Probability and Statistics: MA6.101

Tutorial 1

Topics Covered: Sigma Algebra, Set Theory, Probability Axioms, Conditional Probability, Permutations and Combinations

Q1: You purchase a certain product. The manual states that the lifetime T of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = e^{-t/5}, \quad \text{for all } t \geq 0.$$

For example, the probability that the product lasts more than (or equal to) 2 years is $P(T \geq 2) = e^{-2/5} = 0.6703$. I purchase the product and use it for two years without any problems. What is the probability that it breaks down in the third year?

A: Let A be the event that a purchased product breaks down in the third year. Also, let B be the event that a purchased product does not break down in the first two years. We are interested in $P(A \mid B)$. We have

$$P(B) = P(T \geq 2) = e^{-2/5}.$$

We also have

$$P(A) = P(2 \leq T \leq 3) = P(T \geq 2) - P(T \geq 3) = e^{-2/5} - e^{-3/5}.$$

Finally, since $A \subseteq B$, we have $A \cap B = A$. Therefore,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{e^{-2/5} - e^{-3/5}}{e^{-2/5}} = 0.1813.$$

Q2: There are 30 people in a room. What is the chance that any two of them celebrate their birthday on the same day? Assume 365 days in a year.

A: First, we calculate the probability that no two people share a birthday:

$$P(\text{no shared birthday}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{336}{365}$$

This product can be expressed as:

$$P(\text{no shared birthday}) = \prod_{i=0}^{29} \left(1 - \frac{i}{365}\right)$$

In numerical terms, this evaluates to approximately:

$$P(\text{no shared birthday}) \approx 0.294$$

The probability that at least two people share a birthday is then:

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.294 \approx 0.706$$

Thus, the probability that at least two people in a room of 30 share the same birthday is approximately 70.6%.

Q3: Prove the following inequality without the use of induction:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

A: We know that if $B_1, B_2, B_3, \ldots$ are disjoint subsets of the probability space then

$$\mathbb{P}\left(\bigcup_{i} B_i\right) = \sum_{i} \mathbb{P}(B_i);$$

We can construct the sets B_i from A_i such that they are disjoint,

$$B_i = A_i - \bigcup_{j=1}^{i-1} A_j$$

and show that

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$

Suppose $x \in \bigcup_{i=1}^{\infty} A_i$. Then $x \in A_k$ for some minimum k such that $i < k \implies x \notin A_i$. Therefore $x \in B_k = A_k - \bigcup_{j=1}^{k-1} A_j$. So the first inclusion is true: $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} B_i$.

Next suppose that $x \in \bigcup_{i=1}^{\infty} B_i$. It follows that $x \in B_k$ for some k. And $B_k = A_k - \bigcup_{j=1}^{k-1} A_j$ so $x \in A_k$, and we have the other inclusion: $\bigcup_{i=1}^{\infty} B_i \subseteq \bigcup_{i=1}^{\infty} A_i$.

By construction of each B_i, $B_i \subset A_i$. For $B \subset A \implies \mathbb{P}(B) \le \mathbb{P}(A)$.

So, we can conclude that the desired inequality is true:

$$\mathbb{P}\left(\bigcup_{i} A_i\right) = \mathbb{P}\left(\bigcup_{i} B_i\right) = \sum_{i} \mathbb{P}(B_i) \le \sum_{i} \mathbb{P}(A_i).$$

Q4: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

A:

(a) To prove the first condition: we need to show that $\phi \in \mathcal{G}$ and $\Omega \in \mathcal{G}$. From axioms of probability, $\mathbb{P}(\phi) = 0$ and $\mathbb{P}(\Omega) = 1$. Hence by definition of $\mathcal{G}$, $\phi \in \mathcal{G}$ and $\Omega \in \mathcal{G}$.

(b) To prove the second condition:
Let A $\in \mathcal{G}$
$\implies \mathbb{P}(A) = 0$ OR $\mathbb{P}(A) = 1$.
$\implies 1 - \mathbb{P}(A) = 1$ OR $1 - \mathbb{P}(A) = 0$
$\implies \mathbb{P}(A^C) = 1$ OR $\mathbb{P}(A^C) = 0$
$\implies A^C \in \mathcal{G}$.

