CS 302.1 - Automata Theory

Lecture 02

Shantanav Chakraborty

Center for Quantum Science and Technology (CQST)

Center for Security, Theory and Algorithms (CSTAR)

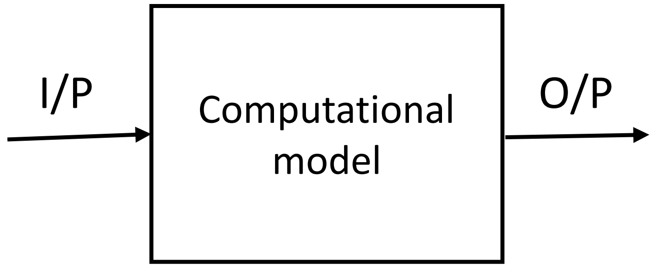
IIIT Hyderabad

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

A quick recap

- Can a given problem be computed by a particular computational model?

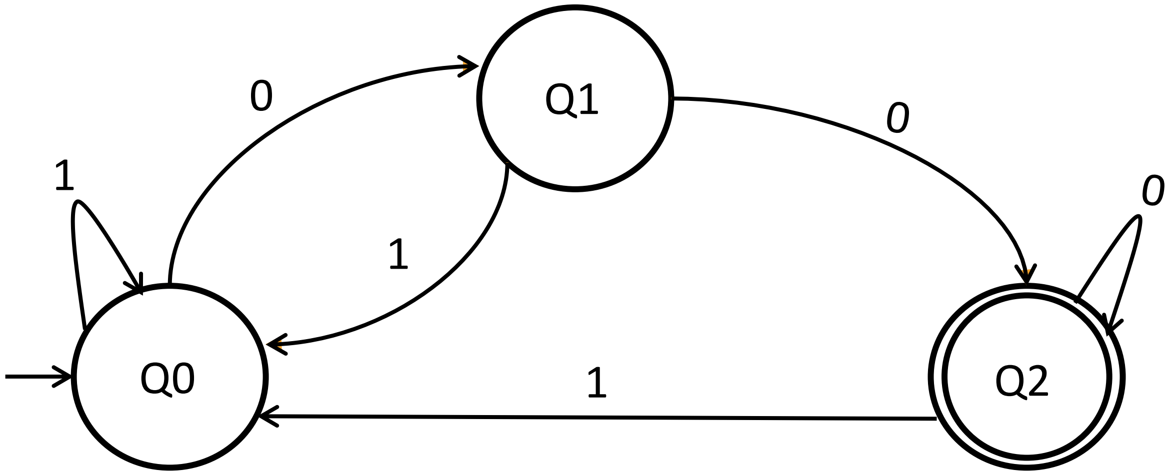


A computational model solves a problem P if,

(i) For all inputs belonging to the YES instance of P, the device outputs **YES**

(ii) For all inputs belonging to the NO instance of P, the device outputs **NO**.

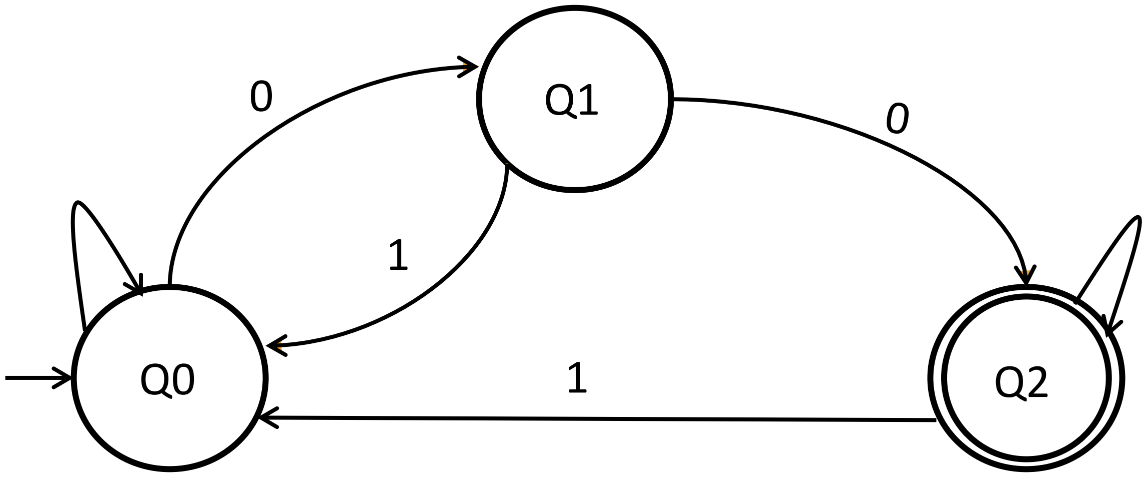
If (i) and (ii) hold, we say that the problem **P is computable** by this computational model.



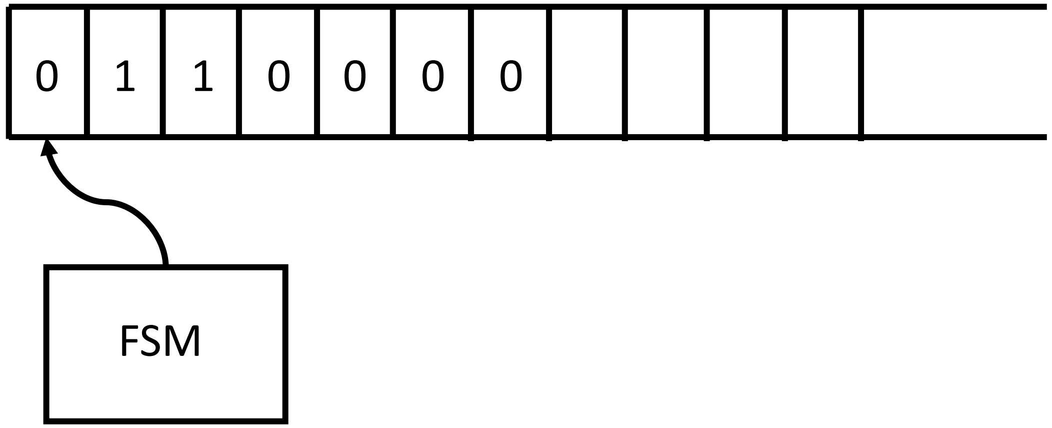
Deterministic Finite Automata (DFA)

Characteristics: (i) Single start State
(ii) Unique Transitions
(iii) Zero or more final states

A quick recap



Deterministic Finite Automata (DFA)



Run:

$$\boldsymbol{Q0} \xrightarrow{0} Q1 \xrightarrow{1} Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q1 \xrightarrow{0} Q2 \xrightarrow{0} Q2 \xrightarrow{0} \boldsymbol{Q2}$$

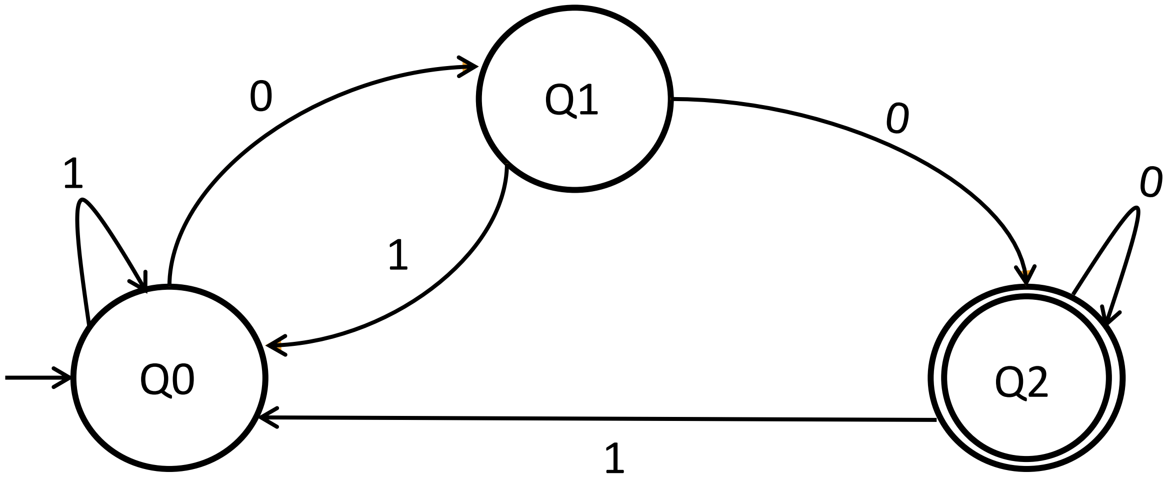
The DFA "accepts" an input string, if it corresponds to a *run* that ends up in the final state Q2. **(Accepting Run)**

The DFA "rejects" an input string, if it corresponds to a *run* that ends up in any non-final state. **(Rejecting Run)**

L(M) = {$\omega|\omega$ results in an accepting run}

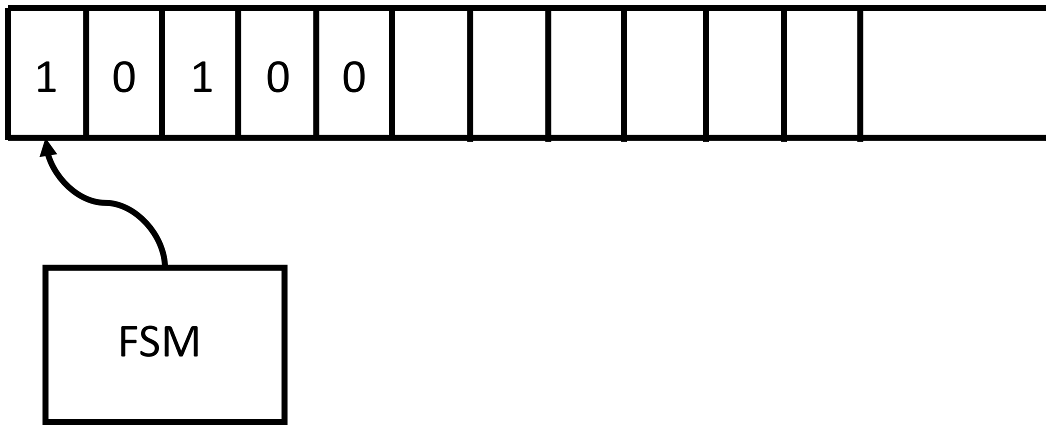
For the example above, **L(M) = {$\omega|\omega$ ends in "00"}**

