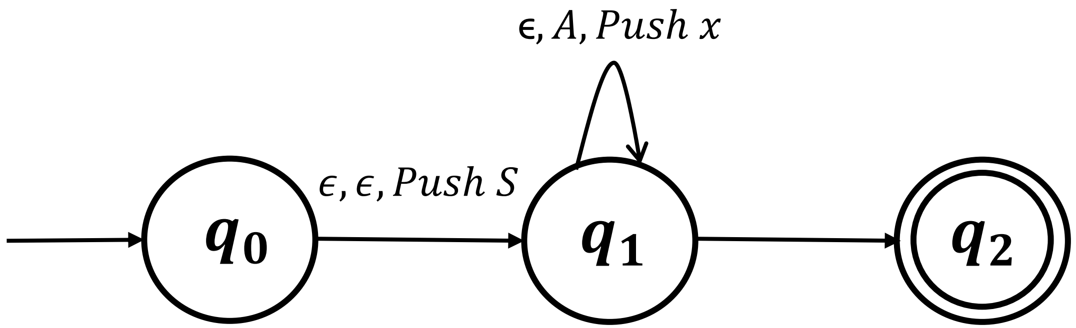

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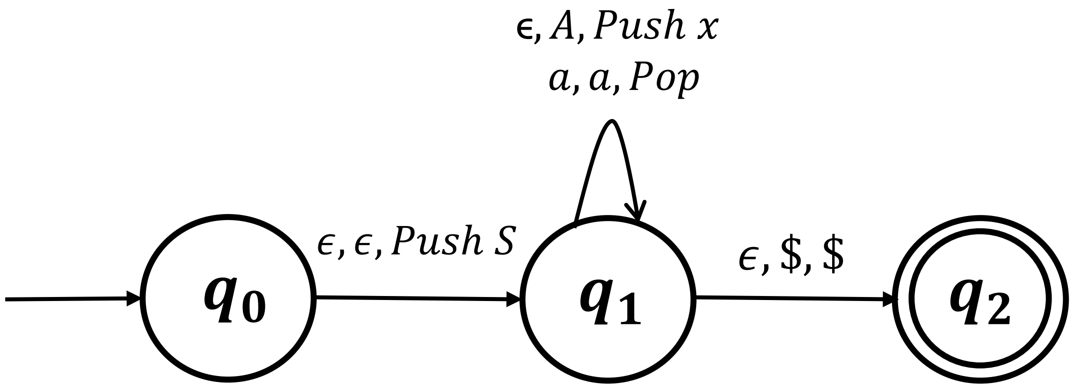

For terminal a

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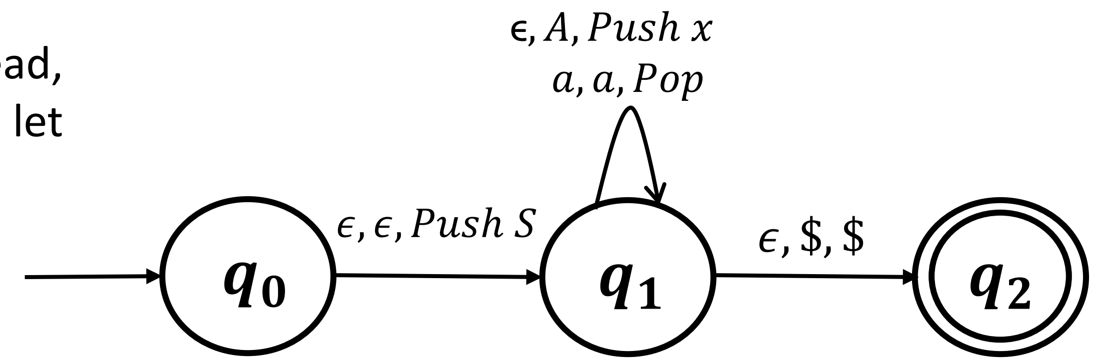
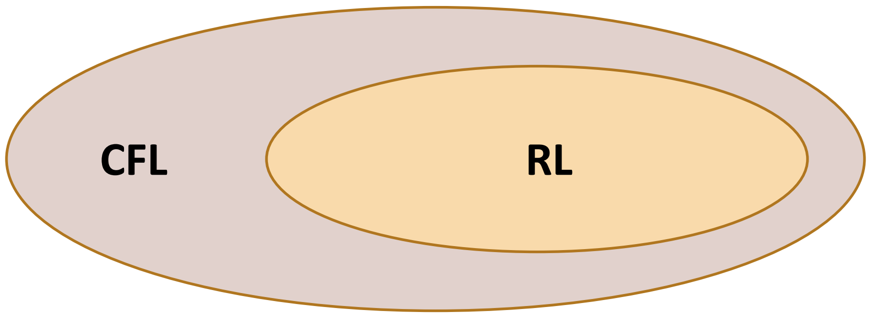


