

The Efficient Use of Conditioning Information in Portfolios

WAYNE E. FERSON and ANDREW F. SIEGEL*

ABSTRACT

We study the properties of unconditional minimum-variance portfolios in the presence of conditioning information. Such portfolios attain the smallest variance for a given mean among all possible portfolios formed using the conditioning information. We provide explicit solutions for n risky assets, either with or without a riskless asset. Our solutions provide insights into portfolio management problems and issues in conditional asset pricing.

SINCE THE PIVOTAL WORK of Markowitz (1959) and Sharpe (1964), mean variance analysis has been a central focus of financial economics. Mean variance theory is used in portfolio analysis, asset pricing, investment performance measurement, and topics in corporate finance. Mean variance analysis also has other important economic applications. Problems involving quadratic objective functions or loss functions generally incorporate a mean variance analysis. Examples include economic policy under uncertainty, labor markets, monetary policy, inventory problems, hedging, resource economics, and a host of other applications.

This paper provides solutions to mean variance optimization problems in the presence of conditioning information. Conditioning information is present when the optimal solution may be a function of information to be received about the probability distribution of future outcomes. For example, empiri-

* Wayne E. Ferson is the Pigott-PACCAR Professor of Finance at the University of Washington and a Research Associate, NBER. Andrew F. Siegel is the Grant I. Butterbaugh Professor of Finance and Management Science and Adjunct Professor of Statistics at the University of Washington. We are grateful to Hank Bessembinder, Jonathan Berk, John Cochrane, Heber Farnsworth, the editor, Richard Green, Bruce Grundy, Lars Hansen, David Ikenberry, Patrice Poncet, Jeff Pontiff, Mark Ready, Ed Rice, Vance Roley, Richard W. Sias, Richard Startz, and an anonymous referee for helpful comments. The paper has also benefited from participants in workshops at the following universities: Arizona, Boston College, California at Berkeley, Duke, Illinois, Miami, New York University, Purdue, Rice, la Sorbonne, Southern California, South Carolina, Stanford, The Stockholm School of Economics, Utah, Wisconsin-Madison, the University of Washington (Finance, Management Science, and Statistics Departments), and Washington State University; also at the 1998 American Finance Association, the 1997 American Statistical Association (Anaheim), 1997 European Finance Association in Vienna, the 1997 Inquire Europe Autumn Seminar, the 1997 Euro Working Group on Financial Modelling Meeting in Venice, April 1997 meeting of the National Bureau of Economic Research Asset Pricing Group, VII International Conference on Stochastic Programming (1998) and the 1997 Western Finance Association meetings.

cal studies in asset pricing identify lagged variables that have some power in forecasting future stock and bond returns (see reviews by Ferson (1995), Keim and Hawawini (1995), or Cochrane (1999)). Such variables represent potential conditioning information. The efficient portfolio solution depends on the conditional expectations.

When there is conditioning information, mean variance efficiency may be defined in terms of the conditional means and variances (*conditionally efficient*), or in terms of unconditional moments. When the objective is to maximize the unconditional mean relative to the unconditional variance, where portfolio strategies may be functions of the information, we have *unconditional mean variance efficiency* with respect to the information.

Unconditional efficiency may be confusing because of the combination of conditioning information, which may be employed by the efficient portfolio, and the use of unconditional expectations about that portfolio's returns. However, such an information structure is quite common, for example, when the agent conducting a mean variance optimization uses more information than is available to the observer of the outcomes. If the observer does not have the conditioning information, he or she can only form unconditional expectations.

The type of information asymmetry that motivates unconditional efficiency is prominent in the literature on portfolio management (e.g., Mayers and Rice (1979)). Dybvig and Ross (1985) consider a portfolio manager who is evaluated based on the unconditional mean and variance of the portfolio return. The manager may use conditioning information about future returns in forming the portfolio. They show that the manager's conditionally efficient portfolio will not appear efficient to the uninformed investor. The unconditionally efficient portfolio we derive is the one that maximizes the measured performance.

The information structure that motivates unconditional efficiency can also occur in economic policy problems, labor markets, and in other agency problems. Unconditional efficiency is also useful when a strategy depends on a signal that will be, but has not yet been revealed to the agent. An econometrician wishing to test an economic model also typically faces the problem of using unconditional (or, less informed) expectations than the agents in the model.

In this paper, we derive unconditionally mean variance efficient portfolio strategies in closed form and illustrate their properties. The solutions are nonlinear functions of the conditional means and conditional covariance matrix of the returns, and may be written in a form analogous to the usual mean variance solutions.

Hansen and Richard (1987) study mean-variance efficient sets and unconditional efficiency with respect to conditioning information. They show that unconditionally efficient portfolios must be conditionally efficient, but not the converse. Given our closed-form solutions, we show that an unconditionally efficient portfolio maximizes a quadratic utility function for an agent who observes the signal. Mean variance efficient sets may then be understood by reference to the utility function that motivates a particular optimal solution.

We find that unconditionally efficient portfolio weights have interesting properties. They behave similarly to other utility maximizing strategies when the realization of the conditioning information is near the center of its distribution. For extreme signals about the returns of risky assets, the efficient solution requires a conservative response. An extremely high expected return presents an opportunity to reduce risk by taking a small position in the risky asset, without compromising the average portfolio performance. This behavior is similar to the redescending influence curves used in robust statistics, and it contrasts with the characteristic sensitivity of traditional portfolio strategies to extreme values.¹ The portfolio manager who is evaluated, as is common in practice, on the basis of unconditional mean return relative to unconditional return volatility (e.g., the Sharpe ratio), may be induced to adopt a conservative response to extreme signals to maximize the measured performance.

The results we develop are also valuable in the context of empirical asset pricing. Bekaert and Liu (1999) note that our solution provides versions of the Hansen-Jagannathan (1991) bounds that are robust to misspecification of the conditional moments. Ferson and Siegel (1999) study the small sample properties of various bounds with conditioning information, and find that our solutions provide empirically attractive bounds. Ferson and Siegel (2000) use our solutions to refine mean variance tests of portfolio efficiency. Because many problems in financial economics involve the determination of minimum variance portfolios, our results should also prove useful in other contexts.

