Probability and Statistics: MA6.101

Tutorial 7

Topics Covered: Moment Generating Functions and Stochastic Simulation

Q1: Let X be an exponential random variable with parameter λ and let Y be a random variable with the Gamma distribution $Y \sim \text{Gamma}(k, \theta)$.

a) Show how to generate X using a uniform random variable U drawn from the interval $[0, 1]$.

b) Show how to generate Y using k uniform random variables drawn from $[0, 1]$.

Note: The Gamma distribution $Y \sim \text{Gamma}(k, \theta)$ can be expressed as the sum of k independent exponential random variables $X_1, X_2, \ldots, X_k$, where each $X_i \sim \text{Exp}\left(\frac{1}{\theta}\right)$. That is:

$$Y = \sum_{i=1}^{k} X_i$$

where X_i are independent and identically distributed.

Q2: Prove that $x \sim f(x) = xe^{-x}; x \geq 0$ has a moment generating function of $\frac{1}{(1-t)^2}$. Hint: Use the change of variable technique to integrate with respect to $w = x(1-t)$ instead of x.

Q3: Use the rejection method to generate a random variable having the $Gamma(\frac{5}{2}, 1)$ density function.
Note: The pdf of $Gamma(k, \theta)$ is given by $f(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}$ and $\Gamma(\frac{5}{2}) = \frac{3}{4}\pi$.
Hint: You need to figure out an appropriate distribution you can already sample from to use in the rejection method.

Q4: What is the expected number of iterations to generate k random numbers from a distribution using the rejection method?

Q5: (a) Let $M_X(s)$ be finite for $s \in [-c, c]$, where $c > 0$. Show that the MGF of Y$= aX + b$ is given by

$$M_Y(s) = e^{sb} M_X(as)$$

and it is finite in $\left[-\frac{c}{|a|}, \frac{c}{|a|}\right]$.

(b) If $X_1, X_2, \ldots, X_n$ are n independent random variables with respective moment-generating functions $M_{X_i}(t) = \mathbb{E}[e^{tX_i}]$ for $i = 1, 2, \ldots, n$, then prove the moment-generating function of the linear combination: $Y = \sum_{i=1}^{n} a_i X_i$ is:

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

Q6: Let $X \sim$ Normal$(Y, 1)$ where $Y \sim$ Exponential(λ). Find the MGF of X.