Probability and Statistics: MA6.101

Midsem Solutions

Section A

Q1: Let X be a continuous random variable with distribution $F_X(.)$ and density $f_X(x)$.Find the probability density and cummulative distribution for Y = X^2 + 4. [Anush]

A: Let $F_Y(y), f_Y(y)$ be the CDF and PDF of Y, and $F_X(x), f_X(x)$ be the CDF and PDF of X, respectively. We have:

1. Cumulative Distribution Function (CDF)

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 + 4 \leq y) \\ &= P(X^2 \leq y - 4) \\ &= P(|X| \leq \sqrt{y-4}) \quad \text{for } y > 4 \\ &= P\left(-\sqrt{y-4} \leq X \leq \sqrt{y-4}\right) \\ &= F_X\left(\sqrt{y-4}\right) - F_X\left(-\sqrt{y-4}\right) \\ &= \int_{-\sqrt{y-4}}^{\sqrt{y-4}} f_X(x)\,dx \end{aligned}$$

$$\therefore F_Y(y) = \begin{cases} \int_{-\sqrt{y-4}}^{\sqrt{y-4}} f_X(x)\,dx, & \text{if } y > 4 \\ 0, & \text{otherwise} \end{cases}$$

2. Probability Density Function (PDF)

Now, to find the PDF $f_Y(y)$, we differentiate the CDF $F_Y(y)$ with respect to y.

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

For $y \leq 4$, we have $F_Y(y) = 0$, so $f_Y(y) = 0$ for $y \leq 4$.

For $y > 4$, we use Leibniz's rule for differentiating an integral with variable limits:

$$f_Y(y) = \frac{d}{dy}\left(\int_{-\sqrt{y-4}}^{\sqrt{y-4}} f_X(x)\,dx\right)$$

Applying Leibniz's rule, we differentiate the integral with respect to y, considering both upper and lower limits:

$$f_Y(y) = f_X\left(\sqrt{y-4}\right) \cdot \frac{d}{dy}\left(\sqrt{y-4}\right) - f_X\left(-\sqrt{y-4}\right) \cdot \frac{d}{dy}\left(-\sqrt{y-4}\right)$$

Now, apply the chain rule to differentiate $\sqrt{y-4}$ and $-\sqrt{y-4}$ with respect to y:

$$\frac{d}{dy}\left(\sqrt{y-4}\right) = \frac{1}{2\sqrt{y-4}}, \quad \frac{d}{dy}\left(-\sqrt{y-4}\right) = -\frac{1}{2\sqrt{y-4}}$$

Substituting these into the expression for $f_Y(y)$, we get:

$$f_Y(y) = \frac{1}{2\sqrt{y-4}} \left(f_X\left(\sqrt{y-4}\right) + f_X\left(-\sqrt{y-4}\right) \right)$$

3. Final Expressions

Thus, the cumulative distribution function $F_Y(y)$ is:

$$F_Y(y) = \begin{cases} 0, & \text{if } y \leq 4 \\ F_X\left(\sqrt{y-4}\right) - F_X\left(-\sqrt{y-4}\right), & \text{if } y > 4 \end{cases}$$

The probability density function $f_Y(y)$ is:

$$f_Y(y) = \begin{cases} 0, & \text{if } y \leq 4 \\ \frac{1}{2\sqrt{y-4}} \left(f_X\left(\sqrt{y-4}\right) + f_X\left(-\sqrt{y-4}\right) \right), & \text{if } y > 4 \end{cases}$$

Grading Criteria

* CDF F_Y in terms of f_X - 3 marks
* PDF f_Y in terms of f_X - 3 marks

Q2: Suppose X and Y are independent and exponential random variables with parameter a and b respectively. Then find $P(X < Y)$. [Ronak]

A: We are given two independent exponential random variables X and Y with parameters a and b, respectively. The goal is to compute $P(X < Y)$.

$$\begin{aligned} f_X(x) &= ae^{-ax}, \quad f_Y(y) = be^{-by}, \quad x, y \geq 0 \\ P(X < Y) &= \int_0^\infty P(X < y \mid Y = y) f_Y(y)\, dy \end{aligned}$$