The rest of the paper is organized as follows. Section I addresses the case of one risky asset and one riskless asset. Section II provides a discussion and interpretation of the efficient solution. The results are extended to the case of n risky assets in Section III. Section IV concludes the paper.

I. An Example with One Risky and One Riskless Asset

Consider a model with two assets: a riskless asset (with rate of return r_f) and a risky asset. The risky asset's return is

$$\tilde{R} = \mu(\tilde{S}) + \tilde{\varepsilon} \quad (1)$$

¹ Traditional mean-variance solutions tend to imply extreme positions in practice [e.g., Michaud (1989), Best and Grauer (1991)]. Recent intertemporal portfolio models, such as Kim and Omberg (1996) and Campbell and Viceira (1999), share this sensitivity. Green and Hollifield (1992) point out the apparent contradiction, in view of asset pricing models such as the CAPM, in which efficient portfolios should be well diversified. They then characterize the relation between diversification and mean-variance efficiency. Several approaches for reducing the sensitivity of mean-variance solutions in practice have been proposed. These include using ad hoc constraints on the portfolio positions (e.g., Frost and Savarino (1988)), and accounting for parameter uncertainty in a Bayesian framework (e.g., Bawa, Brown, and Klein (1979), Kandel and Stambaugh (1996), and Barberis (2000)).

where the conditioning information or signal is \tilde{S} (a random vector) and $\mu(\tilde{S})$ is the conditional expected return. We assume that $\mu(\tilde{S})$ is not identically constant (almost surely). The noise term, $\tilde{\varepsilon}$, is assumed to have conditional mean of zero given \tilde{S} . The conditional variance of the return given \tilde{S} , $\sigma_e^2(\tilde{S})$, is also the conditional variance of $\tilde{\varepsilon}$, which we allow to be a general function of the signal. Thus, our results allow general forms of conditional heteroskedasticity in returns, and normality is not required.

The weight function $x = x(\tilde{S})$ specifies the fraction invested in the risky asset, as a function of the observed signal \tilde{S} . The portfolio has unconditional expected return and variance given by

$$\begin{aligned}\mu_P &= E[r_f + x(\tilde{S})(\tilde{R} - r_f)] \\ &= r_f + E\{x(\tilde{S})[\mu(\tilde{S}) - r_f]\}\end{aligned}\quad (2)$$

and

$$\begin{aligned}\sigma_P^2 &= E\{[x(\tilde{S})(\tilde{R} - r_f)]^2\} - (\mu_P - r_f)^2 \\ &= E\{E[x^2(\tilde{S})(\tilde{R} - r_f)^2 | \tilde{S}]\} - (\mu_P - r_f)^2 \\ &= E\{x^2(\tilde{S})E[(\tilde{R} - r_f)^2 | \tilde{S}]\} - (\mu_P - r_f)^2 \\ &= E\{x^2(\tilde{S})[(\mu(\tilde{S}) - r_f)^2 + \sigma_e^2(\tilde{S})]\} - (\mu_P - r_f)^2,\end{aligned}\quad (3)$$

where simplification follows in each case by iterated expectations, conditioning first on \tilde{S} .

The unconditionally mean variance efficient strategy is provided in the following theorem.

THEOREM 1. *For a given unconditional expected return, μ_P , the unique portfolio having minimum unconditional variance places the following weight on the risky asset:*

$$x(\tilde{S}) = \frac{\mu_P - r_f}{\zeta} \left(\frac{\mu(\tilde{S}) - r_f}{(\mu(\tilde{S}) - r_f)^2 + \sigma_e^2(\tilde{S})} \right), \quad (4)$$

where the constant is

$$\zeta = E \left(\frac{(\mu(\tilde{S}) - r_f)^2}{(\mu(\tilde{S}) - r_f)^2 + \sigma_e^2(\tilde{S})} \right), \quad (5)$$

and the minimized variance is $\sigma_P^2 = (\mu_P - r_f)^2(1/\zeta - 1)$.

Proof: The proof is a special case of Theorem 2, proven in the Appendix.

The constant ζ may be interpreted by its relation to the slopes of both the conditional and unconditional mean-standard-deviation frontiers (see Jagan-

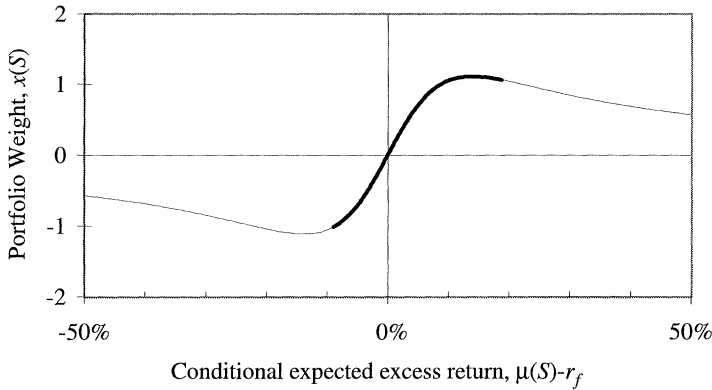


Figure 1. The portfolio weight $x(S)$ as a function of the conditional expected excess return. The darker portion of the curve emphasizes the most likely signal values ($\pm 3\sigma_S$). The expected return is $\mu_P = 11.1$ percent, with signal strength $R^2 = 10$ percent, where $R^2 = \text{Var}(\mu(\tilde{S}))/\text{Var}(\tilde{R})$, and \tilde{R} is the risky asset return.

nathan (1996)). Let the slope of the unconditional mean-standard-deviation frontier, with respect to the information \tilde{S} , be

$$\lambda = \text{Max}_{x(S)} \left[\frac{(\mu_P - r_f)}{\sigma_P} \right] = \left(\frac{1}{\zeta} - 1 \right)^{-1/2}.$$

Let the slope of the conditional frontier, given \tilde{S} , be $\lambda(\tilde{S}) = (\mu(\tilde{S}) - r_f)/\sigma_\varepsilon(\tilde{S})$. Simple algebra shows

$$\zeta = \frac{1}{1 + 1/\lambda^2} = E \left(\frac{1}{1 + 1/\lambda^2(\tilde{S})} \right) \quad (6)$$

which is equation (16) of Jagannathan (1996).

