Probability and Statistics: MA6.101
Quiz 2 Solutions

1 5-Marks Questions

1.1 Question 1

Question: Let A and B be two independent Poisson random variables with parameters a and b respectively. Let C = A + B. Using MGF, show that C is also a Poisson random variable.
Answer: First, let's derive the MGF of a Poisson random variable X with parameter λ.

Given the probability mass function of Poisson distribution:

$$P(X = n) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, 2, ...$$

The moment generating function is defined as:

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \sum_{n=0}^{\infty} e^{tn} P(X = n) \\ &= \sum_{n=0}^{\infty} e^{tn} \frac{\lambda^n e^{-\lambda}}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} \quad \text{(using power series expansion of } e^x) \\ &= e^{\lambda(e^t - 1)}, -\infty < t < \infty \end{aligned}$$

Clearly, the MGF for the Poisson distribution converges for all real values of t. Therefore, the ROC is $t \in (-\infty, \infty)$.

Using the above derivation: (since, $A \sim \text{Poisson}(a)$ and $B \sim \text{Poisson}(b)$)

$$M_A(t) = e^{a(e^t - 1)}$$

$$M_B(t) = e^{b(e^t - 1)}$$

$$\text{where, } -\infty < t < \infty$$

Derivation of MGF of C can be done in any one of the following ways:

Method 1:

$$\begin{aligned}M_C(t) &= M_A(t) \cdot M_B(t) \text{ (MGF of sum of independent random variables, A and B)}\\ &= e^{a(e^t-1)} \cdot e^{b(e^t-1)} = e^{(a+b)(e^t-1)}, -\infty < t < \infty\end{aligned}$$

Method 2:

$$\begin{aligned}M_C(t) &= E[e^{t(A+B)}]\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{t(m+n)} P(A=m, B=n)\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{t(m+n)} P(A=m)P(B=n) \text{ (Since A and B are independent)}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{t(m+n)} \frac{a^m e^{-a}}{m!}\frac{b^n e^{-b}}{n!}\\ &= e^{-(a+b)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(ae^t)^m}{m!}\frac{(be^t)^n}{n!}\\ &= e^{-(a+b)}(e^{ae^t})(e^{be^t}) \quad \text{(using power series expansion of } e^x)\\ &= e^{(a+b)(e^t-1)}\end{aligned}$$

Method 3:

$$\begin{aligned}M_C(t) &= E[e^{t(A+B)}]\\ &= E[e^{tA}e^{tB}]\\ &= E[e^{tA}]E[e^{tB}] \text{ (Since A and B are independent)}\\ &= M_A(t) \cdot M_B(t) = e^{a(e^t-1)}e^{b(e^t-1)} = e^{(a+b)(e^t-1)}\end{aligned}$$

This is the MGF of a Poisson distribution with parameter (a+b). Since moment generating functions uniquely determine distributions, we can conclude that:

$$C = A + B \sim \text{Poisson}(a+b)$$

Therefore, the sum of two independent Poisson random variables with parameters a and b is a Poisson random variable with parameter (a+b).

Marking Scheme:

- Correct derivation of MGF of Poisson distribution [1.5 marks]
  - Setting up MGF definition correctly: $M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} e^{tn} P(X = n)$ [0.5 marks]; getting final form $e^{\lambda(e^t-1)}$ [1 mark]
- Correct derivation of MGF of C [2 marks for any one method]

- **Method 1:** Applying property of MGF for sum of independent variables [1 mark] and getting final expression $M_C(t) = e^{(a+b)(e^t-1)}$ [1 mark]
- **Method 2:** Setting up double sum $E[e^{t(A+B)}]$ [0.5 marks], using independence to split probabilities [0.5 marks] and final simplification to $e^{(a+b)(e^t-1)}$ [1 mark]
- **Method 3:** Using independence to get $E[e^{tA}]E[e^{tB}]$ [1 mark] and substituting MGFs and simplifying to $e^{(a+b)(e^t-1)}$ [1 mark]

- Recognition that $e^{(a+b)(e^t-1)}$ is MGF of Poisson(a+b) [0.5 marks], stating uniqueness property of MGF [0.5 marks], and concluding $C \sim \text{Poisson}(a+b)$ [0.5 marks]

1.2 Question 2

Question: Derive an expression for the k^{th} moment of an exponential random variable with parameter λ using MGF.