Since X and Y are independent, we have:

$$\begin{aligned} P(X < y \mid Y = y) &= P(X < y) \\ P(X < y) &= 1 - e^{-ay} \quad \text{(CDF of } X\text{)} \end{aligned}$$

Thus, the expression for $P(X < Y)$ becomes:

$$\begin{aligned} P(X < Y) &= \int_0^\infty (1 - e^{-ay}) be^{-by}\, dy \\ &= \int_0^\infty be^{-by}\, dy - \int_0^\infty be^{-(a+b)y}\, dy \end{aligned}$$

Now, evaluate each integral:

$$\begin{aligned} \int_0^\infty be^{-by}\, dy &= 1 \\ \int_0^\infty be^{-(a+b)y}\, dy &= \frac{b}{a+b} \end{aligned}$$

Therefore, the final probability is:

$$P(X < Y) = 1 - \frac{b}{a+b}$$
$$= \frac{a}{a+b}$$

Alternate Approach:
We can compute $P(X < Y)$ using the expectation of an indicator variable:

$$P(X < Y) = \mathbb{E}[1_{\{X<Y\}}]$$

This expectation is written as a double integral over the joint distribution of X and Y:

$$P(X < Y) = \int_0^\infty \int_0^\infty 1_{\{x<y\}} f_{XY}(x, y)\, dx\, dy$$

Since X and Y are independent, the joint PDF factors as $f_X(x)f_Y(y)$, and the indicator function $1_{\{x<y\}}$ restricts the limits of the inner integral to $0 \leq x < y$. Thus, we have:

$$P(X < Y) = \int_0^\infty \int_0^y ae^{-ax} be^{-by}\, dx\, dy$$

and proceed similarly.

Grading Criteria

* Setting up integral(s) using conditioning and independence - 3 marks
* Solving the integral(s) and final answer - 3 marks

Q3: We are given random variables $X_1, X_2, \ldots, X_n$ with finite variances, and they are not necessarily independent. We need to find the variance of $S_n = \sum_{i=1}^n X_i$ and then check how the expression changes when the X_i's are independent. [Gopal]

A: We want to find the variance of the sum S_n:

$$S_n = \sum_{i=1}^n X_i$$

The variance of a sum of random variables can be written as:

$$\mathrm{Var}(S_n) = \mathrm{Var}\left(\sum_{i=1}^n X_i\right)$$

Expanding the Variance of the Sum

Using the properties of variance, we expand the above expression as follows:

$$\mathrm{Var}\left(\sum X_i\right) = \mathbb{E}\left[\left(\sum X_i\right)^2\right] - \left(\mathbb{E}\left[\sum X_i\right]\right)^2$$

[1 Marks]

$$= \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_i X_j\right] - \left(\sum_{i=1}^{n}\mathbb{E}[X_i]\right)^2$$

[1 Marks]

$$= \mathbb{E}\left[\sum_{i=1}^{n} X_i^2\right] + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathbb{E}[X_i X_j] - \left(\sum_{i=1}^{n}\mathbb{E}[X_i]\right)^2$$

$$= \sum_{i=1}^{n}\mathbb{E}[X_i^2] + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{E}[X_i X_j] - \sum_{i=1}^{n}(\mathbb{E}[X_i])^2 - 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{E}[X_i]\mathbb{E}[X_j]$$

[1 Marks]

$$= \sum_{i=1}^{n}\mathbb{E}[X_i^2] - \sum_{i=1}^{n}(\mathbb{E}[X_i])^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{E}[X_i X_j] - 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{E}[X_i]\mathbb{E}[X_j]$$

[0.5 Marks for reducing terms to Variance part] [0.5 Marks for reducing terms to Cov part]

$$= \sum_{i=1}^{n}\mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]$$

$$= \sum_{i=1}^{n}\mathrm{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathrm{Cov}(X_i, X_j)$$

[0.5 for correct answer]

B). Special case (when X_i's are independent):

Covariance of independent variables is 0. **[0.5 for writing this condition]**

$$\mathrm{Var}(S_n) = \sum_{i=1}^{n}\mathrm{Var}(X_i)$$

Thus, when the random variables are independent, the covariance terms drop out, leaving just the sum of the individual variances.

