3. Numerical integration

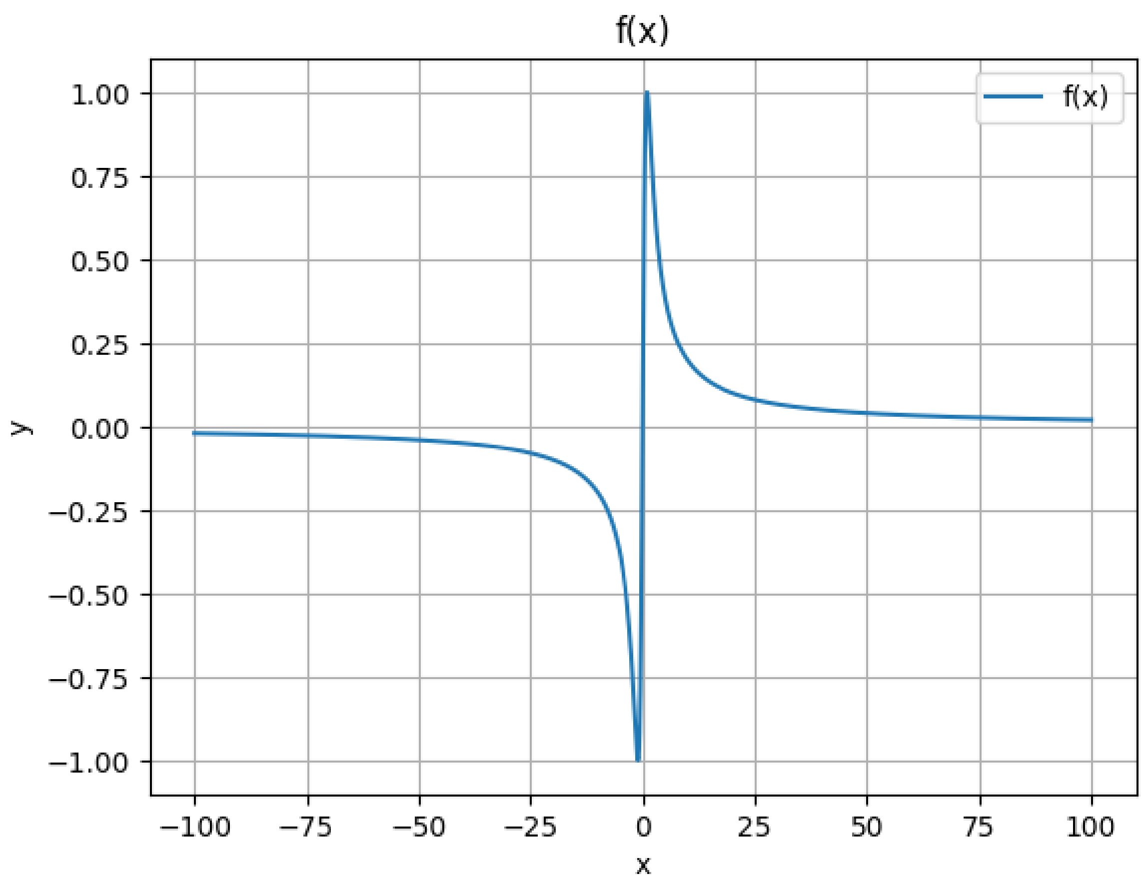
(a)

In [1]:
```
using PyPlot
```

In [2]:
```
f(x)=2*x./(x.^2+1)
```

Out[2]: f (generic function with 1 method)

In [3]:
```
x=linspace(-100.0,100.0,10000)
plot(x,f(x),label="f(x)")
grid("on")
title("f(x)")
xlabel("x")
ylabel("y")
legend()
```



Out[3]: PyObject <matplotlib.legend.Legend object at 0x7fae54030e90>

$I = \int_0^3 f(x)dx$, where $f(x) = \frac{2x}{x^2+1}$

$I = \int_0^3 \frac{2x}{x^2+1}dx = (2x)(x^2+1)^{-1}$

Substitution: $u = x^2 + 1 \wedge \frac{du}{dx} = 2x$

$I = \int_1^{10} u^{-1} \frac{du}{dx} = \left[log(u)\right]_1^{10} = log10 - log1 = log10$

