Probability and Statistics: MA6.101

Homework 10

Topics Covered: Markov Chains

Q1: Consider the Markov chain with three states $S = \{1, 2, 3\}$ and the state transition diagram given below: Suppose $P(X_1 = 1) = \frac{1}{2}$ and $P(X_1 = 2) = \frac{1}{4}$.

(a) Find the state transition matrix for this chain.

(b) Find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

(c) Find $P(X_1 = 3, X_3 = 1)$.

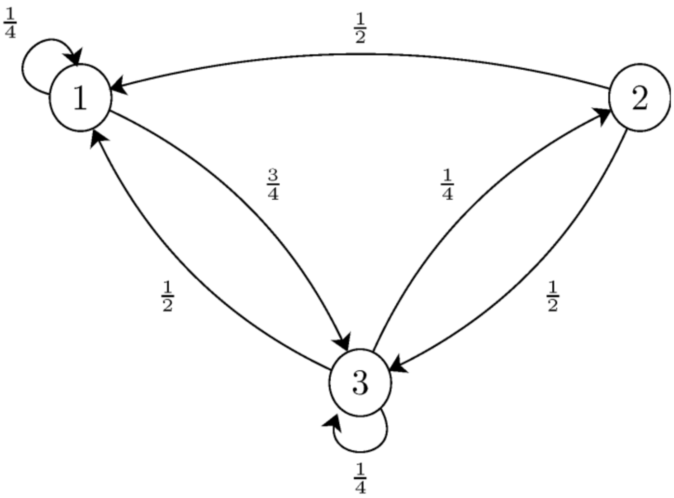


Figure 1: Transition Diagram Q1

A:

(a)

$$P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

(b) First, we obtain

$$P(X_1 = 3) = 1 - P(X_1 = 1) - P(X_1 = 2) = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Hence,

$$P(X_1 = 3, X_2 = 2, X_3 = 1) = P(X_1 = 3) \cdot P_{32} \cdot P_{21} = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{32}.$$

(c) Hence,

$$\begin{aligned}
P(X_1 = 3, X_3 = 1) &= P(X_1 = 3, X_2 = 2, X_3 = 1) \\
&+ P(X_1 = 3, X_2 = 1, X_3 = 1) \\
&+ P(X_1 = 3, X_2 = 3, X_3 = 1) \\
&= P(X_1 = 3) \cdot P_{32} \cdot P_{21} \\
&\quad + P(X_1 = 3) \cdot P_{31} \cdot P_{11} \\
&\quad + P(X_1 = 3) \cdot P_{33} \cdot P_{31} \\
&= \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \\
&\quad + \frac{1}{4} \cdot \frac{1}{2} \cdot 1 \\
&\quad + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{2} \\
&= \frac{1}{32} \\
&+ \frac{1}{8} \\
&+ \frac{3}{32} \\
&= \frac{3}{16}.
\end{aligned}$$

Q2: A transition matrix is said to be double stochastic if $\sum_{i=0}^{M} P_{ij} = 1$ for all states $j = 0, 1 \ldots M$ (i.e. every column sums to 1). Show that such a DTMC has the stationary distribution $\pi_j = \frac{1}{M+1} \forall j$

A:

The stationary distribution π of a DTMC is the unique vector which satisfies

$$\pi = \pi P \tag{1}$$

and

$$\pi \mathbb{1}^T = 1 \tag{2}$$

where $\mathbb{1}$ is the vector containing all 1s

Note that since $\sum_{i=0}^{M} P_{ij} = 1$, we have

$$\begin{aligned}
(\mathbb{1}P)_j &= \sum_{i=0}^{M} \mathbb{1}_i P_{ij} \\
&= \sum_{i=0}^{M} P_{ij} \\
&= 1 \\
\implies \mathbb{1} &= \mathbb{1}P
\end{aligned}$$

If we now normalize $\mathbb{1}$ to satisfy (2), we get π

$$\implies \pi = \frac{1}{|\mathbb{1}|}\mathbb{1}$$

$$\pi_j = \frac{1}{M+1}$$

Q3: Write the transition matrix of the following Markov chains

a) n black balls and n white balls are placed in two urns so that each urn contains n balls. At each stage one ball is selected at random from each urn and the two balls interchange. The state of the system is the number of white balls in the first urn.
b) Consider two urns A and B containing a total of n balls. An experiment is performed in which a ball is selected at random at time $t(t = 1, \dots)$ from among the totality of n balls. Then an urn is selected at random (probability of selecting A is p) and the ball previously drawn is placed in this urn. The state of the system at each trial is the number of balls in A.