II. Interpretation and Discussion

A. Characterizing the Unconditionally Efficient Strategy

The solution given by equations (4) and (5) has a number of interesting features. This section illustrates some of these features, working with the special case of homoskedasticity. In this case, $\sigma_\varepsilon(\tilde{S}) = \sigma_\varepsilon$ is a constant and we denote the standard deviation of the signaled expected return $\mu(\tilde{S})$ as $\sigma_{\mu(\tilde{S})}$. Figure 1 illustrates an example of the efficient weight $x(\tilde{S})$ as a func-

tion of the conditional expected excess return $\mu(\tilde{S}) - r_f$ for a given unconditional mean μ_P equal to 11.1 percent per year.²

Figure 1 illustrates that when the conditional expected risk premium of the risky asset is zero, the weight in the risky asset is zero. For signals indicating a conditional expected excess return near zero, the unconditionally efficient weight appears monotone and nearly linear in $\mu(\tilde{S})$. This is similar to other utility-maximizing strategies.³ For example, assuming a normal distribution, the strategy that maximizes an exponential utility is linear in the conditional expected return $\mu(\tilde{S})$. Kim and Omberg (1996) solve an intertemporal portfolio problem analytically, and find that the portfolio weights are linear in the value function. The approximate solution to the intertemporal portfolio problem for Epstein-Zin (1989) preferences, as presented by Campbell and Viceira (1999), is also linear in the state variable. Thus, traditional solutions to the portfolio optimization problem imply portfolio weights that are sensitive to extreme values of the signal. For example, if the signal is normally distributed, a linear portfolio weight is unbounded. Note, however, that the unconditionally efficient $x(\tilde{S})$ is not monotone in $\mu(\tilde{S})$; in particular, $x(\tilde{S}) \rightarrow 0$ as $\mu(\tilde{S}) \rightarrow \pm\infty$. After a certain point, a more optimistic signal leads to purchasing *less* of the risky asset when the objective is to attain a given unconditional mean return with the smallest unconditional variance.

Figure 1 does not assume normality, but it does assume that $\sigma_\varepsilon(\tilde{S})$ is a constant. Thus the precise shape of the curve depends on the homoskedasticity assumption. However, according to equation (4), if an extreme value of the signal is associated with a large conditional variance, the conservative behavior of the strategy is reinforced relative to the homoskedastic example in the figures. More generally, the solution for the n -asset example discussed below implies that the portfolio weight should be a bounded function of the signal except in pathological cases.

Although the efficient strategy described by (4) holds the unconditional mean fixed at μ_P , the *conditional* expected return of this strategy, $r_f + E[x(\tilde{S})(\tilde{R} - r_f)|\tilde{S}]$, is a function of the signal \tilde{S} . This function is shown in Figure 2, together with the efficient weight, $x(\tilde{S})$, for the case in which the strength of the signal corresponds to $R^2 = 10$ percent, where $R^2 = \text{Var}(\mu(\tilde{S}))/\text{Var}(\tilde{R})$. The conditional expected return of the unconditionally efficient portfolio increases smoothly as the signal moves out of the region where $\mu(\tilde{S}) = 0$, in either direction. In the limit, the amount of risky asset pur-

² The example is matched to data for the period from 1963 to 1994. Over this period, the average one-month Treasury bill rate is $r_f = 6.27$ percent, the average return of the S&P500 is 11.1 percent per year, and the standard deviation is $\sqrt{\sigma_{\mu(S)}^2 + \sigma_\varepsilon^2} = 14.6$ percent. The optimal portfolio with weight function $x(S)$ achieves an expected return of $\mu_P = 11.1$ percent, but with standard deviation reduced to 11.2 percent, with an R^2 value of 10 percent as defined by $R^2 = \sigma_{\mu(S)}^2/(\sigma_{\mu(S)}^2 + \sigma_\varepsilon^2)$.

³ We show below that the efficient strategy $x(\tilde{S})$ maximizes quadratic utility. Markowitz (1991) argues that quadratic utility provides a good approximation to other utility functions in portfolio choice models.

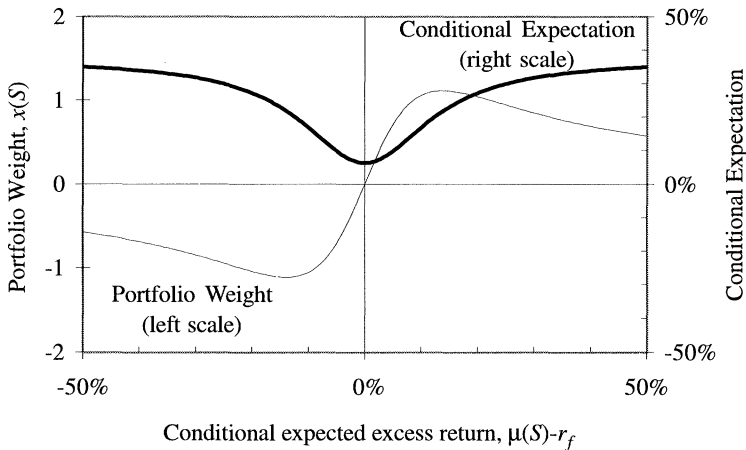


Figure 2. The conditional expectation of the unconditionally efficient portfolio return continues to rise toward a limit (37.1 percent) even as the portfolio weight declines towards zero. The unconditional expected return is $\mu_P = 11.1$ percent.

chased tends to zero in such a way that the conditional expected return approaches the constant $r_f + (\mu_P - r_f)/\zeta$ as $\mu(\tilde{S}) \rightarrow \pm\infty$. This upper limit for the conditional expected return must exceed μ_P in order for the average, or unconditional expected return, to equal μ_P .

