Probability and Statistics: MA6.101

Quiz 1 Solutions

Q1: For a non-negative discrete random variable X, prove that

$$E[X] = \sum_{i=0}^{\infty} P(X > i)$$

[Kushal]

Solution

The students that assumed X to only take integer values (along with the previous constraints) will get the following answer

We know that by definition the formula for a discrete random variable expectation of a random variable X is

$$E[X] = \sum_{x \in \Omega} xP(X = x)$$

where Ω is the set of values the discrete variable X can take.
We also know that

$$P(X > i) = \sum_{x \in \Omega'} P(X = x)$$

where

$$\Omega' = \{\omega \in \Omega : \omega > i\}$$

Now, we know that we can represent x as x = $\sum_0^x (1)$, since X was a non-negative discrete random variable with integer values.

$$E[X] = \sum_{x \in \Omega} (\sum_0^x 1) P(X = x)$$

Since, $P(X = x)$ is always non-negative we can exchange the limits of the summations. And since the range of X was 0 to ∞, so we can replace Ω with that

$$E[X] = \sum_{i=0}^{\infty} \sum_{x > i} P(X = x)$$

$$E[X] = \sum_{i=0}^{\infty} P(X > i)$$

For students who didn't assume that X only took integer values they will get this answer

We know that by definition the formula for a discrete random variable expectation of a random variable X is

$$E[X] = \sum_{x \in \Omega} xP(X = x)$$

where Ω is a countable set.
We also know that

$$P(X > i) = \sum_{x \in \Omega'} P(X = x)$$

where

$$\Omega' = \{\omega \in \Omega : \omega > i\}$$

Now, we know that we can represent x as x = $\int_0^x du$, since X was a non-negative discrete random variable.

$$E[X] = \sum_{x \in \Omega} (\int_0^x du) P(X = x)$$

Since, $P(X = x)$ is always non-negative we can exchange the limits of the summation and the integral.

$$E[X] = \int_0^\infty \sum_{x > u} P(X = x) du$$

$$E[X] = \int_0^\infty P(X > u) du$$

Marking Scheme (5 Marks Total):

- **Correctly stating $E[X]$ and $P(X > i)$ (1 mark)**
  - 0.5 marks each
- **Correctly manipulating the summation of E[X] (3 marks) :**
  - Writing x as a sum or integral with limits 0 to x (1 mark).
  - Swapping the summations or integral and using correct limits (2 marks)
- **Final answer (1 mark) :**
  - Simplifying the double summation or integral to get the final answer

Both answers will be given 5 marks.

Other solutions that don't assume X to be integers will be considered and marked appropriately.

Q2: Consider a Geometric random variable X with parameter p. Derive the expression for its mean, second moment and variance. [Ronak]

Solution

1. Mean (Expected Value) of X

For a Geometric random variable X with parameter p, the probability mass function is given by:

$$P(X = k) = (1-p)^{k-1}p \quad \text{for } k = 1, 2, 3, \ldots$$

The expected value $\mathbb{E}[X]$ is calculated as:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot P(X = k) = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}p$$

This can be simplified using the identity for the sum of an infinite series. Consider the series:

$$S = \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \quad \text{for } |x| < 1$$

Let $x = 1 - p$:

$$\mathbb{E}[X] = p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

Thus, the mean of X is:

$$\mathbb{E}[X] = \frac{1}{p}$$

2. Second Moment of X

The second moment $\mathbb{E}[X^2]$ is calculated as:

$$\mathbb{E}[X^2] = \sum_{k=1}^{\infty} k^2 \cdot P(X = k) = \sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1}p$$

Using the identity for the sum of the series:

$$\sum_{k=1}^{\infty} k^2 x^{k-1} = \frac{1+x}{(1-x)^3}$$

Let $x = 1 - p$:

$$\mathbb{E}[X^2] = p\sum_{k=1}^{\infty} k^2(1-p)^{k-1} = p \cdot \frac{1 + (1-p)}{p^3} = p \cdot \frac{2-p}{p^3} = \frac{2-p}{p^2}$$

Thus, the second moment of X is:

$$\mathbb{E}[X^2] = \frac{2-p}{p^2}$$

3. Variance of X

The variance $\mathrm{Var}(X)$ is calculated as:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Substitute the values from the previous calculations:

$$\mathrm{Var}(X) = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{2-p-1}{p^2} = \frac{1-p}{p^2}$$

Thus, the variance of X is:

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

Alternative Derivation

We will apply the total expectation theorem, with $A_1 = \{X = 1\}$ (the event that the first try is a success) and $A_2 = \{X > 1\}$ (the event that the first try is a failure). This approach leads to a simpler calculation.