(b)

In [4]:
```
a=0
b=3
```

Out[4]: 3

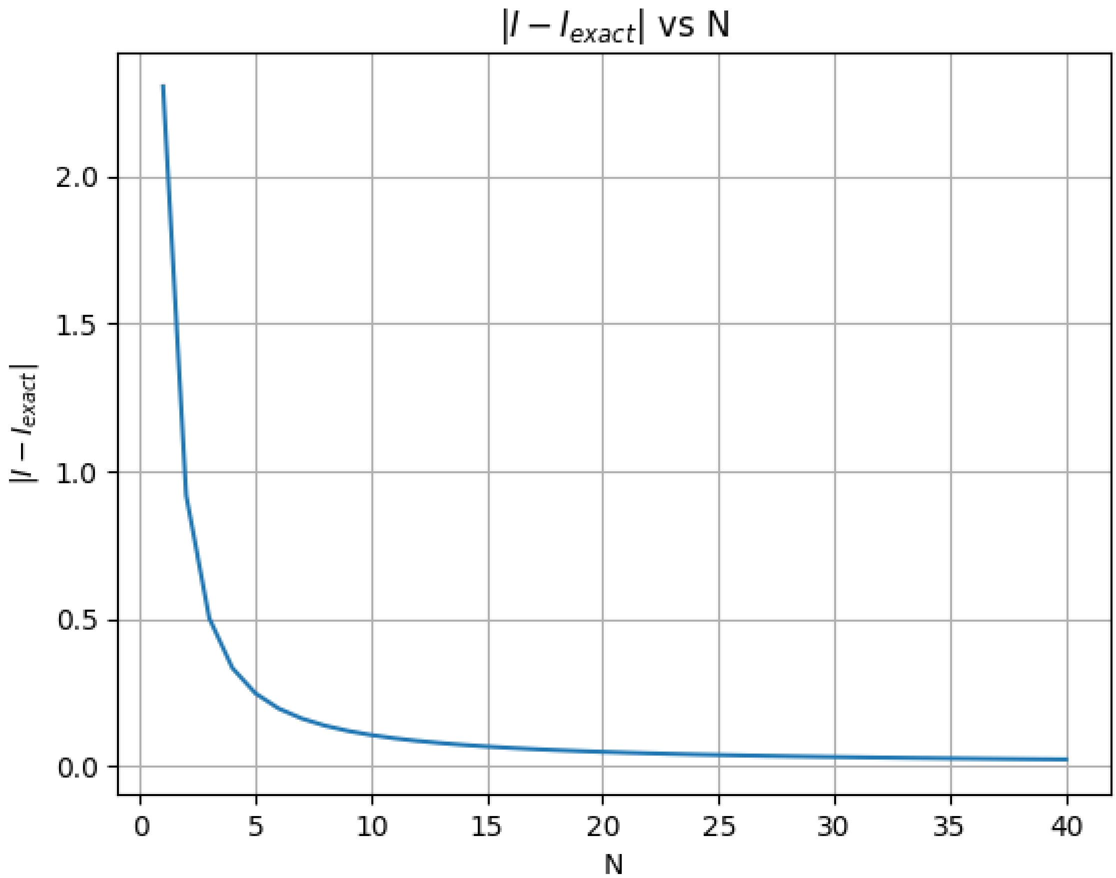
In [5]:
```
function mid(N)
    dx=(b-a)/N
    I=0
    #midpoint algo
    for i in 0:(N-1)
       I+=f(a+2*i/2*dx)
    end
    return(I*dx)
end
```

Out[5]: mid (generic function with 1 method)

In [6]:
```
println("Midpoint results:")
N=[1,5,10,25,50]
for i in 1:length(N)
    println("N= ",N[i],"\t I= ", mid(N[i]))
end
```

```
Midpoint results:
N= 1     I= 0.0
N= 5     I= 2.055045164354843
N= 10    I= 2.1962440125406566
N= 25    I= 2.2639896226127725
N= 50    I= 2.2839368772272812
```