Deterministic Finite Automata (DFA)



State transition diagram of the Finite State Machine

One-way infinite tape



For any language L, we say **M recognizes L if**

$\forall \omega \in L, M(\omega)$ **accepts**

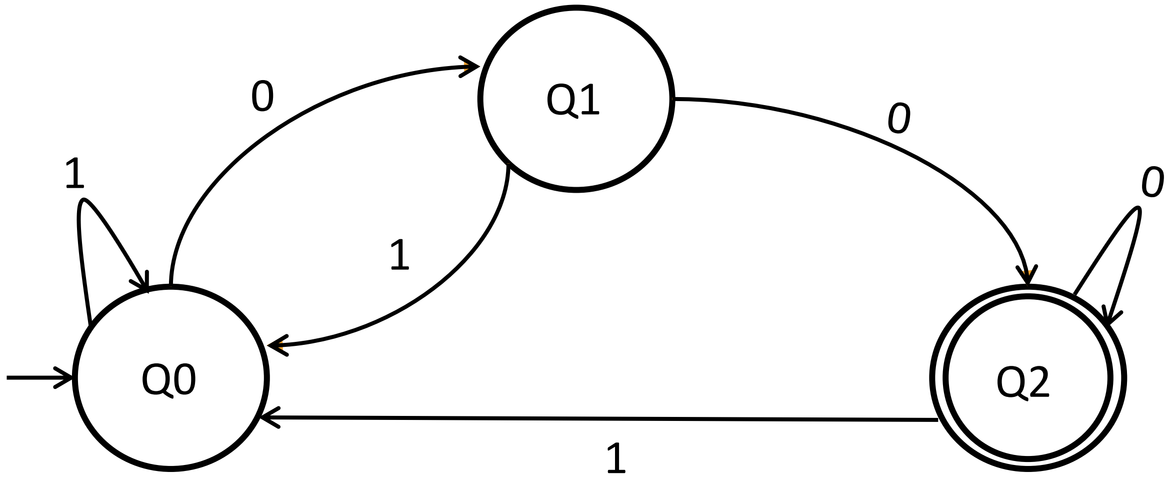
For any language L, we say **M decides L if**

$\forall \omega \in L, M(\omega)$ **accepts**

$\forall \omega \notin L, M(\omega)$ **rejects**

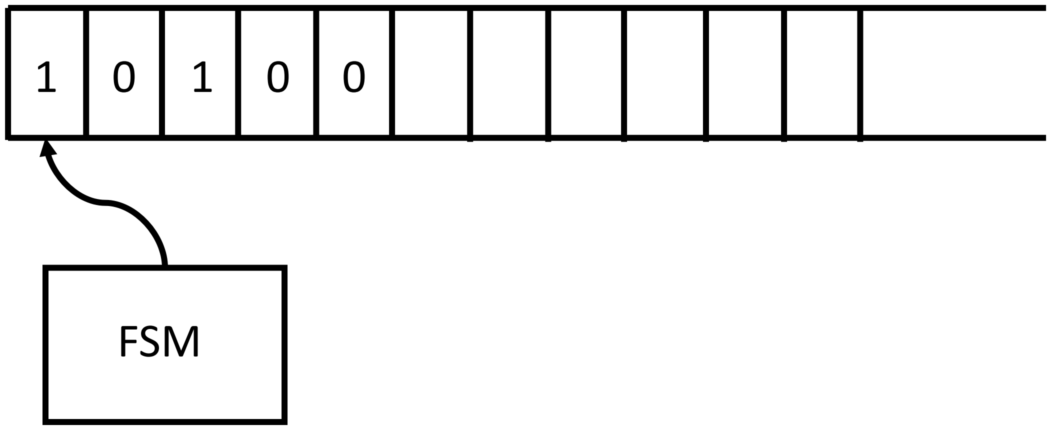
For a DFA, the notions of **deciding a language** and **recognizing a language** are equivalent, but this may not be true for other, more powerful computational models

Deterministic Finite Automata (DFA)



State transition diagram of the Finite State Machine

One-way infinite tape



Characteristics of DFA : (i) Single start state (ii) Unique transitions (iii) Zero or more final states

Formally, a finite automaton M is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

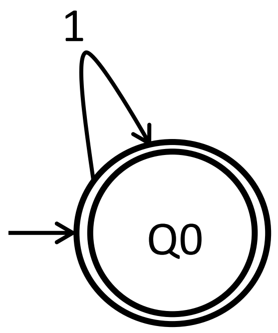
- Q is a finite set called the ***states.*** $\quad Q = \{Q0, Q1, Q2\}$
- Σ is a finite set called the ***alphabet***. $\quad \Sigma = \{0,1\}$
- $\delta : Q \times \Sigma \mapsto Q$ is the ***transition function*** *(unique)*. $\quad (Q0,0) \mapsto Q1;\ (Q0,1) \mapsto Q0, \ldots\ldots, (Q2,1) \mapsto Q0$
- $q_0 \in Q$ is the ***start state***. $\quad q_0 = Q0$
- $F \subseteq Q$ are the ***final/accepting states***. $\quad F = Q2$

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\omega$|$\omega$ has an even number of 0's}**

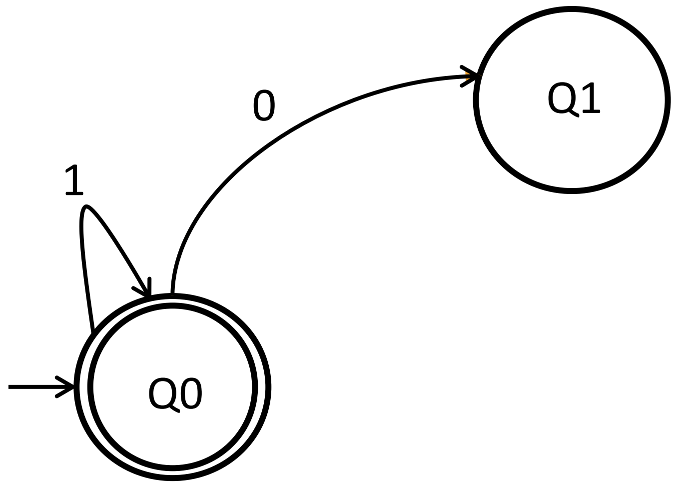
Constructing DFA for a language

Examples: $\boldsymbol{\Sigma = \{0, 1\}}$, **L(M)={$\boldsymbol{\omega | \omega}$ has an even number of 0's}**



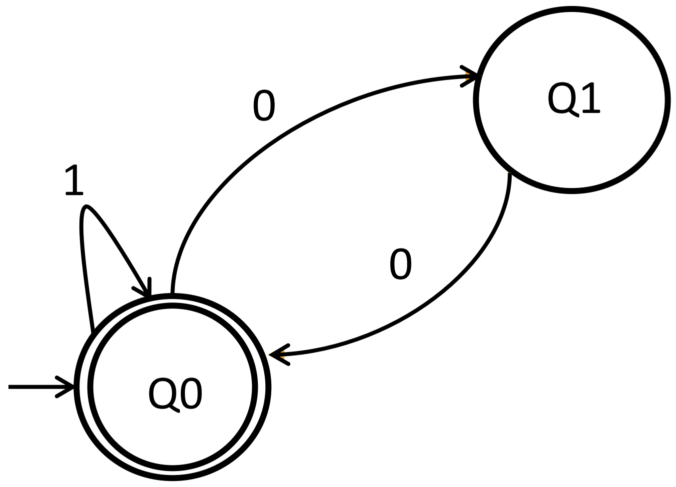
Constructing DFA for a language

Examples: $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\omega$|$\omega$ has an even number of 0's}**



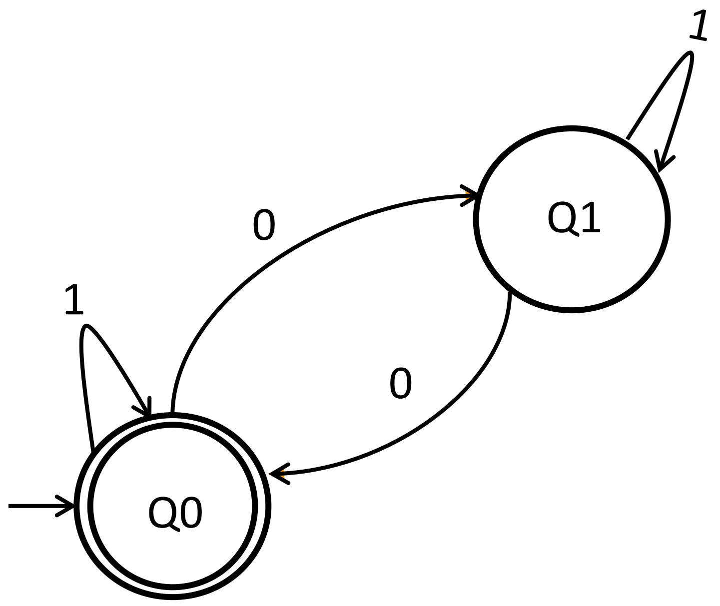
Constructing DFA for a language

Examples: $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\omega$|$\omega$ has an even number of 0's}**



Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\omega|\omega$ has an even number of 0's}**



	0	1
Q0	**Q1**	**Q0**
Q1	**Q0**	**Q1**

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is divisible by 3}**

Any input string would leave three remainders: $0, 1$ or 2.

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is divisible by 3}**

Any input string would leave three remainders: $0, 1$ or 2.