B. Interpreting the Unconditionally Mean Variance Efficient Portfolio

One might be tempted to conclude that a very high signal, because its probability is low, is not to be believed. That, however, is not the explanation for the lack of monotonicity in Figures 1 and 2. In the current example, when a high signal is received, it reliably indicates a high expected return. Why not impose monotonicity on the weight function, and, for example, refuse to let it descend at the right in Figure 1? One answer is because the resulting portfolio would not be efficient. By using the nonmonotone portfolio weight function, one could either attain a higher unconditional expected return for the same standard deviation, or, alternatively, attain a lower standard deviation for the same expected return. The extra expected return that might be achieved by buying aggressively when the signal is high leads to additional risk. In other words, if the objective is to get the smallest unconditional variance for a given average return, then a signal that the expected return is unusually high presents an opportunity to reduce risk by purchasing a smaller amount of the risky asset, while maintaining the portfolio average return.

It is intriguing to note the similarity between the portfolio weight as a function of the signal, as in Figures 1 and 2, and the redescending influence curve in robust M -estimation (Hampel (1974), Goodall (1983), and Carroll

(1989)). Just as a robust estimator limits the statistical influence of an outlying data observation, the unconditionally efficient strategy limits the portfolio influence of an extreme signal observation. The “conservative” response of the unconditionally efficient strategy to an extreme signal suggests that the portfolio rule might be relatively robust to estimation errors.⁴

Another explanation for why the risky asset weight turns back for extreme signals comes from quadratic utility. It can be shown that for each μ_P , there exists a quadratic utility function for which our solution would be the optimal strategy, if the expected utility conditional on the signal is maximized (see the Appendix for a proof). Intuitively, because quadratic utility has increasing relative risk aversion, it aggressively avoids risk given extreme investment returns.

It may be tempting to interpret the nonmonotonicity of the unconditionally efficient strategy in Figures 1 and 2 as mirroring the negative marginal utility of the quadratic utility for extremely high returns. When the weight $x(S)$ given by equation (4) is the optimal choice for an agent with a quadratic utility function, the expected marginal utility of *additional investment* in the risky asset must be zero. Using the correspondence between μ_P and the utility function parameters derived in the Appendix, it may be shown that the expected marginal utility of portfolio return at the point of maximum investment in the risky asset, conditional on the signal value, is

$$\left(\frac{\mu_P - r_f}{\zeta} \right) \left(\frac{\sigma_\varepsilon^2(\tilde{S})}{(\mu(\tilde{S}) - r_f)^2 + \sigma_\varepsilon^2(\tilde{S})} \right),$$

which is positive when $\mu_P - r_f > 0$. The quadratic-utility agent for whom the unconditionally efficient solution is optimal does not become “satiated” as the weight in the risky asset is reduced for extreme signals. This may also be seen in Figure 2, which shows that the conditional mean return of the unconditionally efficient portfolio continues to rise, even as the weight in the risky asset is reduced, for extreme signals.

C. Conditional and Unconditional Efficiency Revisited

Hansen and Richard (1987) show that in the set of returns that can be generated using conditioning information, an unconditionally minimum-variance efficient strategy with respect to the information must be conditionally efficient, but the reverse is not true. The relation of our solution to quadratic utility provides a simple interpretation of this result. The relation between conditional and unconditional efficiency may be understood in terms of the utility functions for which the solutions are optimal.

⁴ An earlier version of this paper presented an out-of-sample experiment confirming this intuition in the context of an investment strategy. Bekaert and Liu (1999) prove the robustness in the context of the Hansen-Jagannathan (1991) bounds, and Ferson and Siegel (1999) confirm it with simulation experiments on the bounds.

We show in the Appendix that our unconditionally efficient portfolio maximizes the conditional expectation of a quadratic utility function in a single-period problem. Because quadratic-utility agents choose mean-variance efficient portfolios, this implies that the unconditionally efficient portfolio must be a conditionally mean-variance efficient portfolio. However, other utility functions can lead to mean-variance efficient portfolios. Consider the exponential utility function, when returns are normally distributed conditional on the signal. This utility maximization delivers a conditionally mean-variance efficient portfolio as the solution. Because our solution for the unconditionally efficient portfolio is unique, the exponential utility solution cannot be the same as our solution. The exponential utility agent chooses a conditionally mean-variance efficient portfolio that is not unconditionally efficient. Thus, conditional efficiency does not imply unconditional efficiency.

III. Multiple Risky Assets

Consider n risky assets with returns R_1, \dots, R_n and a riskless asset returning r_f . The i th asset has rate of return \tilde{R}_i . In $n \times 1$ column-vector notation, we have

$$\tilde{R} = \mu(\tilde{S}) + \tilde{\varepsilon}. \quad (7)$$

The noise term $\tilde{\varepsilon}$ is assumed to have conditional mean of zero given \tilde{S} , and nonsingular conditional covariance matrix $\Sigma_{\varepsilon}(\tilde{S})$, an arbitrary function of the signal \tilde{S} , thus allowing for general forms of conditional heteroskedasticity. The conditional expected return vector $\mu(\tilde{S})$ is permitted to have a singular or nonsingular (unconditional) covariance matrix, so there can be any number of independent signals (more or fewer than n) about the future asset returns.