Equivalence between PDA and CFL

- It can be shown that a language is context free **if and only if** a PDA recognizes it.
  - If L is context free then there exists a PDA that recognizes L. (We proved this)
  - The proof for the other direction (Constructing a CFG that generates L given a PDA that recognizes L) is quite elaborate
  - We won't be covering it in class. But the proof itself is quite easy to understand.
  - Refer to a standard text book (e.g. Sipser)

$$(\boldsymbol{RL \equiv Regular\ Grammar \equiv Regular\ Expressions \equiv NFA \equiv DFA}) \subseteq (\boldsymbol{CFL \equiv CFG \equiv PDA})$$



Deterministic Pushdown Automata

- So far we have considered Non-deterministic PDAs (which are referred to as just PDAs)
- Multiple transitions per input symbol/stack symbol is allowed
- Recall that for regular languages, introducing non-determinism added no extra power to finite automata: NFAs and DFAs were equivalent
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- This enforces deterministic behaviour by ruling out scenarios such as: $\delta(q, a, x) \neq \Phi$ and $\delta(q, a, \epsilon) \neq \Phi$.
- If there is an ϵ-transition for some configuration, no other input consuming move is possible.

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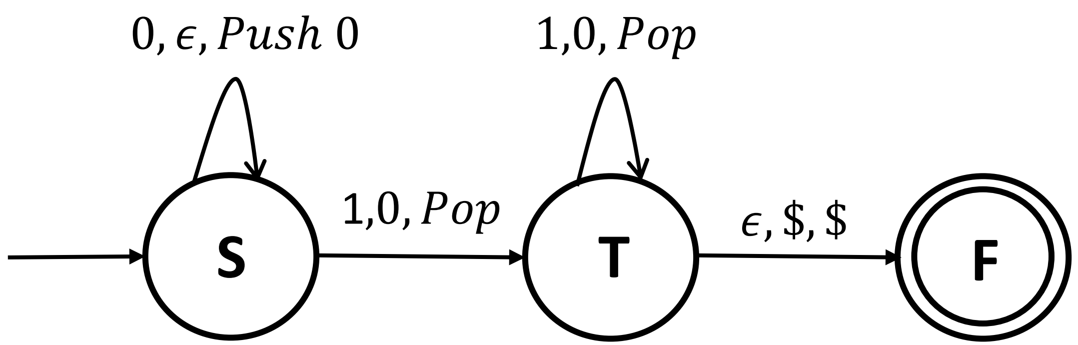