Intuition: Let ω be any substring of the input string divisible by 3, i.e. $\omega = 0(mod\ 3)$

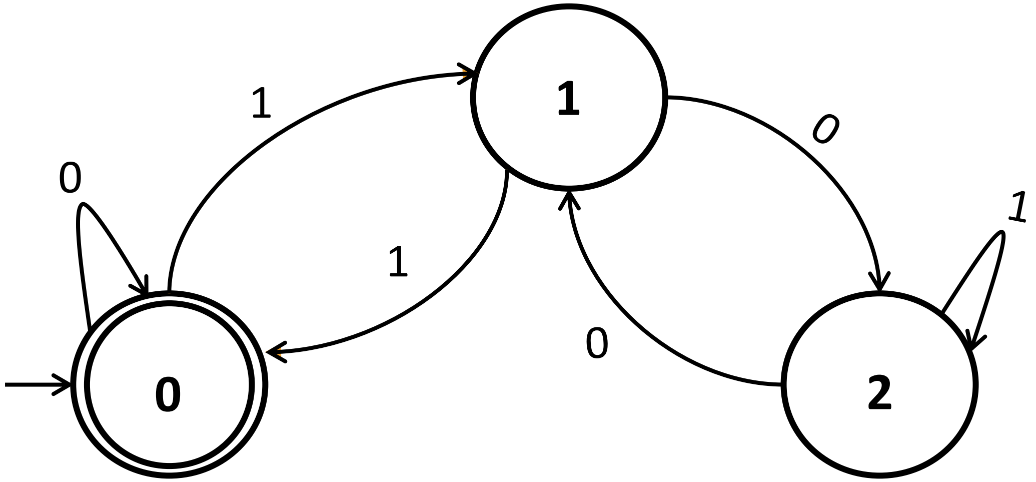
$$\omega\ 0 = 2 \times value\ (\omega) = 0\ (mod\ 3)$$
$$\omega\ 1 = 2 \times value(\omega) + 1 = 1(mod\ 3)$$
$$\omega\ 10 = 2 \times value(\omega 1) = 2(mod\ 3)$$
$$\omega\ 11 = 2 \times value(\omega 1) + 1 = 0(mod\ 3)$$

…. And so on

- The DFA will have three states, each corresponding to the remainder of $value(\omega)/3$.
- The final state $= 0(mod\ 3)$ – the string ω is accepted if after reading it, the DFA ends in this state.

Constructing DFA for a language

Examples: $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\omega$|$\omega$ is divisible by 3}**



Any input string would either leave remainders 0, 1 or 2.

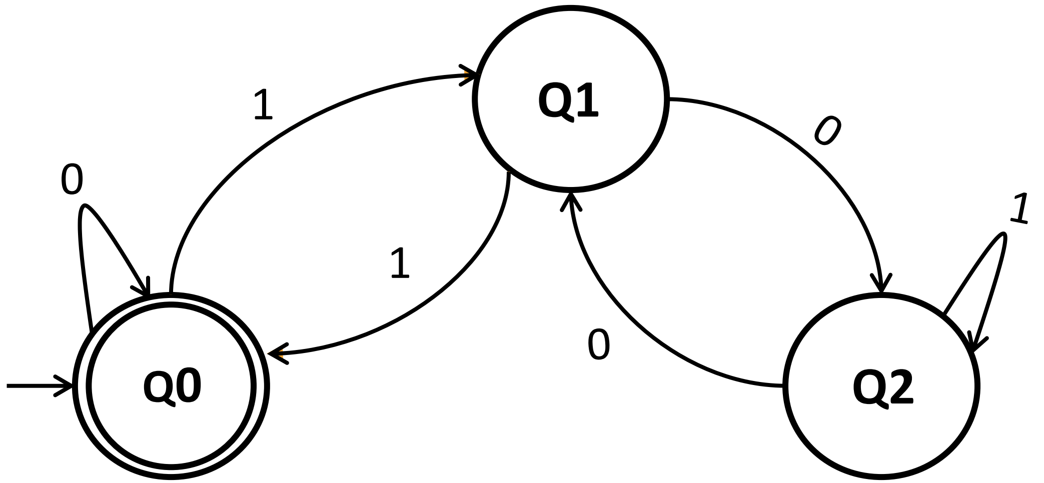
Intuition: Let ω be any substring of the input string divisible by 3, i.e. $\omega = 0(mod\ 3)$

$$\omega\, 0 = 2 \times value\ (\omega) = 0\ (\text{mod}\ 3)$$
$$\omega\, 1 = 2 \times value(\omega) + 1 = 1(mod\ 3)$$
$$\omega\, 10 = 2 \times value(\omega 1) = 2(mod\ 3)$$
$$\omega\, 11 = 2 \times value(\omega 1) + 1 = 0(mod\ 3)$$

…. And so on

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is divisible by 3}**



	0	**1**
Q0	**Q0**	**Q1**
Q1	**Q2**	**Q0**
Q2	**Q1**	**Q2**

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is NOT divisible by 3}**

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma = \{0, 1\}}$, **L(M)={$\boldsymbol{\omega|\omega}$ is NOT divisible by 3}**

Intuition - Construct a **Toggled DFA:** Toggle the final states and the non-final states!

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}$|$\boldsymbol{\omega}$ is NOT divisible by 3}**

Intuition - Construct a **Toggled DFA:** Toggle the final states and the non-final states!

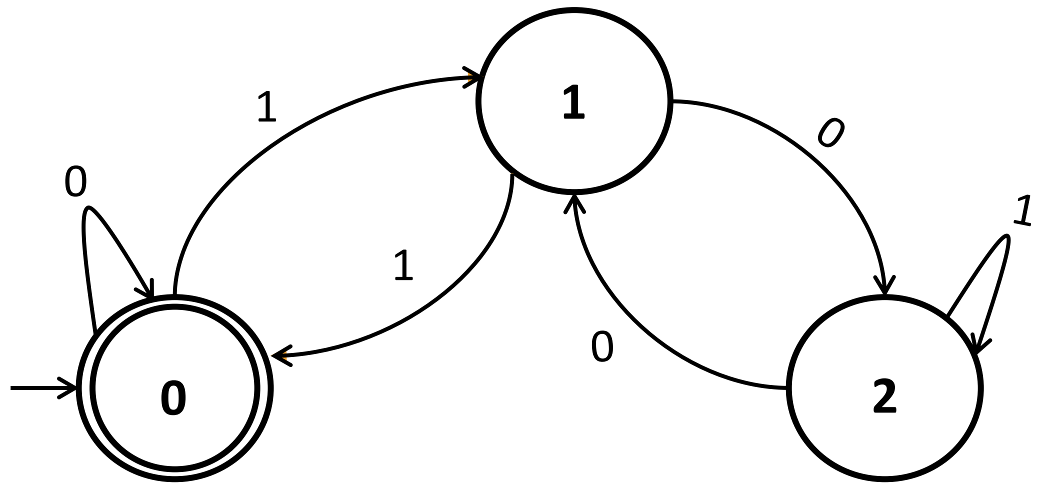
In fact if any DFA accepts $\boldsymbol{L}$, the toggled DFA accepts $\overline{\boldsymbol{L}}$, the complement of $\boldsymbol{L}$

Constructing DFA for a language

Examples: $\mathbf{\Sigma = \{0, 1\}}$, **L(M)={$\boldsymbol{\omega}$|$\boldsymbol{\omega}$ is NOT divisible by 3}**

Intuition - Construct a **Toggled DFA:** Toggle the final states and the non-final states!

In fact if any DFA accepts $\boldsymbol{L}$, the toggled DFA accepts $\boldsymbol{\overline{L}}$, the complement of $\boldsymbol{L}$

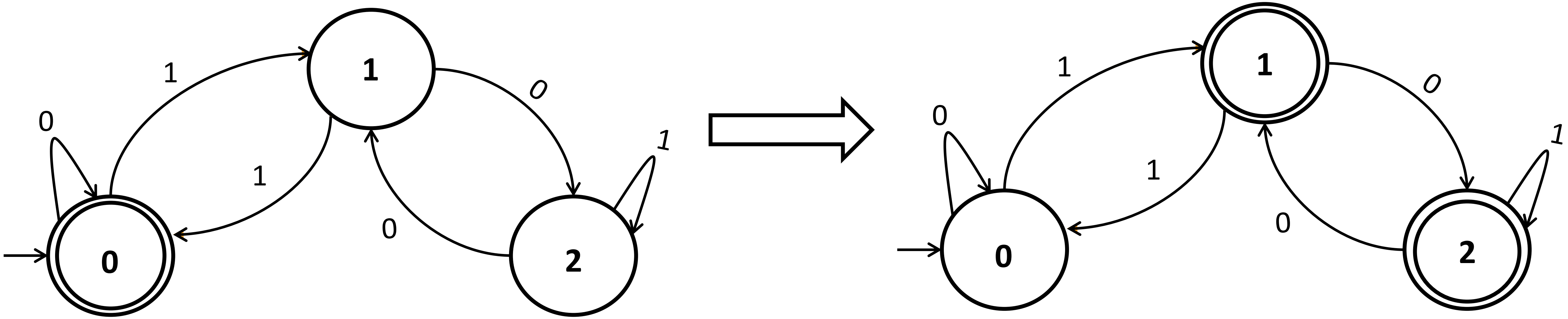


Constructing DFA for a language

Examples: $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is NOT divisible by 3}**

Intuition - Construct a **Toggled DFA:** Toggle the final states and the non-final states!

In fact if any DFA accepts $\boldsymbol{L}$, the toggled DFA accepts $\overline{\boldsymbol{L}}$, the complement of $\boldsymbol{L}$



Non-deterministic Finite Automata (NFA)

Characteristics of DFA : (i) Single start state (ii) Unique transitions (iii) Zero or more final states

Non-deterministic Finite Automata (NFA)

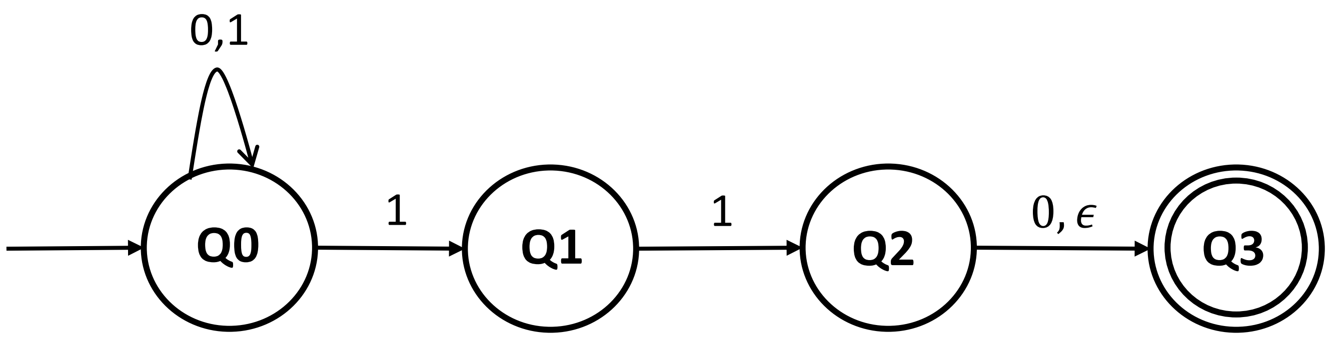
Characteristics of DFA : (i) Single start state (ii) Unique transitions (iii) Zero or more final states

