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Analytical solutions of optimal portfolio rebalancing

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We study optimal portfolio rebalancing in a mean-variance type framework and present new analytical results for the general case of multiple risky assets. We first derive the equation of the no-trade region, and then provide analytical solutions and conditions for the optimal portfolio under several simplifying yet important models of asset covariance matrix: uncorrelated returns, same non-zero pairwise correlation, and a one-factor model. In some cases, the analytical conditions involve one or two parameters whose values are determined by combinatorial, rather than numerical, algorithms. Our results provide useful and interesting insights on portfolio rebalancing, and sharpen our understanding of the optimal portfolio.

Keywords: Optimal portfolio rebalancing; Portfolio optimization; Analytical solutions; no-trade region

JEL Classification: G11, C61

1. Introduction

Portfolio rebalancing is an important part of the portfolio management process. Typically, the investor has a fixed target asset allocation that she seeks to maintain (e.g. a strategic asset allocation policy of a pension plan). Over time, the portfolio's actual asset allocation deviates from the target allocation due to performance differences among the assets or cash flows. This deviation introduces tracking error against the target allocation, and the investor needs to reallocate the portfolio to mitigate this tracking error. In practice, ad-hoc heuristics such as calendar-based periodic rebalancing or setting fixed bands around the target allocation are often used for portfolio rebalancing. However, these heuristics tend to be too mechanical and simplistic. For example, rebalancing only on month ends does not control allocation deviations resulting from intra-month market movements and may bear unnecessary tracking error. Using fixed bands is also unlikely to be optimal because these bands disregard changes in asset volatilities and correlations. For example, during periods of low asset volatilities the bands could be made wider to reduce transaction costs while keeping tracking error the same. Therefore, it is beneficial to model the problem in an analytical framework to determine optimal rebalancing.

One such framework is the mean-variance analysis originated by Markowitz (1952), which seeks the optimal trade-off between portfolio return and risk.† It can naturally be adapted to model portfolio rebalancing because here the goal is similar: finding the optimal trade-off between the transaction costs paid and the tracking error relative to the target. So, for example, if the portfolio deviates enough from the target then it makes sense to pay transaction costs to reduce the tracking error, but it is also not optimal to rebalance all the way back to target because as the portfolio gets really close to the target, the marginal reduction in tracking error becomes too small to justify the marginal transaction costs paid.

In fact, adapting mean-variance analysis for portfolio rebalancing has been studied before (see, e.g. Donohue and Yip 2003, Masters 2003, Dybvig 2005, Mei, DeMiguel and Nogales 2016). However, except for the trivial case of a single risky asset, none of the previous studies has provided any analytical solutions of the optimal portfolio with multiple assets. This is because transaction costs are harder to handle analytically than returns in standard mean-variance analysis. While returns depend linearly on asset weights, transaction costs do not: either increasing or decreasing each asset weight from the current portfolio incurs positive transaction costs. Because of this non-linearity, to the best of our knowledge, there does

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† Risk is variance times a risk aversion parameter.

not exist any analytical solution for portfolio rebalancing with multiple assets in general, even absent any constraints. The main innovation of this paper is to show that analytical solutions do exist when the asset covariance matrix fits into some simplifying yet important financial models of asset returns, such as the one-factor model in CAPM.

Why are analytical solutions and proofs important if numerical procedures are available to find the solutions anyway? First, analytical solutions and proofs can bring useful and interesting insights. For example, when asset returns are uncorrelated and cash is available it is never optimal to rebalance any asset away from its target. This follows easily from the analytical solution derived in this paper. On the other hand, when the asset returns are correlated or cash is not available, in some situations it is optimal to rebalance some assets further away from targets. With each asset covariance model considered in this paper, the analytical solutions and proofs make it very clear why an asset is bought, sold, or not traded in the optimal portfolio. Second, analytical solutions provide a very clear relationship between inputs and outputs. For example, they reveal how the optimal portfolio weights change with varying input parameter values such as volatilities, correlations, transaction costs, investor's Kelly, and betas to the common factor.

It is worth noting that portfolio rebalancing has also been studied in the continuous time model (Leland 1999). Introduced by Merton (1971), the continuous time model has been used in the finance literature to study the closely related problem of optimal consumption and investment with transaction costs (see, e.g. Davis and Norman 1990, Dumas Luciano 1991, Liu 2004). In this paper, we focus on the one-period mean-variance model.

Our contributions. In this paper, we make several contributions to the existing literature on portfolio rebalancing. Our first contribution is to tackle the problem using tools from optimization theory, and derive the analytical equation of the no-trade region for the general case of multiple risky assets. As established in previous research, the no-trade region is the set of initial portfolios from which it is optimal not to trade at all. If the current portfolio is outside the no-trade region, then it is optimal to trade back to the boundary of the non-trade region, not back to the target. Our equation of the no-trade region is similar to the result of Mei, DeMiguel and Nogales (2016), but the model details and proof methods are different.

We then develop a framework to analyze the optimal portfolio, and apply this framework to derive analytical solutions and conditions of the optimal portfolio when asset returns: (1) are uncorrelated, or (2) have the same non-zero pairwise correlation, or (3) follow a one-factor covariance model.

Our second contribution, therefore, is to solve the optimal portfolio analytically when asset returns are uncorrelated and cash is available. While this sounds restrictive, the case of uncorrelated asset returns is of practical interests. For example, in equity factor investing it is common to orthogonalize the factors against each other to make them uncorrelated before allocating among them. Also, funds of hedge funds often seek to allocate to hedge fund strategies that are at least approximately uncorrelated. When cash is not available, there is a parameter in the analytical solution that is determined by a combinatorial (not numerical) algorithm. The algorithm reduces the number of candidate optimal portfolios to a linear function of the number of assets so that they can all be checked efficiently.

Our third contribution is to derive analytical conditions on the optimal portfolio when all assets have the same non-zero pairwise correlation. Such a model has been used to estimate future stock correlations from historical data (see the 'Overall Mean' model in Elton and Gruber 1973). In that paper, it is found that in the case of 5-year estimates the same-pairwise-correlation model outperforms other estimation techniques, and the authors wrote ' ... since these models have a particularly simple structure, their good relative performance suggests that simplified portfolio algorithms might be possible ... '. This is indeed the case with portfolio rebalancing. While the same-pairwise-correlation model is a natural generalization of the uncorrelated case, the mathematical derivation becomes a lot more complicated.

Our final contribution is to derive analytical conditions on the optimal portfolio when the asset returns follow a one-factor covariance model. The one-factor model is a common simplifying assumption in portfolio theory, with the most prominent example being the CAPM in which stocks move together only because of their common exposures to the market factor. The one-factor model is also a generalization of the uncorrelated case, because if each asset has zero beta to the market then all the idiosyncratic asset returns are uncorrelated. The mathematical derivation is also complicated with the one-factor model.

The rest of the paper is organized as follows: Section 2 first presents the analytical framework used for modeling portfolio rebalancing and analyzing the optimal portfolio, and then derives the equation of the no-trade region. Section 3 applies this framework to uncorrelated assets. Section 4 applies the framework to assets with the same non-zero pairwise correlation. Section 5 applies the framework to assets that follow a one-factor model. All sections consider both cases of with and without cash. Section 6 provides some concluding remarks.

2. An analytical approach to portfolio rebalancing

Given a set of n risky assets, we assume that the investor wants to hold them according to a set of target portfolio weights. We also assume that the transaction costs associated with trading each asset is proportional to the dollar amount traded on that asset. To define the portfolio rebalancing problem mathematically, we first define the following variables (here $i = 1, 2, \ldots n$).

x_T	the n by 1 vector of target portfolio weights
$x_{T,i}$	the weight of asset i in the target portfolio (i.e. the ith element of x_T)
x_C	the n by 1 vector of current portfolio weights
$x_{C,i}$	the weight of asset i in the current portfolio (i.e. the ith element of x_C)
tc	the n by 1 vector of proportional transaction costs
tc_i	the proportional costs of trading asset i (i.e. the ith element of tc)
V	the n by n covariance matrix of all risky assets

σ_i^2 variance of asset i(i.e. the ith diagonal element of V)
k the investor's risk aversion towards tracking error to the target portfolio

Note that $k > 0$ and tc_i is positive for each i. If there were no transaction costs then the investor would always rebalance back to the target portfolio. In the real world there are transaction costs, and the investor needs to balance the competing goals of staying close to the target portfolio and not paying excessive transaction costs. Let's say the investor rebalances to a new portfolio x, then the tracking error to the target portfolio is the square root of $(x - x_T)^{\mathrm{T}} V (x - x_T)$, and the transaction costs paid is $|x - x_C|^{\mathrm{T}} \cdot tc$. Here $|x - x_C|$ denotes the elementwise absolute value of $x - x_C$. One way to model the tradeoff between these two terms is to minimize the following objective function:†

$$\min \frac{1}{2k}(x - x_T)^{\mathrm{T}} V (x - x_T) + |x - x_C|^{\mathrm{T}} \cdot tc. \tag{1}$$

This formulation is similar to the Markowitz mean-variance problem, in which the risk aversion parameter k is used to quantify the investor's willingness to trade off risk against expected return. However, deriving analytical solutions for (1) in the general case is a lot more difficult than mean-variance, because the absolute value function is not differentiable at the origin. A standard way to handle this problem is to introduce new variables to replace the absolute value terms and add new constraints (see, e.g. Fabozzi and Markowitz 2011). So we use a n by 1 vector y to replace $|x - x_C|$ and add 2 inequality constraints for each y_i, the ith element of y:

$$y_i \geq x_i - x_{C,i}, \tag{2}$$

$$y_i \geq x_{C,i} - x_i, \tag{3}$$

where x_i is the ith element of x, and the new objective function is:

$$\min \frac{1}{2k}(x - x_T)^{\mathrm{T}} V (x - x_T) + y^{\mathrm{T}} \cdot tc. \tag{4}$$

For this technique to work, equation (4) needs to solve to the same optimal portfolio as (1). Fortunately, this is the case because (4) is a minimization problem and all elements of tc are positive. This means that in the optimal portfolio each y_i is set to the minimum value possible, subject to constraints (2) and (3). This in turn means y_i equals the maximum of $x_i - x_{C,i}$ and $x_{C,i} - x_i$, which is $|x_i - x_{C,i}|$. Therefore, the optimal portfolio of (4) is also the optimal portfolio of (1).

Depending on whether the risk-free asset (i.e. cash) is available or not, there are two cases. When cash is not available to leverage or deleverage the portfolio, the 'fully invested' constraint $\sum_{i=1}^{n} x_i = 1$ applies. In that case, it is assumed that $\sum_{i=1}^{n} x_{C,i} = 1$ and $\sum_{i=1}^{n} x_{T,i} = 1$.