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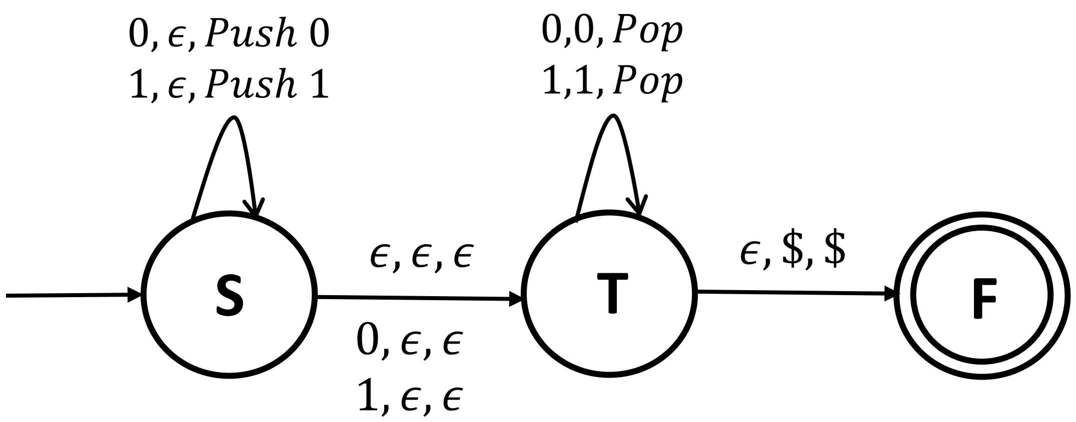

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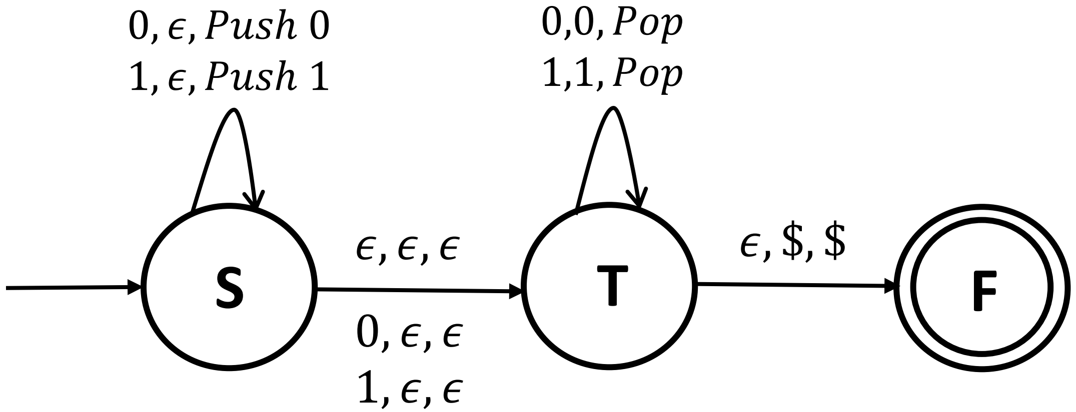
> **Example:** $\boldsymbol{L = \{w | w}$ **is a Palindrome**$\}$
>
> The PDA had to non-deterministically guess when half the string has been read and make a transition.
>
> **So although** $\boldsymbol{L \in CFL, L \notin DCFL}$.