Characteristics of NFA : (i) Single start state (ii) Zero or more final states

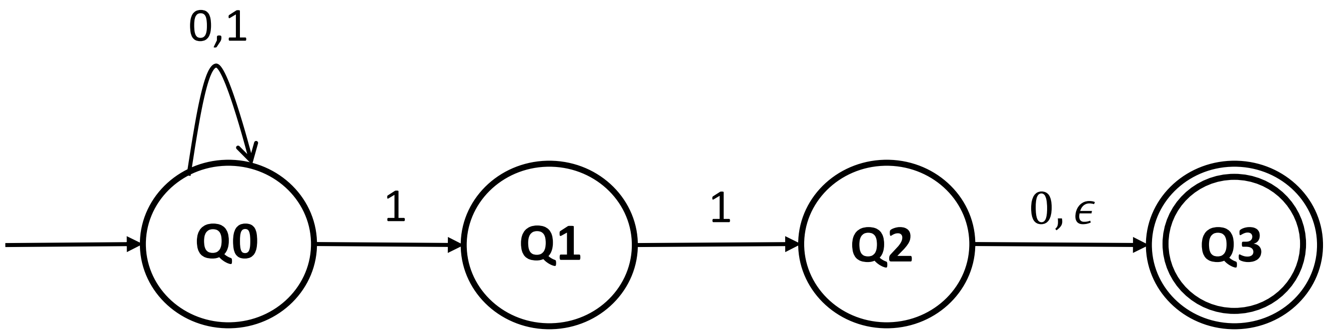
(iii) Multiple transitions are possible on the same input for a state

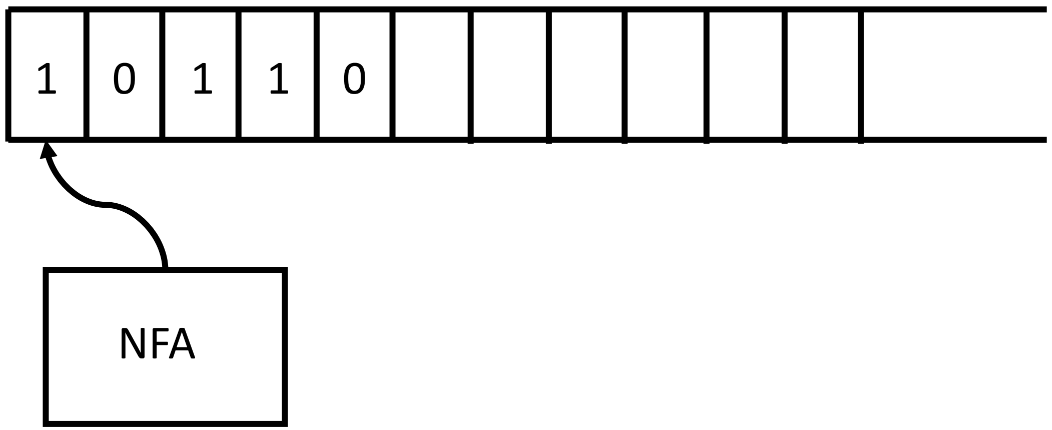
(iv) Some transitions might be missing

(v) ϵ - transitions



Non-deterministic Finite Automata (NFA)



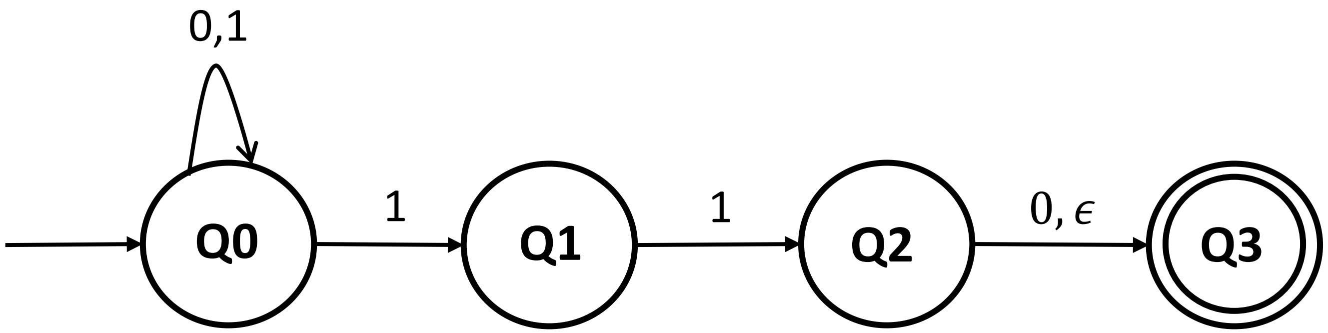


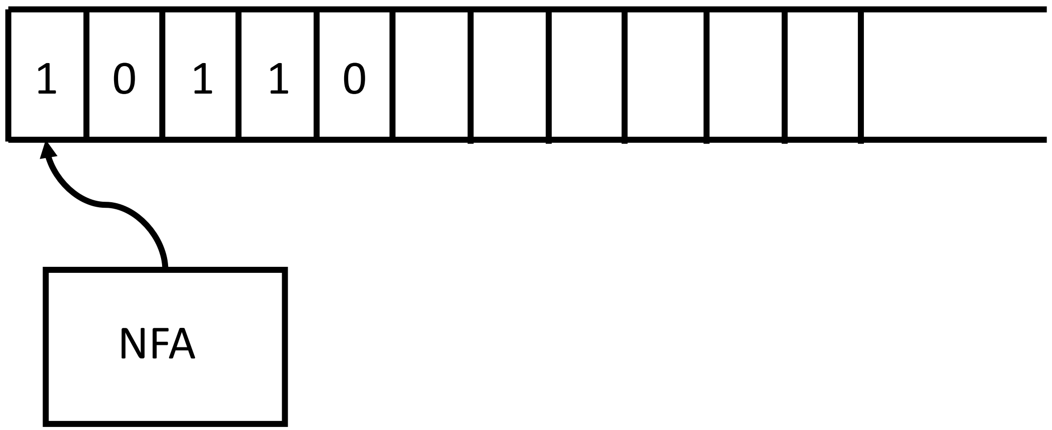
Run 1: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0$ (**REJECT**)

Run 2: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{0} Q3$ (**ACCEPT**)

Multiple **runs** per input is possible

Non-deterministic Finite Automata (NFA)





Run 1: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0$ (**REJECT**)

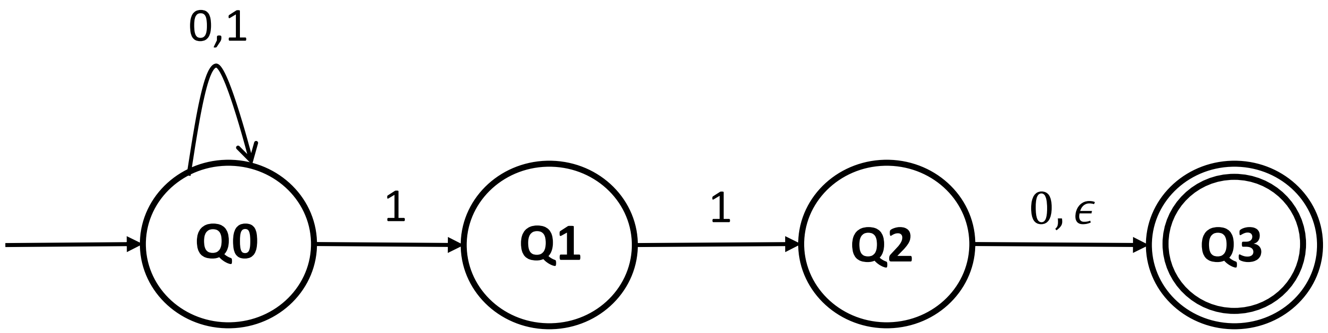
Run 2: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{0} Q3$ (**ACCEPT**)

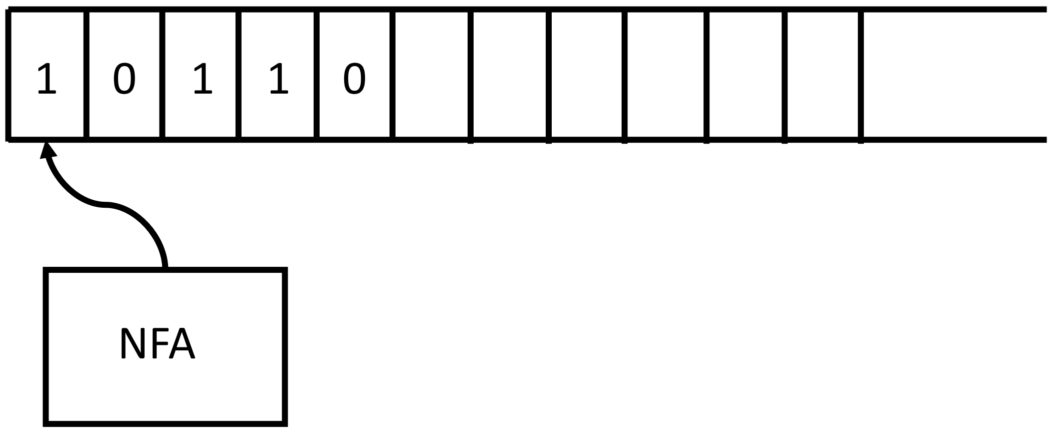
Run 3: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q0 \xrightarrow{1} Q1 \xrightarrow{0}$ **CRASH**

Run 4: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{\epsilon} Q3 \xrightarrow{0}$ **CRASH**

CRASH is a Rejecting Run

Non-deterministic Finite Automata (NFA)





Run 1: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0$ (**REJECT**)

Run 2: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{0} Q3$ (**ACCEPT**)