[1 Marks for getting final expression]

Q4: Let X be a random variable having Binomial distribution with parameters N and p where N is itself a random variable having Poisson distribution with mean λ. Find the probability mass function of the random variable X. Also find $E[X]$.

Let $X \sim \mathcal{B}(N, p)$ and $N \sim \mathrm{Pois}(\lambda)$.

$$
\begin{aligned}
P(X=x) &= \sum_{n=x}^{\infty} P(X=x|N=n)P(N=n) \\
&= \sum_{n=x}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} \lambda^n \frac{e^{-\lambda}}{n!} \\
&= \frac{e^{-\lambda} p^x}{x!} \sum_{n=x}^{\infty} \frac{1}{(n-x)!} (1-p)^{n-x} \lambda^n \\
&= \frac{e^{-\lambda} p^x}{x!} \cdot \frac{\lambda^x}{e^{-\lambda(1-p)}} \sum_{n=x}^{\infty} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^{n-x}}{(n-x)!} \\
&= e^{-\lambda p} \frac{(\lambda p)^x}{x!} \sum_{k=0}^{\infty} e^{-\lambda_1} \frac{\lambda_1^k}{k!}
\end{aligned}
$$

(where $\lambda_1 = \lambda(1-p)$ and $k = n - x$.)

$$
= e^{-\lambda p} \frac{(\lambda p)^x}{x!}
$$

The PMF of X is Poisson with parameter λp. So $E[X] = \lambda p$.

OR

Using the law of iterated expectations, $E[X] = E_N[E_X[X|N]] = E[Np] = pE[N] = \lambda p$.

Marking Scheme Q4 [Shivani]:

(a) 1 mark for statement/usage of probability mass functions for binomial and Poisson distributions (0.5 each).

(b) 2.5 for simplification, 0.5 for final expression for mass function of X.

(c) 1 mark for statement/usage of expected value of Poisson or law of iterated expectations.

(d) 1 mark for final expression for $E[X]$.

Q5: Let X be a standard normal variable (Gaussian with zero mean and unit variance). Let $Z = \sigma X + \mu$. Obtain the pdf and cdf of Z. [Kavin]

Solution 1

To find the CDF:

$$
\begin{aligned}
F_Z(z) &= P(Z \le z) && \text{(1 for formula)} \\
&= P(\sigma X + \mu \le z) \quad \text{(where } X \sim N(0,1)\text{)} && \text{(1 for steps)} \\
&= P\left(X \le \frac{z-\mu}{\sigma}\right) && \text{(1 for expression)} \\
&= \Phi\left(\frac{z-\mu}{\sigma}\right)
\end{aligned}
$$

To find the PDF, we can take the derivative of F_Z:

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) && \text{(1 for formula)} \\
&= \frac{d}{dz}\Phi\left(\frac{z-\mu}{\sigma}\right) \\
&= \frac{1}{\sigma}\Phi'\left(\frac{z-\mu}{\sigma}\right) \quad \text{(chain rule for derivative)} && \text{(1 for steps)} \\
&= \frac{1}{\sigma} f_X\left(\frac{z-\mu}{\sigma}\right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\}. && \text{(1 for expression)}
\end{aligned}
$$

Solution 2

To find the PDF:

$$
\begin{aligned}
& Z = g(X) = \sigma X + \mu \\
\implies & f_Z(z) = \frac{1}{\left|g'(\frac{z-\mu}{\sigma})\right|} f_X(\frac{z-\mu}{\sigma}) && \text{(1 for formula)} \\
\implies & f_Z(z) = \frac{1}{\sigma} f_X(\frac{z-\mu}{\sigma}) && \text{(1 for steps)} \\
\implies & f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\} && \text{(1 for expression)}
\end{aligned}
$$

To find the CDF:

$$
\begin{aligned}
F_Z(z) &= \int_{-\infty}^{z} f_Z(u)\,du && \text{(1 for formula)} \\
\implies F_Z(z) &= \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(u-\mu)^2}{2\sigma^2}\right\} du \\
& && \text{(1 for steps)}
\end{aligned}
$$

