# Probability and Statistics: MA6.101

# Tutorial & Homework 5

Topics Covered: PDF, CDF, Joint random variables

A1: Each time we draw a ball from the box, there is a possibility of a color switch, except when we draw the first time. Hence, there are 49 possible switches. Let $X_i$ be an indicator variable where the value is 1 if the $i$th position has a color change and 0 otherwise, for $i = 1, 2, \ldots, 49$.

Thus, we can write the total number of switches as:

$$S = X_1 + X_2 + \cdots + X_{49} = \sum_{i=1}^{49} X_i$$

And the expected number of color switches is:

$$E[S] = E[X_1 + X_2 + \cdots + X_{49}] = E[X_1] + E[X_2] + \cdots + E[X_{49}]$$

By linearity of expectation, the expected value of the sum of random variables is equal to the sum of their individual expected values. Thus,

$$E[S] = E[X_1] + E[X_2] + \cdots + E[X_{49}]$$

Now, we know that each $X_i$ is identical. This is because all the balls are drawn randomly, the probability distribution is identical for each draw. Thus, the probability of a color switch at any index i is the same, i.e. it does not depend on the index. (Proof at the end of the solution)

Thus, the expected value is the same for each $i$, i.e., $E[X_i] = E[X_1]$. Therefore,

$$E[S] = 49 \cdot E[X_1]$$

We can calculate the expected value $E[X_1]$. There is a color change if the first ball is blue and the second is green, or if the first is green and the second is blue. The probability of either of these events is:

$$P(X_1 = 1) = \frac{1}{2} \cdot \frac{25}{49} + \frac{1}{2} \cdot \frac{25}{49} = \frac{25}{49}$$

Thus, the expected value of $X_1$ is:

$$E[X_1] = \frac{25}{49}$$

And thus the expected number of color changes for the 50 balls is:

$$E[S] = 49 \cdot \frac{25}{49} = 25$$

**Proof:** Proving that each $P(X_i = 1)$ (that is the probability of a color flip at index i) is identical.

We know there can't be a color flip at 1st index. Let's take the case of a

color change at index n. We notice that a color flip requires either a BG or a GB as the last two balls. For all other indices the order of the balls is not relevant to us.
Let's assume that till n-2 index we had picked b blue balls and g green balls. Then for a color flip the n-1th and nth index need to have either a BG or a GB
Now for BG case:

$$P(\text{n-1th index to be Blue} \mid \text{b blue balls and g green balls picked}) = \frac{25-b}{50-b-g}$$

Since 25-b blue balls left and total 50-b-g balls left. And for nth index total balls left is 50-b-g-1. So,
$P($nth index to be Green | b blue balls and g green balls picked and n-1th ball is blue)

$$= \frac{25-g}{50-b-g-1}$$

Thus, $P($n-1th index to be blue and nth index to be Green | b blue balls and g green balls picked till n-2)

$$= \frac{25-b}{50-b-g} \times \frac{25-g}{50-b-g-1}$$

By symmetry we can see that the value for BG and GB case is the same. So, we can calculate using 1 and multiply by 2.
Now, to find $P(X_n = 1)$ we still need to consider all the possible cases of b and g values that sum up till n-2. So, we can write the expression as

$$P(X_n = 1) = \sum_{b+g=n-2} P(b,g) \times 2 \times \frac{25-b}{50-b-g} \times \frac{25-g}{50-b-g-1}$$

where $P(b,g)$ represents the probability of obtaining configuration with b blue balls and g green balls after $n-2$ picks. Now,

$$P(b,g) = \frac{\binom{25}{b}\binom{25}{g}}{\binom{50}{b+g}}$$

Since you pick b blue balls from the 25 blue balls and g green balls from the 25 green balls, and you picked total b+g balls from the 50 balls.
Substituting P(b,g) in the equation:

$$P(X_n = 1) = \sum_{b+g=n-2} \frac{\binom{25}{b}\binom{25}{g}}{\binom{50}{b+g}} \times 2 \times \frac{25-b}{50-b-g} \times \frac{25-g}{50-b-g-1}$$