Define portfolio P by letting the $1 \times n$ row vector $x'(\tilde{S}) = (x_1(\tilde{S}), \dots, x_n(\tilde{S}))$ denote the portfolio share invested in each of the n risky assets, investing (or borrowing) at the riskless rate the amount $1 - x'(\tilde{S})e$, where $e = (1, \dots, 1)'$ denotes the column vector of ones. The observed return on this portfolio will be $r_f + x'(\tilde{S})(\tilde{R} - r_f e)$, with unconditional expectation and variance (after computing conditional expectations given \tilde{S} to eliminate the random noise terms) as follows:

$$\mu_P = r_f + E[x'(\tilde{S})(\mu(\tilde{S}) - r_f e)] \quad (8)$$

$$\begin{aligned} \sigma_P^2 &= E\{x'(\tilde{S})[(\mu(\tilde{S}) - r_f e)(\mu(\tilde{S}) - r_f e)' + \Sigma_{\varepsilon}(\tilde{S})]x(\tilde{S})\} - (\mu_P - r_f)^2 \\ &= E[x'(\tilde{S})\tilde{\Lambda}^{-1}(\tilde{S})x(\tilde{S})] - (\mu_P - r_f)^2 \end{aligned} \quad (9)$$

where we have defined the $n \times n$ matrix

$$\begin{aligned} \Lambda(\tilde{S}) &= \{E[(\tilde{R} - r_f e)(\tilde{R} - r_f e)' | \tilde{S}]\}^{-1} \\ &= [(\mu(\tilde{S}) - r_f e)(\mu(\tilde{S}) - r_f e)' + \Sigma_{\varepsilon}(\tilde{S})]^{-1}. \end{aligned} \quad (10)$$

In the multivariate case, the constant ζ (which does not depend upon the signal \tilde{S}) is

$$\zeta = E[(\mu(\tilde{S}) - r_f e)' \Lambda(\tilde{S})(\mu(\tilde{S}) - r_f e)]. \quad (11)$$

THEOREM 2. *Given the unconditional expected return μ_P , n risky assets, and a riskless asset, the unique portfolio having minimum unconditional variance is determined by the weights*

$$x'(\tilde{S}) = \frac{\mu_P - r_f}{\zeta} (\mu(\tilde{S}) - r_f e)' \Lambda(\tilde{S}). \quad (12)$$

The portfolio variance is

$$\sigma_P^2 = (\mu_P - r_f)^2 \left(\frac{1}{\zeta} - 1 \right). \quad (13)$$

Proof: See the Appendix.

The solution given by Theorem 2 provides portfolio weights that are bounded functions of the conditional mean, $\mu(\tilde{S})$. In particular, $x'(\tilde{S})x(\tilde{S})$ is bounded for any value of the signal \tilde{S} that does not imply a singular conditional covariance matrix. This generalizes the observation in Figure 1 that the weights in the risky asset are conservative given extreme signal realizations.⁵

A. No Riskless Asset

When there is no riskless asset, there are multiple equivalent representations for unconditionally mean-variance efficient portfolios.⁶ Any two portfolios on the unconditional minimum-variance boundary can be combined with *fixed* weights (i.e., not a function of the signal) to generate the entire boundary (Hansen and Richard (1987)). One representation of the solution is

⁵ To see this, note that it can be shown that

$$\begin{aligned} x'(\tilde{S})x(\tilde{S}) &= \left(\frac{\mu_P - r_f}{\zeta} \right)^2 \frac{(\mu(\tilde{S}) - r_f e)' \Sigma_E^{-1}(\tilde{S}) \Sigma_E^{-1}(\tilde{S})(\mu(\tilde{S}) - r_f e)}{[1 + (\mu(\tilde{S}) - r_f e)' \Sigma_E^{-1}(\tilde{S})(\mu(\tilde{S}) - r_f e)]^2} \\ &= \left(\frac{\mu_P - r_f}{\zeta} \right)^2 \frac{\nu'(\tilde{S}) D^{-1}(\tilde{S}) \nu(\tilde{S})}{(1 + \|\nu(\tilde{S})\|^2)^2} \leq \left(\frac{\mu_P - r_f}{\zeta} \right)^2 \frac{\|\nu(\tilde{S})\|^2}{\lambda_{\min}(\tilde{S})(1 + \|\nu(\tilde{S})\|^2)^2} \\ &\leq \frac{(\mu_P - r_f)^2}{4\zeta^2 \lambda_{\min}(\tilde{S})}, \end{aligned}$$

where we have defined $\nu(\tilde{S}) = \sqrt{D^{-1}(\tilde{S})} U(\tilde{S})[\mu(\tilde{S}) - r_f e]$ using the decomposition $\Sigma_E^{-1}(\tilde{S}) = U'(\tilde{S}) D^{-1}(\tilde{S}) U(\tilde{S})$, where D is diagonal and U is orthogonal, $\lambda_{\min}(\tilde{S})$ is the smallest eigenvalue of $\Sigma_E(\tilde{S})$, and $\|\cdot\|$ denotes the L^2 norm. We assume that $\lambda_{\min}(\tilde{S})$ is bounded away from zero for all \tilde{S} , which implies that no signal can indicate a nearly singular conditional covariance matrix.

⁶ A previous working version of this paper provides alternative representations. These results are available on request.

in the same basic form as in the previous section, using the global minimum-variance portfolio in place of the riskless asset. We use the same notation as in the previous section, except that we redefine the matrix Λ as follows:

$$\Lambda(\tilde{S}) = [E(\tilde{R}\tilde{R}'|\tilde{S})]^{-1} = [\mu(\tilde{S})\mu'(\tilde{S}) + \Sigma_{\varepsilon}(\tilde{S})]^{-1}. \quad (14)$$

Define portfolio P by letting $x' = x'(\tilde{S}) = (x_1(\tilde{S}), \dots, x_n(\tilde{S}))$ denote the shares invested in each of the n risky assets, with the constraint that $x'e = 1$. The observed return on this portfolio, $R_P = x'(\tilde{S})\tilde{R}$, has expectation and variance (after computing conditional expectations given \tilde{S} to eliminate the random noise terms) as follows:

$$\begin{aligned} \mu_P &= E[x'(\tilde{S})\mu(\tilde{S})], \\ \sigma_P^2 &= E\{x'(\tilde{S})[\mu(\tilde{S})\mu'(\tilde{S}) + \Sigma_{\varepsilon}(\tilde{S})]x(\tilde{S})\} - \mu_P^2. \end{aligned} \quad (15)$$

Define the following portfolio constants:

$$\alpha_1 = E\left(\frac{1}{e'\Lambda(\tilde{S})e}\right), \quad (16)$$

$$\alpha_2 = E\left(\frac{e'\Lambda(\tilde{S})\mu(\tilde{S})}{e'\Lambda(\tilde{S})e}\right), \quad (17)$$

$$\alpha_3 = E\left[\mu'(\tilde{S})\left(\Lambda(\tilde{S}) - \frac{\Lambda(\tilde{S})ee'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right]. \quad (18)$$

The parameters α_1 , α_2 , and α_3 may be considered analogues to the “efficient set constants” in the classical mean-variance efficiency analysis that ignores conditioning information (e.g., Ingersoll (1987)).

