Probability and Statistics: MA6.101

Homework 6

Topics Covered: Conditional Probability, Conditional Expectation

Q1: Let X and Y be two jointly continuous random variables with joint probability density function (PDF) given by:

$$f_{XY}(x,y) = \begin{cases} \frac{x^2}{6} + \frac{y^2}{6} + \frac{xy}{8}, & 0 \leq x \leq 2, 0 \leq y \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

For $0 \leq y \leq 3$, find:

- $\mathbb{E}[X \mid Y = 2]$
- $\mathrm{Var}(X \mid Y = 2)$

Q2: Let X and Y be two independent $N(0,1)$ random variables, and $U = X + Y$.

(a) Find the conditional PDF of U given $X = x$, $f_{U|X}(u|x)$.

(b) Find the PDF of U, $f_U(u)$.

(c) Find the conditional PDF of X given $U = u$, $f_{X|U}(x|u)$.

(d) Find $E[X|U = u]$, and $\mathrm{Var}(X|U = u)$.

Q3: Fraser runs a dolphin-watch business. Every day, he is unable to run the trip due to bad weather with probability p, independently of all other days. Fraser works every day except the bad-weather days, which he takes as holiday.

Let Y be the number of consecutive days Fraser has to work between bad-weather days. Let X be the total number of customers who go on Fraser's trip in this period of Y days. Conditional on Y, the distribution of X is

$$(X|Y) \sim \mathrm{Poisson}(\mu Y).$$

(a) Name the distribution of Y, and state $E(Y)$ and $\mathrm{Var}(Y)$.

(b) Find the expectation and the variance of the number of customers Fraser sees between bad-weather days, $E(X)$ and $\mathrm{Var}(X)$.

[Poisson(λ) Random Variable is a discrete random variable with mean λ and variance λ.]

Q4: A surface is ruled with parallel lines, which are at distance d from each other See Figure 1. Suppose that we throw a needle of length l on the surface at random. What is the probability that the needle will intersect one of the lines?

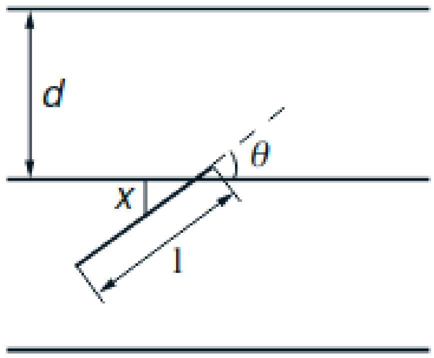

Figure 1: Figure for question 4

Q5: The following is one formulation of a famous "two envelope" paradox.
Jill is a money-loving individual who, given two options, invariably chooses the one that gives her the most money in expectation. One day Harry, a trusted (and capable of delivering) individual, offers her the following deal as a gift. He will secretly toss a fair coin until the first time that it comes up tails. If there are n heads before the first tails, he will place 10^n dollars in one envelope and 10^{n+1} dollars in the second envelope.(Thus, the probability that one envelope has 10^n dollars and the other has 10^{n+1} dollars is 2^{-n-1} for $n \geq 0$.) Harry will then hand Jill the pair of envelopes (randomly ordered, indistinguishable from the outside) and invite her to choose one. After Jill chooses an envelope she will be allowed to open it. Once she does, she will be allowed to either keep the money in the first envelope or switch to the second envelope and keep whatever amount of money is in the second envelope. However, if she decides to switch envelopes, she has to pay a one dollar "switching fee."

(a) If Jill finds 100 dollars in the first envelope she opens, what is the conditional probability that the other envelope contains 1000 dollars? What is the conditional probability that the other envelope contains 10 dollars?

(b) If Jill finds 100 dollars in the first envelope she opens, how much money does Jill expect to win from the game if she does not switch envelopes? (Answer: 100 dollars.) How much does she expect to win (net, after the switching fee) if she does switch envelopes?

(c) Generalize the answers above to the case that the first envelope contains 10^n dollars (for $n \geq 0$) instead of 100.