The method of Lagrange multipliers is the standard way to tackle constrained optimization problems (see, e.g. Boyd and Vandenberghe 2004 Chapter 5). In this case, the augmented objective function is:

$$\begin{aligned} L = {} & \frac{1}{2k}(x - x_T)^{\mathrm{T}} V (x - x_T) + y^{\mathrm{T}} \cdot tc \\ & + \sum_{i=1}^{n} \lambda_i (x_i - y_i - x_{C,i}) + \sum_{i=1}^{n} \bar{\lambda}_i (-x_i - y_i + x_{C,i}) \\ & + \lambda_0 \left(1 - \sum\nolimits_{i=1}^{n} x_i\right). \end{aligned}$$

Here λ_i is the Lagrange multiplier associated with constraint (2), $\bar{\lambda}_i$ is the Lagrange multiplier associated with constraint (3), and λ_0 is the Lagrange multiplier associated with the fully invested constraint.‡ Applying the Karush–Kuhn–Tucker (KKT) conditions and using λ (resp. $\bar{\lambda}$) to denote the n by 1 vector of all λ_i (resp. $\bar{\lambda}_i$), we get the following set of equations as optimality conditions:

$$\frac{\partial L}{\partial x} = \frac{1}{k} V (x - x_T) + \lambda - \bar{\lambda} - \lambda_0 I = 0, \tag{5}$$

$$\frac{\partial L}{\partial y_i} = tc_i - (\lambda_i + \bar{\lambda}_i) = 0, \tag{6}$$

$$\lambda_i (x_i - y_i - x_{C,i}) = 0, \tag{7}$$

$$\bar{\lambda}_i (-x_i - y_i + x_{C,i}) = 0, \tag{8}$$

$$\lambda_i \geq 0, \quad \bar{\lambda}_i \geq 0, \tag{9}$$

$$\frac{\partial L}{\partial \lambda_0} = 1 - \sum_{i=1}^{n} x_i = 0. \tag{10}$$

Here I is the n by 1 vector of ones. Note that equations (6) through (9) are repeated for each i, while (5) is a vector equation. Equations (7) and (8) are called 'complementary slackness' and (9) is called 'dual feasibility' in the KKT conditions (see Boyd and Vandenberghe 2004 Chapter 5).

The complementary slackness conditions mean that, if constraint (2) (resp. (3)) is a strict inequality then λ_i (resp. $\bar{\lambda}_i$) must be 0. On the other hand, by (6) $\lambda_i + \bar{\lambda}_i = tc_i > 0$ so λ_i and $\bar{\lambda}_i$ cannot both be 0. This means that for each i, at least one of the two constraints (2) and (3) is an equality.§ Therefore, there are three cases according to whether (2) and/or (3) are equalities.

(1) Both are equalities: $y_i = x_i - x_{C,i}$ and $y_i = -x_i + x_{C,i}$ which implies $x_i = x_{C,i}$. There is no trading on asset i in the solution and it is a 'no-trade' asset. In this case we will need to solve for λ_i and $\bar{\lambda}_i$, and check that dual feasibility (9) holds.

(2) Constraint (2) is an equality and (3) is a strict inequality: $y_i = x_i - x_{C,i}$ and $y_i > x_{C,i} - x_i$ and so $x_i > x_{C,i}$. The weight of asset i increases and it is a 'buy' asset. Note that by (8) and (6) we have $\bar{\lambda}_i = 0$ and $\lambda_i = tc_i$.

† Another frequently used objective function is $(x - x_T)^{\mathrm{T}} V (x - x_T) + \tau |x - x_C|^{\mathrm{T}} \cdot tc$, where τ is the transaction costs multiplier. It is easy to see that this is essentially the same objective function as (1). It is also possible to include a term for expected return but that doesn't change the essence of the problem.

‡ Throughout this paper all derivations are done for the no-cash case (i.e. with the fully invested constraint). The other case follows easily by removing the term for the fully invested constraint.

§ This is another way to see why y_i equals the maximum of $x_i - x_{C,i}$ and $x_C, i - x_i$.

(3) Constraint (2) is a strict inequality and (3) is an equality: $y_i > x_i - x_{C,i}$ and $y_i = x_{C,i} - x_i$ and so $x_i < x_{C,i}$. The weight of asset i decreases and it is a 'sell' asset. Note that by (7) and (6) we have $\lambda_i = 0$ and $\bar{\lambda}_i = tc_i$.

Let $\boldsymbol{N}$, $\boldsymbol{B}$, and $\boldsymbol{S}$ denote, respectively, the set of no-trade, buy and sell assets in the optimal portfolio. Now we are ready to derive the equation of the no-trade region and analyze the optimal portfolio.

2.1. Equation of the no-trade region

Here is the observation needed to derive the equation of the no-trade region: this region is the set of current portfolio weights x_C for which the optimal portfolio of the rebalancing problem is also x_C. In other words, the no-trade set $\boldsymbol{N}$ includes all assets, and $\boldsymbol{B}$ and $\boldsymbol{S}$ are empty sets. It is also not necessary to include the $\lambda_0 I$ term in (5) in the derivation, because x_C already satisfies the fully invested constraint. Rewriting (6) as a vector equation $\bar{\lambda} = tc - \lambda$ and plugging it into (5) we have $(1/k)V(x - x_T) + 2\lambda - tc = 0$, or, $\lambda = -(1/2k)V(x - x_T) + (1/2)tc$. Applying the dual feasibility conditions (9) we have:

$$0 \leq \lambda = -\frac{1}{2k}V(x - x_T) + \frac{1}{2}tc. \tag{11}$$

$$0 \leq \bar{\lambda} = tc - \lambda = \frac{1}{2k}V(x - x_T) + \frac{1}{2}tc. \tag{12}$$

Combining (11) and (12) and rearranging, we get the following equation of the no-trade region when cash is available†:

$$\frac{1}{k}|V(x - x_T)| \leq tc. \tag{13}$$

Note that both sides of equation (13) are n by 1 vectors with the absolute value function on the left side being elementwise, and $\leq$ denotes elementwise less-than-or-equal-to. The left side measures the deviations from the target portfolio, transformed by the covariance matrix and scaled by risk aversion. When every transformed deviation is smaller than or equal to the corresponding transaction cost, no rebalancing is needed.

Geometrically, equation (13) defines a multi-dimensional parallelogram centered at the target portfolio x_T. When the assets are uncorrelated, V is diagonal and the no-trade region is a n dimensional rectangular box

$$\left[x_{T,1} - \frac{k \cdot tc_1}{\sigma_1^2}, x_{T,1} + \frac{k \cdot tc_1}{\sigma_1^2}\right] \times \cdots .. \times \left[x_{T,n} - \frac{k \cdot tc_n}{\sigma_n^2}, x_{T,n} + \frac{k \cdot tc_n}{\sigma_n^2}\right].$$

Equation (13) reveals how the different parameters affect the shape and size of the no-trade region:

(1) The no-trade region expands as the transaction costs increase. This is intuitive: the larger the transaction costs, the greater the hurdle the investor has to overcome in order to rebalance.
(2) The no-trade region shrinks as the risk aversion parameter k decreases. This is intuitive: the smaller the k, the more risk averse the investor is towards tracking error to the target portfolio, and more willing to rebalance.
(3) The no-trade region shrinks as asset volatilities increase. This is intuitive: all else being equal, the higher the asset volatilities, the higher the tracking error to the target portfolio, and so the investor is more willing to rebalance.
(4) The dependence of the no-trade region on asset correlations is more complicated: both its size and its shape are impacted by correlations, with more noticeable impact on its shape. Here we use the case of $n = 2$ (and with cash) to illustrate this point. When the two assets are uncorrelated, the no-trade region is a rectangle; when the assets are positively correlated, the top-left and bottom-right corners of the no-trade region move away from each other and the parallelogram leans to the left (figure 1, left); when the assets are negatively correlated, the bottom-left and top-right corners of the no-trade region move away from each other and the parallelogram leans to the right (figure 1, right). These are also intuitive: when the two assets are positively correlated, they tend to substitute each other, and so it is less likely to rebalance when one is overweight and the other is underweight relative to the target portfolio. Similarly, when the two assets are negatively correlated, they tend to offset each other, and so overweighting or underweighting both relative to the target portfolio requires less rebalancing than otherwise.

2.2. A framework to analyze the optimal portfolio

In this section, we describe a general framework to analyze the optimal portfolio. When the current portfolio is outside of the no-trade region, we derive analytical conditions for an asset to be in $\boldsymbol{B}$, $\boldsymbol{S}$ or $\boldsymbol{N}$ of the optimal portfolio. Note that once these sets are fixed, the transaction costs in (1) reduce to linear terms, and the solution follows easily by standard mean-variance optimization.

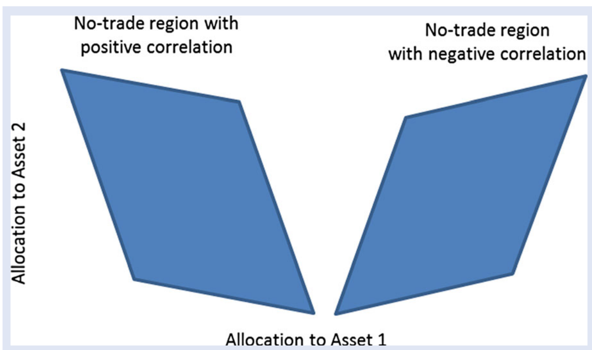

Figure 1. Shape of the no-trade region with positive and negative correlation. This figure illustrates the shape of the no-trade region for two risky assets with positive or negative correlation. The left shape results from positive asset correlation, and the right shape results from negative asset correlation.

† When cash is not available, the no-trade region is the intersection of the region defined by (13)—a multi-dimensional parallelogram—and the hyperplane $\sum_{i=1}^{n} x_i = 1$.

Let's suppose for the moment that we knew $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ of the optimal portfolio. We can re-write equation (5) to separate $\boldsymbol{B}$ and $\boldsymbol{S}$ from $\boldsymbol{N}$. Use n_N to denote the number of no-trade assets, and use $n_{B\backslash S} = n - n_N$ to denote the number of buy and sell assets. Let V_N be the n_N by n_N submatrix of $V$of no-trade assets in the optimal portfolio; let $V_{B\backslash S}$ be the $n_{B\backslash S}$ by $n_{B\backslash S}$ submatrix of $V$of buy and sell assets in the optimal portfolio; and let $V_{N,B\backslash S}$ be the n_N by $n_{B\backslash S}$ submatrix of $V$with covariance terms between no-trade assets and buy or sell assets in the optimal portfolio. We denote the subset of elements associated with no-trade assets in a vector by attaching a subscript $\boldsymbol{N}$ to that vector. For example, λ_N is the vector of Lagrange multipliers for no-trade assets, $x_{C,N}$ is the vector of current portfolio weights for no-trade assets, and so on. Similarly, we denote the subset of elements associated with buy and sell assets in a vector by attaching a subscript ***B§*** to that vector. For example, $x_{B\backslash S}$ is the vector of optimal portfolio weights for buy and sell assets, and $x_{T,B\backslash S}$ is the vector of target portfolio weights for buy and sell assets, and so on.