THEOREM 3: *Given unconditional expected return μ_P , n risky assets, and no riskless asset, the unique portfolio having minimum unconditional variance is determined by the weights*

$$x'(\tilde{S}) = \frac{e'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3} \mu'(\tilde{S}) \left(\Lambda(\tilde{S}) - \frac{\Lambda(\tilde{S})ee'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e} \right). \quad (19)$$

The variance of the portfolio defined by $x(\tilde{S})$ is

$$\sigma_P^2 = \left(\alpha_1 + \frac{\alpha_2^2}{\alpha_3} \right) - \frac{2\alpha_2}{\alpha_3} \mu_P + \frac{1 - \alpha_3}{\alpha_3} \mu_P^2. \quad (20)$$

Proof: See the Appendix.

COROLLARY: *The global minimum-variance portfolio has the following mean and variance:*

$$\mu^* = \alpha_2 / (1 - \alpha_3). \quad (21)$$

$$(\sigma^*)^2 = \alpha_1 - \alpha_2^2 / (1 - \alpha_3). \quad (22)$$

Proof: This follows immediately by minimizing the quadratic function for σ_P^2 as a function of μ_P , as given in (20). Q.E.D.

IV. Summary and Conclusions

We derive unconditionally mean-variance efficient portfolios when the weights may be a function of conditioning information. Specifically, we provide the functional forms of the portfolio weights for three situations: (1) an economy with one risky asset and one riskless asset, (2) an economy with n risky assets and one riskless asset, and (3) an economy with n risky assets but no riskless asset.

We demonstrate that the unconditionally efficient portfolio weight is not monotone in the realization of the signal about future returns, but is "conservative" in the face of an extreme signal. An extremely high conditional mean return presents an opportunity to reduce the portfolio risk without compromising its average performance. The solution implies an interesting form of robustness. Our results are useful for asset pricing, portfolio performance measurement, and other problems.

Appendix

Proof of Theorem 2: That the portfolio mean is μ_P follows directly from equations (8) and (12) and the definition of ζ . The portfolio variance can be found by substituting for just one $x(\tilde{S})$ term, then recognizing and substituting for the portfolio mean as follows:

$$\begin{aligned} \sigma_P^2 &= E[x'(\tilde{S})\tilde{\Lambda}^{-1}(\tilde{S})x(\tilde{S})] - (\mu_P - r_f)^2 \\ &= \frac{\mu_P - r_f}{\zeta} E[x'(\tilde{S})(\mu(\tilde{S}) - r_f e)] - (\mu_P - r_f)^2 \\ &= \frac{(\mu_P - r_f)^2}{\zeta} - (\mu_P - r_f)^2 = (\mu_P - r_f)^2 \left(\frac{1}{\zeta} - 1 \right). \end{aligned} \quad (23)$$

Now suppose that $y(\tilde{S})$ defines another portfolio Y that has the same mean as P defined by $x(\tilde{S})$, so that $\mu_P = \mu_Y$. Consider the portfolio whose weights are defined by $y(\tilde{S}) - x(\tilde{S})$; in particular, its variance cannot be negative. We then have

$$\begin{aligned} \sigma_P^2 &\leq \sigma_P^2 + \text{Var}[y'(\tilde{S})(\tilde{R} - r_f e) - x'(\tilde{S})(\tilde{R} - r_f e)] \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2\text{Cov}[y'(\tilde{S})(\tilde{R} - r_f e), x'(\tilde{S})(\tilde{R} - r_f e)] \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2E[y'(\tilde{S})(\tilde{R} - r_f e)(\tilde{R} - r_f e)'x(\tilde{S})] + 2(\mu_P - r_f)^2. \end{aligned} \quad (24)$$

Substituting for \tilde{R} , using iterated expectations given \tilde{S} , then substituting for x , we find

$$\begin{aligned}
 \sigma_P^2 &\leq \sigma_Y^2 + 2\sigma_P^2 - 2E[y'(\tilde{S})\tilde{\Lambda}^{-1}(\tilde{S})x(\tilde{S})] + 2(\mu_P - r_f)^2 \\
 &= \sigma_Y^2 + 2\sigma_P^2 - 2\frac{\mu_P - r_f}{\zeta} E[y'(\tilde{S})(\mu(\tilde{S}) - r_f e)] + 2(\mu_P - r_f)^2 \\
 &= \sigma_Y^2 + 2\sigma_P^2 - 2\frac{(\mu_P - r_f)^2}{\zeta} + 2(\mu_P - r_f)^2 \\
 &= \sigma_Y^2 + 2\sigma_P^2 - 2\sigma_P^2 = \sigma_Y^2,
 \end{aligned} \tag{25}$$

which establishes that $\sigma_P^2 \leq \sigma_Y^2$ and completes the proof. Q.E.D.