Answer:

Let X be an exponential random variable with rate parameter λ. The probability density function (PDF) of X is given by:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

The moment generating function $M_X(t)$ of X is:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx$$

$$M_X(t) = \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx$$

For $\lambda > t$, this integral evaluates to:

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

The k-th moment of X, denoted by $\mathbb{E}[X^k]$, can be obtained by differentiating the MGF $M_X(t)$ k times with respect to t and then evaluating the result at $t = 0$:

$$\mathbb{E}[X^k] = \left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0}$$

We have:

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

Now, differentiate this expression k times:

- First derivative:
$$\frac{d}{dt}M_X(t) = \frac{\lambda}{(\lambda - t)^2}$$
- Second derivative:
$$\frac{d^2}{dt^2}M_X(t) = \frac{2\lambda}{(\lambda - t)^3}$$
- In general, the k-th derivative is:
$$\frac{d^k}{dt^k}M_X(t) = \frac{k!\lambda}{(\lambda - t)^{k+1}}$$

Evaluating this at $t = 0$, we get:

$$\mathbb{E}[X^k] = \left.\frac{k!\lambda}{(\lambda - t)^{k+1}}\right|_{t=0} = \frac{k!}{\lambda^k}$$

Thus, the k-th moment of an exponential random variable with rate parameter λ is:

$$\mathbb{E}[X^k] = \frac{k!}{\lambda^k}$$

Marking Scheme:

- Correct MGF - 1.5 Mark
- Correct ROC $t < \lambda$ - 1 Mark
- Correct k^{th} moment - 2.5 marks (1.5 marks for steps + 1 mark for final expression)
- No marks for deriving without the use of MGF

1.3 Question 3

Question: Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables, where $X_i \sim \text{Uniform}(-1, 1)$. Define the sequence of random variables:

$$Y_n = \frac{X_n}{n}$$

Show that Y_n converges in probability to a limit, and identify the limit.

Answer: We are given that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables, where $X_i \sim \text{Uniform}(-1, 1)$. We need to investigate the convergence of $Y_n = \frac{X_n}{n}$ as $n \to \infty$.

Since $X_n \sim \text{Uniform}(-1, 1)$, the random variable X_n takes values in the interval $[-1, 1]$. Hence, for any n, $Y_n = \frac{X_n}{n}$ is bounded by:

$$-\frac{1}{n} \leq Y_n \leq \frac{1}{n}.$$

As $n \to \infty$, the bounds $\frac{1}{n}$ and $-\frac{1}{n}$ both converge to 0. This suggests that Y_n becomes arbitrarily close to 0 as n grows. So, the limit is 0.

To show that Y_n converges in probability to 0, we need to prove that for every $\epsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - 0| \geq \epsilon) = 0.$$

This is equivalent to:

$$\mathbb{P}(|Y_n| \geq \epsilon) = \mathbb{P}(Y_n \geq \epsilon) + \mathbb{P}(Y_n \leq -\epsilon).$$

Here, the random variable $Y_n = \frac{X_n}{n}$

$$\mathbb{P}(|X_n| \geq n\epsilon) = \mathbb{P}(X_n \geq n\epsilon) + \mathbb{P}(X_n \leq -n\epsilon).$$

So WLOG for some ϵ there exists two cases :
Case 1: $n\epsilon > 1$, Case 2: $n\epsilon \leq 1$ because X_n is a uniform random variable whose realisations can lie in the interval (-1,1)

Case 1: $n > \frac{1}{\epsilon}$

For this case, we compute the probability:

$$\mathbb{P}(|X_n| \geq n\epsilon) = \mathbb{P}(X_n \geq n\epsilon) + \mathbb{P}(X_n \leq -n\epsilon).$$

Since $n\epsilon > 1$, and we know that $|X_n| \leq 1$ always holds (since $X_n \in [-1, 1]$), and hence:

$$\mathbb{P}(X_n \geq n\epsilon) = 0.$$

and similarly,

$$\mathbb{P}(X_n \leq -n\epsilon) = 0.$$

Thus, for any $n > \frac{1}{\epsilon}$, we have:

$$\mathbb{P}(|Y_n| \geq \epsilon) = 0.$$

Case 2: $n \leq \frac{1}{\epsilon}$

For this case, again we start with the probability:

$$\mathbb{P}(|X_n| \geq n\epsilon) = \mathbb{P}(X_n \geq n\epsilon) + \mathbb{P}(X_n \leq -n\epsilon).$$

Here, the random variable X_n lies between -1 and 1. However, when $n\epsilon \leq 1$, it is possible that X_n lies between $n\epsilon$ and 1, or between -1 and $-n\epsilon$.