A:
a): Let A_n denote the stochastic process of the number of white balls in the first urn. We want to find the transition probabilities $P(A_{n+1} = j|A_n = i)$ for $i = 0, 1, \dots n$

Suppose at stage n there are i white balls in the first urn, and $n - i$ white balls in the second urn ($i = 0, 1, \dots n$). Since the total number of balls in each urn is n, there will be $n - i$ black balls in the first urn and i black balls in the second urn.

Then at stage $n + 1$, there can be i white balls (if the two balls selected from each urn are both white or black), $i - 1$ white balls (if we select a white ball from the first urn and a black ball from the second urn) or $i + 1$ (if we select a black ball from the first urn and a white ball from the second urn). Therefore

$$P(A_{n+1} = j|A_n = i) = \begin{cases} \left(\frac{i}{n}\right)^2 & j = i - 1 \\ \frac{2i(n-i)}{n^2} & j = i \\ \left(\frac{n-i}{n}\right)^2 & j = i + 1 \\ 0 & \text{o.w.} \end{cases}$$

for $i = 1, \cdots, n - 1$. Similarly, for the special cases $i = 0$ and $i = n$ we have

$$P(A_{n+1} = j|A_n = 0) = \begin{cases} 1 & j = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(A_{n+1} = j|A_n = n) = \begin{cases} 1 & j = n - 1 \\ 0 & \text{o.w.} \end{cases}$$

b) Let A_n denote the stochastic process of the number of balls in urn A. We want to find the transition probabilities $P(A_{n+1} = j|A_n = i)$ for $i = 0, 1, \dots n$

Suppose at stage n there are i balls in urn A and. Then at stage $n + 1$ there can there can be i balls (select ball from urn A, place in urn A, or select ball from urn B, place in urn B), $i - 1$ balls (select ball from urn A, place in urn B) or $i + 1$ balls (select from urn B, place in urn A). Therefore,

$$P(A_{n+1} = j|A_n = i) = \begin{cases} \frac{i}{n}(1 - p) & j = i - 1 \\ \frac{i}{n}p + \frac{n-i}{n}(1 - p) & j = i \\ \left(\frac{n-i}{n}\right)p & j = i + 1 \\ 0 & \text{o.w.} \end{cases}$$

for $i = 1, \cdots, n - 1$. Similarly, for the special cases $i = 0$ and $i = n$ we have

$$P(A_{n+1} = j|A_n = 0) = \begin{cases} 1 - p & j = 0 \\ p & j = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(A_{n+1} = j | A_n = n) = \begin{cases} 1-p & j = n-1 \\ p & n \\ 0 & \text{o.w.} \end{cases}$$

Q4: Consider a spinner with numbers 1 through 4 that is spun repeatedly. Define the following processes:

(a) Let S_n represent the highest number observed on the spinner up to the n-th spin.

(b) At the n-th spin, let T_n denote the number of spins required to observe the next "4."

Prove that both S_n and T_n follow the Markov property, and determine the transition probabilities for each process.

A:

(a) The largest number up to the n-th spin is a number from the set $\{1, 2, 3, 4\}$. Since $S_{n+1} = \max\{S_n, \text{outcome of the } (n+1)\text{th spin}\}$, S_{n+1} depends only on S_n and not on the past. Therefore, S_n satisfies the Markov property.
The transition probabilities are:

$$P(S_{n+1} = j \mid S_n = i) = \begin{cases} \frac{i}{4} & \text{if } j = i \\ \frac{1}{4} & \text{if } j > i \\ 0 & \text{otherwise} \end{cases}$$

where $i, j \in \{1, 2, 3, 4\}$.