Proof of Theorem 3: The expected return of the portfolio defined by x may be found as follows:

$$\begin{aligned}
 E(x'(\tilde{S})\tilde{R}) &= E[x'(\tilde{S})\mu(\tilde{S})] \\
 &= E\left[\frac{e'\Lambda(\tilde{S})\mu(\tilde{S})}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3} \mu'(\tilde{S})\left(\Lambda(\tilde{S}) - \frac{\Lambda(\tilde{S})ee'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right] \\
 &= \alpha_2 + \frac{\mu_P - \alpha_2}{\alpha_3} \alpha_3 = \mu_P.
 \end{aligned} \tag{26}$$

The variance of the portfolio is computed by substituting for $x(\tilde{S})$ and using the fact $x'e = 1$:

$$\begin{aligned}
 \sigma_P^2 &= E[x'(\tilde{S})\Lambda^{-1}(\tilde{S})x(\tilde{S})] - \mu_P^2 \\
 &= E\left\{x'(\tilde{S})\Lambda^{-1}(\tilde{S})\left[\frac{\Lambda(\tilde{S})e}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3}\left(\Lambda(\tilde{S}) - \frac{\Lambda(\tilde{S})ee'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right]\right\} \\
 &\quad - \mu_P^2 \\
 &= E\left[\frac{x'(\tilde{S})e}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3}\left(x'(\tilde{S}) - \frac{x'(\tilde{S})ee'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right] - \mu_P^2 \\
 &= E\left[\frac{1}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3}\left(x'(\tilde{S}) - \frac{e'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right] - \mu_P^2.
 \end{aligned} \tag{27}$$

Continuing by taking the expectations, we find

$$\begin{aligned}
 \sigma_P^2 &= \alpha_1 + \frac{\mu_P - \alpha_2}{\alpha_3} \mu_P - \frac{\mu_P - \alpha_2}{\alpha_3} \alpha_2 - \mu_P^2 \\
 &= \left(\alpha_1 + \frac{\alpha_2^2}{\alpha_3}\right) - \frac{2\alpha_2}{\alpha_3} \mu_P + \frac{1 - \alpha_3}{\alpha_3} \mu_P^2.
 \end{aligned} \tag{28}$$

Next, suppose that $y(\tilde{S})$ defines another portfolio Y that has the same mean as P defined by x , so that $\mu_P = \mu_Y$. Because variance cannot be negative, we have⁷

$$\begin{aligned}\sigma_P^2 &\leq \sigma_P^2 + \text{Var}[y'(\tilde{S})\tilde{R} - x'(\tilde{S})\tilde{R}] \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2\text{Cov}[y'(\tilde{S})\tilde{R}, x'(\tilde{S})\tilde{R}] \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2E[y'(\tilde{S})\tilde{R}\tilde{R}'x(\tilde{S})] + 2\mu_P^2.\end{aligned}\quad (29)$$

Substituting for \tilde{R} , taking conditional expectation given \tilde{S} , then substituting for x , we find (using $y'e = 1$)

$$\begin{aligned}\sigma_P^2 &\leq \sigma_Y^2 + 2\sigma_P^2 - 2E[y'(\tilde{S})\Lambda^{-1}(\tilde{S})x(\tilde{S})] + 2\mu_P^2 \\ &= \sigma_Y^2 + 2\sigma_P^2 \\ &\quad - 2E\left\{y'(\tilde{S})\Lambda^{-1}(\tilde{S})\left[\frac{\Lambda(\tilde{S})e}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3}\left(\Lambda(\tilde{S}) - \frac{\Lambda(\tilde{S})ee'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right]\right\} \\ &\quad + 2\mu_P^2 \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2E\left[\frac{y'(\tilde{S})e}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3}\left(y'(\tilde{S}) - \frac{y'(\tilde{S})ee'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right] \\ &\quad + 2\mu_P^2 \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2E\left[\frac{1}{e'\Lambda(\tilde{S})e} + \frac{\mu_P - \alpha_2}{\alpha_3}\left(y'(\tilde{S}) - \frac{e'\Lambda(\tilde{S})}{e'\Lambda(\tilde{S})e}\right)\mu(\tilde{S})\right] + 2\mu_P^2 \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2\left[\alpha_1 + \frac{\mu_P - \alpha_2}{\alpha_3}\mu_P - \frac{\mu_P - \alpha_2}{\alpha_3}\alpha_2\right] + 2\mu_P^2 \\ &= \sigma_Y^2 + 2\sigma_P^2 - 2\sigma_P^2 = \sigma_Y^2,\end{aligned}\quad (30)$$

which establishes that $\sigma_P^2 \leq \sigma_Y^2$. Q.E.D.

Proof of uniqueness: We argue that the solution $x(S)$ for the unconditionally efficient portfolio weight is unique, assuming that the conditional covariance of returns is nonsingular. For a given value of μ_P , assume that there is another solution, $y(S)$, with $E[y'(S)R] = \mu_P$ and $\text{Var}[y'(S)R] = \text{Var}[x'(S)R] \equiv \sigma^2$. Both portfolios plot on the unconditional mean-standard deviation boundary at the same point. Consider a portfolio combining these two with a fixed weight, α . Any fixed-weight portfolio, combining two points on the boundary, must also plot on the boundary (e.g., Hansen and Richard (1987)); in this case, they plot at the same point because the portfolio mean is still μ_P . Thus, $\sigma^2 = \alpha^2\sigma^2 + (1 - \alpha)^2\sigma^2 + 2\alpha(1 - \alpha)\sigma^2\rho$, implying that the

⁷ Note that weights $\bar{y} - \bar{x}$ define an "arbitrage portfolio" return because they sum to zero, not one. Of course, they do define a random variable that must have nonnegative variance.

correlation, ρ , between $x'(S)R$ and $y'(S)R$ must be 1. That is, if we “regress” one portfolio on the other, the error term is almost surely (a.s.) zero: $y'(S)R = a + b [x'(S)R]$ (a.s.). Equating the unconditional means and variances implies $a = 0$ and $b = 1$. Thus $y'(S)R = x'(S)R$ (a.s.). The assumption that the conditional covariance matrix of R is nonsingular then implies that $y'(S) = x'(S)$ (a.s.). This follows from observing that if $\text{Cov}(R|S)$ is nonsingular, then, because $\text{Cov}(R|S)$ is positive definite, $\text{Cov}(R|S) + E(R|S)E(R'|S)$ is positive definite. Finally, using the fact that $y'(S)R = x'(S)R$ (a.s.), it follows that

$$\begin{aligned} 0 &= E\{[(y'(S) - x'(S))R][(y'(S) - x'(S))R']^T | S\} \\ &= E\{(y'(S) - x'(S))RR'(y(S) - x(S)) | S\} \\ &= (y'(S) - x'(S))[\text{Cov}(R|S) + E(R|S)E(R'|S)](y(S) - x(S)). \end{aligned} \quad (31)$$

Because this is a quadratic form with a positive definite matrix, it follows that $y(S) = x(S)$ (a.s.). Q.E.D.