Finally, we use $tc_{B\backslash -S}$ to denote the vector of transaction costs for buy and sell assets, but with all costs for sell assets multiplied by -1. The reason for this definition is that based on our earlier discussion, $\lambda_i - \bar{\lambda}_i$ equals tc_i for a buy asset, and equals $-tc_i$ for a sell asset. Therefore, the definition of $tc_{B\backslash -S}$ means $tc_{B\backslash -S} = \lambda_{B\backslash S} - \bar{\lambda}_{B\backslash S}$. Note that $\lambda_N - \bar{\lambda}_N = 2\lambda_N - tc_N$ by (6). Plugging these two equalities into (5) and rewriting with the notations defined above give:

$$\frac{1}{k}\begin{bmatrix} V_{B\backslash S} & V_{N,B\backslash S}^{\mathrm{T}} \\ V_{N,B\backslash S} & V_N \end{bmatrix}\left(\begin{bmatrix} x_{B\backslash S} \\ x_{C,N}\end{bmatrix} - \begin{bmatrix} x_{T,B\backslash S} \\ x_{T,N}\end{bmatrix}\right) + \begin{bmatrix} tc_{B\backslash -S} \\ 2\lambda_N - tc_N \end{bmatrix} - \lambda_0 I = 0. \tag{14}$$

Here the unknowns are $x_{B\backslash S}$, λ_N and λ_0. The first row of (14) is:

$$\frac{1}{k}(V_{B\backslash S}(x_{B\backslash S} - x_{T,B\backslash S}) + V_{N,B\backslash S}^{\mathrm{T}}(x_{C,N} - x_{T,N})) + tc_{B\backslash -S} - \lambda_0 I_{B\backslash S} = 0. \tag{15}$$

Note that (15) is a set of $n_{B\backslash S}$ linear equations. We re-arrange it to solve for $x_{B\backslash S}$ by the other variables, and apply the constraint $x_i > x_{C,i}$ for each buy asset i, and the constraint $x_i < x_{C,i}$ for each sell asset i. This creates the necessary conditions for the buy and sell assets. The second row of (14) is:

$$\frac{1}{k}(V_{N,B\backslash S}(x_{B\backslash S} - x_{T,B\backslash S}) + V_N(x_{C,N} - x_{T,N})) + 2\lambda_N - tc_N - \lambda_0 I_N = 0. \tag{16}$$

Note that (16) is a set of n_N linear equations. We re-arrange it to solve for λ_N by the other variables, including $x_{B\backslash S}$ solved from (15). We then apply the dual feasibility constraints $\lambda_i \geq 0$ and $\bar{\lambda}_i \geq 0$ (or equivalently, $\lambda_i \leq tc_i$) on each no-trade asset i. This creates the necessary conditions for the no-trade assets.

To be clear: what we described above is not an algorithm to find the optimal portfolio, because the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ are not known. Rather, it is an analysis of the properties that the optimal portfolio must have. While it seems that for a general covariance matrix Vno concrete results can be obtained, for some special classes of covariance matrices this analysis leads to interesting analytical solutions and insights of the optimal portfolio. In the following sections, we will show that this is the case for two special classes of covariance matrices: when all assets have the same pairwise correlation, or when they follow a one-factor model. We will show that for these classes of covariance matrices, the necessary conditions derived either can be applied to every asset without knowing the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$, or involve one or two unknown parameters that depend on $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ but can be determined efficiently. In the former case, explicit analytical solutions exist; in the latter case, the analytical conditions limit the possible $\boldsymbol{B}$, $\boldsymbol{S}$, $\boldsymbol{N}$ combinations to a small number so that they can all be checked quickly.† In both cases the necessary conditions for assets in $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ are mutually exclusive, and so they are also sufficient conditions.

3. Uncorrelated assets

We start with the simple case that all asset correlations are zero. When all assets are uncorrelated, $V_{N,B\backslash S}$ is a matrix of zeros, and $V_{B\backslash S}$ and V_N are diagonal matrices. These simplify things greatly, since each linear equation in (15) and (16) only involves one asset. For a buy asset i, its equation in (15) is:

$$\frac{1}{k}\sigma_i^2(x_i - x_{T,i}) + tc_i - \lambda_0 = 0. \tag{17}$$

Solving for x_i and applying the constraint $x_i > x_{C,i}$ give the necessary condition:

$$\frac{1}{k}\sigma_i^2(x_{C,i} - x_{T,i}) + tc_i < \lambda_0. \tag{18}$$

Similarly, solving for a sell asset i in (15) and applying the constraint $x_i < x_{C,i}$ give the necessary condition:

$$\frac{1}{k}\sigma_i^2(x_{C,i} - x_{T,i}) - tc_i > \lambda_0. \tag{19}$$

Note that because $tc_i > 0$, conditions (18) and (19) are exclusive of each other. This should be the case because it wouldn't make sense to buy and sell the same asset simultaneously. For a no-trade asset i, its equation in (16) is:

$$\frac{1}{k}\sigma_i^2(x_{C,i} - x_{T,i}) + 2\lambda_i - tc_i - \lambda_0 = 0. \tag{20}$$

Solving for λ_i and applying the constraints $\lambda_i = 0$ and $\lambda_i \leq tc_i$ give the necessary conditions for a no-trade asset:

$$\frac{1}{k}\sigma_i^2(x_{C,i} - x_{T,i}) - tc_i \leq \lambda_0, \tag{21}$$

$$\frac{1}{k}\sigma_i^2(x_{C,i} - x_{T,i}) + tc_i \geq \lambda_0. \tag{22}$$

† The number of combinations is a linear function of n when cash is available, and a quadratic function of n when cash is not available. Without these analytical conditions, the total number of possible $\boldsymbol{B}$, $\boldsymbol{S}$, $\boldsymbol{N}$ combinations is an exponential function of n.

Table 1. Optimal portfolio weights of uncorrelated assets with cash.

Asset	Volatility	T-Costs	Target weights	Current weights	$x_{T,i} - \dfrac{k \cdot tc_i}{\sigma_i^2}$	$x_{T,i} + \dfrac{k \cdot tc_i}{\sigma_i^2}$	Action	Optimal weights
1	5%	0.01%	10%	5%	8.0%	12.0%	Buy	8.0%
2	10%	0.20%	10%	15%	0.0%	20.0%	No-Trade	15.0%
3	15%	0.30%	10%	5%	3.3%	16.7%	No-Trade	5.0%
4	20%	0.30%	10%	15%	6.3%	13.8%	Sell	13.8%
5	25%	0.01%	10%	5%	9.9%	10.1%	Buy	9.9%
6	5%	0.40%	10%	15%	-70.0%	90.0%	No-Trade	15.0%
7	10%	0.50%	10%	5%	-15.0%	35.0%	No-Trade	5.0%
8	15%	0.50%	10%	15%	-1.1%	21.1%	No-Trade	15.0%
9	20%	0.60%	10%	5%	2.5%	17.5%	No-Trade	5.0%
10	25%	0.60%	10%	15%	5.2%	14.8%	Sell	14.8%

Notes: This table shows the calculation of optimal portfolio weights using equation (24) when all assets are uncorrelated and cash is available. Kelly is 0.5.

Note that condition (21) is the opposite of (19), and condition (22) is the opposite of (18). In other words, all the necessary conditions for buy/sell/no-trade assets are mutually exclusive. Therefore, they are also sufficient conditions. Next, we use these conditions to find the optimal solution. There are two cases depending on whether cash is available or not.

3.1. *Cash is available*

When cash is available, λ_0 drops out of the equations. Combining (19), (19), (21) and (22) we have:

$$x_{C,i} \begin{cases} < x_{T,i} - k \cdot tc_i/\sigma_i^2 & \text{for buy assets,} \\ \in [x_{T,i} - k \cdot tc_i/\sigma_i^2, & \\ \quad x_{T,i} + k \cdot tc_i/\sigma_i^2] & \text{for no - trade assets,} \\ > x_{T,i} + k \cdot tc_i/\sigma_i^2 & \text{for sell assets.} \end{cases} \tag{23}$$

And the optimal solution is†:

$$x_i = \begin{cases} x_{T,i} - k \cdot tc_i/\sigma_i^2 & \text{for buy assets,} \\ x_{C,i} & \text{for no - trade assets,} \\ x_{T,i} + k \cdot tc_i/\sigma_i^2 & \text{for sell assets.} \end{cases} \tag{24}$$

Equations (23) and (24) together show that buy and sell assets are always rebalanced to the boundary of the no-trade region. This in turn implies that it is never optimal to rebalance an asset away from its target, i.e. $x_i < x_{C,i} < x_{T,i}$, or $x_i > x_{C,i} > x_{T,i}$. Later we will see that this is only true for uncorrelated assets with cash. If the assets are correlated or if cash is not available, then it may be optimal to rebalance some assets away from their targets. Table 1 illustrates the calculation of optimal portfolio weights with 10 hypothetical assets.

3.2. *Cash is not available*

When cash is not available we need to solve for λ_0. While it seems there are infinitely many possible values of λ_0, the important observation is that what really matters is the set of $2n$ numbers on the left-hand side of (18) and (19), namely $\sigma_i^2(x_{C,i} - x_{T,i})/k + tc_i$ and $\sigma_i^2(x_{C,i} - x_{T,i})/k - tc_i$ for $i = 1, 2, \ldots n$. We call these 'critical numbers'. Given a pair of adjacent critical numbers, as long as λ_0 falls somewhere in between them, the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ are completely fixed regardless of the exact value of λ_0. Therefore, there are only $4n - 1$ cases to consider regarding λ_0: when it coincides with one of the $2n$ critical numbers, and when it falls in between one of the $2n - 1$ adjacent pairs.‡ In each case, we express every x_i using λ_0 as in equation (25) (which is equation (24) with λ_0 added back), and back out λ_0 using $\sum_{i=1}^n x_i = 1$. We then check if λ_0 indeed coincides with the critical number, or falls in between the two adjacent critical numbers under consideration, and if it does, evaluate the objective function (1). We keep the solution that minimizes the objective function across all cases.

$$x_i = \begin{cases} x_{T,i} + k \cdot (\lambda_0 - tc_i)/\sigma_i^2 & \text{for buy assets,} \\ x_{C,i} & \text{for no - trade assets,} \\ x_{T,i} + k \cdot (\lambda_0 + tc_i)/\sigma_i^2 & \text{for sell assets.} \end{cases} \tag{25}$$

The above discussion leads to the following algorithm:

Algorithm 1

1. Sort the critical numbers in ascending order and break ties arbitrarily. Initialize the optimal objective function value to infinity.
2. Iterate through each critical number:
   (1) Assume that λ_0 coincides with this critical number, determine the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ and back out λ_0. If λ_0 equals this critical number, calculate all x_i and evaluate the objective function (1).
   (2) Assume that λ_0 is larger than this critical number but smaller than the next critical number, determine the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ and back out λ_0. If λ_0 falls in between this and the next critical number, calculate all x_i and evaluate the objective function (1).
   (3) If in (1) or (2) the objective function evaluates to a number smaller than the current optimal objective

† Equation (17) gives the optimal solution for buy assets. Optimal solution for sell assets is derived similarly.

‡ There is no need to consider the open interval from minus infinity to the smallest critical number because that leads to the impossible case of selling all assets. Similarly, there is no need to consider the open interval from the largest critical number to infinity because that leads to the impossible case of buying all assets.

Table 2. The execution of Algorithm 1 on assets in Table 1.