In [7]:
```
x=[]
y=[]
for i in 1:40
    push!(x,i)
    push!(y,abs(log(10)-mid(i)))
end
plot(x,y)
grid("on")
title(L"$|I-I_{exact}|$ vs N")
xlabel("N")
ylabel(L"$|I-I_{exact}|$")
legend()
```




```
/usr/local/lib/python2.7/dist-packages/matplotlib/axes/_axes.py:545: User
Warning: No labelled objects found. Use label='...' kwarg on individual p
lots.
  warnings.warn("No labelled objects found. "
```

(c)

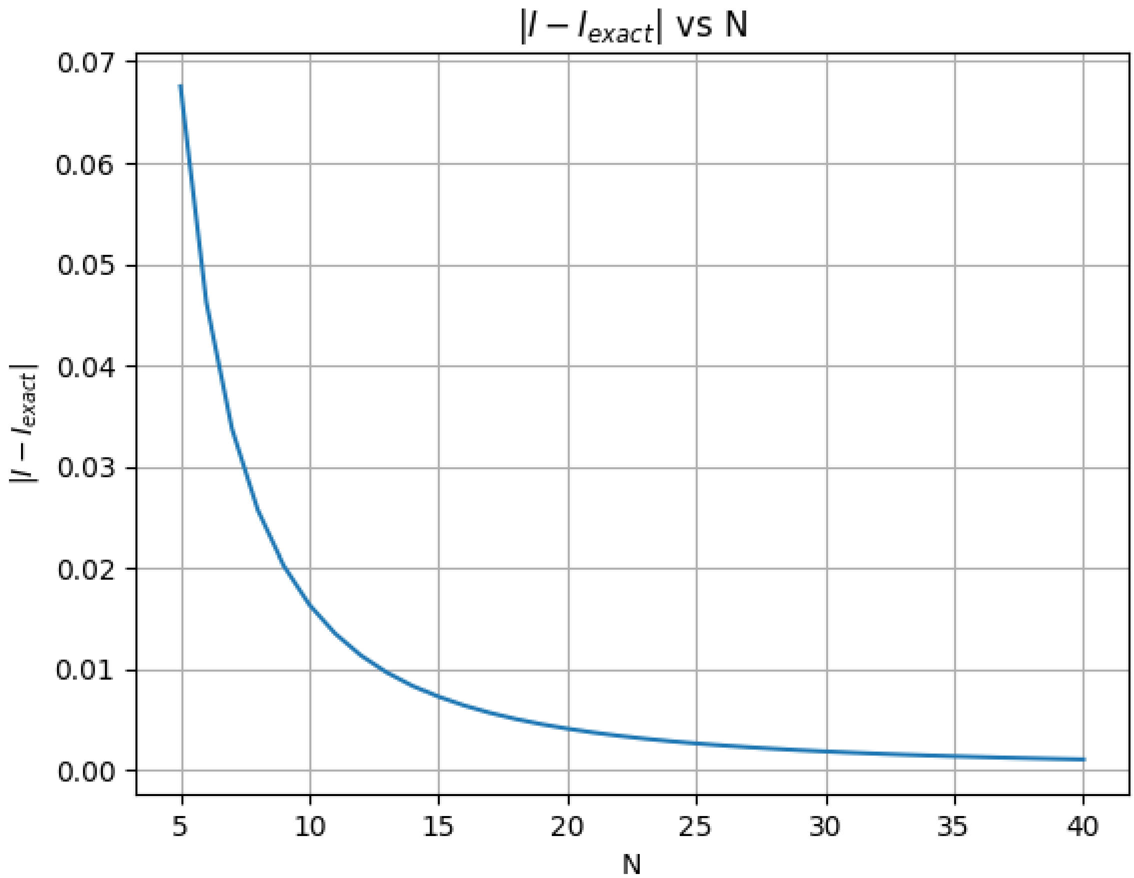
In [8]:
```
function trapezoid(N)
    dx=(b-a)/N
    I=0
    #trapezoid algo
    for i in 1:N
       I+=f(a+(i-1)*dx)+f(a+i*dx)
    end
    return(I*dx/2)
end
```

Out[8]: trapezoid (generic function with 1 method)