And we know $\binom{n}{r} = \binom{n}{n-r}$. So,

$$P(X_n = 1) = \sum_{b+g=n-2} \frac{\binom{25}{25-b}\binom{25}{25-g}}{\binom{50}{50-b-g}} \times 2 \times \frac{25-b}{50-b-g} \times \frac{25-g}{50-b-g-1}$$

Using property that $\binom{n}{r} = \frac{n}{r} \times \binom{n-1}{r-1}$.

$$P(X_n = 1) = \sum_{b+g=n-2} \frac{25^2 \times \binom{24}{24-b}\binom{24}{24-g}}{50 \times 49 \times \binom{48}{48-b-g}} \times 2$$

$$P(X_n = 1) = \frac{25}{49} \sum_{b+g=n-2} \frac{\binom{24}{24-b}\binom{24}{24-g}}{\binom{48}{48-b-g}}$$

Also, $48 - b - g = 48 - (b + g)$. So, it's constant with respect to summing over all values of b,g such that b+g=n-2. Taking it out of summation.

$$P(X_n = 1) = \frac{25}{49} \frac{\sum_{b+g=n-2} \binom{24}{24-b}\binom{24}{24-g}}{\binom{48}{48-b-g}}$$

We can write the summation as $\sum_{b=0} b = n - 2$ and substitute $g = n - 2 - b$. On doing that

$$P(X_n = 1) = \frac{25}{49} \frac{\sum_{b=0}^{n-2} \binom{24}{24-b}\binom{24}{24-(n-2-b)}}{\binom{48}{48-b-(n-2-b)}}$$

We can write this as

$$P(X_n = 1) = \frac{25}{49} \frac{\sum_{b=0}^{n-2} \binom{24}{b}\binom{24}{(n-2)-b}}{\binom{48}{n-2}}$$

We know by the property that

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}$$

Here $n = m = 24$ and $r = b$ and $k = n - 2$. Thus, expression is

$$P(X_n = 1) = \frac{25}{49} \frac{\binom{48}{n-2}}{\binom{48}{n-2}}$$

Thus, for any index n the probability of a color flip at that index is

$$P(X_n = 1) = \frac{25}{49}$$

A2: (a) Finding the value of k:

$$\int_{-\infty}^{\infty} k^2 e^{-|kx|} dx = 1$$

$$k^2 \int_{-\infty}^{\infty} e^{-|kx|} dx = 1$$

Since, it's an odd function

$$2k^2 \int_{0}^{\infty} e^{-|kx|} dx = 1$$

If $k > 0$

$$2k^2 \int_0^\infty e^{-kx} dx = 1$$

$$2k^2 \left[ -\frac{1}{k} e^{-kx} \right]_0^\infty = 1$$

$$2k^2 \left( 0 + \frac{1}{k} \right) = 1$$

$$k = \frac{1}{2}$$

If $k < 0$

$$2k^2 \int_0^\infty e^{kx} dx = 1$$

$$2k^2 \left[ \frac{1}{k} e^{kx} \right]_0^\infty = 1$$

Since $k < 0$ $\quad kx = -\infty$ at upper limit

$$2k^2 \left( 0 - \frac{1}{k} \right) = 1$$

$$k = -\frac{1}{2}$$

(b) Finding the mean (E(X)):

$$E(X) = \int_{-\infty}^\infty x f(x) dx = 0$$

$$E(X) = \int_{-\infty}^\infty k^2 x e^{-|kx|} dx = 0 \quad \text{(Since Odd function)}$$

(c) Finding the variance (V(X)):

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^\infty x^2 f(x) dx$$

$$E(X^2) = k^2 \int_{-\infty}^\infty x^2 e^{-|kx|} dx$$

Since it's an even function

$$E(X^2) = 2k^2 \int_0^\infty x^2 e^{-|kx|} dx$$

Now, an important thing to note here is while removing the mod we know $x > 0$. But, k takes values $\frac{1}{2}$ and $-\frac{1}{2}$, So $|kx| = \frac{x}{2}$ in both cases.