		Assuming λ_0 equals the current critical number				Assuming λ_0 falls between the current and the next critical number			
Critical number	Asset i	Update to *B/S/N*	λ_0	Equals critical number?	Objective function	Update to *B/S/N*	λ_0	Falls in expected range?	Objective function
− 1.00%	9	9 from S to N							
− 0.64%	5	5 from S to N							
− 0.62%	5					5 from N to B	− 0.24%	No	
− 0.60%	7	7 from S to N	− 0.21%	No			− 0.21%	No	
− 0.53%	3	3 from S to N	− 0.19%	No			− 0.19%	No	
− 0.38%	6	6 from S to N	− 0.07%	No			− 0.07%	No	
− 0.28%	8	8 from S to N	− 0.06%	No			− 0.06%	No	
− 0.10%	2	2 from S to N	− 0.05%	No			− 0.05%	Yes	0.05%
− 0.04%	1	1 from S to N	− 0.12%	No			− 0.12%	No	
− 0.02%	1		− 0.12%	No		1 from N to B	− 0.03%	No	
0.02%	10	10 from S to N	− 0.03%	No			− 0.03%	No	
0.08%	3		− 0.03%	No		3 from N to B	− 0.02%	No	
0.10%	4	4 from S to N							
0.20%	9					9 from N to B			
0.30%	2					2 from N to B			
0.40%	7					7 from N to B			
0.43%	6					6 from N to B			
0.70%	4					4 from N to B			
0.73%	8					8 from N to B			
1.23%	10					10 from N to B			

Notes: This table illustrates the execution of Algorithm 1 on the assets in Table 1 when cash is not available. Kelly is 0.5. At the beginning of the algorithm, ***B*** and ***N*** are initialized as empty sets and ***S*** is initialized to include all assets.

function value, update the optimal objective function value to that number and save the set of x_i as the optimal portfolio.

Note that Algorithm 1 can be further optimized in several ways. For example, we can initialize ***B*** and ***N*** to empty sets and ***S*** to include all assets before the first iteration. As the algorithm iterates from one critical point to the next, the sets ***B***, ***S*** and ***N*** are updated incrementally (instead of re-calculated every time), because at most one asset could move from one set to another. Table 2 illustrates such an execution of Algorithm 1 on the assets in table 1. In this example, λ_0 falls in the expected range only once across all iterations and that gives the optimal solution. Also, note that all the iterations after asset 4 moves from ***S*** to ***N*** can be skipped, because at that point ***S*** is already empty while ***B*** is not. It is impossible to buy some assets without selling others.

Table 3 shows the optimal portfolio weights resulting from the execution in table 2. Note that it is optimal to sell asset 1 from 5% down to 2.84%, even though its target is 10%. Why is that? When cash is available, the total optimal portfolio weight in table 1 is 106.5%. Without cash, the total must be 100%. Assets 1 and 5 are the candidates to trade to make up the difference given their small transaction costs. Between asset 1 and 5, asset 1 is used because of its lower volatility which leads to a smaller impact on the tracking error when traded away from target.

4. Same non-zero pairwise correlation

In this section, we consider the case that all assets have the same non-zero pairwise correlation. We have the following theorem whose proof is given in the appendix.

Table 3. Optimal portfolio weights of uncorrelated assets without cash.

Asset	Volatility	T-Costs	Target weights	Current weights	Action	Optimal weights
1	5%	0.01%	10%	5%	Sell	2.84%
2	10%	0.20%	10%	15%	No-Trade	15.00%
3	15%	0.30%	10%	5%	No-Trade	5.00%
4	20%	0.30%	10%	15%	Sell	13.18%
5	25%	0.01%	10%	5%	Buy	9.55%
6	5%	0.40%	10%	15%	No-Trade	15.00%
7	10%	0.50%	10%	5%	No-Trade	5.00%
8	15%	0.50%	10%	15%	No-Trade	15.00%
9	20%	0.60%	10%	5%	No-Trade	5.00%
10	25%	0.60%	10%	15%	Sell	14.43%

Notes: This table shows optimal portfolio weights when all assets are uncorrelated and cash is not available. Kelly is 0.5. The calculation of optimal weights is illustrated in Table 2.

Table 4. The execution of Algorithm 1 on assets with pairwise correlation 0.5.

		Assuming C equals the current critical number				Assuming C falls between the current and the next critical number			
Critical number	Asset i	Update to *B/S/N*	C	Equals critical number?	Objective function	Update to *B/S/N*	C	Falls in expected range?	Objective function
− 7.75%	6	6 from S to N	− 1.97%	No			− 1.97%	No	
− 5.50%	7	7 from S to N	− 1.58%	No			− 1.58%	No	
− 4.00%	9	9 from S to N	− 1.28%	No			− 1.28%	No	
− 2.75%	3	3 from S to N	− 1.07%	No			− 1.07%	No	
− 2.58%	8	8 from S to N	− 0.82%	No			− 0.82%	No	
− 1.50%	2	2 from S to N	− 0.68%	No			− 0.68%	No	
− 1.29%	5	5 from S to N	− 0.53%	No			− 0.53%	No	
− 1.21%	5		− 0.53%	No		5 from N to B	− 0.66%	No	
− 1.15%	10	10 from S to N	− 0.54%	No			− 0.54%	Yes	0.03%
− 0.50%	4	4 from S to N	− 0.55%	No			− 0.55%	No	
− 0.45%	1	1 from S to N	− 0.61%	No			− 0.61%	No	
− 0.05%	1		− 0.61%	No		1 from N to B	− 0.42%	No	
1.25%	3		− 0.42%	No		3 from N to B	0.00%	No	
2.00%	9		0.00%	No		9 from N to B	0.40%	No	
2.50%	4		0.40%	No		4 from N to B	0.75%	No	
2.50%	2		0.75%	No		2 from N to B	1.00%	No	
3.65%	10		1.00%	No		10 from N to B	1.33%	No	
4.08%	8		1.33%	No		8 from N to B	1.64%	No	
4.50%	7		1.64%	No		7 from N to B	1.92%	No	
8.25%	6		1.92%	No		6 from N to B	2.50%	No	

Notes: This table illustrates the execution of Algorithm 1 on the assets in Table 1 with pairwise correlation 0.5. Cash is available and Kelly is 0.5. At the beginning of the algorithm, ***B*** and ***N*** are initialized as empty sets and ***S*** is initialized to include all assets.

THEOREM 4.1 *Let ρ be the common correlation among all assets. There exists a constant C such that the optimal solutions are:*

$$x_i = \begin{cases} x_{T,i} + C/\sigma_i + k(\lambda_0 - tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for buy assets,} \\ x_{C,i} & \text{for no - trade assets,} \\ x_{T,i} + C/\sigma_i + k(\lambda_0 + tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for sell assets.} \end{cases} \tag{26}$$

And the necessary and sufficient conditions for an asset i to be a buy, sell or no-trade asset are:

$$C \begin{cases} > (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i} & \text{for buy assets,} \\ \in \left[(x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i}, \; (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i} \right] & \text{for no - trade assets,} \\ < (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i} & \text{for sell assets.} \end{cases} \tag{27}$$

If $\rho = 0$ then C is 0, in which case equation (26) reduces to (25), and equation (27) reduces to (18), (19), (21) and (22), as expected. When cash is available, λ_0 disappears in (26) and (27).

The first condition in (27) implies that, if all assets are ranked by $(x_{C,i} - x_{T,i})\sigma_i - (k(\lambda_0 - tc_i)/(1-\rho)\sigma_i)$ and if a given asset is bought in the optimal portfolio, then any lower-ranked asset must also be bought in the optimal portfolio. Similarly, if all assets are ranked by $(x_{C,i} - x_{T,i})\sigma_i - (k(\lambda_0 + tc_i)/(1-\rho)\sigma_i)$ and if a given asset is sold in the optimal portfolio, then any higher-ranked asset must also be sold in the optimal portfolio. Next, we show how to use Theorem 4.1 to find the optimal portfolio, starting with the case that cash is available.

4.1. Cash is available

When cash is available, λ_0 disappears and we only need to determine C. Similar to the case of uncorrelated assets without cash in Section 3.2, we focus on the following $2n$ critical numbers from (27) by setting $\lambda_0 = 0 : (x_{C,i} - x_{T,i})\sigma_i \mp k \cdot tc_i/((1-\rho)\sigma_i)$: for $i = 1, 2, \ldots n$. The algorithm to search for C is basically the same as Algorithm 1 in Section 3.2, but applied to a different set of critical numbers.† Within each iteration, the sets ***B***, ***S*** and ***N*** are first determined and then C is calculated.‡ This is followed by checking if all conditions in (27) are satisfied, and if they are, evaluating the objective function and keeping the minimum across all iterations. Table 4 illustrates the execution of the algorithm on the set of assets in table 1, assuming pairwise correlation 0.5.

Table 5 shows the optimal portfolio weights resulting from the execution in table 4. Note that it is optimal to sell asset 1 from 5% down to 3.2%, even though its target weight is 10%. The reason is a correlation effect: with a positive correlation among the assets, further underweighting asset 1 offsets the tracking error from overweighting assets such as 8 and 10, which are not traded because of their high transaction costs. Asset 1 has low volatility and small transaction costs, and so

† Another difference with Algorithm 1 is that here cash is available, so it is possible for ***B*** to be empty while ***S*** is not, or vice versa.
‡ See the proof of Theorem 4.1 in the appendix for how to calculate C.

Table 5. Optimal portfolio weights with pairwise correlation 0.5 (cash available).

Asset	Volatility	T-Costs	Target weights	Current weights	Action	Optimal weights
1	5%	0.01%	10%	5%	Sell	3.20%
2	10%	0.20%	10%	15%	No-Trade	15.00%
3	15%	0.30%	10%	5%	No-Trade	5.00%
4	20%	0.30%	10%	15%	Sell	14.80%
5	25%	0.01%	10%	5%	Buy	7.68%
6	5%	0.40%	10%	15%	No-Trade	15.00%
7	10%	0.50%	10%	5%	No-Trade	5.00%
8	15%	0.50%	10%	15%	No-Trade	15.00%
9	20%	0.60%	10%	5%	No-Trade	5.00%
10	25%	0.60%	10%	15%	No-Trade	15.00%

Notes: This table shows optimal portfolio weights when all assets have pairwise correlation 0.5 and cash is available. Kelly is 0.5. The calculation of optimal weights is illustrated in Table 4.

the negative impact on tracking error and the extra transaction costs incurred by underweighting it further away from target are more than offset by the correlation effect. Recall that in table 3, asset 1 is also rebalanced away from target because of the '100% invested' constraint. Here that constraint doesn't apply, but the correlation effect drives asset 1 away from target. As we mentioned in the last section, 'uncorrelated assets with cash' is the only situation where assets are never rebalanced away from target.

4.2. *Cash is not available*

When cash is not available, λ_0 also needs to be determined. Imagine λ_0 changes from negative infinity to positive infinity while keeping other parameters fixed, then each critical number in (27), i.e. $(x_{C,i} - x_{T,i})\sigma_i - k(\lambda_0 \mp tc_i)/((1-\rho)\sigma_i)$, moves on a line. These lines induce a division of the plane into faces, edges and vertices, which are collectively called a *line arrangement* (see de Berg et. al. 2008, Chapter 8). Naturally, the pair (λ_0,C) represents a point on the plane. Like the observation that leads to Algorithm 1, here we note that when this point falls inside a face or on an edge of the line arrangement, the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ are fixed regardless of the exact location of the point. There are $2n$ lines, and so the total number of faces, edges and vertices of the line arrangement is on the order of n^2. We traverse the line arrangement and dynamically update the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ for each face, edge and vertex along the way. In each step, once the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ are determined, each x_i only depends on λ_0.† So we can back out λ_0 using $\sum_{i=1}^{n} x_i = 1$, calculate every x_i, and check if $x_i > x_{C,i}$ for $i \in B$ and $x_i < x_{C,i}$ for $i \in S$. If the solution passes these checks, then we evaluate the objective function (1) and keep the solution that minimizes it across all steps.