In [9]:
```
println("Trapezoid results:")
N=[1,5,10,25,50]
for i in 1:length(N)
    println("N= ",N[i],"\t I= ", trapezoid(N[i]))
end
```

```
Trapezoid results:
N= 1     I= 0.8999999999999999
N= 5     I= 2.2350451643548426
N= 10    I= 2.286244012540656
N= 25    I= 2.299989622612773
N= 50    I= 2.301936877227281
```

In [10]:
```
x=[]
y=[]
for i in 5:40
    push!(x,i)
    push!(y,abs(log(10)-trapezoid(i)))
end
plot(x,y)
grid("on")
title(L"$|I-I_{exact}|$ vs N")
xlabel("N")
ylabel(L"$|I-I_{exact}|$")
legend()
```



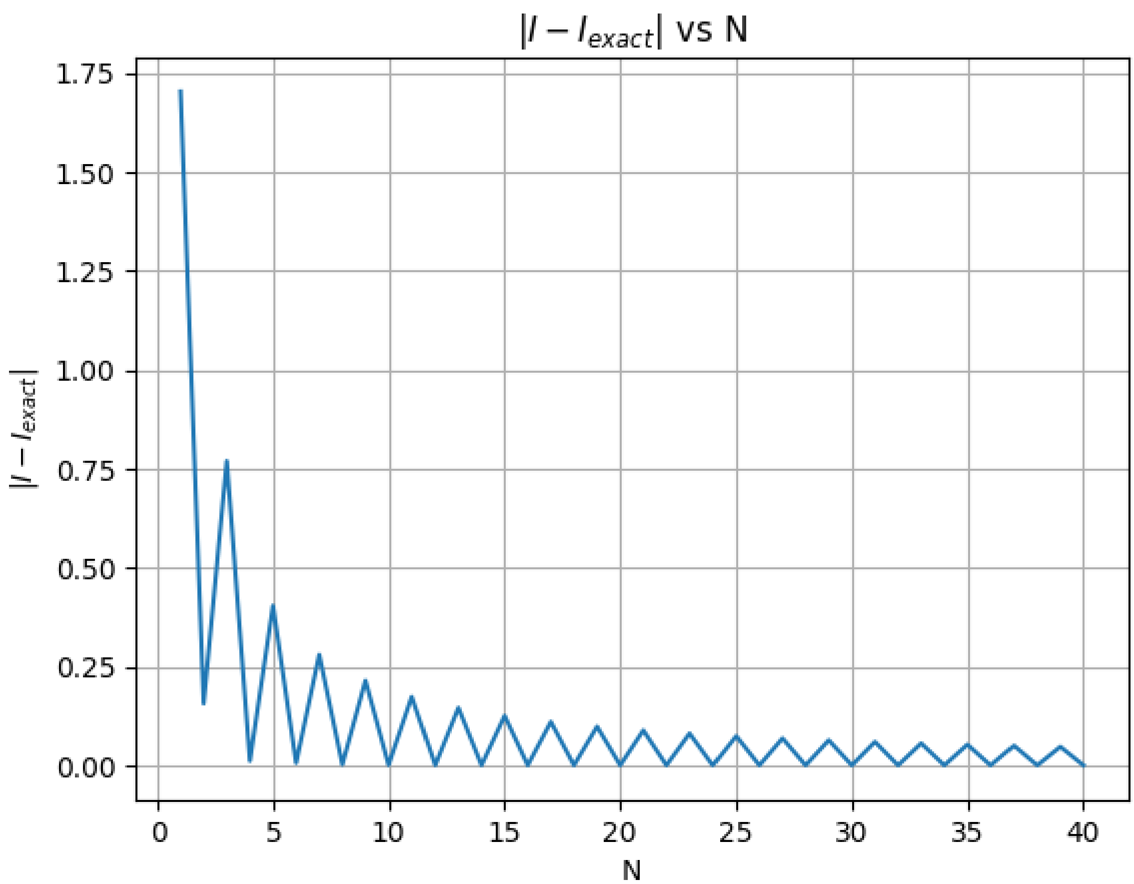
In [11]:
```
function simpson(N)
    dx=(b-a)/N
    #start & end value
    I=f(a)+f(b)
    #pair
    for i in 1:2:N-1
        I+=4*f(a+i*dx)
    end
    #impair
    for i in 2:2:N-2
        I+=2*f(a+i*dx)
    end
    return(I*dx/3)
end
```