$$E(X^2) = 2k^2 \int_0^\infty x^2 e^{-\frac{x}{2}} dx$$

Let $t = \frac{x}{2}$ so $dt = \frac{dx}{2}$. Thus,

$$E(X^2) = 2k^2 \int_0^\infty (2t)^2 e^{-t} 2dt$$

$$E(X^2) = 16k^2 \int_0^\infty t^2 e^{-t} dt$$

We know that

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

and $\Gamma(n) = (n-1)!$ when $n \in N$

$$E(X^2) = 16k^2 \times \Gamma(3)$$

for both $k = \frac{1}{2}$ and $k = -\frac{1}{2}$, $k^2 = \frac{1}{4}$

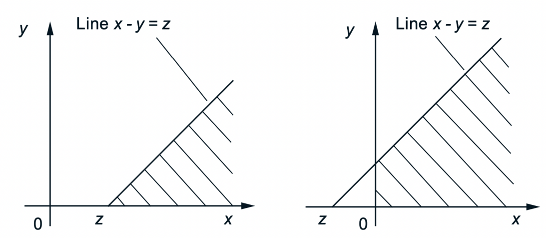
$$E(X^2) = 16k^2 \times 2! = 16 \times \frac{1}{4} \times 2 = 8$$

Thus,

$$Variance = 8 - 0^2 = 8$$

Therefore, the value of k is $\frac{1}{2}$ or $-\frac{1}{2}$, the mean is 0, and the variance is 8.

A3: We want to find the PDF of Z = X - Y. We will first calculate the CDF $F_Z(z)$ by considering separately the cases $z \geq 0$ and $z < 0$. Note that since $x$ and $y$ represent defect amounts, they must always satisfy $x, y \geq 0$.




$$F_Z(z) = \mathbf{P}(X - Y \leq z)$$
$$= 1 - \mathbf{P}(X - Y > z)$$

To obtain the value $\mathbf{P}(X - Y > z)$ we must integrate the joint PDF $f_{X,Y}(x, y)$ over the shaded area in the above figures, which correspond to $z \geq 0$ (left side of diagram) and $z < 0$ (right side of diagram).
For $z \geq 0$,

$$F_Z(z) = 1 - \mathbf{P}(X > z + Y)$$
$$= 1 - \int_0^\infty \left( \int_{z+y}^\infty f_{X,Y}(x, y) dx \right) dy$$

In the above equation, we are fixing $y$ and considering all possible values of $x$ starting from $z + y$ to infinity, since we are interested in the event

$X > z + Y$. After that, we integrate over all possible values of $y$ from 0 to infinity.

$$= 1 - \int_0^{\infty} \lambda e^{-\lambda y} \left( \int_{z+y}^{\infty} \lambda e^{-\lambda x} dx \right) dy$$
$$= 1 - \int_0^{\infty} \lambda e^{-\lambda y} \left( e^{-\lambda(z+y)} \right) dy$$
$$= 1 - \lambda e^{-\lambda z} \int_0^{\infty} \left( e^{-\lambda(2y)} \right) dy = 1 - \frac{1}{2} e^{-\lambda z}$$

The integral above cannot be applied for $z < 0$ because, for some values of $y$ in the range $0 \leq y < \infty$, $z + y = x$ may become negative (as shown in the right side of diagram).

For the case $z < 0$, we can use a similar calculation, but we can also just use symmetry. The symmetry of the situation implies that the random variables Z = X - Y and -Z = Y - X have the same distribution.

$$F_Z(z) = \mathbf{P}(Z \leq z) = \mathbf{P}(-Z \geq -z) = \mathbf{P}(Z \geq -z) = 1 - F_Z(-z)$$
$$= 1 - \left( 1 - \frac{1}{2} e^{-\lambda(-z)} \right) = \frac{1}{2} e^{\lambda z}$$

Combining the two cases we obtain,

$$F_Z(z) = \begin{cases} 1 - \frac{1}{2} e^{-\lambda z} & \text{if } z \geq 0, \\ \frac{1}{2} e^{\lambda z} & \text{if } z < 0, \end{cases}$$

We can calculate the PDF by differentiating the CDF:

$$f_Z(z) = \begin{cases} \frac{\lambda}{2} e^{-\lambda z} & \text{if } z \geq 0, \\ \frac{\lambda}{2} e^{\lambda z} & \text{if } z < 0, \end{cases}$$
$$f_Z(z) = \frac{\lambda}{2} e^{-\lambda |z|}$$

(Note: This is known as a two-sided exponential PDF, also known as the Laplace PDF.)