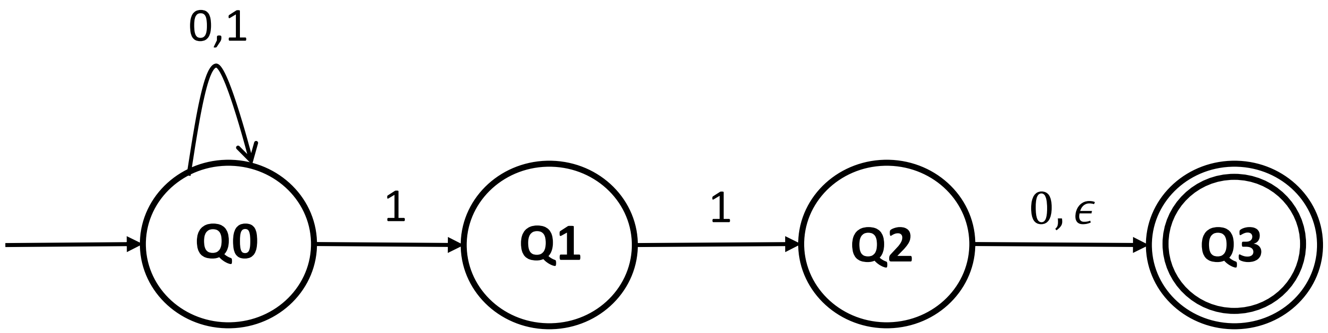
Run 3: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q0 \xrightarrow{1} Q1 \xrightarrow{0}$ CRASH **(REJECT)**

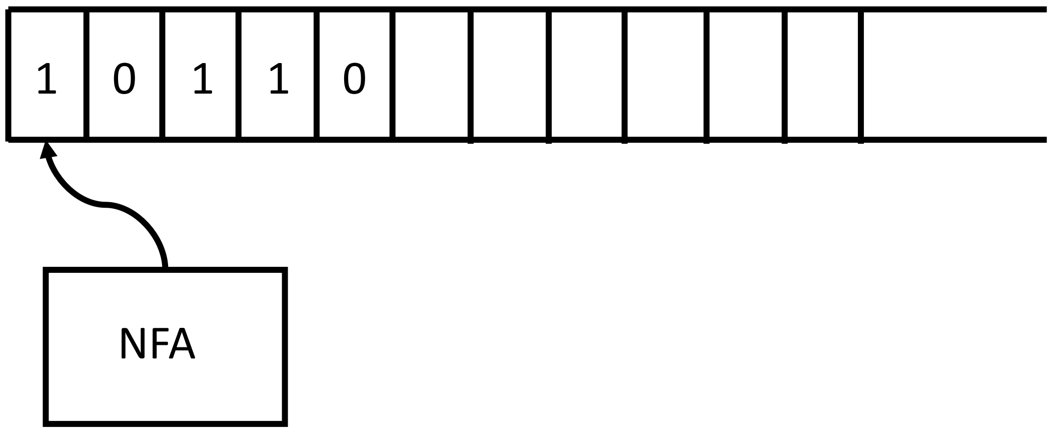
Run 4: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{\epsilon} Q3 \xrightarrow{0}$ CRASH **(REJECT)**

The NFA “accepts” an input string, if it at **least one *run* ends up in the final state**. **(Accepting Run)**

The NFA “rejects” an input string, if there are **no runs that end up in a final state**. **(Rejecting Run)**

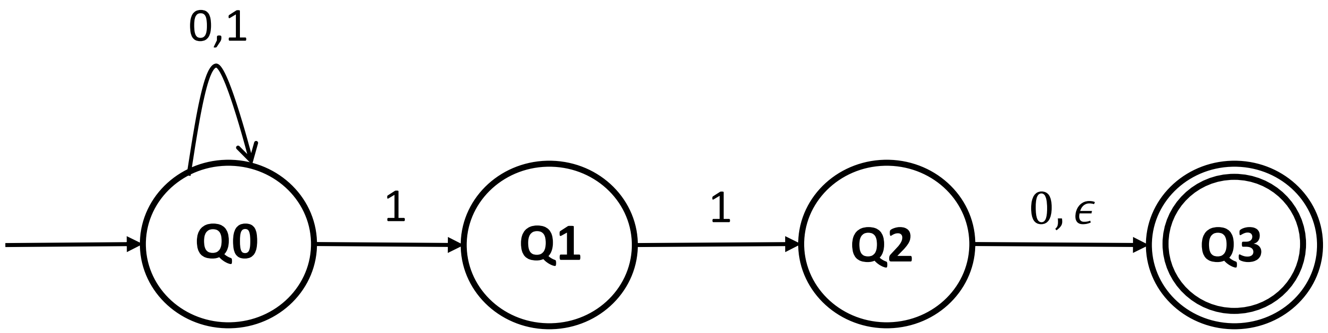
Non-deterministic Finite Automata (NFA)

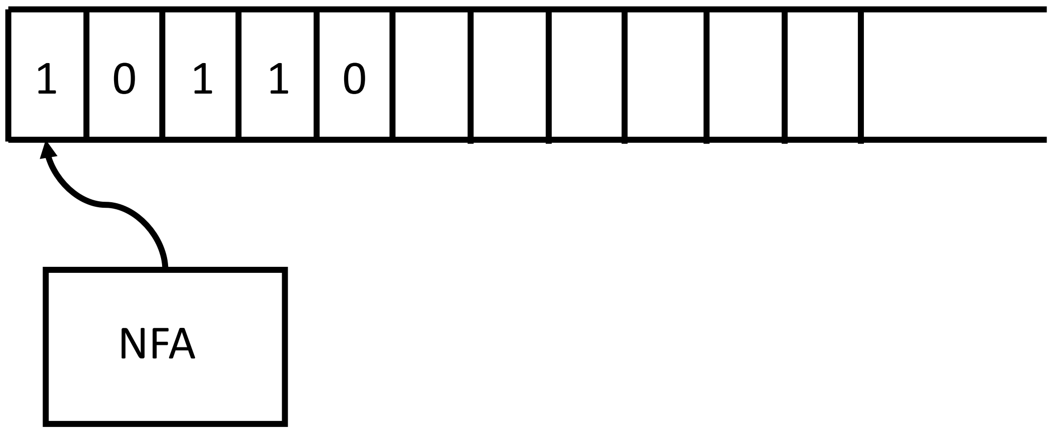




	0	**1**	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

Non-deterministic Finite Automata (NFA)





Formally, a NFA M is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set called the ***states.***
- Σ is a finite set called the ***alphabet***.
- $\delta: Q \times \Sigma \mapsto P(Q)$ is the ***transition function***. $P(Q)$ is the power set of Q
- $q_0 \in Q$ is the ***start state***.
- $F \subseteq Q$ is the set of ***final/accepting states***.

	0	**1**	$\boldsymbol{\epsilon}$
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

NFA vs DFA

- Are NFAs more powerful than DFAs?
- Intuitively, non-determinism seems to be adding more “power”.

NFA vs DFA

- Are NFAs more powerful than DFAs?
- Intuitively, non-determinism seems to be adding more “power”.
- Let L_1 be the language accepted by NFAs and L_2 be the language accepted by DFAs
- Is $L_2 \subseteq L_1$?

NFA vs DFA

- Are NFAs more powerful than DFAs?
- Intuitively, non-determinism seems to be adding more “power”.
- Let L_1 be the language accepted by NFAs and L_2 be the language accepted by DFAs
- Is $L_2 \subseteq L_1$? Clearly true, because a DFA is just a special case of an NFA.

NFA vs DFA

- Are NFAs more powerful than DFAs?
- Intuitively, non-determinism seems to be adding more “power”.
- Let L_1 be the language accepted by NFAs and L_2 be the language accepted by DFAs
- Is $L_2 \subseteq L_1$? Clearly true, because a DFA is just a special case of an NFA.
- Surprisingly, what we will show next is that $\boldsymbol{L_1 \subseteq L_2}$!
- That is, **given an NFA, we can convert it to a DFA that accepts the same language**.
- Such a DFA is called a “**Remembering DFA**”.

Thus, DFAs and NFAs are completely equivalent and $\boldsymbol{L_1 = L_2}$!

Converting an NFA to a DFA

Intuitive idea for the construction of a Remembering DFA from an NFA:

- Let R be the Remembering DFA corresponding to an NFA N.
- R on an input enters a state that is labelled by all possible states that N can enter on that input.
- Note that this “trims away” the non-determinism of the NFA N without “losing” the language it accepts.
- Also note that if N has k states, then R has at most 2^k states. Why?

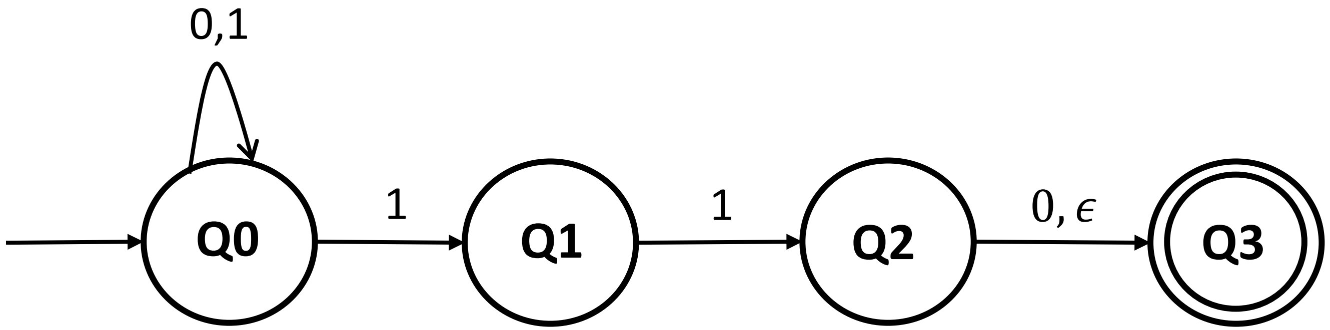
Converting an NFA to a DFA

Intuitive idea for the construction of a Remembering DFA from an NFA:

- Let R be the Remembering DFA corresponding to an NFA N.
- R on an input enters a state that is labelled by all possible states that N can enter on that input.
- Note that this “trims away” the non-determinism of the NFA N without “losing” the language it accepts.
- Also note that if N has k states, then R has at most 2^k states. Why?
- Any label in the Remembering DFA is a subset of $\{Q_0,\ Q_1,\ Q_2, \dots \dots, Q_{k-1}\}$, where Q_i = State of the NFA.
- There are at most 2^k labels for the DFA.

Converting an NFA to a DFA

- R on an input enters a state that is labelled by all possible states that N can enter on that input.

