Probability and Statistics: MA6.101

Homework 9

Topics Covered: CLT, Random Vectors

Q1: Forty nine measurements are recorded to several decimal places. Each of these 49 numbers is rounded off to the nearest integer. The sum of the original 49 numbers is approximated by the sum of those integers. Assume that the errors made in rounding off are independent, identically distributed random variables with a uniform distribution over the interval $(-0.5, 0.5)$. Compute approximately the probability that the sum of the integers is within two units of the true sum.

Q2: Let X_1 be a uniform random variable with support $R_{X_1} = [1, 2]$ and probability density function

$$f_{X_1}(x_1) = \begin{cases} 1 & \text{if } x_1 \in R_{X_1} \\ 0 & \text{if } x_1 \notin R_{X_1} \end{cases}$$

Let X_2 be a continuous random variable, independent of X_1, with support $R_{X_2} = [0, 2]$ and probability density function

$$f_{X_2}(x_2) = \begin{cases} \frac{3}{8}x_2^2 & \text{if } x_2 \in R_{X_2} \\ 0 & \text{if } x_2 \notin R_{X_2} \end{cases}$$

Let

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1^2 \\ X_1 + X_2 \end{bmatrix}$$

Find the joint probability density function of the random vector $\mathbf{Y}$.

Q3: Find the Kullback–Leibler divergence between two multivariate gaussian distributions, $P \sim N(\mu_1, \Sigma_1)$ and $Q \sim N(\mu_2, \Sigma_2)$, both n dimensional. KL divergence between two distributions P and Q of a continuous random variable is given by

$$\mathrm{KL}[P \,||\, Q] = \int_{\mathcal{X}} p(x) \ln \frac{p(x)}{q(x)} \, \mathrm{d}x$$

Q4: Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ belong to a bivariate normal distribution $\mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right)$. Show that $x_1 | x_2 \sim \mathcal{N}(\mu_{1|2}, \Sigma_{1|2})$ where

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Q5: X and Y are said to be bivariate normal if $aX + bY$ is normal for all a and b. If X and Y are bivariate normal with 0 mean, variance of 1, and ρ correlation, then their joint pdf is:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(x^2 - 2\rho xy + y^2 \right) \right)$$

Find the joint pdf of $X + Y$ and $X - Y$.

Q6: Let $\mathbf{Z} = [Z_1, Z_2]^T$ be a normal random vector with the following mean and covariance matrices: [Gopal]

$$\mathbf{m} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}.$$

Let also:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \mathbf{AZ} + \mathbf{b}.$$

Answer the following:

1. Find $P(Z_2 > 1)$. 2. Find the expected value vector of $\mathbf{W}$, denoted as $\mathbf{m_W} = \mathbb{E}[\mathbf{W}]$. 3. Find the covariance matrix of $\mathbf{W}$, denoted as $\mathbf{C_W}$. 4. Find $P(W_3 \leq 4)$.