A4: (a) The joint range is

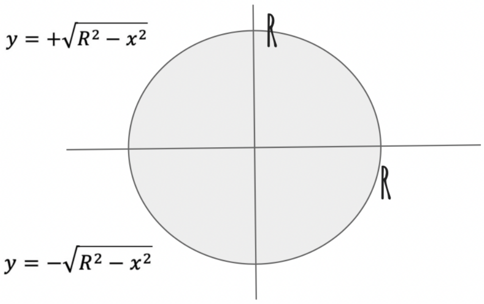
$$R_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

since the values must be within the circle of radius $R$.

(b) Since joint PDF of the random variables X and Y is uniform:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\text{Area of Circle}} = \frac{1}{\pi R^2} & x, y \in R_{X,Y} \\ 0 & \text{otherwise} \end{cases}$$

$X$ can range from $-R$ to $R$, or $R_X = [-R, R]$. For $x$ not in $R_X$, $f_X(x)$ is zero. We can calculate the $f_X(x)$ in $R_X$ as follows:




$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$
$$= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \left(\frac{1}{\pi R^2}\right) dy = \frac{2}{\pi R^2}\sqrt{R^2 - x^2}$$

Note that the $f_X(x)$ is not a uniform PDF.

(c)
$$Z = \sqrt{X^2 + Y^2}$$

We can see that $Z$ will take on any value from 0 to $R$, since the point could be at the origin and as far as $R$. This gives, $R_Z = [0, R]$.
Using LOTUS:

$$\mathbb{E}[Z] = \mathbb{E}\left[\sqrt{X^2 + Y^2}\right] = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{x^2 + y^2} f_{X,Y}(x,y)dydx$$

(as $X$ ranges from $-R$ to $R$, $Y$ will range from $-\sqrt{R^2 - x^2}$ to $\sqrt{R^2 - x^2}$)
(Note that we could've set up this integral $dxdy$.)

(d) No, they are not independent. This is because $X$ and $Y$ both have marginal range from $-R$ to $R$, but the joint range is not a rectangle of this region (it is a circle). So, $R_{X,Y} \neq R_X \times R_Y$. More explicitly, take a point very close to the top right corner of the square $(R, R)$, say $(0.99R, 0.99R)$. The joint PDF is defined to be 0 at this point (not in the circle), but the marginal PDFs of both $X$ and $Y$ are nonzero at $0.99R$ (since $0.99R$ is in the marginal range of both). So, we get $f_{X,Y}(0.99R, 0.99R) \neq f_X(0.99R)f_Y(0.99R)$.

A5: The two methods of computing $\mathbb{E}[X]$ are equivalent.

(a) Using LOTUS (taking g(X,Y) = X),

$$\mathbb{E}[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)dxdy = \int_0^1\int_0^1 x\left(x + \frac{3}{2}y^2\right)dxdy$$

(b) For $0 \leq x \leq 1$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^1 \left(x + \frac{3}{2}y^2\right)dy = x + \frac{1}{2}$$

Thus, the marginal PDF $f_X(x)$ is:

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \left( x + \frac{1}{2} \right) dx$$

Both result in

$$\mathbb{E}[X] = \frac{7}{12}$$

A6: We need to find the probability:

$$P(X_3 < X_1 \text{ and } X_3 < X_2).$$

Since the random variables $X_1, X_2, X_3$ are independent and exponentially distributed, the joint probability distribution of $X_1, X_2, X_3$ can be written as the product of their individual density functions.
We condition on $X_3 = x_3$ and integrate over all possible values of $X_3$, applying the law of total probability. Therefore:

$$P(X_3 < X_1 \text{ and } X_3 < X_2) = \int_0^{\infty} P(X_1 > x_3 \text{ and } X_2 > x_3 \mid X_3 = x_3) f_{X_3}(x_3)\, dx_3.$$