If the first try is successful, then $X = 1$, and we have:

$$\mathbb{E}[X \mid X = 1] = 1.$$

If the first try fails ($X > 1$), we have wasted one try, and we are back where we started. Therefore, the expected number of remaining tries is $\mathbb{E}[X]$, so:

$$\mathbb{E}[X \mid X > 1] = 1 + \mathbb{E}[X].$$

By the total expectation theorem:

$$\mathbb{E}[X] = P(X = 1)\mathbb{E}[X \mid X = 1] + P(X > 1)\mathbb{E}[X \mid X > 1]$$

$$= p \cdot 1 + (1-p) \cdot (1 + \mathbb{E}[X]).$$

Expanding and solving for $\mathbb{E}[X]$:

$$\mathbb{E}[X] = p + (1-p)(1 + \mathbb{E}[X]),$$

$$\mathbb{E}[X] = p + (1-p) + (1-p)\mathbb{E}[X],$$

$$\mathbb{E}[X] - (1-p)\mathbb{E}[X] = 1,$$

$$\mathbb{E}[X](p) = 1,$$

$$\mathbb{E}[X] = \frac{1}{p}.$$

With similar reasoning, we can derive $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2 \mid X = 1] = 1,$$

$$\mathbb{E}[X^2 \mid X > 1] = \mathbb{E}[(1 + X)^2],$$

$$\mathbb{E}[(1 + X)^2] = 1 + 2\mathbb{E}[X] + \mathbb{E}[X^2].$$

Using the total expectation theorem again:

$$\mathbb{E}[X^2] = p \cdot 1 + (1 - p) \cdot \left(1 + 2\mathbb{E}[X] + \mathbb{E}[X^2]\right),$$

$$\mathbb{E}[X^2] = p + (1 - p) + 2(1 - p)\mathbb{E}[X] + (1 - p)\mathbb{E}[X^2],$$

$$\mathbb{E}[X^2] - (1 - p)\mathbb{E}[X^2] = p + 2(1 - p)\mathbb{E}[X],$$

$$\mathbb{E}[X^2](p) = 1 + 2(1 - p)\mathbb{E}[X],$$

$$\mathbb{E}[X^2] = \frac{1 + 2(1 - p)\mathbb{E}[X]}{p}.$$

Substituting $\mathbb{E}[X] = \frac{1}{p}$:

$$\mathbb{E}[X^2] = \frac{1 + 2(1 - p)\frac{1}{p}}{p} = \frac{1 + 2(1 - p) \cdot \frac{1}{p}}{p},$$

$$\mathbb{E}[X^2] = \frac{1 + \frac{2(1-p)}{p}}{p} = \frac{p + 2(1 - p)}{p^2},$$

$$\mathbb{E}[X^2] = \frac{2 - p}{p^2}.$$

So,

$$\mathbb{E}[X^2] = \frac{2}{p^2} - \frac{1}{p}.$$

Alternate Variance Derivation

$$\mathrm{Var}[X] = E[X(X - 1)] + E[X] - (E[X])^2$$

$$E[X(X - 1)] = 2(1 - p)p^2$$

Marking Scheme (5 Marks Total):

- **Derivation of the Mean (2 marks):**
  - Correct setup of the expectation formula (1 mark).
  - Correct derivation and final expression for $\mathbb{E}[X] = \frac{1}{p}$ (1 mark).
- **Derivation of the Second Moment (2 marks):**
  - Correct setup of the second moment formula (1 mark).
  - Correct derivation and final expression for $\mathbb{E}[X^2] = \frac{2-p}{p^2}$ (1 mark).
- **Derivation of the Variance (1 mark):**
  - Correct derivation and final expression for $\text{Var}(X) = \frac{1-p}{p^2}$ (1 mark).

