

Probability and Statistics: MA6.101

Homework 3

Topics Covered: Random Variables, Expectation, Functions on Random Variables, Discrete Random Variables.

Q1: St. Petersburg Paradox: “A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The pot starts at 1 dollar and is doubled every time a head appears. The first time a tail appears, the game ends and the player wins whatever is in the pot. Thus the player wins 1 dollar if a tail appears on the first toss, 2 dollars if a head appears on the first toss and a tail on the second, 4 dollars if a head appears on the first two tosses and a tail on the third, 8 dollars if a head appears on the first three tosses and a tail on the fourth, and so on. In short, the player wins 2^{k-1} dollars if the coin is tossed k times until the first tail appears. What would be a fair price to pay the casino for entering the game?”

- Let X be the amount of money (in dollars) that the player wins. Find $E[X]$.
- Now suppose that the casino only has a finite amount of money. Specifically, suppose that the maximum amount of the money that the casino will pay you is 2^{30} dollars (around 1.07 billion dollars). That is, if you win more than 2^{30} dollars, the casino is going to pay you only 2^{30} dollars. Let Y be the money that the player wins in this case. Find $E[Y]$.

Q2: Let X be a random variable with mean $E[X] = \mu$. Define the function $f(\alpha)$ as

$$f(\alpha) = E[(X - \alpha)^2].$$

Find the value of α that minimizes f . Can you intuitively explain why did we get that result?

Q3: : Let X be a binomial random variable with parameters (n, p) . What value of p maximizes $P\{X = k\}, k = 0, 1, 2, \dots, n$?

Q4: For each of the following random variables, find $P(X > 7)$ and $P(3 < X \leq 8)$:

- $X \sim \text{Geometric}(0.25)$
- $X \sim \text{Binomial}(12, 0.4)$

Q5: Suppose you participate in a quiz competition that consists of 15 multiple-choice questions. Each question has 5 possible options. You know the correct answer to 7 questions, but for the remaining 8 questions, you guess randomly. Let your total score Z on the quiz be the number of correct answers you get.

- Find the PMF of Z .
- What is $P(Z > 10)$?

Q6: Let X denote a discrete random variable that can take the values -2, -1, M and 2. Given that X has probability distribution function $f(X) = \frac{X+4}{16}$, find the variance of X .