(b) For T_n, if $T_n \in \{2, 3, 4, \ldots\}$, then $T_{n+1} = T_n - 1$. For example, at time n, if it is given that the next "4" is going to appear in the 8th spin from now, then at time $n+1$, we know that the next "4" will appear in the 7th spin from that point.
The only non-trivial case is when $T_n = 1$, meaning that the next "4" will appear on the next spin (i.e., the $n+1$-th spin). In this case, the process restarts at time $n+1$ and T_{n+1} follows a Geometric distribution with probability $\frac{1}{4}$.
The transition probabilities for T_n are:

$$P(T_{n+1} = j \mid T_n = i) = \begin{cases} 1 & \text{if } j = i-1 \text{ and } i \geq 2 \\ \frac{1}{4}\left(\frac{3}{4}\right)^{j-1} & \text{if } i = 1 \text{ and } j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Q5: A digital signal processing device operates using only two signals, represented by the values -1 and 1. The device is designed to transmit one of these signals through multiple stages. However, at each stage, there is a probability r that the signal entering the stage will be flipped when it exits, and a probability $s = 1 - r$ that it will remain unchanged.

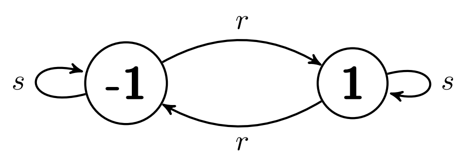
(a) Construct a Markov chain to model the transmission process.

(b) Determine the transition probability matrix.

(c) Assuming that both signals are equally likely at the initial stage, calculate the probability that the device outputs the signal -1 after passing through two stages.

A:

(a) The Markov chain for this process is shown below:



(b) The transition probability matrix P for this process is:

$$P = \begin{pmatrix} s & r \\ r & s \end{pmatrix}$$

where:

(c) Let X_0 be the initial state of the signal and X_2 be the state of the signal after two stages. We are asked to find $P(X_2 = -1)$ given that both signals -1 and 1 have an equal probability at the initial stage, i.e., $P(X_0 = -1) = P(X_0 = 1) = \frac{1}{2}$.
The probability distribution after two stages is given by P^2, the square of the transition matrix P:

$$P^2 = \begin{pmatrix} s & r \\ r & s \end{pmatrix}^2 = \begin{pmatrix} s^2 + r^2 & 2rs \\ 2rs & s^2 + r^2 \end{pmatrix}$$

The probability of being in state -1 after two stages is:

$P(X_2 = -1) = P(X_0 = -1) \cdot P(\text{-1 to -1 in two stages}) + P(X_0 = 1) \cdot P(\text{1 to -1 in two stages})$

Since $P(X_0 = -1) = P(X_0 = 1) = \frac{1}{2}$, we have:

$$P(X_2 = -1) = \frac{1}{2} \cdot (s^2 + r^2) + \frac{1}{2} \cdot (2rs)$$

Simplifying, we get:

$$P(X_2 = -1) = \frac{s^2 + r^2 + 2rs}{2} = \frac{1}{2}(s + r)^2 = \frac{1}{2}$$

Thus, after two stages, the probability of obtaining the signal -1 is $\frac{1}{2}$, regardless of r and s, provided $s + r = 1$.

Q6: Consider the Markov chain in figure below.

(a) Find the recurrent classes R_1 and R_2.

(b) Assuming $X_0 = 3$, find the probability that the chain gets absorbed in R_1.