(c) To prove the third condition:
Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of sets in G. We need to show that $\bigcup_{i=1}^{\infty} A_i \in G$.
There are two possibilities to consider for each A_i:

- $P(A_i) = 0$
- $P(A_i) = 1$

First, let's consider the case where $\bigcup_{i=1}^{\infty} P(A_i) = 0$:
If $P(A_i) = 0$ for all i, then the union $\bigcup_{i=1}^{\infty} A_i$ is also of measure 0. This follows from the countable subadditivity property of measures:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i) = 0.$$

Hence, $\bigcup_{i=1}^{\infty} A_i \in G$.
Now, let's consider the case where $\bigcup_{i=1}^{\infty} P(A_i) = 1$:
If there exists at least one A_i such that $P(A_i) = 1$, then the union $\bigcup_{i=1}^{\infty} A_i$ will be of measure 1. This follows because if any set in a countable collection has measure 1, the union of the entire collection must also have measure 1:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \geq P(A_i) = 1.$$

Hence, $\bigcup_{i=1}^{\infty} A_i \in G$.

Q5: A permutation σ is called a *derangement* if $\forall i, \sigma(i) \neq i$. Consider a uniform random permutation σ of $\{1, \ldots, n\}$, and let D_n be the event that σ is a derangement. Use the inclusion-exclusion principle to find a formula for the number of derangements, and show that $\mathbb{P}(D_n) \xrightarrow{n\to\infty} e^{-1}$.

A: Problem 6

Q6: A 6-sided die is rolled n times. What is the probability all faces have appeared? (Hint: Use Principle of Inclusion and Exclusion)

A: First, we calculate the total number of sequences of length n with digits from 1 to 6:

$$\text{Total number of sequences} = 6^n$$

Let A_i be the set of sequences of length n that do not contain the digit i. Using the Principle of Inclusion and Exclusion (PIE), we find the number of sequences that do not contain any one of the digits:

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = \sum_{i=1}^{6} |A_i| - \sum_{1\le i<j\le 6} |A_i \cap A_j| + \sum_{1\le i<j<k\le 6} |A_i \cap A_j \cap A_k| - \cdots$$
$$+(-1)^{6}|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6|$$

We calculate the intersections as follows:

$$|A_i| = 5^n, \quad |A_i \cap A_j| = 4^n, \quad |A_i \cap A_j \cap A_k| = 3^n, \quad \ldots, \quad |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| = 0$$

This simplifies to:
$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = \binom{6}{1}5^n - \binom{6}{2}4^n + \binom{6}{3}3^n - \binom{6}{4}2^n + \binom{6}{5}1^n$$

The set of sequences that contain all digits from 1 to 6 is the complement of the above set, So, Number of sequences containing all digits:

$$6^n - \left(\binom{6}{1}5^n - \binom{6}{2}4^n + \binom{6}{3}3^n - \binom{6}{4}2^n + \binom{6}{5}1^n\right)$$

Finally, the probability that a sequence of length n contains all digits from 1 to 6 is:

$$\text{Probability} = \frac{6^n - \left(\binom{6}{1}5^n - \binom{6}{2}4^n + \binom{6}{3}3^n - \binom{6}{4}2^n + \binom{6}{5}1^n\right)}{6^n}$$

Q7: Let $E_1, E_2, \ldots, E_n$ be n events, each with positive probability. Prove that

$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = \mathbb{P}\{E_1\} \cdot \mathbb{P}\{E_2 \mid E_1\} \cdot \mathbb{P}\{E_3 \mid E_1 \cap E_2\} \cdots \mathbb{P}\left\{E_n \mid \bigcap_{i=1}^{n-1} E_i\right\}.$$

A: By expanding the right-hand side using the definition of conditional probability, we get:

$$\mathbb{P}\{E_1\} \cdot \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_1)} \cdot \frac{\mathbb{P}(E_1 \cap E_2 \cap E_3)}{\mathbb{P}(E_1 \cap E_2)} \cdots \frac{\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right)}{\mathbb{P}\left(\bigcap_{i=1}^{n-1} E_i\right)}.$$

After cancelling terms, we are left with only the numerator of the last fraction, which is equal to the left-hand side.

Q8: Queueville Airlines knows that on average 5% of the people making flight reservations do not show up. (They model this information by assuming that each person independently does not show up with probability of 5%.) Consequently, their policy is to sell 52 tickets for a flight that can only hold 50 passengers. What is the probability that there will be a seat available for every passenger who shows up? ‘

A: Probability that everyone gets a seat = probability that at most 50 people show up = 1− probability that 51 or 52 people show up.

The probability of any given passenger showing up is 0.95. Since the arrival of each passenger is an independent event, the probability that all 52 show up is $(0.95)^{52}$. Similarly, the probability that 51 people show up is $\binom{52}{1}(0.95)^{51}(0.05)$.

Probability of everyone getting a seat $= 1 - (0.95)^{52} - \binom{52}{1}(0.95)^{51}(0.05)$