Now, how to traverse the line arrangement and update the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ for each face, edge and vertex? As shown in figure 2, one way to do this is to start with the vertical line $\lambda_0 = 0$, move it horizontally to the right (i.e. $\lambda_0 > 0$) and then to the left (i.e. $\lambda_0 < 0$). At its starting position $\lambda_0 = 0$, the line intersects with the edges of the line arrangement at a set of points whose y-coordinates are the critical numbers in the previous section's 'cash available' case. These points, shown by black dots in figure 2, are called 'critical points'. Using the

† Each x_i depends on λ_0 and C by (26), but C depends on λ_0 as shown in the appendix.

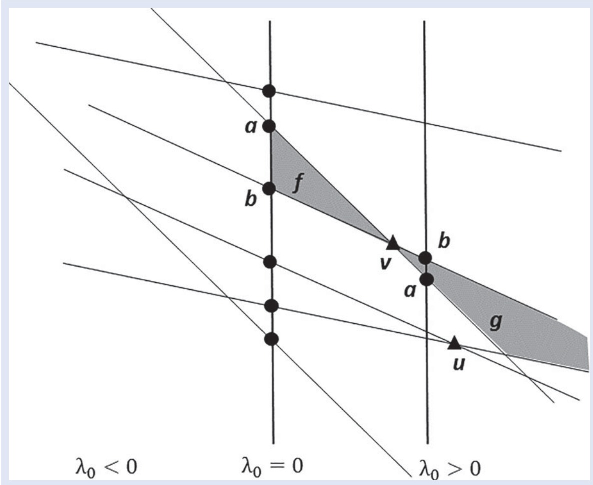

Figure 2. A hypothetical line arrangement. This figure shows a hypothetical line arrangement, the vertical line $\lambda_0 = 0$, and a new line as it moves to the right and passes the vertex $\boldsymbol{v}$. The black dots on the vertical line are the critical points. As the vertical line moves to the right, the critical point $\boldsymbol{a}$ (resp. $\boldsymbol{b}$) traces the edge $\boldsymbol{av}$ (resp. $\boldsymbol{bv}$), and the segment $\boldsymbol{ab}$ sweeps the face $\boldsymbol{f}$. After the vertical line passes $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ cross on the vertical line.

process as illustrated in table 4, we determine the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ for each critical point and each segment between two adjacent critical points. As the vertical line moves left or right, these critical points move up or down on the vertical line to trace the edges of the line arrangement. However, until the vertical line hits the next vertex of the line arrangement, the orderings of these critical points on the vertical line remain fixed, and so do the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ of these critical points and the segments. Therefore, we can assign the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ of every critical point to the edge of the line arrangement that it traces as the vertical line moves. For example, in figure 2 as the vertical line $\lambda_0 = 0$ moves to the right, the critical point $\boldsymbol{a}$ moves towards the vertex $\boldsymbol{v}$, but its sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ remain fixed until it reaches $\boldsymbol{v}$. So, we assign the sets of $\boldsymbol{a}$ to the edge $\boldsymbol{av}$. Similarly, we assign the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ of each segment to the face of the line arrangement it sweeps. For example, in figure 2 as the vertical line $\lambda_0 = 0$ moves to the right, the segment $\boldsymbol{ab}$ sweeps the triangular face $\boldsymbol{f}$. So, we assign the sets of $\boldsymbol{ab}$ to the face $\boldsymbol{f}$. When the vertical line hits $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ meet at $\boldsymbol{v}$ and we create the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ for $\boldsymbol{v}$ by copying those of $\boldsymbol{a}$ and moving at most two assets from $\boldsymbol{B}$ or $\boldsymbol{S}$ to $\boldsymbol{N}$. As the vertical line moves past $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ cross

Table 6. Optimal portfolio weights with pairwise correlation 0.9 (cash not available).

Asset	Volatility	T-Costs	Target weights	Current weights	Optimal weights
1	5%	0.01%	10%	5%	4.53%
2	10%	0.20%	10%	15%	15.00%
3	15%	0.30%	10%	5%	5.00%
4	20%	0.30%	10%	15%	15.00%
5	25%	0.01%	10%	5%	5.47%
6	5%	0.40%	10%	15%	15.00%
7	10%	0.50%	10%	5%	5.00%
8	15%	0.50%	10%	15%	15.00%
9	20%	0.60%	10%	5%	5.00%
10	25%	0.60%	10%	15%	15.00%

Notes: This table shows optimal portfolio weights as calculated by the line arrangement algorithm when all assets have pairwise correlation 0.9 and cash is not available. Kelly is 0.5.

each other on the vertical line, and we update their sets ***B***, ***S*** and ***N*** to reflect the crossing of their positions. These updates also involve at most two assets. Finally, we create the sets ***B***, ***S*** and ***N*** for the newly created segment ***ba*** by copying those of ***a*** and moving at most two assets among ***B***, ***S*** and ***N***. These sets are then assigned to the face ***g*** that the new segment ***ba*** sweeps as the vertical line moves further to the right. All the updates and creation of new sets are done efficiently because of the small number of assets involved. After the vertical line moves past ***v***, we repeat the above process as it moves toward the next vertex ***u***, and so on. Note that we don't need to store the sets ***B***, ***S*** and ***N*** for all the faces, edges and vertices of the line arrangement explicitly. At any given point, we only need to store these sets for the (at most $4n+1$) faces, edges and vertices intersected by the vertical line. Table 6 shows the result of executing this line arrangement algorithm on the set of assets in table 1, assuming pairwise correlation 0.9.

5. The one-factor covariance matrix model

Factor models are often used to estimate and predict asset returns and risks. A factor model decomposes an asset's return or risk into factors common to all assets plus the asset's own specific return or risk. The simplest model is when there is only one factor, such as the market factor in the CAPM. In this case, we use σ_M to denote the volatility of the common factor, β_i to denote the beta of asset i to the common factor, σ_i to denote the total volatility of asset i, and $\sigma_{\varepsilon,i}$ to denote the idiosyncratic risk of asset i. Then $\sigma_i^2 = \beta_i^2\sigma_M^2 + \sigma_{\varepsilon,i}^2$. We assume that $\sigma_{\varepsilon,i} > 0$ for each i.

We have the following Theorem 5.1 to characterize the optimal portfolio under the one-factor covariance matrix model. The proof is given in the appendix.

THEOREM 5.1 *There exists a constant C such that the optimal solutions are:*

$$x_i = \begin{cases} x_{T,i} + C\beta_i/\sigma_{\varepsilon,i}^2 \\ +k(\lambda_0 - tc_i)/\sigma_{\varepsilon,i}^2 & \text{for buy assets,} \\ x_{C,i} & \text{for no - trade assets,} \\ x_{T,i} + C\beta_i/\sigma_{\varepsilon,i}^2 \\ +k(\lambda_0 + tc_i)/\sigma_{\varepsilon,i}^2 & \text{for sell assets.} \end{cases} \tag{28}$$

And the necessary and sufficient conditions for an asset i to be a buy, sell or no-trade asset are:

$$\begin{aligned} &C \begin{cases} > \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i > 0, \\ \in \left[\dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i}, \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i}\right] & \text{for no - trade assets with } \beta_i > 0, \\ < \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i > 0, \end{cases} \\ &C \begin{cases} < \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i < 0, \\ \in \left[\dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i}, \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i}\right] & \text{for no - trade assets with } \beta_i < 0, \\ > \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i < 0. \end{cases} \end{aligned} \tag{29}$$

If $\beta_i = 0$, then equation (28) reduces to (25), and equation (29) is replaced by (18), (19), (21) and (22). When cash is available, λ_0 disappears in (28) and (29).

The first condition in (29) of Theorem 5.1 implies that if all assets with positive betas are ranked by $((x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k\cdot(\lambda_0 - tc_i))/\beta_i$ and if a given asset is bought in the optimal portfolio, then any lower-ranked asset must also be bought in the optimal portfolio. Similarly, if all assets with positive betas are ranked by $((x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k\cdot(\lambda_0 + tc_i))/\beta_i$ and if a given asset is sold in the optimal portfolio, then any higher-ranked asset must also be sold in the optimal portfolio. Similar statements can be made for assets with negative betas. Next, we show how to use Theorem 5.1 to find the optimal portfolio. In the discussion below we assume that all β_i are non-zero.

5.1. Cash is available

When cash is available, λ_0 disappears and we only need to determine C. Similar to the case of uncorrelated assets without cash in Section 3.2, we focus on the following $2n$ critical numbers from (29) when $\lambda_0 = 0$: $((x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 \mp k\cdot tc_i)/\beta_i$ for $i = 1, 2, \ldots n$. So, the algorithm is essentially the same as Algorithm 1. A subtlety here is that at each critical point, the updates made to the sets ***B***, ***S*** and ***N*** depend on the sign of

Table 7. Optimal portfolio weights of one-factor model (cash available).

Asset	Beta	Idiosyncratic risk	T-Costs	Target weights	Current weights	Action	Optimal weights
1	1.2	5%	0.01%	10%	5%	Sell	− 6.19%
2	1.1	10%	0.20%	10%	15%	No-Trade	15.00%
3	1	15%	0.30%	10%	5%	No-Trade	5.00%
4	0.9	20%	0.30%	10%	15%	Sell	12.90%
5	0.8	25%	0.01%	10%	5%	Buy	9.44%
6	1.2	5%	0.40%	10%	15%	No-Trade	15.00%
7	− 0.3	10%	0.50%	10%	5%	No-Trade	5.00%
8	1	15%	0.50%	10%	15%	No-Trade	15.00%
9	− 0.5	20%	0.60%	10%	5%	No-Trade	5.00%
10	0.8	25%	0.60%	10%	15%	Sell	14.32%

Notes: This table shows 10 hypothetical assets that follow a one-factor model but otherwise have the same transaction costs, target weights, and current weights as in Table 1. Their idiosyncratic risks are the same as volatilities in Table 1. Assets 7 and 9 have negative betas. The common factor volatility is 15%, cash is available, and Kelly is 0.5.

the asset's beta. For example as we move to a critical point $((x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k \cdot tc_i)/\beta_i$ from below, if $\beta_i > 0$ then asset i moves from $\boldsymbol{S}$ to $\boldsymbol{N}$, but if $\beta_i < 0$then asset $i$stays in $\boldsymbol{N}$ and no action is taken. The initializations of $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ are also different. In Algorithm 1 we initialize $\boldsymbol{B}$ and $\boldsymbol{N}$ to be empty and $\boldsymbol{S}$ to be the set of all assets; here we initialize $\boldsymbol{B}$ to be the set of assets with $\beta_i < 0$ and $\boldsymbol{S}$ to be the set of assets with $\beta_i > 0$. Table 7 shows 10 hypothetical assets with the same transaction costs, target weights, and current weights as in table 1. Their idiosyncratic risks are the same as volatilities in table 1. Assets 7 and 9 have negative betas. The common factor volatility is 15% and Kelly is 0.5.

Table 8 illustrates the calculation of optimal portfolio weights, which are shown in the last column of table 7. Note that it is optimal to sell asset 1 from 5% down to -6.19%, even though its target weight is 10%. The reason is again a correlation effect: further underweighting asset 1 offsets the tracking error from overweighting positive beta assets such as 6 and 8 which are positively correlated with asset 1 through the common factor, and offsets the tracking error from underweighting negative beta assets such as 7 and 9 which are negatively correlated with asset 1. While every asset can potentially be traded to offset tracking error from other assets, asset 1 is the best choice here because of its low idiosyncratic risk and small transaction costs.