Because $X_1, X_2$ are independent of $X_3$, we have:

$$P(X_1 > x_3 \text{ and } X_2 > x_3 \mid X_3 = x_3) = P(X_1 > x_3 \text{ and } X_2 > x_3).$$

Since $X_1, X_2$ are independent too,

$$P(X_1 > x_3 \text{ and } X_2 > x_3) = P(X_1 > x_3) \cdot P(X_2 > x_3).$$

Thus, we can rewrite the equation as:

$$P(X_3 < X_1 \text{ and } X_3 < X_2) = \int_0^{\infty} P(X_1 > x_3) \cdot P(X_2 > x_3) \cdot f_{X_3}(x_3) dx_3.$$

CDF of exponential is:

$$P(X_i > x) = e^{-\lambda_i x}, \quad x \geq 0.$$

Therefore:

$$P(X_3 < X_1 \text{ and } X_3 < X_2) = \int_0^{\infty} \lambda_3 e^{-\lambda_3 x_3} e^{-\lambda_1 x_3} e^{-\lambda_2 x_3}\, dx_3.$$

$$P(X_3 < X_1 \text{ and } X_3 < X_2) = \int_0^{\infty} \lambda_3 e^{-(\lambda_1 + \lambda_2 + \lambda_3) x_3}\, dx_3.$$

$$P(X_3 < X_1 \text{ and } X_3 < X_2) = \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}.$$

A7: Since the joint probability density function $f_{X,Y}(x,y) = c$ is constant over the region where $x \geq 0$, $y \geq 0$, and $x + y \leq 3$, we can solve for $c$ by using the property that the total probability must equal 1:

$$1 = \int_0^3 \int_0^{3-x} f_{X,Y}(x,y)\, dy\, dx = \int_0^3 \int_0^{3-x} c\, dy\, dx.$$

Either by solving this integral or by finding the area $(1/2 \cdot 3 \cdot 3)$ which is shaded we get:

$$1 = c \times \frac{9}{2}.$$

$$c = \frac{2}{9}.$$

To find the marginal pdf of $X$, we integrate the joint pdf over all possible values of $y$:

$$f_X(x) = \int_0^{3-x} f_{X,Y}(x,y)\, dy = \int_0^{3-x} \frac{2}{9}\, dy.$$

Evaluating this integral:

$$f_X(x) = \frac{2}{9}(3-x), \quad 0 \leq x \leq 3.$$

Thus, the marginal pdf of $X$ is:

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x), & 0 \leq x \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the marginal pdf of $Y$ is:

$$f_Y(y) = \begin{cases} \frac{2}{9}(3-y), & 0 \leq y \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

A8:

$$\frac{X^2 + Y^2}{XY} = \frac{X^2}{XY} + \frac{Y^2}{XY} = \frac{X}{Y} + \frac{Y}{X}$$

Using the linearity of expectation and the fact that $X$ and $Y$ are independent:

$$E\left[\frac{X}{Y} + \frac{Y}{X}\right] = E\left[\frac{X}{Y}\right] + E\left[\frac{Y}{X}\right]$$

Since $X$ and $Y$ are identically distributed:

$$E\left[\frac{X}{Y}\right] = E\left[\frac{Y}{X}\right]$$

Thus:

$$E\left[\frac{X}{Y}+\frac{Y}{X}\right]=2E\left[\frac{X}{Y}\right]$$

Since $X$ and $Y$ are independent:

$$2E\left[\frac{X}{Y}\right]=2E\left[X\right]E[1/Y]$$

For geometric random variables, we have:

$$E\left[X\right]=\frac{1}{p}$$

Now we find $E\left[\frac{1}{Y}\right]$:

$$E\left[\frac{1}{Y}\right]=\sum_{k=1}^{\infty}\frac{1}{k}(1-p)^{k-1}p$$

$$E\left[\frac{1}{Y}\right]=\frac{p}{1-p}\sum_{k=1}^{\infty}\frac{(1-p)^{k}}{k}$$