Let $t = \dfrac{(u-\mu)}{\sigma}$, then $\sigma dt = du$

$$
\begin{aligned}
\implies F_Z(z) &= \int_{-\infty}^{\frac{z-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2}\,dt && \text{(1 for expression)} \\
\implies F_Z(z) &= \Phi\left(\frac{z-\mu}{\sigma}\right)
\end{aligned}
$$

Section B

Q1: Suppose U_1 and U_2 are independent uniform random variables on the segments $[-1, 1]$ and $[0, 1]$ respectively. Let $Z = U_1 + U_2$. Derive an expression for the pdf and cdf of Z.

A: Let $X = U_1$ and $Y = U_2$. Using convolution,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(z-y)\,dy$$

The integration is performed over y, but the parallel case for x will be similar, except the limits will change during the integration (the result will not change).

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(z-y)\,dy$$

The pdf for $X \sim U[a,b]$ (where $a < b$) is:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

Since the support of Y is $[0,1]$, for values of y outside this range, $f_Y(y) = 0$. Hence, modifying the limits of integration, we get:

$$f_Z(z) = \int_0^1 f_Y(y) \cdot f_X(z-y)\,dy$$

$$= \int_0^1 1 \cdot f_X(z-y)\,dy$$

$$= \int_0^1 f_X(z-y)\,dy$$

We haven't yet substituted the value of $f_X(z-y)$ as $\frac{1}{2}$ because we don't know whether $z-y$ will lie in the range of the random variable X. Hence, to get a non-zero value of $f_X(z-y)$:

$$\boxed{-1 \leq z - y \leq 1}$$

The range of $Z = X + Y$ is:

$$-1 \leq X \leq 1 \quad \text{and} \quad 0 \leq Y \leq 1$$

$$\implies -1 \leq X + Y \leq 2$$

$$\implies -1 \leq Z \leq 2$$

We will now break z into 3 cases:

- **Case 1:** $z \in [-1, 0]$

$$\implies -2 \leq z - y \leq 0$$

But we need $z - y \geq -1$, which gives:

$$\implies y \leq z + 1$$

This will act as the new upper limit of our integration. For $y \geq z+1$, the pdf will take 0 as its value.

$$\begin{aligned} f_Z(z) &= \int_0^{z+1} f_X(z-y)\,dy + \int_{z+1}^1 f_X(z-y)\,dy \\ &= \int_0^{z+1} f_X(z-y)\,dy + 0 \\ &= \int_0^{z+1} \frac{1}{2}\,dy \\ &= \frac{1}{2}\cdot(z+1) \end{aligned}$$

- **Case 2:** $z \in [0,1]$

$$\implies -1 \leq z-y \leq 1$$

This is already within the range of X, so the pdf of X will take a non-zero value for this entire interval.

$$\begin{aligned} f_Z(z) &= \int_0^1 f_X(z-y)\,dy \\ &= \int_0^1 \frac{1}{2}\,dy \\ &= \frac{1}{2} \end{aligned}$$

- **Case 3:** $z \in [1,2]$

$$\implies 0 \leq z-y \leq 2$$

But we need $z-y \leq 1$, which gives:

$$\implies y \geq z-1$$

This will act as the new lower limit of our integration. For $y \leq z-1$, the pdf will take 0 as its value.

$$\begin{aligned} f_Z(z) &= \int_0^{z-1} f_X(z-y)\,dy + \int_{z-1}^1 f_X(z-y)\,dy \\ &= 0 + \int_{z-1}^1 \frac{1}{2}\,dy \\ &= \frac{1}{2}\cdot(2-z) \end{aligned}$$

Combining all three cases, the pdf of Z is:

$$f_Z(z) = \begin{cases} \frac{1}{2}(z+1) & \text{for } z \in [-1,0] \\ \frac{1}{2} & \text{for } z \in [0,1] \\ \frac{1}{2}(2-z) & \text{for } z \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

The cdf $F_Z(z)$ is obtained by integrating the pdf in each case:

$$F_Z(z) = \begin{cases} 0 & \text{for } z < -1 \\ \frac{(z+1)^2}{4} & \text{for } z \in [-1, 0] \\ \frac{1}{2}z + \frac{1}{4} & \text{for } z \in [0, 1] \\ 1 - \frac{(2-z)^2}{4} & \text{for } z \in [1, 2] \\ 1 & \text{for } z > 2 \end{cases}$$

Note:

- For finding cdf, if you plot the graph of pdf and find the area under it for suitable limits, then you can reduce your calculations.
- There are other methods to solve this question too. The overall distribution of correct answer would be same for all the methods, but the partial marking would depend on the coherence of the method used (if different from the one given above).