Out[11]:
```
simpson (generic function with 1 method)
```

In [12]:
```
println("Simpson results:")
N=[1,5,10,25,50]
for i in 1:length(N)
    println("N= ",N[i],"\t I= ", simpson(N[i]))
end
```

```
Simpson results:
N= 1     I= 0.6
N= 5     I= 1.898570258910864
N= 10    I= 2.3033102952692612
N= 25    I= 2.228637242525305
N= 50    I= 2.302585962098784
```

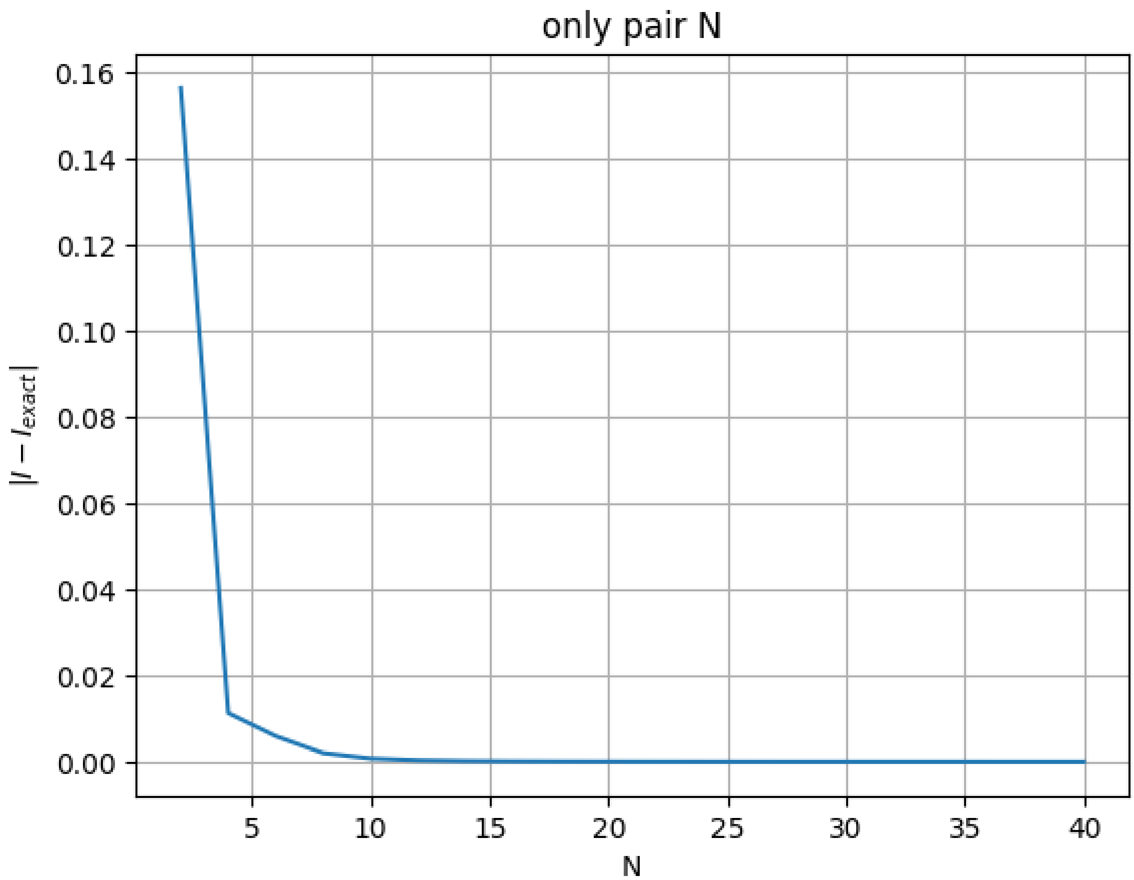
In [13]:
```
x=[]
y=[]
for i in 1:40
    push!(x,i)
    push!(y,abs(log(10)-simpson(i)))
end
plot(x,y)
grid("on")
title(L"$|I-I_{exact}|$ vs N")
xlabel("N")
ylabel(L"$|I-I_{exact}|$")
legend()
```



to work properly, we must take into account only even N-values, therefore see the following plot