The series $\sum_{k=1}^{\infty}\frac{x^k}{k}=-\ln(1-x)$ (from Maclaurin series):

$$\sum_{k=1}^{\infty}\frac{(1-p)^{k}}{k}=-\ln(p)$$

Thus:

$$E\left[\frac{1}{Y}\right]=\frac{p}{1-p}\cdot(-\ln(p))=\frac{p}{1-p}\ln(1/p)$$

Now:

$$E\left[\frac{X}{Y}\right]=\frac{1}{p}\cdot\frac{p}{1-p}\ln(1/p)=\frac{\ln(1/p)}{1-p}$$

Finally:

$$E\left[\frac{X^2+Y^2}{XY}\right]=2\frac{\ln(1/p)}{1-p}$$

A9: (a) Marginal PMF of $X$ and $Y$

The marginal PMF of $X$ is obtained by summing over all values of $Y$:

$$p_X(x)=\sum_{y}p(x,y).$$

Similarly, the marginal PMF of $Y$ is obtained by summing over all values of $X$:

$$p_Y(y)=\sum_{x}p(x,y).$$

For $X$:

$$p_X(1) = p(1,1) + p(1,2) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6},$$
$$p_X(2) = p(2,1) + p(2,2) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6},$$
$$p_X(3) = p(3,1) + p(3,2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Thus, the marginal PMF of $X$ is:

$$p_X(x) = \begin{cases} \frac{1}{6} & \text{for } x = 1, 2, \\ \frac{1}{2} & \text{for } x = 3. \end{cases}$$

For $Y$:

$$p_Y(1) = p(1,1) + p(2,1) + p(3,1) = \frac{1}{12} + \frac{1}{12} + \frac{1}{4} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12},$$

$$p_Y(2) = p(1,2) + p(2,2) + p(3,2) = \frac{1}{12} + \frac{1}{12} + \frac{1}{4} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

Thus, the marginal PMF of $Y$ is:

$$p_Y(y) = \begin{cases} \frac{5}{12} & \text{for } y = 1, 2. \end{cases}$$

(b) Using the Law of the Unconscious Statistician (LOTUS), calculate $E[X^2 + Y^2]$

By the Law of the Unconscious Statistician (LOTUS), we calculate $E[X^2 + Y^2]$ as:

$$E[X^2 + Y^2] = \sum_x \sum_y (x^2 + y^2) p(x, y).$$

Now, calculate each term:

$$E[X^2 + Y^2] = \sum_x \sum_y (x^2 + y^2) p(x, y)$$
$$= (1^2 + 1^2) \cdot \frac{1}{12} + (1^2 + 2^2) \cdot \frac{1}{12} + (2^2 + 1^2) \cdot \frac{1}{12} + (2^2 + 2^2) \cdot \frac{1}{12}$$
$$+ (3^2 + 1^2) \cdot \frac{1}{4} + (3^2 + 2^2) \cdot \frac{1}{4}.$$

Now, substitute the values:

$$= (1 + 1) \cdot \frac{1}{12} + (1 + 4) \cdot \frac{1}{12} + (4 + 1) \cdot \frac{1}{12} + (4 + 4) \cdot \frac{1}{12}$$
$$+ (9 + 1) \cdot \frac{1}{4} + (9 + 4) \cdot \frac{1}{4}$$
$$= \frac{2}{12} + \frac{5}{12} + \frac{5}{12} + \frac{8}{12} + \frac{10}{4} + \frac{13}{4}.$$

Simplify the fractions:

$$= \frac{2+5+5+8}{12} + \frac{10+13}{4} = \frac{20}{12} + \frac{23}{4} = \frac{5}{3} + \frac{23}{4}.$$