Conditional quadratic utility maximization: We show that maximizing the conditional expected value of a quadratic utility function is equivalent to finding an unconditionally efficient portfolio for the case of n risky assets and a riskless asset. (The extension to n risky assets and no riskless asset is straightforward.) The problem is to maximize $E[u(\tilde{R}_P)|S]$ where $\tilde{R}_P = r_f + x'(S)(\tilde{R} - r_f e)$. The utility function is $u(R_P) = a + bR_P - (c/2)R_P^2$. The first-order conditions, which are necessary and sufficient for the maximization, are $E\{(b - cR_P)[\mu(S) - r_f + \tilde{\varepsilon}] | S\} = 0$. Substituting the expression for R_P and solving for $x(S)$, we obtain $x'(S) = (b/c - r_f)\mu'(S)\Lambda^{-1}(S)$. This solution is equivalent to the solution given by (12), in the sense that for a given μ_P in (12), there exists a quadratic utility with $b/c = r_f + [(\mu_P - r_f)/\xi]$ that would choose the same portfolio.

REFERENCES

- Barberis, Nicholas, 2000, Investing for the long run when returns are predictable, *Journal of Finance* 55, 225–264.
- Bawa, Vijay, Stephen Brown, and Roger Klein, 1979, *Estimation Risk and Optimal Portfolio Choice* (North Holland, Amsterdam).
- Bekaert, Geert, and Jun Liu 1999, Conditioning information and variance bounds on pricing kernels, Working paper, Stanford University.
- Best, Michael J., and R. Grauer, 1991, On the sensitivity of mean-variance-efficient portfolios to changes in asset means: Some analytical and computational results, *Review of Financial Studies* 4, 315–342.
- Campbell, John Y., and Luis M. Viceira, 1999, Consumption and portfolio decisions when expected returns are time-varying, Working paper, Harvard University.
- Carroll, Raymond J., 1989, Redescending M -Estimators; in *Encyclopedia of Statistical Sciences, Supplement Volume*, pp. 134–137 (John Wiley, New York).
- Cochrane, John H., 1999, Asset pricing, Working paper, University of Chicago.
- Dybvig, Philip H., and Stephen A. Ross, 1985, Differential information and performance measurement using a security market line, *Journal of Finance* 60, 383–399.
- Epstein, Larry G., and Stanley E. Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

- Ferson, Wayne E., 1995, Theory and empirical testing of asset pricing models; in *Handbooks in Operations Research and Management Science*, Vol. 9, Chapter 5 (Elsevier, Amsterdam).
- Ferson, Wayne E., and Andrew F. Siegel, 1999, Stochastic discount factor bounds with conditioning information, Working paper, University of Washington.
- Ferson, Wayne E., and Andrew F. Siegel, 2000, Testing portfolio efficiency with conditioning information, Working paper, University of Washington.
- Frost, Peter, and James E. Savarino, 1988, For better performance: Constrain portfolio weights, *Journal of Portfolio Management* 15, 29–34.
- Goodall, Colin, 1983, *M*-estimators of location: An outline of the theory; in David C. Hoaglin, Frederick Mosteller, and John W. Tukey, eds.: *Understanding Robust and Exploratory Data Analysis* (John Wiley, New York).
- Green, Richard C., and Burton Hollifield, 1992, When will mean-variance efficient portfolios be well-diversified? *Journal of Finance* 47, 1785–1809.
- Hampel, Frank R., 1974, The influence curve and its role in robust estimation, *Journal of the American Statistical Association* 69, 383–393.
- Hansen, Lars P., and Ravi Jagannathan, 1991, Implications of security market data for models of dynamic economies, *Journal of Political Economy* 99, 225–262.
- Hansen, Lars P., and Scott F. Richard, 1987, The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models, *Econometrica* 55, 587–613.
- Ingersoll, Jonathan E., Jr., 1987, *Financial Decision Making* (Rowman & Littlefield, Savage, MD).
- Jagannathan, Ravi, 1996, Relation between the slopes of the conditional and unconditional mean-standard deviation frontiers of asset returns; in Sasumu Saito, Katsushige Sawaki, and Keiichi Kubota, eds.: *Modern Portfolio Theory and its Applications: Inquires into Asset Valuation Problems* (Center for Academic Societies, Osaka, Japan).
- Kandel, Shmuel A., and Robert F. Stambaugh, 1996, On the predictability of stock returns: An asset allocation perspective, *Journal of Finance* 51, 385–424.
- Keim, Donald B., and Gabriel Hawawini, 1995, On the predictability of common stock returns: World-wide evidence; in Robert A. Jarrow, Vojislav Maksimovic, and William T. Ziemba, eds.: *Finance, Handbooks in Operations Research and Management Science* (Elsevier, Amsterdam).
- Kim, Tong Suk, and Edward Omberg, 1996, Dynamic nonmyopic portfolio behavior, *Review of Financial Studies* 9, 141–161.
- Markowitz, Harry M., 1959, *Portfolio Selection: Efficient Diversification of Investments* (John Wiley, New York).
- Markowitz, Harry M., 1991, Foundations of portfolio theory, *Journal of Finance* 46, 469–478.
- Mayers, David, and Edward M. Rice, 1979, Measuring portfolio performance and the empirical content of asset pricing models, *Journal of Financial Economics* 7, 3–28.
- Michaud, Richard O., 1989, The Markowitz optimization enigma: Is optimized optimal? *Financial Analysts Journal* 45, 31–42.
- Sharpe, William F., 1964, Capital asset prices: A theory of market equilibrium under conditions of risk, *Journal of Finance* 19, 425–442.