In [16]:

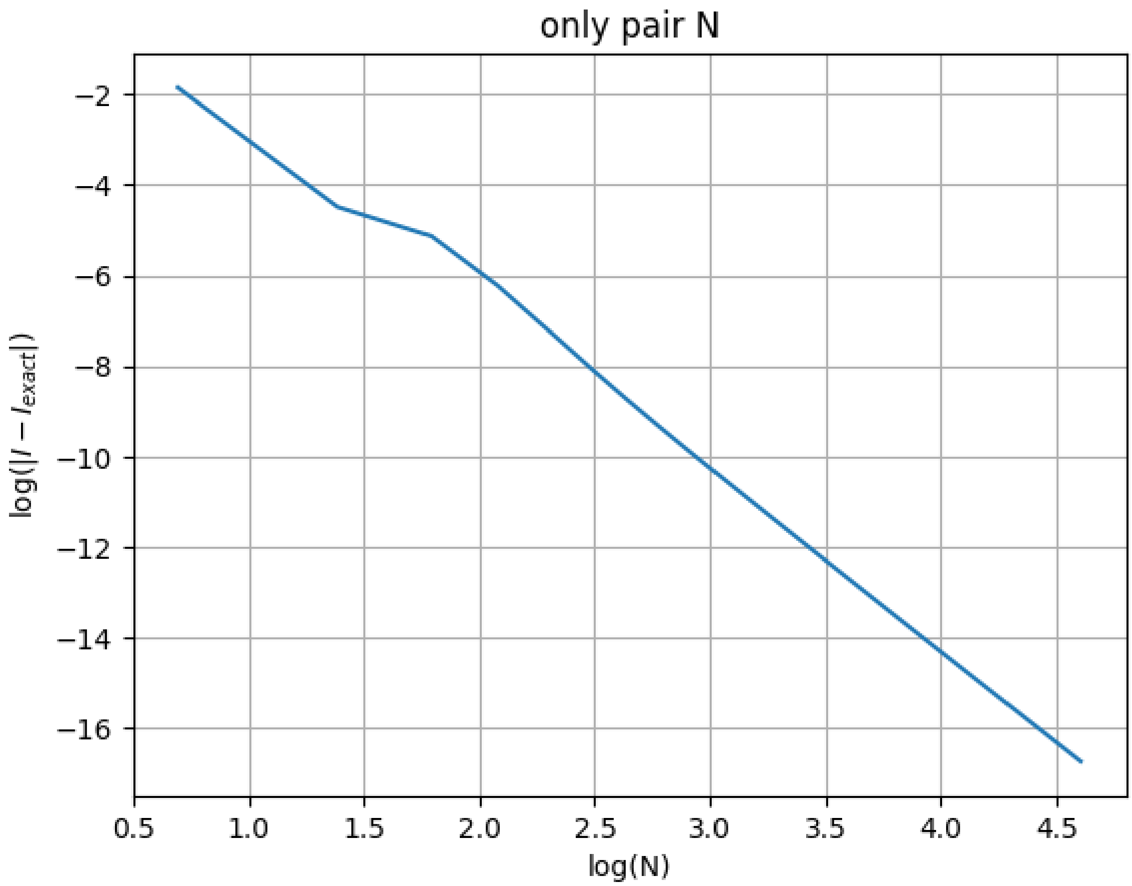
```
#only pairs
x=[]
y=[]
for i in 0:2:40
    push!(x,i)
    push!(y,abs(log(10)-simpson(i)))
end
plot(x,y)
grid("on")
title("only pair N")
xlabel("N")
ylabel(L"$|I-I_{exact}|$")
legend()
```



it makes more sense, to plot log/log

In [15]:

```
#only pairs
x=[]
y=[]
for i in 0:2:100
    push!(x,log(i))
    push!(y,log(abs(log(10)-simpson(i))))
end
plot(x,y)
grid("on")
title("only pair N")
xlabel("log(N)")
ylabel(L"log($|I-I_{exact}|$)")
legend()
```



$$g(x) = ax^3 + bx^2 + cx + d$$

~~$f_1 = f(0), f_2 = f(h), f_3 = f(2h), f_4 = f(3h)$~~

$f_1 = f(0), f_2 = f(h), f_3 = f(2h), f_4 = f(3h)$

$f_1 = f(0) = d$

Numeration

$f_2 = ah^3 + bh^2 + ch + d$ —— (2)