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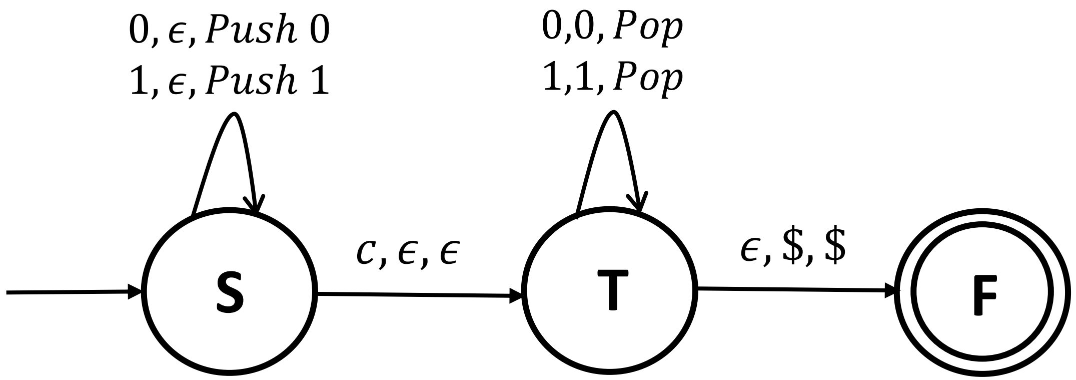
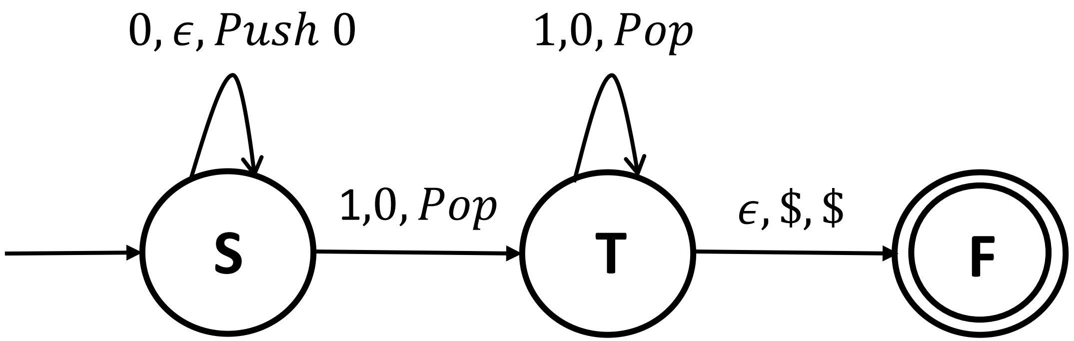
$$DCFL \subseteq CFL$$

- $\boldsymbol{\Sigma = \{0, 1, c\}}$
- $\boldsymbol{L_1 = \{wcw^R | w \in \{0, 1\}^+\}}$
- $\boldsymbol{L_1 \in DCFL}$.



- $\boldsymbol{L_2 = \{0^n1^n, n \geq 1\}}$
- $\boldsymbol{L_2 \in DCFL}$.



Deterministic Pushdown Automata

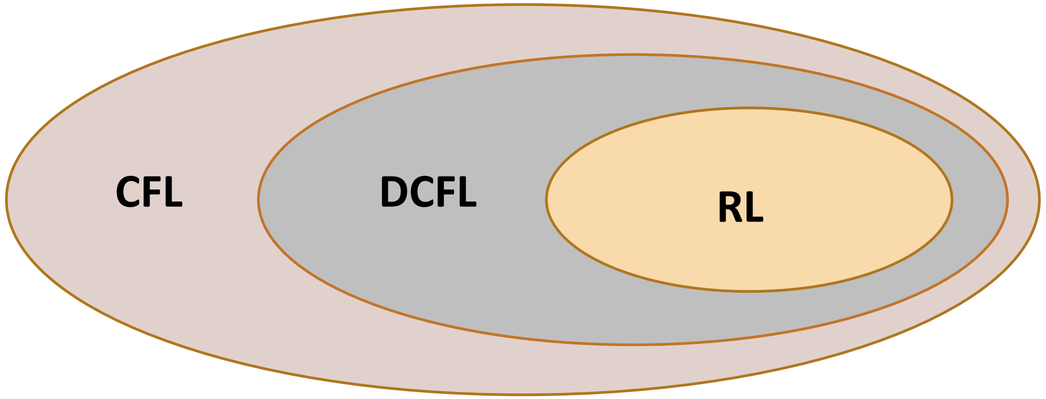
- The language recognized by DPDAs known as **Deterministic CFLs (DCFLs)** are **a proper subset of CFLs**, i.e.

$$\boldsymbol{DCFL \subseteq CFL}$$

$$\boldsymbol{(RL \equiv Regular\ Grammar \equiv Regular\ Expressions \equiv NFA \equiv DFA) \subseteq (DCFL \equiv DPDA) \subseteq (CFL \equiv CFG \equiv PDA)}$$

Next lecture:

- Pumping lemma for CFLs
- Closure properties of CFLs



Thank You!