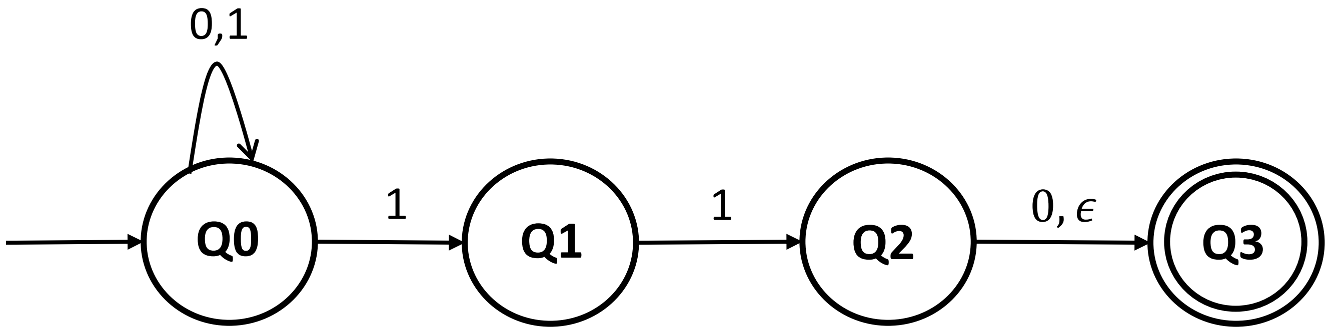


NFA N

	0	1	ϵ
Q0	Q0	Q0, Q1	
Q1		Q2	
Q2	Q3		Q3
Q3			

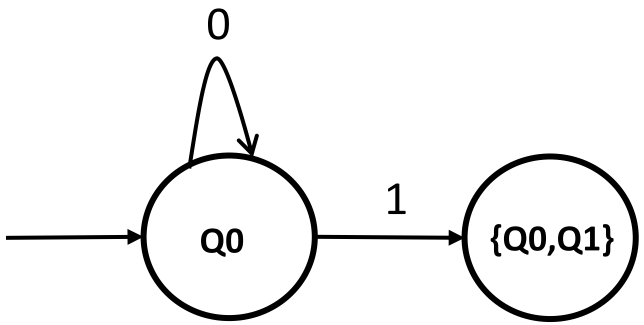
Converting an NFA to a DFA

- R on an input enters a state that is labelled by all possible states that N can enter on that input.



NFA N

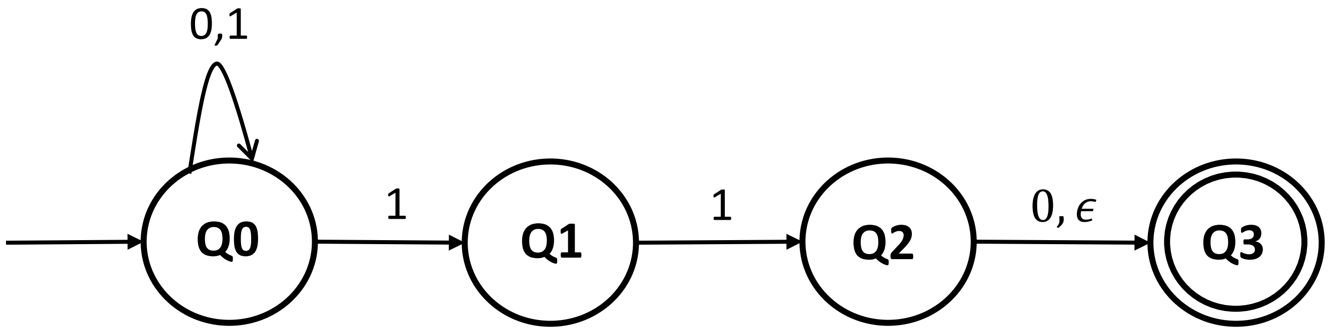
	0	1	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			



Remembering DFA R

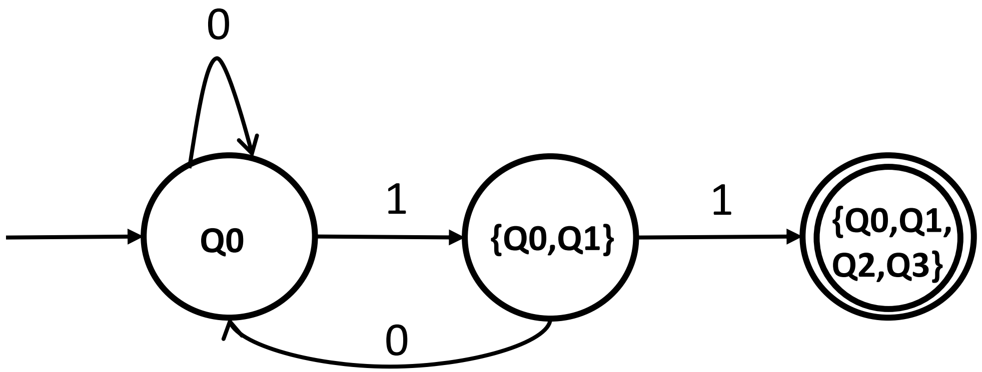
Converting an NFA to a DFA

- R on an input enters a state that is labelled by all possible states that N can enter on that input.



NFA N

	0	**1**	$\boldsymbol{\epsilon}$
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

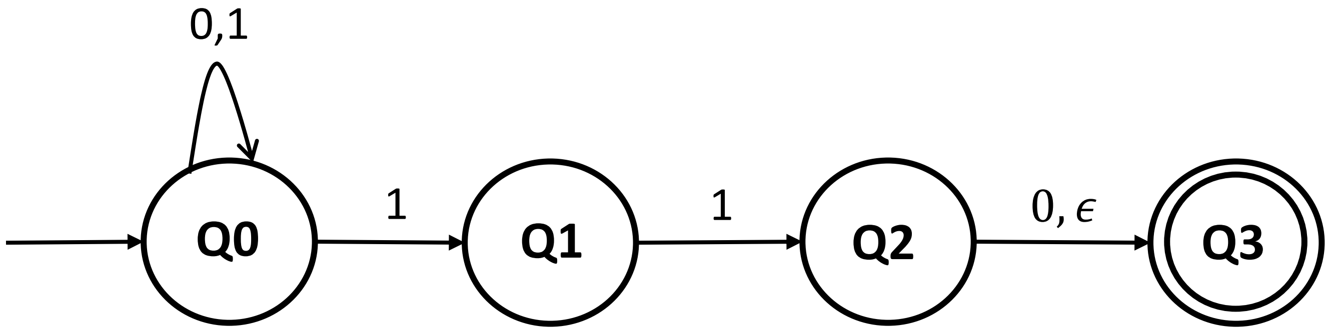


Remembering DFA R

Any state of R that contains in its label, an accepting state of N is an accepting state of R.

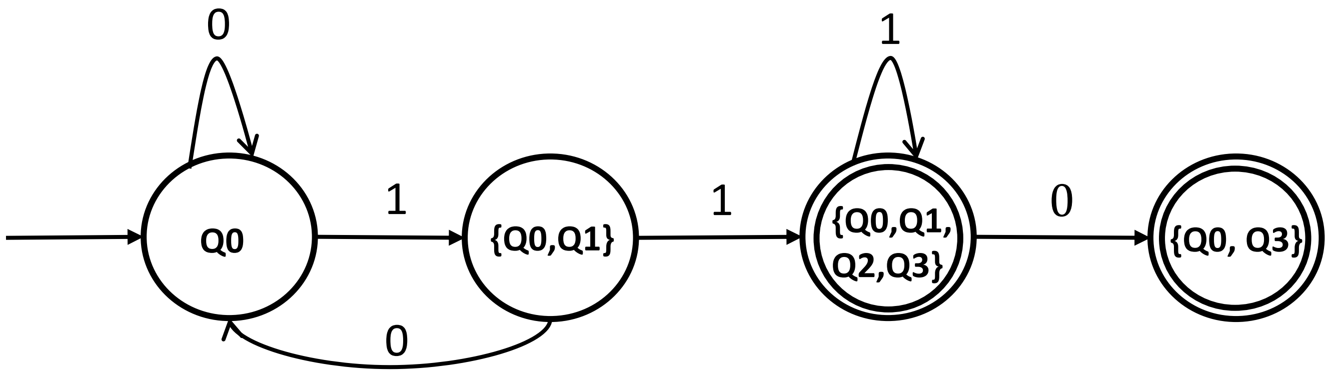
Converting an NFA to a DFA

- R on an input enters a state that is labelled by all possible states that N can enter on that input.



NFA N

	0	**1**	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

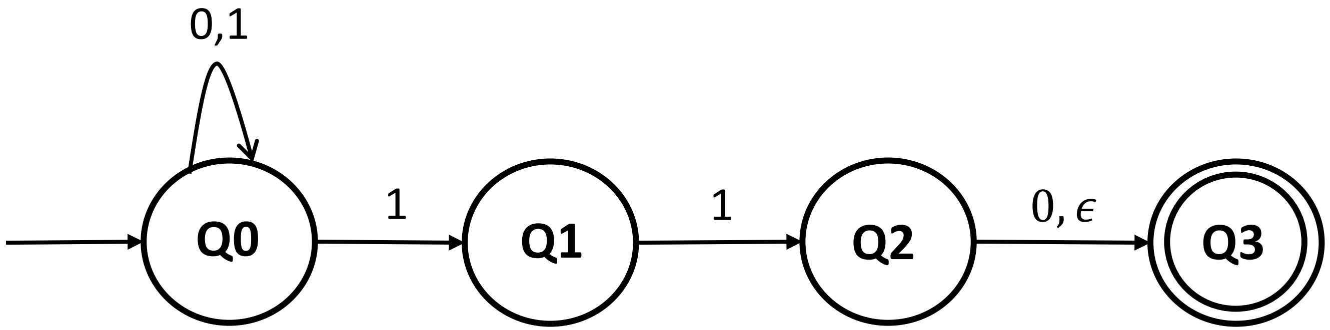


Remembering DFA R

Any state of R that contains in its label, an accepting state of N is an accepting state of R.