Since $X_n \sim \text{Uniform}(-1, 1)$, we have:

$$\mathbb{P}(X_n \geq n\epsilon) = \frac{1 - n\epsilon}{2}, \quad \text{for } n\epsilon \leq 1.$$

Similarly,

$$\mathbb{P}(X_n \leq -n\epsilon) = \frac{1 - n\epsilon}{2}, \quad \text{for } n\epsilon \leq 1.$$

Thus, the total probability is:

$$\mathbb{P}(|Y_n| \geq \epsilon) = \frac{1 - n\epsilon}{2} + \frac{1 - n\epsilon}{2} = 1 - n\epsilon, \quad \text{for } n\epsilon \leq 1.$$

So on combining the cases, for an ϵ, we get,

$$\mathbb{P}(|Y_n| \geq \epsilon) = \begin{cases} 1 - n\epsilon & \text{if } 0 \leq n \leq 1/\epsilon, \\ 0 & \text{if } n > 1/\epsilon. \end{cases}$$

For sufficiently large n ($n \to \infty$), n will be greater than any fixed k $= 1/\epsilon$ where $\epsilon > 0$. Hence we have,

$$\lim_{n\to\infty} \mathbb{P}(|Y_n| \geq \epsilon) = 0.$$

Therefore,

$$Y_n \xrightarrow{p} 0 \quad .$$

Marking Scheme:

- 2 marks for finding the limit
- 3 marks for proving convergence in probability.
- Any other valid method will be considered and given marks appropriately

1.4 Question 4

Question: Let $X_1, X_2, \ldots$ be a sequence of random variables such that:

$$F_{X_n}(x) = \frac{e^{n(x-1)}}{1 + e^{n(x-1)}} \quad \text{for } x > 0.$$

Show that X_n converges in distribution. Identify the limiting random variable.

Answer: A sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X, denoted by:

$$X_n \xrightarrow{d} X,$$

if:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \text{for all } x \text{ where } F_X(x) \text{ is continuous.}$$

We do not require the CDF to converge at points where $F_X(x)$ is discontinuous.

Case 1: $x < 1$

For $x < 1$, as $n \to \infty$, $n(x-1) \to -\infty$. Therefore:

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{e^{n(x-1)}}{1+e^{n(x-1)}} = \frac{0}{1+0} = 0.$$

Thus:

$$\lim_{n\to\infty} F_{X_n}(x) = 0.$$

Case 2: $x > 1$

For $x > 1$, as $n \to \infty$, $n(x-1) \to \infty$. Therefore:

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \frac{1}{1+e^{-n(x-1)}} = \frac{1}{1+0} = 1.$$

Thus:

$$\lim_{n\to\infty} F_{X_n}(x) = 1.$$

Case 3: $x = 1$

At $x = 1$, the CDF value for each n is:

$$\lim_{n\to\infty} F_{X_n}(1) = \frac{e^{n(1-1)}}{1+e^{n(1-1)}} = \frac{1}{2}.$$

We should define the limiting CDF $F_X(x)$ from $\lim_{n\to\infty} F_{X_n}(x)$ such that:

$$F_X(x) = \lim_{n\to\infty} F_{X_n}(x) \quad \text{for all } x \text{ where } F_X(x) \text{ is continuous.}$$

So,

$$F_X(x) = \lim_{n\to\infty} F_{X_n}(x) \quad \text{for } x \neq 1.$$

Since directly assigning the value $\lim_{n\to\infty} F_{X_n}(1)$ would violate the right-continuous property of CDFs, we instead assign:

$$F_X(1) = 1 \quad \text{for } x = 1,$$

to ensure $F_X(x)$ is a valid CDF.

