Probability and Statistics: MA6.101

Tutorial & Homework 5

Topics Covered: PDF, CDF, Joint random variables

Q1: You are given an box with 50 balls (25 blue and 50 green). You pick balls from the box one by one without replacements. A blue followed by a green or a green followed by a blue is "a color change." Calculate the expected number of color changes if the balls are being picked randomly from the box till the box is empty.

Q2: The probability density function of a random variable X is

$$f(x) = k^2 e^{-|kx|} \quad -\infty < x < \infty$$

Find the value of k. Find the mean and variance of the random variable.

Q3: Two factories produce a certain product, and the amount of defect in each product is modeled by independent random variables, X and Y, following exponential distribution with parameter λ. What is the PDF of the difference between the amount of their defects?

Q4: A diver is about to enter a circular swimming pool with a radius R. When the diver jumps off the diving board, they randomly land anywhere within the pool, assuming they always hit the water. Each point of impact (x, y) on the surface of the pool is equally likely, so that the joint PDF of the random variables X and Y is uniform.

(a) Find and sketch the joint range $R_{X,Y}$.

(b) Write an expression for $f_X(x)$, the marginal PDF for X.

(c) Now, let Z be the Euclidean distance from the center that the dart falls. Write an expression for $\mathbb{E}[Z]$.

(d) Are X and Y are independent?

Q5: Given the joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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compute expectation $\mathbb{E}[X]$ using

(a) joint PDF of X and Y, $f_{X,Y}(x, y)$

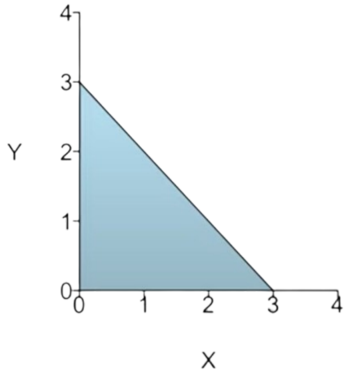
(b) marginal PDF of X, $f_X(x)$

Q6: For $i = 1, 2, 3$, suppose X_i is an exponential random variable with parameter λ_i, and the X_i's are independent. Obtain the probability that X_3 is less than both X_1 and X_2, i.e., $P(X_3 < X_1 \text{ and } X_3 < X_2)$.

Q7: A pair of jointly continuous random variables X and Y have a joint probability density function given by:

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{in the shaded region of the figure,} \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the value of the constant c.
2. Find the marginal pdf of X and Y.



Q8: Given X and Y to be two independent geometric random variables with parameter p. Find the expectation:

$$E\left[\frac{X^2+Y^2}{XY}\right]$$

Q9: Let X and Y be discrete random variables with the joint PMF:

$$p(x,y) = \begin{cases} \frac{1}{12} & \text{for } (x,y) = (1,1),(1,2),(2,1),(2,2) \\ \frac{1}{4} & \text{for } (x,y) = (3,1),(3,2), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal PMF of X and Y.
(b) Using the Law of the Unconscious Statistician (LOTUS), calculate $E[X^2+Y^2]$.
(c) Determine if X and Y are independent.

Q10: Consider three jointly continuous random variables X, Y, and Z, characterized by the following joint probability density function (PDF):

$$f_{XYZ}(x,y,z) = \begin{cases} x+y, & \text{if } 0 \le x,y,z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the joint PDF of X and Y.
(b) Find the marginal PDF of X.
(c) Find the expected value $E[Y]$.

Q11: Suppose the random variables X and Y have the joint density function defined by

$$f_{XY}(x,y) = \begin{cases} c(2x+y), & \text{if } 2 < x < 6,\ 0 < y < 5 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find c.

(b) Find the CDF of X and CDF of Y.

(c) Find the PDF of X and PDF of Y.

(d) Find $P(X+Y<4)$.

(e) Are X and Y independent?

Q12: For two discrete random variables X and Y, show that

(a) $E[X+Y] = E[X] + E[Y]$

(b) $E[f(X)+h(Y)] = E[f(X)] + E[h(Y)]$
$f(X)$ and $h(Y)$ are arbitrary functions of the respective random variables

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