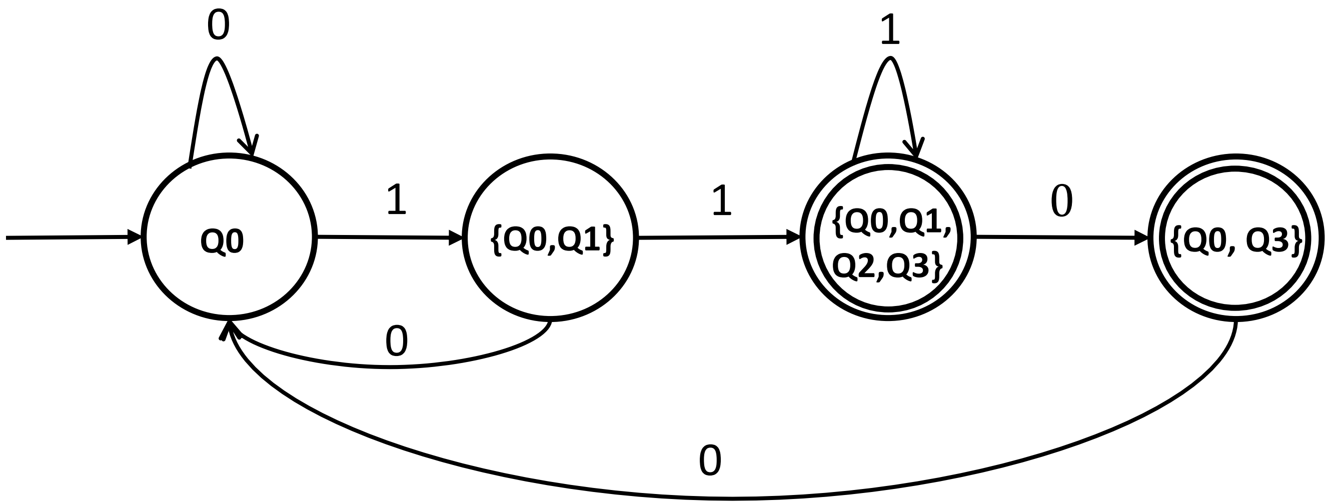
Converting an NFA to a DFA

- M_2 on an input enters a state that is labelled by all possible states that M_1 can enter on that input.



NFA N

	0	1	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

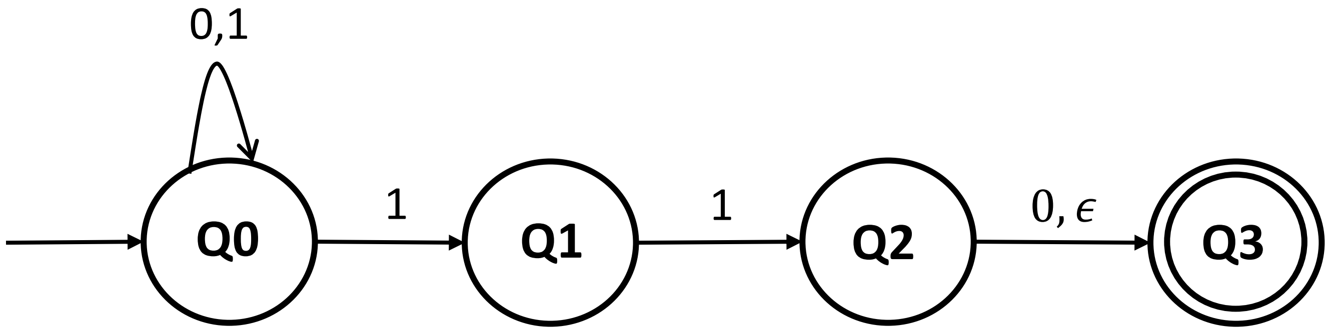


Remembering DFA R

Any state of R that contains in its label, an accepting state of N is an accepting state of R.

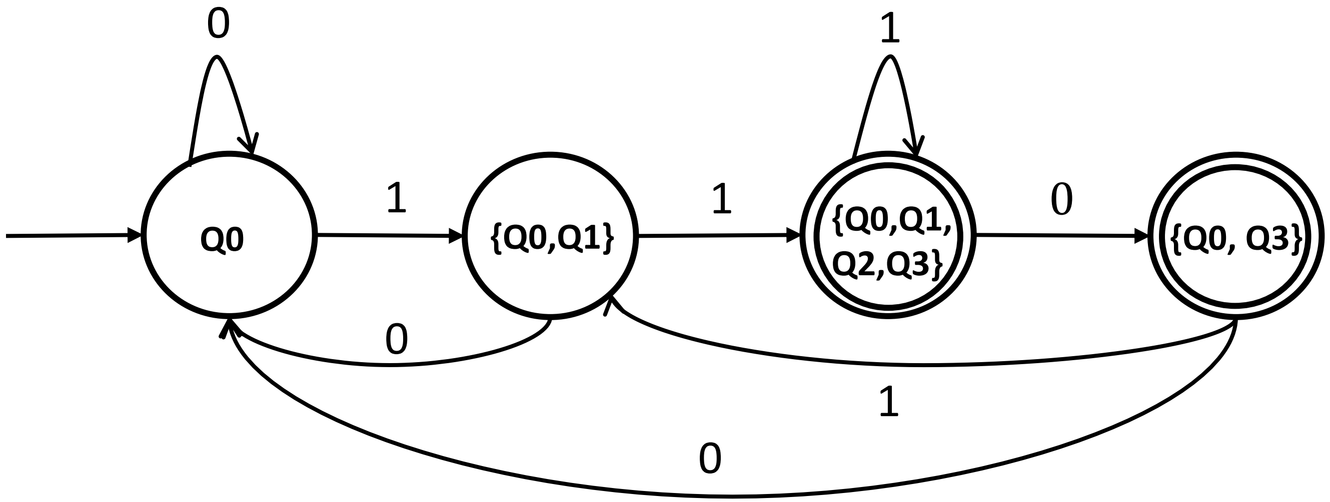
Converting an NFA to a DFA

- M_2 on an input enters a state that is labelled by all possible states that M_1 can enter on that input.



NFA N

	0	1	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

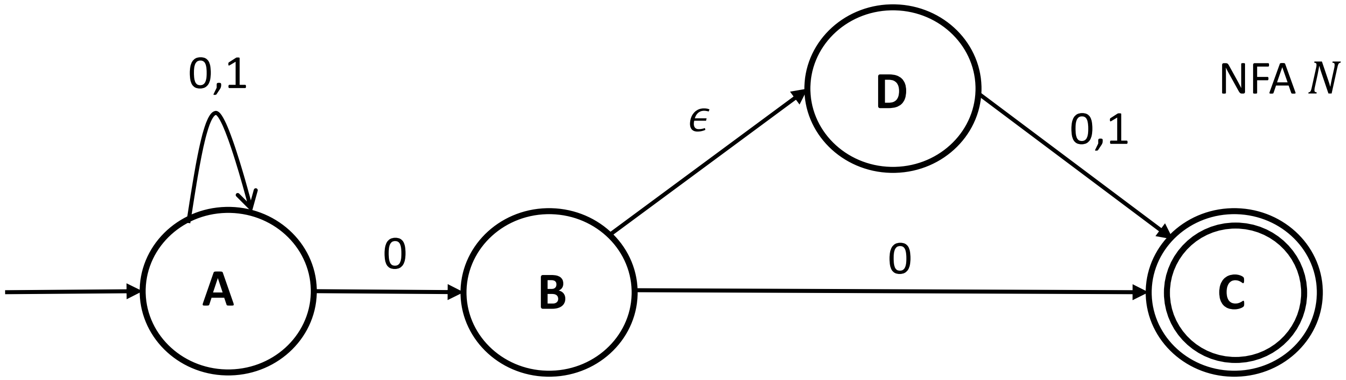


Remembering DFA R

Any state of R that contains in its label, an accepting state of N is an accepting state of R.

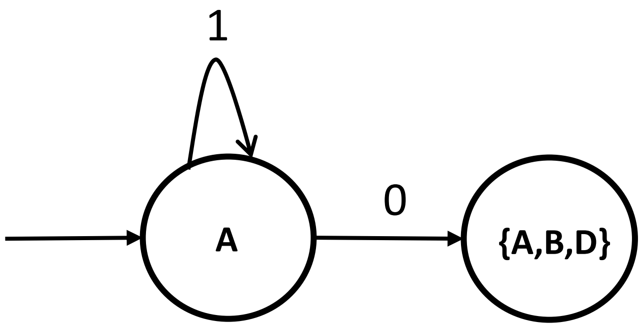
Converting an NFA to a DFA

- M_2 on an input enters a state that is labelled by all possible states that M_1 can enter on that input.



NFA N

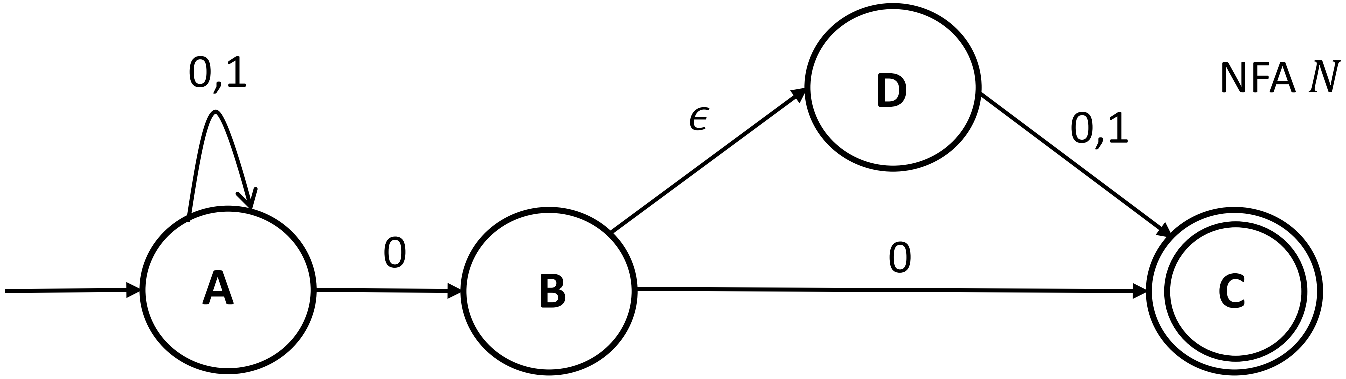
	0	**1**	ϵ
A	**A, B**	**A**	
B	**C**		**D**
C			
D	**C**	**C**	



