Probability and Statistics: MA6.101

Tutorial 10

Topics Covered: Markov Chains

Q1: Suppose a machine can be either operational or in maintenance, and the machine's status on successive days follows a Markov chain with stationary transition probabilities. Suppose the transition matrix is as follows:

	Operational	Maintenance
Operational	0.7	0.3
Maintenance	0.6	0.4

(a) If the machine is in maintenance on a given day, what is the probability that it will also be in maintenance the next day?

(b) If the machine is operational on a given day, what is the probability that it will remain operational for the next two days?

(c) If the machine is in maintenance on a given day, what is the probability that it will be operational on at least one of the next three days?

Q2: A gambler begins with an initial fortune of i dollars. Each time he plays, he has the possibility of winning 1 dollar with a probability p or losing 1 dollar with a probability $1-p$. The gambler will only stop playing if he either accumulates N dollars or loses all of his money. What is the probability that he will end up with N dollars ?

Q3: Purpose-flea zooms around the vertices of the transition diagram shown below. Let X_t represent Purpose-flea's state at time t (where $t = 0, 1, \ldots$).

(a) Find the transition matrix P.

(b) Find $P(X_2 = 3 \mid X_0 = 1)$.

(c) Suppose that Purpose-flea is equally likely to start on any vertex at time 0. Find the probability distribution of X_1.

(d) Suppose that Purpose-flea begins at vertex 1 at time 0. Find the probability distribution of X_2.

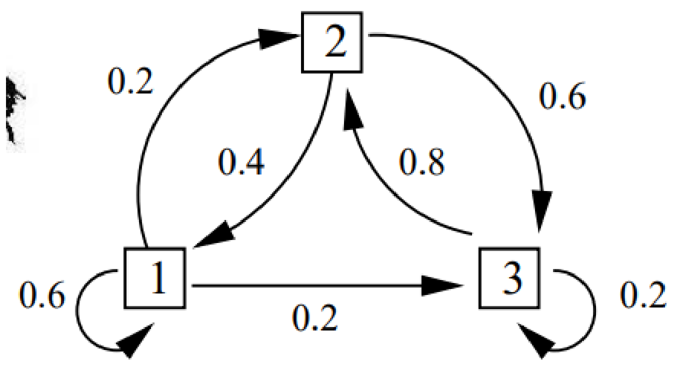

Figure 1: Transition Diagram of Purpose-flea's Movement

Q4: Simulate a markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.9 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \end{bmatrix}$$

and find its limiting distribution.

Q5: A person walks along a straight line and, at each time period, takes a step to the right with probability b and a step to the left with probability $1 - b$. The person starts in one of the positions $1, 2, \ldots, m$, but if they reach position 0 (or position $m + 1$), their step is instantly reflected back to position 1 (or position m, respectively). Equivalently, we may assume that when the person is in positions 1 or m, they will stay in that position with probability $1 - b$ and b, respectively.

(a) Find the transition probability matrix P.

(b) Find the stationary distribution using the formula $\pi = \pi P$.

Q6: Consider a Markovian Coin, $S = \{0, 1\}$. Where 0 denotes Head and 1 denotes Tails. Suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e.,

$$\pi^{(0)} = [P(X_0 = 0) \quad P(X_0 = 1)] = [\alpha \quad 1 - \alpha],$$

where $\alpha \in [0, 1]$.

(a) How does transition matrix define the nature of the coin.

(b) Using induction (or any other method), show that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

(c) Show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

(d) Show that

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

Q7: For the Markovian coin described above:

(a) Calculate the stationary distribution. What do you observe?

(b) Find the mean return times, r_0 and r_1, for this Markov chain. Do you observe anything?

(c) Can you intuitively explain the result above?