SCHOOL OF EARTH AND ATMOSPHERIC SCIENCES

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EARTH SYSTEM MODELING (EAS 4610/6310) FALL 2017

Problem Sheet # 4 Return date: Thursday, 21 September (before 09:30 am)

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5. **Newton's law of cooling** **(10 points)**
The rate of temperature loss $\mathrm{d}T(t)/\mathrm{d}t$ of a body is proportional to the difference in temperatures between the body and its environment:

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -\kappa \left[T(t) - T_e\right] \qquad . \tag{1}$$

The heat transfer coefficient κ and the temperature T_e of the environment are constants.

(a) Show that the analytical solution of equation (1) reads $T(t) = T_e + [T(0) - T_e] \exp(-\kappa t)$, where $T(0) > T_e$ is the temperature of the body at the beginning of the cooling process ($t = 0$).

(b) Give the marching equation $T(m\Delta t) = \ldots$ for the Euler-Forward scheme to solve equation (1). One way to obtain this equation is by expressing (1) in terms of the new variable $u(t) = T(t) - T_e$ and applying the results from the lecture.

(c) Determine the stability criterion of the Euler-Forward scheme for equation (1).

(d) Write a script that solves equation (1) by using the Euler-Forward method. Select a set of parameters $(\kappa, \Delta t)$ that guarantees a numerically stable solution (see part (c)). Compute the numerical solution for $T(0) = 80$ and $T_e = 15$ until the temperature $T(t)$ is smaller than $T_e + 0.1$. Generate a plot that shows both, the numerical and the analytical solution.

(e) Give the marching equation $T(m\Delta t) = \ldots$ for the Euler-Backward scheme to solve equation (1). Write a script that solves equation (1) with the Euler-Backward scheme for the same parameters as in part (d). Plot both, the numerical and the analytical solution.

(f) Calculate the first-order Taylor polynomial $P_1(t)$ and the second-order Taylor polynomial $P_2(t)$ of the analytical solution at point $t = 0$. Include these two functions in your plots from parts (d) and (e). Discuss the results.

6. **General stability condition** **(graduate students only, + 5 points)**
We consider a differential equation of the type

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = f\left(N(t)\right) \qquad , \tag{2}$$

where $f(N(t))$ is an *arbitrary* function of $N(t)$.

(a) Show that in this general case the stability criterion for the Euler-Forward scheme reads

$$\left|1 + \Delta t \frac{\mathrm{d}f}{\mathrm{d}N}\right| < 1 \qquad .$$

To do this, impose a *weak* perturbation $N(t) \to N(t) + \delta N(t)$ (where $\delta N(t)$ is small) on the "ideal" numerical solution $N(t)$ and study the evolution of $\delta N(t)$ in time.

(b) Evaluate condition (2) for $f(N) = N^2$. Make a sketch of $f(N)$ that illustrates where the Euler-Forward scheme is stable/unstable in this example.