Remembering DFA R

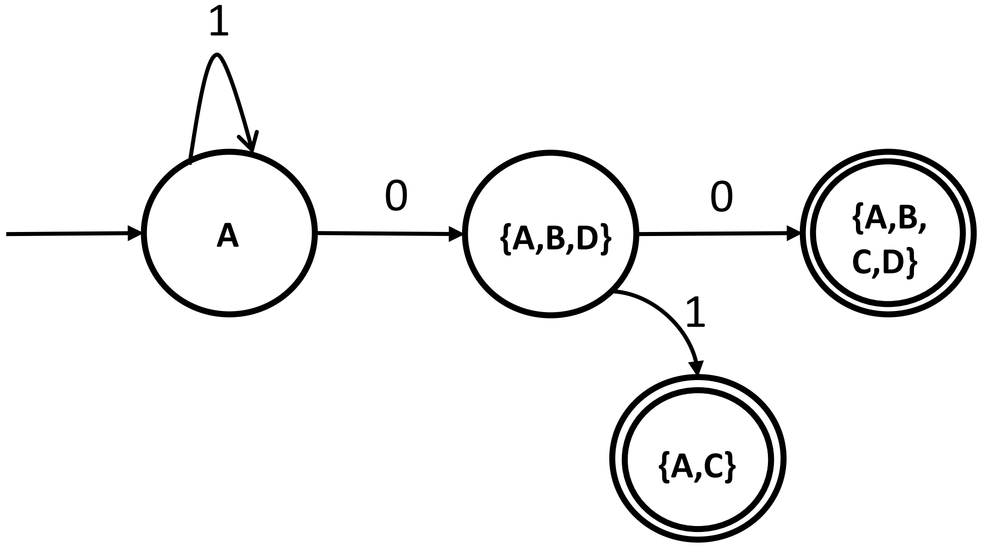
Converting an NFA to a DFA

- M_2 on an input enters a state that is labelled by all possible states that M_1 can enter on that input.



NFA N

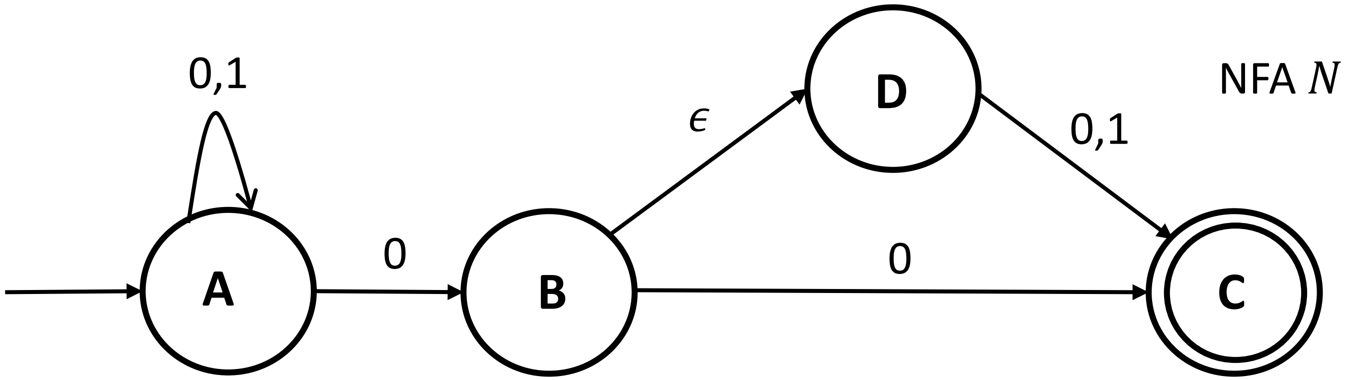
	0	1	ϵ
A	A, B	A	
B	C		D
C			
D	C	C	



Remembering DFA R

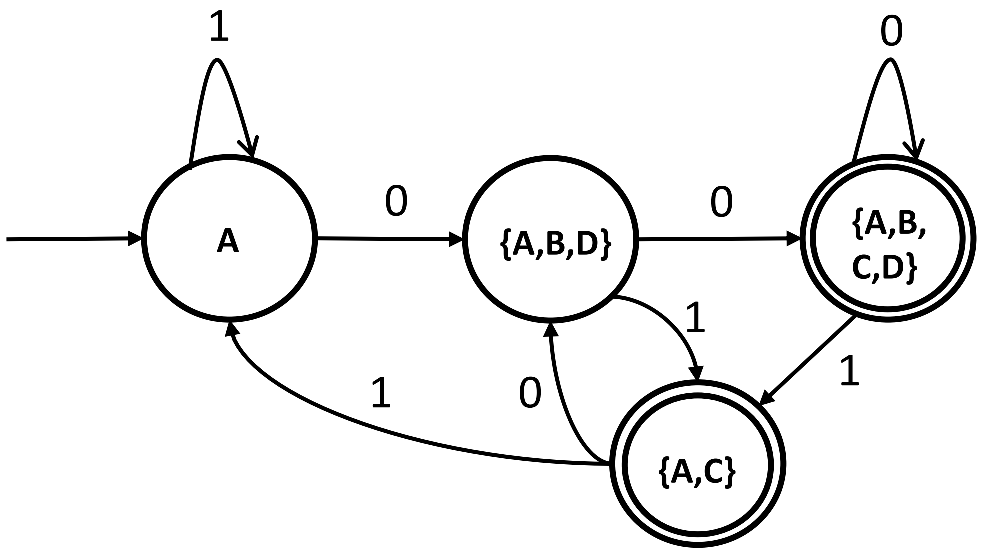
Converting an NFA to a DFA

- M_2 on an input enters a state that is labelled by all possible states that M_1 can enter on that input.



NFA N

	0	1	ϵ
A	A, B	A	
B	C		D
C			
D	C	C	



Remembering DFA R

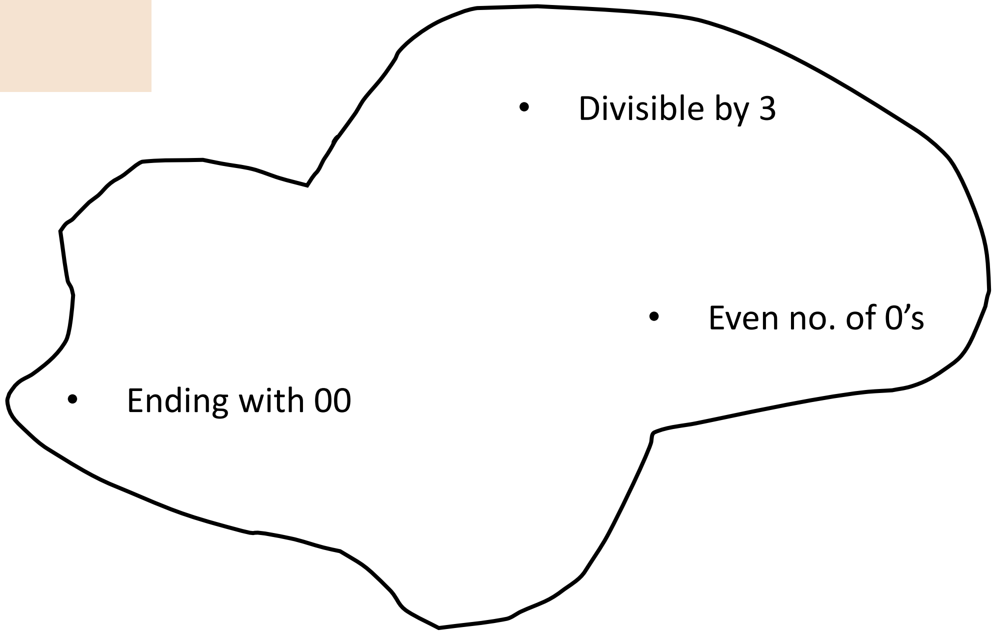
Regular Languages

A language is called a **Regular Language** if there exists some finite automata recognizing it.

If M be a finite automaton (DFA/NFA) and,

$$L(M) = \{\omega | \omega \text{ is accepted by } M\}$$

$\boldsymbol{L(M)}$ **is regular.**



Set of all regular Languages

Regular Languages

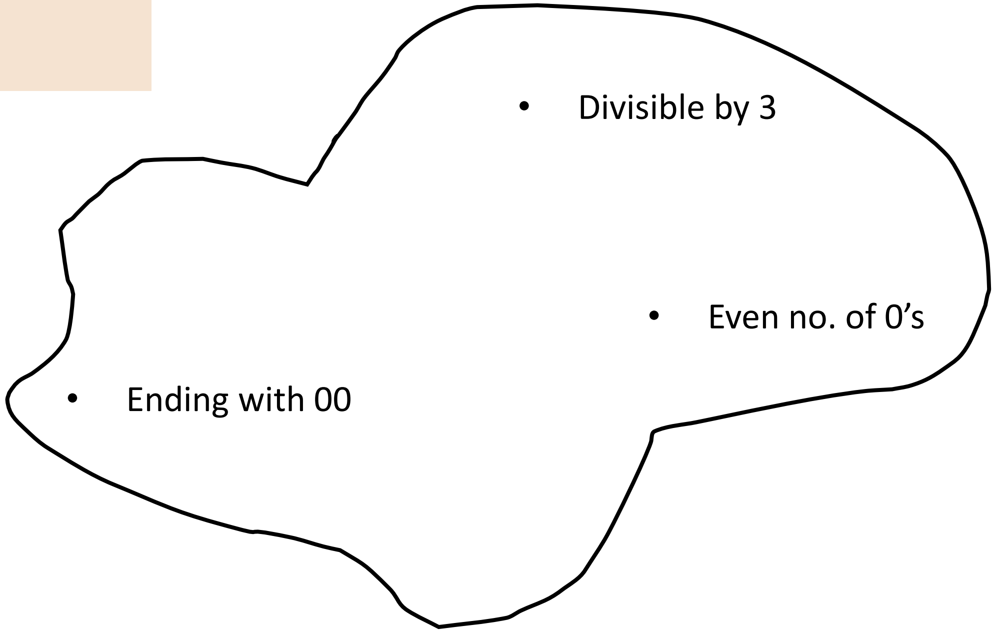
A language is called a **Regular Language** if there exists some finite automata recognizing it.

If M be a finite automaton (DFA/NFA) and,

$$L(M) = \{\omega | \omega \text{ is accepted by } M\}$$

$\boldsymbol{L(M)}$ **is regular.**

- Any language has associated with it, a set of operations that can be performed on it.
- These operations help us to understand the properties of that language, e.g. closure properties
- For regular languages, this will help us prove that certain languages are non-regular and hence we cannot hope to design a finite automaton for them



Set of all regular Languages

Thank You!