Q3: Suppose that we roll a die twice. Consider the following three events

- A = Second roll is 4
- B = Difference between the two rolls is 4
- C = Difference between the two rolls is 3

Are the three events pairwise independent? Are they also mutually independent? [Abhinav]

Solution

$P(A) = 1/6$
For event B, favourable outcomes are (1, 5), (5, 1), (2, 6), (6, 2) and hence $P(B) = 4/36 = 1/9$
For event C, favourable outcomes are (1, 4), (4, 1), (2, 5), (5, 2), (3, 6), (6, 3) and hence $P(C) = 6/36 = 1/6$
$P(A \bigcap B) = 0$ as the difference between 2 dice can't be 4 with the second roll being 4, whereas $P(A)P(B) = 1/54$
$P(B \bigcap C) = 0$ as the difference between 2 dice can't be 4 and 3 simultaneously, whereas $P(B)P(C) = 1/54$
$P(A \bigcap C) = 1/36$ as there is only one possibility (1, 4)
Since only $P(A \bigcap C) = 1/36 = P(A) \cdot P(C)$, A & C are independent. A & B and B & C are not.
Since all the pairs are not independent, the 3 events are not pairwise independent.
Since all subsets of the set of events (A, B and C) are not independent, the events are not mutually independent.

Marking Scheme (5 Marks Total):

- Correctly finding the probabilities of events and their intersection (3 marks)
- Pairwise independence with reason (1 mark)

- Mutual independence with reason (1 mark)
- If the difference is considered as $1^{st}die - 2^{nd}die$, changes will be as follows:
  - $P(B) = 1/18$, $P(C) = 1/12$
  - $P(A \bigcap C) = 0$ and still the events will not be both pairwise and mutually independent.
  - This notion of difference will also be considered if it is clearly mentioned in the answer. Also holds for $2^{nd}die - 1^{st}die$
- Partial marks will be given depending on the correctness of the answer

Q4: Let X denote a Gaussian random variable with parameters c and d. Let Y denote a Binomial random variable with parameters n and p. Derive the expression for their respective variance

Solution

Derivation for Gaussian

Note: Replaced d by d^2 here for simplicity
The pdf of a Gaussian random variable $X \sim \mathcal{N}(c, d^2)$ is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right)$$

Let $I(n)$ denote the following family of integrals

$$I(n) = \int_{-\infty}^{\infty} \frac{(x-c)^n}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx$$

Applying integration by parts, taking $u = \exp\left(-\frac{(x-c)^2}{2d^2}\right)$, $dv = \frac{(x-c)^n dx}{\sqrt{2\pi d^2}}$, the indefinite integral is given by

$$\begin{aligned}
\int udv &= uv - \int vdu \\
&= \exp\left(-\frac{(x-c)^2}{2d^2}\right) \frac{(x-c)^{n+1}}{(n+1)\sqrt{2\pi d^2}} - \int \frac{(x-c)^{n+1}}{(n+1)\sqrt{2\pi d^2}} \frac{-(x-c)}{d^2} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx \\
&= \exp\left(-\frac{(x-c)^2}{2d^2}\right) \frac{(x-c)^{n+1}}{(n+1)\sqrt{2\pi d^2}} + \frac{1}{(n+1)d^2} \int \frac{(x-c)^{n+2}}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx
\end{aligned}$$

Applying the limits from $-\infty$ to ∞, the first term simplifies to 0 and we get

$$I(n) = \frac{I(n+2)}{(n+1)d^2}$$

or

$$I(n) = (n-1)d^2 I(n-2)$$

$$\begin{aligned}
I(0) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx \\
&= \int_{-\infty}^{\infty} f_X(x)\, dx \\
&= 1
\end{aligned}$$

To calculate the mean, we observe that

$$\begin{aligned}
I(1) &= \int_{-\infty}^{\infty} \frac{(x-c)}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx - c\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx \\
&= E[X] - c
\end{aligned}$$

$$I(1) = \int_{-\infty}^{\infty} \frac{(x-c)}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx$$

Substituting $t = x - c$, we get

$$I(1) = \int_{-\infty}^{\infty} \frac{t}{\sqrt{2\pi d^2}} \exp\left(-\frac{t^2}{2d^2}\right) dt$$

Since this is an odd function, the integral is 0

$$\begin{aligned}
&\implies I(1) = 0 \\
&\implies E[X] - c = 0 \\
&\implies E[X] = c
\end{aligned}$$

The variance is given by

$$\begin{aligned}
Var(X) &= E[(X - E[X])^2] \\
&= \int_{-\infty}^{\infty} (x-c)^2 f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} \frac{(x-c)^2}{\sqrt{2\pi d^2}} \exp\left(-\frac{(x-c)^2}{2d^2}\right) dx \\
&= I(2)
\end{aligned}$$