Finding a common denominator:

$$= \frac{20}{12} + \frac{69}{12} = \frac{89}{12}.$$

Thus, $E[X^2 + Y^2] = \frac{89}{12}$.
(c) Are $X$ and $Y$ independent?
Two random variables $X$ and $Y$ are independent if and only if the joint PMF can be factored as the product of their marginal PMFs, i.e.,

$$p(x,y) = p_X(x)p_Y(y) \quad \text{for all } x \text{ and } y.$$

We compare $p(x,y)$ with $p_X(x)p_Y(y)$.
For $(x,y) = (1,1)$:

$$p_X(1)p_Y(1) = \frac{1}{6} \cdot \frac{5}{12} = \frac{5}{72}, \quad p(1,1) = \frac{1}{12}.$$

Clearly, $p_X(1)p_Y(1) \neq p(1,1)$, so $X$ and $Y$ are not independent.

A10: (a)

$$\begin{aligned} f_{XY}(x,y) &= \int_0^1 f_{XYZ}(x,y,z)\,dz \\ &= \begin{cases} \int_0^1 (x+y)\,dz, & \text{for } 0 \le x, y \le 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} x+y, & \text{if } 0 \le x, y \le 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} f_X(x) &= \int_0^1 f_{XY}(x,y)\,dy \\ &= \begin{cases} \int_0^1 (x+y)\,dy, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} x+\frac{1}{2}, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(c) The expected value $E[Y]$ is given by:

$$
\begin{aligned}
E[Y] &= \int_0^1 \int_0^1 \int_0^1 y f_{XYZ}(x,y,z)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^1 \int_0^1 y(x+y)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^1 y(x+y)\,dx\,dy \quad \text{(integrating over } z \text{ gives a factor of 1)} \\
&= \int_0^1 y \left( \int_0^1 (x+y)\,dx \right) dy \\
&= \int_0^1 y \left( \left[ \frac{x^2}{2} + yx \right]_0^1 \right) dy \\
&= \int_0^1 y \left( \frac{1}{2} + y \right) dy \\
&= \left[ \frac{y^2}{4} + \frac{y^3}{3} \right]_0^1 \\
&= \frac{1}{4} + \frac{1}{3} = \frac{7}{12}.
\end{aligned}
$$

A11: (a) To find the constant $c$, we use

$$
\begin{aligned}
1 = \text{total probability} &= \int_2^6 \int_0^5 c(2x+y)\,dy\,dx \\
&= \int_2^6 c\left(2xy + \frac{y^2}{2}\right)\Bigg|_0^5 dx \\
&= \int_2^6 c\left(10x + \frac{25}{2}\right) dx = 210c.
\end{aligned}
$$

So $c = \frac{1}{210}$.

(b) The marginal CDFs for $X$ and $Y$ are given by

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 2 \\ \int_2^x \left( \int_0^5 \frac{2x+y}{210}\,dy \right) dx = \frac{2x^2+5x-18}{84}, & \text{if } 2 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases}$$

$$F_Y(y) = P(Y \le y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(x,y)\,dy\,dx$$

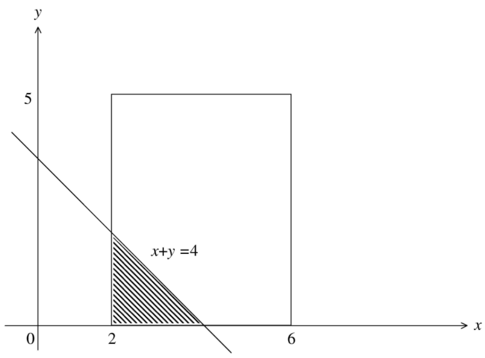
$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \int_2^6 \left( \int_0^y \frac{2x+y}{210}\,dy \right) dx = \frac{y^2+16y}{105}, & \text{if } 0 \le y < 5 \\ 1, & \text{if } y \ge 5 \end{cases}$$

(c) PDF of $X$: $f_X(x) = \frac{d}{dx}(F_X(x)) = \begin{cases} \frac{4x+5}{84}, & \text{if } 2 < x < 6 \\ 0, & \text{otherwise} \end{cases}$