Conclusion:

The limiting CDF $F_X(x)$ is given by:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1, \\ 1 & \text{for } x \geq 1. \end{cases}$$

This corresponds to a constant random variable equal to 1. Therefore:

$$X_n \xrightarrow{d} 1$$

Marking Scheme:

- 2 marks for correctly determining the limiting CDF for $x < 1$.
- 2 marks for correctly determining the limiting CDF for $x > 1$.
- 1 mark for correctly identifying the limiting random variable.

2 10-Marks Questions

2.1 Question 1

Question: Given samples $x_1, \ldots x_n$ from an exponential random variable X with parameter λ, convert it into samples from another exponential random variable with parameter μ. Explain the procedure in detail with justifications
Answer: The solution involves 2 steps:

- Sampling a uniform $U[0,1]$ random variable from X. This involves converting samples of X - $x_1 \ldots x_n$ to samples of U- $u_1 \ldots u_n$
- Sampling Y from $U[0,1]$. This involves converting samples of U - $u_1 \ldots u_n$ to samples of Y- $y_1 \ldots y_n$

Sampling $U[0,1]$ from X

Let $\hat{U} = F_X(X)$. Then the cdf of $\hat{U}$ is $F_U(.)$
Proof:

The CDF of $\hat{U}$ is given by

$$\begin{aligned} F_{\hat{U}}(u) &= \mathbb{P}[\hat{U} \leq u] \\ &= \mathbb{P}[F_X(X) \leq u] \\ &= \mathbb{P}[X \leq F_X^{-1}(u)] \\ &= F_X(F_X^{-1}(u)) \\ &= u = F_U(u) \end{aligned}$$

The cdf of $X \sim exp(\lambda)$ is given by

$$\begin{aligned} F_X(x) &= 1 - e^{-\lambda x} \\ \implies \hat{U} = F_X(X) &= 1 - e^{-\lambda X} \end{aligned}$$

We can convert samples of X to samples of U

$$u_i = 1 - e^{-\lambda x_i}, \quad i = 1 \dots n$$

Sampling Y from $\hat{U}$

Now using the samples of $\hat{U}$, we generate samples of Y using inverse transform sampling
Let $\hat{Y} = F_Y^{-1}(\hat{U})$. Then the cdf of $\hat{Y}$ is $F_Y(.)$
Proof:

$$\begin{aligned} F_{\hat{Y}}(y) &= \mathbb{P}[\hat{Y} \leq y] \\ &= \mathbb{P}[F_Y^{-1}(\hat{U}) \leq y] \\ &= \mathbb{P}[\hat{U} \leq F_Y(y)] \\ &= F_{\hat{U}}(F_Y(y)) \\ &= F_Y(y) \quad \text{since } 0 \leq F_Y(y) \leq 1, \end{aligned}$$

The CDF of $Y \sim exp(\mu)$ is given by

$$\begin{aligned} F_Y(y) &= 1 - e^{-\mu y} \\ \implies F_Y^{-1}(y) &= \frac{-1}{\mu} \ln(1 - y) \end{aligned}$$

$$
\begin{aligned}
\hat{Y} &= F_Y^{-1}(\hat{U}) \\
&= \frac{-1}{\mu}\ln(1-\hat{U}) \\
&= \frac{-1}{\mu}\ln(1-(1-e^{-\lambda X})) \\
&= \frac{-1}{\mu} ln(e^{-\lambda X}) \\
&= \frac{\lambda}{\mu} X
\end{aligned}
$$

We can convert samples of $X : x_1 \dots x_n$ to samples of Y

$$
\begin{aligned}
y_i &= \frac{-1}{\mu}\ln(1-u_i) \\
\implies y_i &= \frac{\lambda}{\mu} x_i, \quad i = 1 \dots n
\end{aligned}
$$

Marking Scheme:

- Correct proof for $\hat{U}$ - 2.5 marks
- Correct formula for $\hat{U}$ - 2.5 marks
- Correct proof for $\hat{Y}$ -2.5 marks
- Correct formula for $\hat{Y}$ - 2.5 marks
- Marks will also be awarded for accept reject method