As it turns out, if we reduce the idiosyncratic risk of asset 5, another asset with very low transaction costs, from 25% to 5% then its optimal weight changes from 9.44% to 2.57%. That changes its status from buy to sell and makes it rebalance away from target as well. At the same time, the optimal weight of asset 1 moves closer to target from -6.19% to -2.14%. If we then increase the beta of asset 5 to 1.2 to match with asset 1,

Table 8. The execution of Algorithm 1 on assets in Table 7.

		Assuming C equals the current critical number				Assuming C falls between the current and the next critical number			
Critical number	Asset i	Update to $B/S/N$	C	Equals critical number?	Objective function	Update to $B/S/N$	C	Falls in expected range?	Objective function
− 0.67%	7	7 from B to N	− 0.10%	No			− 0.10%	No	
− 0.40%	5	5 from S to N	− 0.09%	No			− 0.09%	No	
− 0.38%	5		− 0.09%	No		5 from N to B	− 0.10%	No	
− 0.26%	3	3 from S to N	− 0.09%	No			− 0.09%	No	
− 0.20%	9	9 from B to N	− 0.09%	No			− 0.09%	No	
− 0.16%	6	6 from S to N	− 0.04%	No			− 0.04%	No	
− 0.14%	8	8 from S to N	− 0.04%	No			− 0.04%	No	
− 0.05%	2	2 from S to N	− 0.04%	No			− 0.04%	Yes	0.06%
− 0.01%	1	1 from S to N	− 0.20%	No			− 0.20%	No	
− 0.01%	1		− 0.20%	No		1 from N to B	− 0.03%	No	
0.02%	10	10 from S to N	− 0.03%	No			− 0.03%	No	
0.04%	3		− 0.03%	No		3 from N to B	− 0.03%	No	
0.06%	4	4 from S to N	− 0.03%	No			− 0.03%	No	
0.14%	2		− 0.03%	No		2 from N to B	0.00%	No	
0.18%	6		0.00%	No		6 from N to B	0.07%	No	
0.36%	8		0.07%	No		8 from N to B	0.08%	No	
0.39%	4		0.08%	No		4 from N to B	0.09%	No	
0.77%	10		0.09%	No		10 from N to B	0.09%	No	
1.00%	7		0.09%	No		7 from N to S	0.10%	No	
1.00%	9		0.10%	No		9 from N to S	0.10%	No	

Notes: This table illustrates the calculation of optimal portfolio weights using Algorithm 1 on the assets in Table 7. Cash is available. The common factor volatility is 15% and Kelly is 0.5. At the beginning of the algorithm, $\boldsymbol{N}$ is initialized as an empty set, $\boldsymbol{B}$ is initialized to include assets 7 and 9, and $\boldsymbol{S}$ is initialized to include all other assets.

Table 9. Optimal portfolio weights of one-factor model (cash not available).

Asset	Beta	Idiosyncratic risk	T-costs	Target weights	Current weights	Action	Optimal weights
1	1.2	5%	0.01%	10%	5%	Sell	– 2.46%
2	1.1	10%	0.20%	10%	15%	No-Trade	15.00%
3	1	15%	0.30%	10%	5%	No-Trade	5.00%
4	0.9	20%	0.30%	10%	15%	Sell	14.00%
5	0.8	25%	0.01%	10%	5%	Buy	10.33%
6	1.2	5%	0.40%	10%	15%	No-Trade	15.00%
7	– 0.3	10%	0.50%	10%	5%	No-Trade	5.00%
8	1	15%	0.50%	10%	15%	No-Trade	15.00%
9	– 0.5	20%	0.60%	10%	5%	Buy	8.14%
10	0.8	25%	0.60%	10%	15%	No-Trade	15.00%

Notes: This table shows the optimal weights of assets in Table 7 when cash is not available. The common factor volatility is 15% and Kelly is 0.5.

then both assets end up having the same optimal weight of 1.24%. If we further reduce the idiosyncratic risk of asset 5 down to 3%, then it has an optimal weight of -3.06% while asset 1 stays at 5%, as now it makes sense to only rebalance asset 5 away from target.

5.2. *Cash is not available*

When cash is not available, λ_0 also needs to be determined. Like the case of same pair-wise correlation without cash in Section 4.2, we run the line arrangement algorithm to search for the point (λ_0, C) by traversing the line arrangement defined by (29). Again, the subtlety here is that at each vertex, edge or face of the line arrangement, the updates made to the sets $\boldsymbol{B}$, $\boldsymbol{S}$ and $\boldsymbol{N}$ depend on the sign of the asset's beta, and $\boldsymbol{B}$ (resp. $\boldsymbol{S}$) is initialized to be the set of assets with $\beta_i < 0$ (resp.$\beta_i > 0$). Table 9 shows the optimal portfolio weights in the last column when cash is not available.

6. Concluding remarks

In this paper, we study optimal portfolio rebalancing by modeling it as a mean-variance type problem and present new analytical results for the general case of multiple risky assets. We first derive the equation of the no-trade region, and then provide a framework to analyze the optimal portfolio. We apply this framework to create, to the best of our knowledge, the first set of analytical solutions and conditions of the optimal portfolio under several simplifying yet important models of asset covariance matrix: uncorrelated returns, same non-zero pairwise correlation, and the one-factor model. Each case is analyzed with and without cash. In situations where unknown parameters are involved in the analytical conditions, they are determined by combinatorial rather than numerical algorithms. Our analytical results provide interesting insights, such as when it might be optimal to rebalance some assets further away from target, and how the optimal portfolio changes with the various parameters.

The analytical approach of this paper can also be applied to a hierarchically nested covariance matrix to model more complex portfolios. Consider, for example, a portfolio that has US equities, international developed equities, emerging markets equities, and several uncorrelated market-neutral hedge funds. We cannot apply the results for uncorrelated assets because all the regional equities are correlated. However, we can first aggregate all the regional equities into global equities and apply the results for uncorrelated assets to global equities plus the market-neutral hedge funds. In the next level, we apply the one-factor model to the regional equities using global equities as the common factor. By modeling the portfolio this way, analytical results can be obtained using the approach presented in this paper.

It would be interesting to see if the approach in this paper can be applied to the optimal consumption and investment problem with transaction costs in the continuous time model. Another direction for future research is to consider other types of transaction costs. While the linear transaction costs model used in this paper seems to be the most commonly used, there are other models such as fixed transaction costs, fixed plus linear transaction costs, piece-wise linear transaction costs, and quadratic transaction costs.

Disclosure statement

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Technical appendix

THEOREM 4.1 *Let ρ be the common correlation among all assets. There exists a constant C such that the optimal solutions are:*

$$x_i = \begin{cases} x_{T,i} + C/\sigma_i + k(\lambda_0 - tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for buy assets} \\ x_{C,i} & \text{for no - trade assets} \\ x_{T,i} + C/\sigma_i + k(\lambda_0 + tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for sell assets} \end{cases} \tag{A1}$$

And the necessary and sufficient conditions for an asset i to be a buy, sell or no-trade asset are:

$$C \begin{cases} > (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i} & \text{for buy assets} \\ \in \left[(x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i}, \right. \\ \quad \left. (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i}\right] & \text{for no - trade assets} \\ < (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i} & \text{for sell assets} \end{cases} \tag{A2}$$

If $\rho = 0$ then C is 0, in which case equation (A1) *reduces to* (25) *and equation* (A2) *reduces to* (18), (19), (21) *and* (22), *as expected. When cash is available λ_0 disappears in* (A1) *and* (A2).

Proof We apply the general framework presented in the main body of the paper to analyze the optimal portfolio. Let's suppose for the moment that ***B, S*** and ***N*** of the optimal portfolio were known. Re-arranging equation (15) gives:

$$\begin{aligned} x_{B\backslash S} &= x_{T,B\backslash S} + V_{B\backslash S}^{-1} V_{N,B\backslash S}^T (x_{T,N} - x_{C,N}) \\ &\quad + k\, V_{B\backslash S}^{-1}(\lambda_0 I_{B\backslash S} - tc_{B\backslash -S}). \end{aligned} \tag{A3}$$

Recall that $V_{B\backslash S}$ is the $n_{B\backslash S}$ by $n_{B\backslash S}$ submatrix of V with variance and covariance terms among buy and sell assets; $V_{N,B\backslash S}$ is the n_N by $n_{B\backslash S}$ submatrix of V with covariance terms between no-trade and buy or sell assets; n_N is the number of no-trade assets; and $n_{B\backslash S} = n - n_N$ is the number of buy and sell assets.

To proceed with equation (A3), we first determine the (i,j) entry of $V_{B\backslash S}^{-1}$ for any buy or sell assets i and j. The $n_{B\backslash S}$ by $n_{B\backslash S}$ correlation matrix among all buy and sell assets is:

$$\begin{pmatrix} 1 & \rho & \cdots & \rho & \rho \\ \rho & 1 & \cdots & \rho & \rho \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho & \rho & \cdots & 1 & \rho \\ \rho & \rho & \cdots & \rho & 1 \end{pmatrix}.$$

It is easy to check that, for $a = \frac{(n_{B\backslash S}-2)\rho+1}{(1-\rho)((n_{B\backslash S}-1)\rho+1)}$ and $b = \frac{-\rho}{(1-\rho)((n_{B\backslash S}-1)\rho+1)}$, the inverse of this correlation matrix is†:

$$\begin{pmatrix} a & b & \cdots & b & b \\ b & a & \cdots & b & b \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & a & b \\ b & b & \cdots & b & a \end{pmatrix}.$$

Here a appears on the diagonal, and b appears on off-diagonal entries. Therefore, for two buy or sell assets i and j, the (i,j) entry of $V_{B\backslash S}^{-1}$ is $b/\sigma_i\sigma_j$ if $i \neq j$, and a/σ_i^2 if $i = j$.

Next, note that for a no-trade asset l, the (j,l) entry of $V_{N,B\backslash S}^T$ is $\rho\sigma_j\sigma_l$. Therefore, the (i,l) entry of $V_{B\backslash S}^{-1} V_{N,B\backslash S}^T$ is:

$$\begin{aligned} &\sum_{j\in B\cup S, j\neq i} \frac{b}{\sigma_i\sigma_j}\cdot\rho\sigma_j\sigma_l + \frac{a}{\sigma_i^2}\cdot\rho\sigma_i\sigma_l = \sum_{j\in B\cup S, j\neq i} \frac{\rho b\sigma_l}{\sigma_i} + \frac{\rho a\sigma_l}{\sigma_i} \\ &= \sigma_l\rho\frac{(n_{B\backslash S}-1)b + a}{\sigma_i}. \end{aligned}$$

So, the ith position of the vector $V_{B\backslash S}^{-1} V_{N,B\backslash S}^T(x_{T,N} - x_{C,N})$ in equation (A3) is:

$$\sum_{l\in N}\sigma_l(x_{T,l} - x_{C,l})\rho\frac{(n_{B\backslash S}-1)b + a}{\sigma_i} = C_N\rho\frac{(n_{B\backslash S}-1)b + a}{\sigma_i}. \tag{A4}$$

Here $C_N = \sum_{l\in N}\sigma_l(x_{T,l} - x_{C,l})$ is a constant that only depends on assets in N.