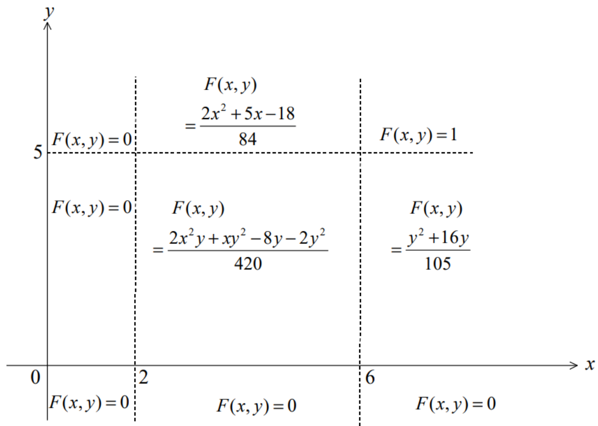
PDF of $Y$: $f_Y(y) = \frac{d}{dy}(F_Y(y)) = \begin{cases} \frac{2y+16}{105}, & \text{if } 0 < y < 5 \\ 0, & \text{otherwise} \end{cases}$

(d)

$$P(X+Y<4) = \frac{1}{210}\int_2^4 \int_0^{4-x} (2x+y)\,dy\,dx = \frac{2}{35}$$




(e) Joint distribution function




Suppose $(x, y)$ is located in $\{(x, y) : x > 6, 0 < y < 5\}$, then

$$F(x, y) = \int_2^6 \int_0^y \frac{2x+y}{210}\,dy\,dx = \frac{y^2+16y}{105}$$

and $f_{XY}(x, y) = \frac{2y+16}{105}$.
Note that for this density $f_{XY}(x, y)$, we have

$$f_{XY}(x, y) \neq f_X(x) \cdot f_Y(y)$$

So $X$ and $Y$ are not independent.

A12: (a) Let $g(X, Y) = X + Y$. Using LOTUS, we have

$$E[X+Y] = \sum_{(x_i, y_j) \in R_{XY}} (x_i + y_j) P_{XY}(x_i, y_j)$$

On splitting the sum

$$= \sum_{(x_i, y_j) \in R_{XY}} x_i P_{XY}(x_i, y_j) + \sum_{(x_i, y_j) \in R_{XY}} y_j P_{XY}(x_i, y_j)$$

Separating the summation

$$= \sum_{x_i \in R_X} x_i \sum_{y_j \in R_Y} P_{XY}(x_i, y_j) + \sum_{y_j \in R_Y} y_j \sum_{x_i \in R_X} P_{XY}(x_i, y_j)$$

$$= \sum_{x_i \in R_X} x_i P_X(x_i) + \sum_{y_j \in R_Y} y_j P_Y(y_j) \quad \text{( by propery of marginal PMF)}$$

$$= E[X] + E[Y].$$

(b) Let $g(X,Y) = f(X) + h(Y)$. Using LOTUS, we have:

$$E[f(X) + h(Y)] = \sum_{(x_i, y_j) \in R_{XY}} (f(x_i) + h(y_j))\, P_{XY}(x_i, y_j)$$

On splitting the sum

$$= \sum_{(x_i, y_j) \in R_{XY}} f(x_i) P_{XY}(x_i, y_j) + \sum_{(x_i, y_j) \in R_{XY}} h(y_j) P_{XY}(x_i, y_j)$$

Separating the summation

$$= \sum_{x_i \in R_X} f(x_i) \sum_{y_j \in R_Y} P_{XY}(x_i, y_j) + \sum_{y_j \in R_Y} h(y_j) \sum_{x_i \in R_X} P_{XY}(x_i, y_j)$$

$$= \sum_{x_i \in R_X} f(x_i) P_X(x_i) + \sum_{y_j \in R_Y} h(y_j) P_Y(y_j) \quad \text{( by propery of marginal PMF)}$$

Finally by LOTUS, this is:

$$= E[f(X)] + E[h(Y)]$$