From the recursive relation, we get

$$\begin{aligned}
I(2) &= d^2 I(0) \\
&= d^2(1) \\
&= d^2 \\
\implies Var(X) &= d^2
\end{aligned}$$

Derivation for Binomial

The pmf of a binomial random variable $Y \sim \mathcal{B}(n, p)$ is given by

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

The mean is given by

$$\begin{aligned}E[Y] &= \sum_{k=0}^{n} k p_Y(k) \\ &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}\end{aligned}$$

To calculate the mean, we take the derivative of the binomial sum

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

Differentiating w.r.t p

$$\sum_{k=0}^{n} \binom{n}{k} \left(kp^{k-1}(1-p)^{n-k} - (n-k)p^k(1-p)^{n-k-1}\right) = 0$$

$$\implies \sum_{k=0}^{n} \binom{n}{k} kp^{k-1}(1-p)^{n-k-1}(p+1-p) = n\sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k-1}$$

$$\implies \frac{1}{p(1-p)} \sum_{k=0}^{n} k\binom{n}{k} p^k(1-p)^{n-k}(p+1-p) = \frac{n}{1-p}\sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k}$$

$$\implies \frac{E[Y]}{p(1-p)} = \frac{n}{1-p}$$

$$\implies E[Y] = np$$

To calculate $E[Y^2]$, we take the derivative of E[Y]

$$E[Y] = \sum_{k=0}^{n} k\binom{n}{k} p^k(1-p)^{n-k} = np$$

Differentiating w.r.t p

$$\sum_{k=0}^{n} k\binom{n}{k} \left(kp^{k-1}(1-p)^{n-k} - (n-k)p^k(1-p)^{n-k-1}\right) = n$$

$$\implies \sum_{k=0}^{n} k^2\binom{n}{k} p^{k-1}(1-p)^{n-k-1}\,(1-p+p) - n\sum_{k=0}^{n} k\binom{n}{k} p^k(1-p)^{n-k-1} = n$$

$$\implies \frac{E[Y^2]}{p(1-p)} - n\frac{E[Y]}{(1-p)} = n$$

$$\implies E[Y^2] = np(1-p) + npE[Y]$$

$$\implies E[Y^2] = np(1-p) + n^2p^2$$

The variance is given by

$$\begin{aligned}Var(Y) &= E[Y^2] - E[Y]^2 \\ &= np(1-p) + n^2p^2 - n^2p^2 \\ Var(Y) &= np(1-p)\end{aligned}$$

Alternative derivation for binomial

We consider $Y \sim \mathcal{B}(n, p)$ as sum of n independent bernoulli random variables Z_i with parameter p

$$\begin{aligned} Var(Y) &= Var(\sum_{i=1}^{n} Z_i) \\ &= \sum_{i=1}^{n} Var(Z_i) \\ &= nVar(Z) \end{aligned}$$

Z takes value 1 with probability p and 0 with probability $1 - p$
The expected value of Z is given by

$$\begin{aligned} E[Z] &= \sum_{z} z \cdot P(Z = z) \\ &= 0 \cdot P(Z = 0) + 1 \cdot P(Z = 1) \\ &= 0 \cdot (1 - p) + 1 \cdot p = p \end{aligned}$$

$$\begin{aligned} E[Z^2] &= \sum_{z} z^2 \cdot P(Z = z) \\ &= 0^2 \cdot P(Z = 0) + 1^2 \cdot P(Z = 1) \\ &= 0 \cdot (1 - p) + 1 \cdot p = p \end{aligned}$$

$$\begin{aligned} Var(Z) &= E[Z^2] - E[Z]^2 \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$

$$\begin{aligned} Var(Y) &= nVar(Z) \\ &= np(1 - p) \end{aligned}$$

Marking Scheme (5 Marks Total):

- **Derivation for Gaussian (2.5 marks)**
  - Correct derivation of mean (0.5 marks).
  - Correct derivation of variance (2 marks).
- **Derivation for Binomial (2.5 marks):**
  - Correct derivation of mean (1 mark)
  - Correct derivation of variance (1.5 marks).
- **Alternative Derivation for binomial (2.5 marks)**
  - Formula for variance of sum of i.i.d random variables (0.5 marks)
  - Correct derivation of mean (1 mark)

- Correct derivation of variance (1 mark)

Note:

- No marks will be given for directly writing the result without proof.
- All integrals need proper proofs, except gaussian integral for which equating integral of pdf with 1 will suffice.