Using $(\lambda_0 \mp tc_j)$ as the notation for $(\lambda_0 - tc_j)$ if $j \in B$ and $(\lambda_0 + tc_j)$ if $j \in S$ (and similarly with $(\lambda_0 \mp tc_i)$), the ith position of the vector $V_{B\backslash S}^{-1}(\lambda_0 I_{B\backslash S} - tc_{B\backslash -S})$ is:

$$\begin{aligned} &\sum_{j\in B\cup S, j\neq i}\frac{b}{\sigma_i\sigma_j}\cdot(\lambda_0 \mp tc_j) + \frac{a}{\sigma_i^2}\cdot(\lambda_0 \mp tc_i) = \\ &\sum_{j\in B\cup S, j\neq i}\frac{b}{\sigma_i\sigma_j}\cdot(\lambda_0 \mp tc_j) + \frac{b}{\sigma_i^2}\cdot(\lambda_0 \mp tc_i) \\ &-\frac{b}{\sigma_i^2}\cdot(\lambda_0 \mp tc_i) + \frac{a}{\sigma_i^2}\cdot(\lambda_0 \mp tc_i) = \\ &\sum_{j\in B\cup S}\frac{b}{\sigma_i\sigma_j}\cdot(\lambda_0 \mp tc_j) + \frac{(a-b)}{\sigma_i^2}\cdot(\lambda_0 \mp tc_i) \\ &= \frac{b}{\sigma_i}C_{B,S} + \frac{(a-b)}{\sigma_i^2}\cdot(\lambda_0 \mp tc_i). \end{aligned} \tag{A5}$$

Here $C_{B,S} = \sum_{j\in B\cup S}(1/\sigma_j)\cdot(\lambda_0 \mp tc_j)$ is a constant that only depends on assets in B and S.

Plugging (A4) and (A5) into (A3) gives x_i for a buy or sell asset i:

$$\begin{aligned} x_i &= x_{T,i} + C_N\rho\frac{(n_{B\backslash S}-1)b + a}{\sigma_i} + k\frac{b}{\sigma_i}C_{B,S} \\ &\quad + k\frac{(a-b)}{\sigma_i^2}\cdot(\lambda_0 \mp tc_i). \end{aligned}$$

Let $C = C_N\rho((n_{B\backslash S}-1)b + a) + kbC_{B,S}$ be a constant that depends on the optimal solution, and note that $a - b = 1/(1-\rho)$, then:

$$x_i = \begin{cases} x_{T,i} + C/\sigma_i + k(\lambda_0 - tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for buy assets,} \\ x_{T,i} + C/\sigma_i + k(\lambda_0 + tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for sell assets.} \end{cases} \tag{A6}$$

This is the same as equation (A1). Applying $x_i > x_{C,i}$ for buy assets and $x_i < x_{C,i}$ for sell assets and re-arranging (A6), we derive

† Simply check that these two matrices multiply to the identity matrix.

the following necessary conditions:

$$C \begin{cases} > (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i} & \text{for buy assets,} \\ < (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i} & \text{for sell assets.} \end{cases} \tag{A7}$$

Note that $k \cdot tc_i/((1-\rho)\sigma_i)$ is positive, so in (A7) the buy condition lower bound is always higher than the sell condition upper bound and no asset can be labeled both as a buy and a sell asset. Equation (A7) covers buy and sell assets in (A2). Next, we prove the conditions for no-trade assets in (A2). We start by re-arranging equation (16) to:

$$\lambda_N - \frac{1}{2}(tc_N + \lambda_0 I_N) = -\frac{1}{2k} V_{N,B\setminus S}(x_{B\setminus S} - x_{T,B\setminus S}) - \frac{1}{2k} V_N(x_{C,N} - x_{T,N}). \tag{A8}$$

Let's first focus on the right-hand side of (A8). Given a no-trade asset l, the lth position of the first term on the right-hand side of (A8) is $-(\rho\sigma_l/2k)\sum_{i\in B\cup S}(\sigma_i(x_i - x_{T,i}))$. Plugging in x_i from (A6) it becomes:

$$\begin{aligned} &-\frac{\rho\sigma_l}{2}\sum_{i\in B\cup S}\left(\frac{C}{k} + \frac{\lambda_0 \mp tc_i}{(1-\rho)\cdot\sigma_i}\right) \\ &= -\frac{\rho\sigma_l}{2}\left(\frac{n_{B\setminus S}C}{k} + \frac{1}{(1-\rho)}\sum_{i\in B\cup S}\frac{\lambda_0 \mp tc_i}{\sigma_i}\right) \\ &= -\frac{\rho\sigma_l}{2}\left(\frac{n_{B\setminus S}C}{k} + \frac{C_{B,S}}{(1-\rho)}\right). \end{aligned}$$

Note that the last step uses the earlier definition $C_{B,S} = \sum_{i\in B\cup S} 1/\sigma_i \cdot (\lambda_0 \mp tc_i)$. The lth position of the second term on the right-hand side of (A8) is:

$$\begin{aligned} &-\frac{\rho\sigma_l}{2k}\sum_{j\in N, j\neq l}(\sigma_j(x_{C,j} - x_{T,j})) - \frac{\sigma_l^2}{2k}(x_{C,l} - x_{T,l}) \\ &= \frac{\rho\sigma_l}{2k}\sum_{j\in N, j\neq l}(\sigma_j(x_{T,j} - x_{C,j})) - \frac{\sigma_l^2}{2k}(x_{C,l} - x_{T,l}) \\ &= \frac{\rho\sigma_l}{2k}(C_N - \sigma_l(x_{T,l} - x_{C,l})) - \frac{\sigma_l^2}{2k}(x_{C,l} - x_{T,l}) \\ &= \frac{\rho\sigma_l}{2k}C_N - \frac{(1-\rho)\sigma_l^2}{2k}(x_{C,l} - x_{T,l}). \end{aligned}$$

So, the lth position of the right-hand side of (A8) is:

$$\begin{aligned} &-\frac{\rho\sigma_l}{2}\left(\frac{n_{B\setminus S}C}{k} + \frac{C_{B,S}}{(1-\rho)}\right) + \frac{\rho\sigma_l}{2k}C_N \\ &- \frac{(1-\rho)\sigma_l^2}{2k}(x_{C,l} - x_{T,l}) = \\ &-\frac{\rho\sigma_l}{2}\left(\frac{C_{B,S}}{(1-\rho)} - \frac{C_N}{k}\right) - \frac{\rho\sigma_l n_{B\setminus S}C}{2k} \\ &- \frac{(1-\rho)\sigma_l^2}{2k}(x_{C,l} - x_{T,l}). \end{aligned}$$

Now we claim that:

$$-\frac{\rho\sigma_l}{2}\left(\frac{C_{B,S}}{(1-\rho)} - \frac{C_N}{k}\right) = \frac{((n_{B\setminus S} - 1)\rho + 1)\sigma_l}{2k}C. \tag{A9}$$

To verify (A9), note that $(n_{B\setminus S} - 1)b + a = (1-\rho)/(1-\rho)((n_{B\setminus S} - 1)\rho + 1)$. So,

$$\begin{aligned} C &= C_N\rho((n_{B\setminus S} - 1)b + a) + kbC_{B,S} \\ &= \frac{\rho(1-\rho)C_N}{(1-\rho)((n_{B\setminus S} - 1)\rho + 1)} - \frac{k\rho C_{B,S}}{(1-\rho)((n_{B\setminus S} - 1)\rho + 1)} \\ &= \frac{k\rho}{((n_{B\setminus S} - 1)\rho + 1)}\left(\frac{C_N}{k} - \frac{C_{B,S}}{(1-\rho)}\right). \end{aligned}$$

This leads to (A9) after re-arranging. Using (A9), the lth position of the right-hand side of (A8) becomes:

$$\begin{aligned} &\frac{((n_{B\setminus S} - 1)\rho + 1)\sigma_l C}{2k} - \frac{\rho\sigma_l n_{B\setminus S}C}{2k} - \frac{(1-\rho)\sigma_l^2}{2k}(x_{C,l} - x_{T,l}) \\ &= \frac{(1-\rho)\sigma_l C}{2k} - \frac{(1-\rho)\sigma_l^2}{2k}(x_{C,l} - x_{T,l}). \end{aligned}$$

Recall that $\lambda_l \geq 0$ and $\lambda_l \leq tc_l$ are the dual feasibility constraints in (9), so the lth position of the left-hand side (and therefore the right-hand side) of (A8) must lie between $-(1/2)(tc_l + \lambda_0)$ and $(1/2)(tc_l - \lambda_0)$. In other words,

$$\begin{aligned} &-\frac{1}{2}(tc_l + \lambda_0) \leq \frac{(1-\rho)\sigma_l C}{2k} - \frac{(1-\rho)\sigma_l^2}{2k}(x_{C,l} - x_{T,l}) \\ &\leq \frac{1}{2}(tc_l - \lambda_0). \end{aligned} \tag{A10}$$

Re-arranging (A10) gives the necessary conditions for no-trade assets in (A2), where the lower bound for no-trade assets coincides with the upper bound for sell assets, and the upper bound for no-trade assets coincides with the lower bound for buy assets! In other words, the necessary conditions for buy, sell, and no-trade assets are mutually exclusive while jointly covering the entire real line. Therefore, they are also sufficient conditions.

THEOREM 5.1 *Let β_i and $\sigma^2_{\varepsilon,i}$ be as defined in the paper. There exists a constant C such that the optimal solutions are:*

$$x_i = \begin{cases} x_{T,i} + C\beta_i/\sigma^2_{\varepsilon,i} + k(\lambda_0 - tc_i)/\sigma^2_{\varepsilon,i} & \text{for buy assets,} \\ x_{C,i} & \text{for no - trade assets,} \\ x_{T,i} + C\beta_i/\sigma^2_{\varepsilon,i} + k(\lambda_0 + tc_i)/\sigma^2_{\varepsilon,i} & \text{for sell assets.} \end{cases} \tag{A11}$$

And the necessary and sufficient conditions for an asset i to be a buy, sell or no-trade asset are:

$$C \begin{cases} > \dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i > 0, \\ \in \left[\dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 + tc_i)}{\beta_i}, \dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 - tc_i)}{\beta_i}\right] & \text{for no - trade assets with } \beta_i > 0, \\ < \dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i > 0, \end{cases}$$

$$C \begin{cases} < \dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i < 0, \\ \in \left[\dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 - tc_i)}{\beta_i}, \dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 + tc_i)}{\beta_i}\right] & \text{for no - trade assets with } \beta_i < 0, \\ > \dfrac{(x_{C,i} - x_{T,i})\sigma^2_{\varepsilon,i} - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i < 0. \end{cases} \tag{A12}$$

If $\beta_i = 0$, *then equation* (A11) *reduces to* (25) *and equation* (A12) *is replaced by* (18), (19), (21) *and* (22). *When cash is available*, λ_0 *disappears in* (A11) *and* (A12).

Proof We apply the general framework presented in the main body of the paper to analyze the optimal portfolio. Let's suppose for the moment that ***B, S*** and ***N*** of the optimal portfolio were known. Re-arranging equation (15) gives:

$$x_{B\setminus S} = x_{T,B\setminus S} + V_{B\setminus S}^{-1}V_{N,B\setminus S}^T(x_{T,N} - x_{C,N}) + k\,V_{B\setminus S}^{-1}(\lambda_0 I_{B\setminus S} - tc_{B\setminus -S}). \tag{A13}$$

Recall that $V_{B\setminus S}$ is the $n_{B\setminus S}$ by $n_{B\setminus S}$ submatrix of V with variance and covariance terms among buy and sell assets; $V_{N,B\setminus S}$ is the n_N by $n_{B\setminus S}$ submatrix of V with covariance terms between no-trade and buy or sell assets; n_N is the number of no-trade assets; and $n_{B\backslash S} = n - n_N$ is the number of buy and sell assets.

