SCHOOL OF EARTH AND ATMOSPHERIC SCIENCES

GEORGIA INSTITUTE OF TECHNOLOGY

Sven Simon
Jacob J. Buffo

EARTH SYSTEM MODELING (EAS 4610/6310) FALL 2017

Problem Sheet # 2 Return date: Thursday, 07 September (before 09:30 am)

---

3. **Numerical integration** **(10 points)**
In this problem we will apply the methods introduced in class to calculate numerical approximations of definite integrals. As an example, we consider the integral

$$I = \int_0^3 f(x)\,\mathrm{d}x \qquad , \quad \text{where} \qquad f(x) = \frac{2x}{x^2+1} \qquad .$$

(a) Generate a plot of the function $f(x)$. Show that the *exact* value of the integral is given by $I_{exact} = \ln(10)$.

(b) Write a Matlab script that implements the *Midpoint Rule* for numerical integration of the function $f(x)$. The script should be designed to ask the user for the lower bound a of the integral, the upper bound b, and the number N of subintervals into which $[a;b]$ will be divided. Use this script to numerically evaluate the above integral I for $N = 1, 5, 10, 25, 50$. For each N, give the numerical value of the integral. Plot the difference $|I - I_{exact}|$ as a function of N.

(c) In a similar way, write a script that implements the *Trapezoid Rule*. Again, the script should ask the user for the lower bound a of the integral, the upper bound b, and the number N of subintervals into which $[a;b]$ will be divided. Use this script to numerically evaluate the integral I for $N = 1, 5, 10, 25, 50$. For each N, give the numerical value of the integral. Plot the difference $|I - I_{exact}|$ as a function of N.

(d) We again divide the interval $[a;b]$ into N subintervals. We now would like to approximate $f(x)$ in *each of these N subintervals* with a parabola of second order and apply *Simpson's Rule* to evaluate the integral $\int_a^b f(x)\,\mathrm{d}x$. Write a Matlab script that performs numerical integration with Simpson's Rule and meets the same requirements as outlined in problems (b) and (c). Please note that you will also need to take into account the *midpoint* of each subinterval. Use this script to numerically evaluate the integral I for $N = 1, 5, 10, 25, 50$. For each N, give the numerical value of the integral. Plot the difference $|I - I_{exact}|$ as a function of N.

(e) *(graduate students only, + 5 points)*
In class we derived Simpson's rule by approximating the integrand of $\int_0^{2h} f(x)\,\mathrm{d}x$ with a parabola of second order. We now would like to improve this method by approximating $f(x)$ with a parabola of *third* order, i.e.,

$$g(x) = ax^3 + bx^2 + cx + d \qquad ,$$

with four constants a, b, c, d. Consider an integral of the type $J = \int_0^{3h} f(x)\,\mathrm{d}x$ and determine the constants a, b, c, d such that $f(x) = g(x)$ is fulfilled at $x = 0, h, 2h, 3h$. Show that this leads to the following approximation for the integral:

$$J \approx \frac{3h}{8}\left[f(0) + 3f(h) + 3f(2h) + f(3h)\right] \qquad .$$