Marking Scheme Q1 [Aanvik]:

(a) For calculation of pdf, 6 marks will be given.

- 3 marks if no calculation for each case given and only the final expression is given.
- 2 marks for each case solved (1 mark for the limit simplification and 1 mark for the correct answer.)
- 1.5 mark for writing the formulas for convolution and pdf of X and Y correctly. This will only be considered if there is no simplification of any cases mentioned above.

(b) For calculation of cdf, 4 marks will be given.

- 2 marks if no calculation for each case given and only the final expression is given.
- 1 mark combined for mentioning sub case 1 and 5, 1 mark each for the remaining sub cases. 0.5 marks for working and 0.5 marks for correct expression.
- 0.5 marks for writing the formula for cdf in terms of pdf. This will only be considered if there is no simplification of any cases mentioned above.

Note: The answering scheme is as generic as possible, the split for marks and the weightage of each component will remain same across all the methods.

Q2: Let X and Y be jointly continuous random variables with joint probability density function (PDF):

$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We are required to solve the following:

(a) Find $E[X]$ and $E[Y]$.

(b) Are X and Y independent? Justify.

(c) Find $E[Y|X > 2]$.

(d) Find $P(X > Y)$.

A:

(a)

The marginal PDF of X, denoted as $f_X(x)$, is found by integrating the joint PDF $f_{X,Y}(x, y)$ over all values of y:

$$f_X(x) = \int_0^\infty f_{X,Y}(x, y)\, dy$$

Substitute the joint PDF $f_{X,Y}(x, y) = 6e^{-(2x+3y)}$ for $x, y \geq 0$:

$$f_X(x) = \int_0^\infty 6e^{-(2x+3y)}\, dy$$

Evaluate the integral:

$$f_X(x) = 6e^{-2x} \int_0^\infty e^{-3y}\, dy$$

The integral $\int_0^\infty e^{-3y}\, dy = \frac{1}{3}$, so:

$$f_X(x) = 6e^{-2x} \cdot \frac{1}{3} = 2e^{-2x}$$

Thus, the marginal PDF of X is:

$$f_X(x) = 2e^{-2x}, \quad x \geq 0$$

The expectation $E[X]$ is given by:

$$E[X] = \int_0^\infty x f_X(x)\, dx$$

Substitute $f_X(x) = 2e^{-2x}$:

$$E[X] = \int_0^\infty x \cdot 2e^{-2x}\, dx$$

Using integration by parts, we get:

$$E[X] = \frac{1}{2}$$

ALT: The marginal pdf calculated is the PDF of an exponential random variable with rate parameter $\lambda = 2$.

The expectation of an exponential random variable is given by:

$$E[X] = \frac{1}{\lambda}$$

Thus, the expectation of X is:

$$E[X] = \frac{1}{2}$$

Similarly, the marginal PDF of Y, denoted as $f_Y(y)$, is found by integrating the joint PDF $f_{X,Y}(x, y)$ over all values of x:

$$f_Y(y) = \int_0^\infty f_{X,Y}(x, y)\, dx$$

Substitute $f_{X,Y}(x, y) = 6e^{-(2x+3y)}$:

$$f_Y(y) = \int_0^\infty 6e^{-(2x+3y)}\, dx$$

Evaluate the integral:

$$f_Y(y) = 6e^{-3y} \int_0^\infty e^{-2x}\, dx$$

The integral $\int_0^\infty e^{-2x}\, dx = \frac{1}{2}$, so:

$$f_Y(y) = 6e^{-3y} \cdot \frac{1}{2} = 3e^{-3y}$$

Thus, the marginal PDF of Y is:

$$f_Y(y) = 3e^{-3y}, \quad y \geq 0$$

$$E[Y] = \frac{1}{\lambda}$$
$$E[Y] = \frac{1}{3}$$

Ans:

$$E[X] = \frac{1}{2}; E[Y] = \frac{1}{3}$$

(b)