In the one-factor covariance model, the (i, i) entry of V is $\sigma_i^2 = \beta_i^2\sigma_M^2 + \sigma_{\varepsilon,i}^2$, and the (i,j) entry of V is $\beta_i\beta_j\sigma_M^2$. Let $\beta_{B\backslash S}$ be the $n_{B\backslash S}$ by 1 vector of betas of buy and sell assets, and let $V_{\varepsilon,B\backslash S}$ be the $n_{B\backslash S}$ by $n_{B\backslash S}$ diagonal matrix of idiosyncratic variances of buy and sell assets (i.e. the (i, i) entry of $V_{\varepsilon,B\backslash S}$ is $\sigma_{\varepsilon,i}^2$), then $V_{B\backslash S}$ equals $\sigma_M^2(\beta_{B\backslash S}\beta_{B\backslash S}^T) + V_{\varepsilon,B\backslash S}$. Applying the Woodbury matrix identity,† we have $V_{B\backslash S}^{-1} = V_{\varepsilon,B\backslash S}^{-1} - C_{B,S}(\beta_{B\backslash S,\sigma^2}\beta_{B\backslash S,\sigma^2}^T)$, where $\beta_{B\backslash S,\sigma^2}$ is the $n_{B\backslash S}$ by 1 vector with its ith element being $\beta_i/\sigma_{\varepsilon,i}^2$, and $C_{B,S} = (1/\sigma_M^2 + \sum_{i\in B\cup S}(\beta_i^2/\sigma_{\varepsilon,i}^2))^{-1}$ is a positive scalar. Therefore, for two buy or sell assets i and j, the (i,j) entry of $V_{B\backslash S}^{-1}$ is $-C_{B,S}(\beta_i/\sigma_{\varepsilon,i}^2)(\beta_j/\sigma_{\varepsilon,j}^2)$ if $i \neq j$, and $-C_{B,S}(\beta_i^2/\sigma_{\varepsilon,i}^4) + (1/\sigma_{\varepsilon,i}^2)$ if $i = j$.

Next, note that for a no-trade asset l, the (j, l) entry of $V_{N,B\backslash S}^T$ is $\beta_j\beta_l\sigma_M^2$. Therefore, the (i, l) entry of $V_{B\backslash S}^{-1}V_{N,B\backslash S}^T$ is:

$$\sum_{j\in B\cup S}\left(-C_{B,S}\frac{\beta_i}{\sigma_{\varepsilon,i}^2}\frac{\beta_j}{\sigma_{\varepsilon,j}^2}\beta_j\beta_l\sigma_M^2\right) + \frac{1}{\sigma_{\varepsilon,i}^2}\beta_i\beta_l\sigma_M^2$$
$$= \frac{1}{\sigma_{\varepsilon,i}^2}\beta_i\beta_l\sigma_M^2\left(1 - C_{B,S}\sum_{j\in B\cup S}\frac{\beta_j^2}{\sigma_{\varepsilon,j}^2}\right).$$

Note that $\sum_{j\in B\cup S}(\beta_j^2/\sigma_{\varepsilon,j}^2) = (1/C_{B,S}) - (1/\sigma_M^2)$, and the equation above simplifies to $C_{B,S}\beta_i\beta_l/\sigma_{\varepsilon,i}^2$.

So, the ith position of the vector $V_{B\backslash S}^{-1}V_{N,B\backslash S}^T(x_{T,N} - x_{C,N})$ in equation (A13) is:

$$\sum_{l\in N}\frac{C_{B,S}\beta_i\beta_l}{\sigma_{\varepsilon,i}^2}(x_{T,l} - x_{C,l}) = \frac{C_{B,S}\beta_i C_N}{\sigma_{\varepsilon,i}^2}. \tag{A14}$$

Here $C_N = \sum_{l\in N}\beta_l(x_{T,l} - x_{C,l})$ is a constant that only depends on assets in N.

Using $(\lambda_0 \mp tc_j)$ as the notation for $(\lambda_0 - tc_j)$ if $j \in B$ and $(\lambda_0 + tc_j)$ if $j \in S$ (and similarly with $(\lambda_0 \mp tc_i)$), the ith position of the vector $V_{B\backslash S}^{-1}(\lambda_0 I_{B\backslash S} - tc_{B\backslash -S})$ is:

$$\sum_{j\in B\cup S}\left(-C_{B,S}\frac{\beta_i}{\sigma_{\varepsilon,i}^2}\frac{\beta_j}{\sigma_{\varepsilon,j}^2}\cdot(\lambda_0 \mp tc_j)\right) + \frac{1}{\sigma_{\varepsilon,i}^2}\cdot(\lambda_0 \mp tc_i)$$
$$= -\frac{C_{B,S}\beta_i D_{B,S}}{\sigma_{\varepsilon,i}^2} + \frac{(\lambda_0 \mp tc_i)}{\sigma_{\varepsilon,i}^2}. \tag{A15}$$

Here $D_{B,S} = \sum_{j\in B\cup S}((\beta_j/\sigma_{\varepsilon,j}^2)\cdot(\lambda_0 \mp tc_j))$ is a constant that only depends on assets in B and S.

Plugging (A14) and (A15) into (A13) gives x_i for a buy or sell asset i:

$$x_i = x_{T,i} + \frac{C_{B,S}\beta_i C_N}{\sigma_{\varepsilon,i}^2} - k\frac{C_{B,S}\beta_i D_{B,S}}{\sigma_{\varepsilon,i}^2} + k\frac{(\lambda_0 \mp tc_i)}{\sigma_{\varepsilon,i}^2}.$$

Let $C = C_{B,S}C_N - kC_{B,S}D_{B,S}$ be a constant that depends on the optimal solution, then:

$$x_i = \begin{cases} x_{T,i} + C\beta_i/\sigma_{\varepsilon,i}^2 + k(\lambda_0 - tc_i)/\sigma_{\varepsilon,i}^2 & \text{for buy assets,} \\ x_{T,i} + C\beta_i/\sigma_{\varepsilon,i}^2 + k(\lambda_0 + tc_i)/\sigma_{\varepsilon,i}^2 & \text{for sell assets.} \end{cases} \tag{A16}$$

This is the same as equation (A11). Applying $x_i > x_{C,i}$ for buy assets and $x_i < x_{C,i}$ for sell assets and re-arranging (A16), we derive the following necessary conditions‡:

$$C\begin{cases} > \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i > 0, \\ < \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i > 0, \\ < \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i < 0, \\ > \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i < 0. \end{cases} \tag{A17}$$

Note that $k \cdot tc_i$ is positive, so in (A17) when $\beta_i > 0$ the buy condition lower bound is always higher than the sell condition upper bound, and when $\beta_i < 0$ the buy condition upper bound is always lower than the sell condition lower bound. Therefore, no asset can be labeled both as a buy and a sell asset. Equation (A17) covers buy and sell assets in (A12). Next, we prove the conditions for no-trade assets in (A12). We start by re-arranging equation (16) to:

$$\lambda_N - \frac{1}{2}(tc_N + \lambda_0 I_N) = -\frac{1}{2k}V_{N,B\backslash S}(x_{B\backslash S} - x_{T,B\backslash S})$$
$$- \frac{1}{2k}V_N(x_{C,N} - x_{T,N}). \tag{A18}$$

Let's first focus on the right-hand side of (A18). Given a no-trade asset l, the lth position of the first term on the right-hand side of (A18) is $-(\beta_l\sigma_M^2/2k)\sum_{i\in B\cup S}(\beta_i(x_i - x_{T,i}))$. Plugging in x_i from (A16) it becomes: $-(\beta_l\sigma_M^2/2k)(\sum_{i\in B\cup S}(C\beta_i{}^2/\sigma_{\varepsilon,i}^2) + k\cdot\sum_{i\in B\cup S}\beta_i(\lambda_0 \mp tc_i)/\sigma_{\varepsilon,i}^2)$, or $-(\beta_l\sigma_M^2/2k)(\sum_{i\in B\cup S}(C\beta_i{}^2/\sigma_{\varepsilon,i}^2) + kD_{B,S})$. The lth position of the second term on the right-hand side of (A18) is $-(\beta_l\sigma_M^2/2k)\sum_{i\in N}\beta_i(x_{C,i} - x_{T,i}) - (\sigma_{\varepsilon,l}^2/2k)(x_{C,l} - x_{T,l})$, or $(\beta_l\sigma_M^2/2k)C_N - (\sigma_{\varepsilon,l}^2/2k)(x_{C,l} - x_{T,l})$. So, the lth position of the right-hand side of (A18) is: $-(\beta_l\sigma_M^2/2k)(\sum_{i\in B\cup S}(C\beta_i{}^2/\sigma_{\varepsilon,i}^2) + kD_{B,S} - C_N) - (\sigma_{\varepsilon,l}^2(x_{C,l} - x_{T,l})/2k)$. But $kD_{B,S} - C_N = -C/C_{B,S}$ by the earlier definition of C, so this is $-(\beta_l\sigma_M^2 C/2k)(\sum_{i\in B\cup S}(\beta_i{}^2/\sigma_{\varepsilon,i}^2) - (1/C_{B,S})) - (\sigma_{\varepsilon,l}^2(x_{C,l} - x_{T,l})/2k)$, which simplifies to $(\beta_l C/2k) - (\sigma_{\varepsilon,l}^2(x_{C,l} - x_{T,l})/2k)$ because $C_{B,S} = (1/\sigma_M^2 + \sum_{i\in B\cup S}(\beta_i^2/\sigma_{\varepsilon,i}^2))^{-1}$.

Recall that $\lambda_l \geq 0$ and $\lambda_l \leq tc_l$ are the dual feasibility constraints in (9), so the lth position of the left-hand side (and therefore the right-hand side) of (A18) must lie between $-(1/2)(tc_l + \lambda_0)$ and $(1/2)(tc_l - \lambda_0)$. In other words,

$$-\frac{1}{2}(tc_l + \lambda_0) \leq \frac{\beta_l C}{2k} - \frac{\sigma_{\varepsilon,l}^2(x_{C,l} - x_{T,l})}{2k} \leq \frac{1}{2}(tc_l - \lambda_0). \tag{A19}$$

Re-arranging (A19) gives the necessary conditions for no-trade assets in (A12), where the lower bound for no-trade assets coincides with the upper bound for sell assets with $\beta_i > 0$ or buy assets with $\beta_i < 0$, and the upper bound for no-trade assets coincides with the lower bound for buy assets with $\beta_i > 0$ or sell assets with $\beta_i < 0$! In other words, the necessary conditions for buy, sell, and no-trade assets are mutually exclusive while jointly covering the entire real line. Therefore, they are also sufficient conditions.

† The Woodbury matrix identity is: $(UCV + A)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$, where $\boldsymbol{A}$, $\boldsymbol{C}$, $\boldsymbol{U}$, $\boldsymbol{V}$ are matrices of conformable sizes.

‡ For assets with zero beta, the necessary conditions are the same as those for uncorrelated assets in equations (18) and (19).