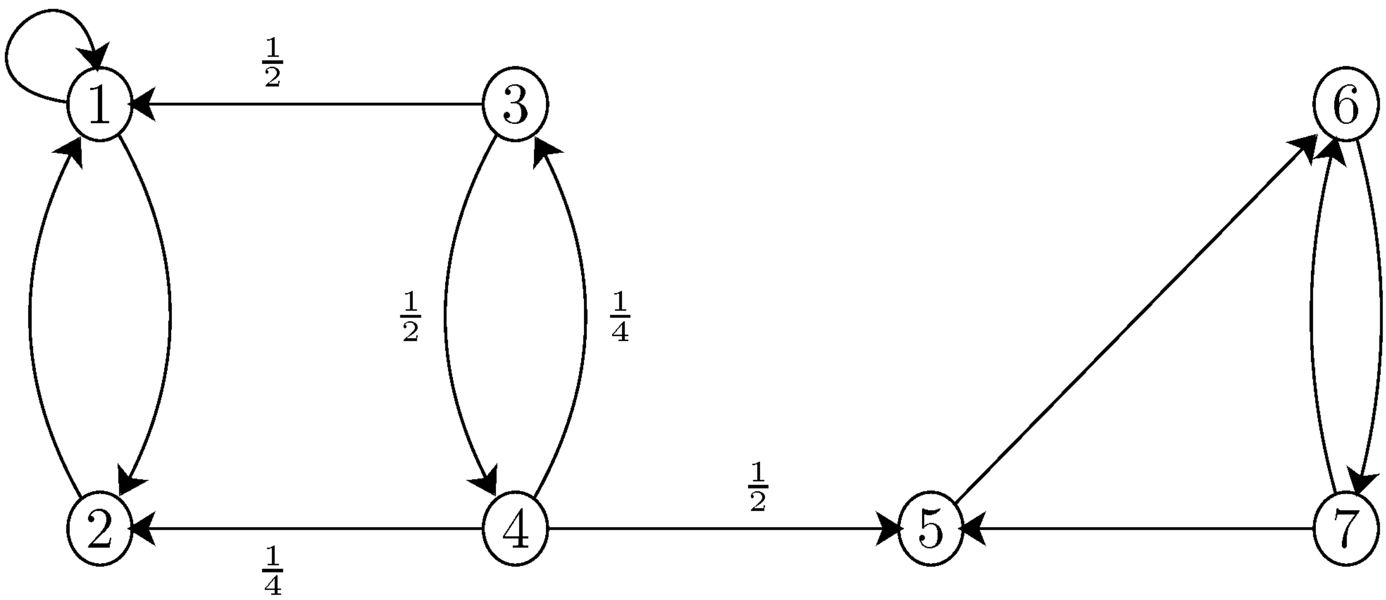


Figure 2: Markov Chain

A:

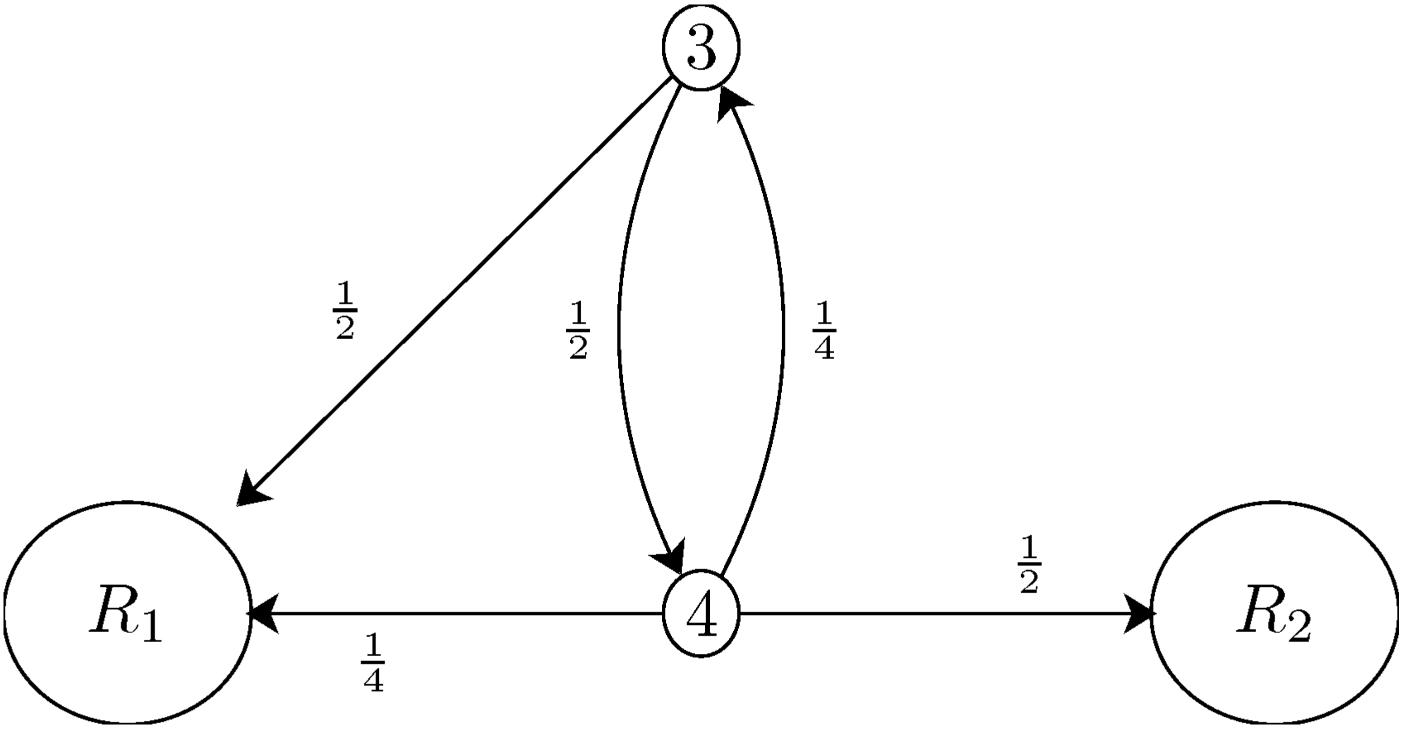


Figure 3: Reduced Markov Chain

(a) There are two recurrent classes, $R_1 = \{1, 2\}$, and $R_2 = \{5, 6, 7\}$.

(b) Here, we can replace each recurrent class with one absorbing state. The resulting state diagram is shown below. Now we can apply our standard methodology to find the probability of absorption in state R_1. In particular, define

$$a_i = P(\text{absorption in } R_1 \mid X_0 = i), \quad \text{for all } i \in S.$$

By the above definition, we have $a_{R_1} = 1$, and $a_{R_2} = 0$. To find the unknown values of a_i's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \quad \text{for } i \in S.$$

We obtain

$$a_3 = \frac{1}{2}a_{R_1} + \frac{1}{2}a_4 = \frac{1}{2} + \frac{1}{2}a_4,$$

$$a_4 = \frac{1}{4}a_{R_1} + \frac{1}{4}a_3 + \frac{1}{2}a_{R_2} = \frac{1}{4} + \frac{1}{4}a_3.$$

Solving the above equations, we obtain

$$a_3 = \frac{5}{7}, \quad a_4 = \frac{3}{7}.$$

Therefore, if $X_0 = 3$, the chain will end up in class R_1 with probability $a_3 = \frac{5}{7}$.

Q7: Let $X = [X_0, X_1, \dots]$ be a Markov chain having transition matrix P. Recall that for any non-negative integer n, we have $P_{ij}^{(n)} = \Pr(X_n = j \mid X_0 = i)$. Then for any $m \geq 0$ and $n \geq 0$, we get

$$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)}.$$

This equation is called the *Chapman-Kolmogorov equation*. Prove this.

If X has a finite state space, we can write $P^{(n)}$ as a matrix, called the n-step transition matrix. Then $P^{(n)} = P^n$ for all $n \geq 0$.

A: By the definition of $P^{(n)}$, we get that

$$P^{(0)}(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Hence, the Chapman-Kolmogorov equations are trivially true if $m = 0$ or $n = 0$.
Now let $m \geq 1$ and $n \geq 1$. Then

$$P_{ij}^{(m+n)} = \Pr(X_{m+n} = j \mid X_0 = i)$$
$$= \sum_k \Pr(X_{m+n} = j, X_m = k \mid X_0 = i)$$
$$= \sum_k \Pr(X_{m+n} = j \mid X_m = k, X_0 = i) \Pr(X_m = k \mid X_0 = i) \quad \text{(marginalisation)}$$
$$= \sum_k \Pr(X_n = j \mid X_m = k) \Pr(X_m = k \mid X_0 = i) \quad \text{(Markov Property)}$$
$$= \sum_k P_{ik}^{(m)} P_{kj}^{(n)} \quad \text{(We can consider m as the starting point of another markov chain.)}$$
$$= [P^{(m)} P^{(n)}]_{ij}$$

When X has a finite state space, the Chapman-Kolmogorov equation can be expressed in matrix form: $P^{(m+n)} = P^{(m)} P^{(n)}$ for all $m \geq 0$ and $n \geq 0$.

$$P^{(0)} = I \text{ and } P^{(1)} = P \text{ by the definition of } P^{(n)}.$$

The Chapman-Kolmogorov equation gives us $P^{(n+1)} = P^{(n)} P^{(1)} = P^{(n)} P$. Using mathematical induction, we can prove that $P^{(n)} = P^n$ for all $n \geq 0$.