To check if X and Y are independent, we check if the joint PDF can be factored as the product of marginal PDFs:

$$f_X(x) = \int_0^\infty f_{X,Y}(x, y)\, dy = 6e^{-2x} \cdot \frac{1}{3} = 2e^{-2x}$$

$$f_Y(y) = \int_0^\infty f_{X,Y}(x, y)\, dx = 6e^{-3y} \cdot \frac{1}{2} = 3e^{-3y}$$

The joint PDF $f_{X,Y}(x, y) = 6e^{-(2x+3y)}$ can be written as $f_X(x) \cdot f_Y(y) = 2e^{-2x} \cdot 3e^{-3y}$. Thus, X and Y are independent.

(c)

Since X and Y are independent, we have

$$E[Y|X > 2] = E[Y]$$

Ans:

$$E[Y|X > 2] = 1/3$$

(d)

To find $P(X > Y)$, we compute the following double integral:

$$P(X > Y) = \int_0^\infty \int_0^x f_{X,Y}(x, y)\, dy\, dx = \int_0^\infty \int_0^x 6e^{-(2x+3y)}\, dy\, dx$$

The inner integral evaluates to:

$$\int_0^x e^{-3y}\, dy = \frac{1 - e^{-3x}}{3}$$

Thus, the probability is:

$$P(X > Y) = \int_0^\infty 6e^{-2x} \cdot \frac{1 - e^{-3x}}{3}\, dx$$

Splitting the integral:

$$P(X > Y) = 2\int_0^\infty e^{-2x}\, dx - 2\int_0^\infty e^{-5x}\, dx = 1 - \frac{2}{5} = \frac{3}{5}$$

ALT:

We are given that X and Y are independent exponential random variables with parameters $\lambda_X = 2$ and $\lambda_Y = 3$. The probability $P(X > Y)$ can be computed as:

$$P(X > Y) = \int_0^\infty P(X > y \mid Y = y) f_Y(y)\, dy$$

Since $P(X > y \mid Y = y) = P(X > y) = e^{-2y}$ (because X and Y are independent and $X \sim \text{Exp}(2)$) and the PDF of Y is $f_Y(y) = 3e^{-3y}$ (since $Y \sim \text{Exp}(3)$), we have:

$$P(X > Y) = \int_0^\infty e^{-2y} \cdot 3e^{-3y}\, dy$$

Simplifying the integrand:

$$P(X > Y) = 3\int_0^\infty e^{-5y}\, dy$$

The integral of e^{-5y} is:

$$\int_0^\infty e^{-5y}\, dy = \frac{1}{5}$$

Thus, the probability is:

$$P(X > Y) = 3 \times \frac{1}{5} = \frac{3}{5}$$

Marking Scheme Q2 [Divyaraj]

(a) 2 marks for E[X] and 2 marks for E[Y]

- 1 mark for showing integration steps to reach E[X].
- 1 mark for correct answer of E[X].
- The above is also true for E[Y].

(b) For part (b) following Rubric is used:-

- 1 mark for writing that X and Y are independent if the joint PDF can be factored as the product of marginal PDFs.
- 1 mark for showing it for this Joint PDF.

(c) For part (c) following Rubric is used:-

- 1 mark for mentioning the fact that since X and Y are independent,

$$E[Y|X > 2] = E[Y]$$

.

- 1 mark for correctly writing answer.

(d) Following rubric is used if you solved by double integral:-

- 1 mark for writing the expression of double integral of $P(X > Y)$.
- 0.5 mark for solving correctly each of integral of double integration.

Following rubric is used if you solved by conditioning on Y:-

- 1 mark for writing expression of $P(X > Y)$ with conditioning on Y.
- 0.5 mark for mentioning how independence is used to simplify expression of $P(X > Y)$.
- 0.5 mark for solving final answer correctly.