$f_3 = 8ah^3 + 4bh^2 + 2ch + d$ —— (3) $\quad | - 8 \cdot (2)$

$f_4 = 27ah^3 + 9bh^2 + 3ch + d$ —— (4) $\quad | - 27 \cdot (2)$

$f_2 = ah^3 + bh^2 + ch + d$

$f_3 - 8f_2 = -4bh^2 - 6ch - 7d$

$f_4 - 27f_2 = -18bh^2 - 24ch - 26d \quad | \cdot 2 - 9 \cdot (3)$

~~$2f_4 - 54f_2 = 9f_3$~~

$f_2 = ah^3 + bh^2 + ch + d$ (***)

$f_3 - 8f_2 = -4bh^2 - 6ch - 7d$ (**)

$2f_4 - 54f_2 - 9f_3 + 72f_2 = 6ch + 11d$ (*)

insert d

$(*) \to 2f_4 + 18f_2 - 9f_3 = 6ch + 11f_1$

$$\boxed{c = \frac{2f_4 + 18f_2 - 9f_3 - 11f_1}{6h}}$$

c & d

$(**)$

$$f_3 - 8f_2 = -4bh^2 - 2f_4 - 18f_2 + 9f_3 + 11f_1 - 7f_1$$

$$\frac{-8f_3 + 10f_2 - 4f_1 + 2f_4}{4h^2} = -b \to b = \frac{4f_1 - 10f_2 + 8f_3 - 2f_4}{4h^2}$$

$$\boxed{b = \left(f_1 - \frac{5f_2}{2} + 2f_3 - \frac{1}{2}f_4\right)\frac{1}{h^2}}$$

b & c & d

$(***)$

$$f_2 = ah^3 + f_1 - \frac{5}{2}f_2 + 2f_3 - \frac{1}{2}f_4 + \frac{1}{3}f_4 + 3f_2 - \frac{3}{2}f_3 - \frac{11}{6} + f_1$$

$$-\frac{1}{6}f_1 + \frac{1}{2}f_2 - \frac{1}{2}f_3 + \frac{1}{6}f_4 = ah^3$$

$$\boxed{a = \left(-\frac{1}{6}f_1 + \frac{1}{2}f_2 - \frac{1}{2}f_3 + \frac{1}{6}f_4\right)\frac{1}{h^3}}$$

$$\int_0^{3h} g(x) = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_0^{3h} = \boxed{\frac{81h^4}{4}a + 9bh^3 + \frac{9ch^2}{2} + 3hd}$$

insert a, b, c & d

$$= \frac{81}{4}h\left(-\frac{1}{6}f_1 + \frac{1}{2}f_2 - \frac{1}{2}f_3 + \frac{1}{6}f_4\right) + 9h\left(f_1 - \frac{5}{2}f_2 + 2f_3 - \frac{1}{2}f_4\right)$$

$$+ \frac{9h}{2}\left(\frac{2}{6}f_4 + 18f_2 - \frac{9}{6}f_3 - \frac{11}{6}f_1\right) + 3hf_1$$

$$= h\left(-\frac{81}{24}f_1 + \frac{81}{8}f_2 - \frac{81}{8}f_3 + \frac{81}{24}f_4 + 9f_1 - \frac{45}{2}f_2 + 18f_3 - \frac{9}{2}f_4\right.$$

$$\left. + \frac{9}{6}f_4 + \frac{81f_2}{6} - \frac{27f_3}{4} - \frac{33}{4}f_1 + 3f_1\right)$$

$$= h\left(\frac{9}{24}f_1 + \frac{27}{24}f_2 + \frac{9}{8}f_3 + \frac{9}{24}f_4\right) = \frac{3h}{8}\left(f_1 + 3f_2 + 3f_3 + f_4\right)$$

$$= \frac{3h}{8}\left(f(0) + 3f(h) + 3f(2h) + f(3h)\right)$$

$\boxtimes$