DYNAMIC PORTFOLIO SELECTION
BY AUGMENTING THE ASSET SPACE

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Dynamic Portfolio Selection by Augmenting the Asset Space
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NBER Working Paper No. 10372
March 2004
JEL No. G0, G1

## ABSTRACT

We present a novel approach to dynamic portfolio selection that is no more difficult to implement than the static Markowitz model. The idea is to expand the asset space to include simple (mechanically) managed portfolios and compute the optimal static portfolio in this extended asset space. The intuition is that a static choice among managed portfolios is equivalent to a dynamic strategy. We consider managed portfolios of two types: "conditional" and "timing" portfolios. Conditional portfolios are constructed along the lines of Hansen and Richard (1987). For each variable that affects the distribution of returns and for each basis asset, we include a portfolio that invests in the basis asset an amount proportional to the level of the conditioning variable. Timing portfolios invest in each basis asset for a single period and therefore mimic strategies that buy and sell the asset through time. We apply our method to a problem of dynamic asset allocation across stocks, bonds, and cash using the predictive ability of four conditioning variables.

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# 1  Introduction

Several studies have pointed out the importance of dynamic trading strategies to exploit the predictability of the first and second moments of asset returns and to hedge changes in the investment opportunity set. However, computing these optimal dynamic investment strategies has proven to be a rather formidable problem. Because closed-form solutions are only available for a few cases, researchers have explored a variety of numeric methods, including solving partial differential equations, discretizing the state-space, and using Monte Carlo simulation. Unfortunately, these techniques are out of reach for most practitioners and have therefore remained largely in the ivory tower. The workhorse of portfolio optimization in industry remains the static Markowitz approach.

Our paper presents a novel approach to dynamic portfolio selection that is no more difficult to implement than the static Markowitz model. The idea is to expand the asset space to include simple (mechanically) managed portfolios and compute the optimal *static* portfolio within this extended asset space. The intuition is that a static choice of managed portfolios is equivalent to a dynamic strategy. The optimal dynamic strategy can therefore be expressed as a fixed combination of mechanically managed portfolios. We consider managed portfolios of two types: "conditional" and "timing" portfolios. Conditional managed portfolios are constructed along the lines of Hansen and Richard (1987).[1] For each variable that affects the distribution of returns and for each basis asset, we consider a portfolio that invests in the basis asset an amount proportional to the level of the conditioning variable. Timing portfolios invest in each asset for a single period and in the risk-free rates in all other periods. Timing portfolios mimic strategies that buy and sell the asset through time. For example, holding a constant amount of all the timing portfolios related to a single asset approximates a strategy that holds a constant proportion of wealth in the asset. In contrast, hedging demands induce the investor to hold different amounts of the timing portfolios through time.

Having expanded the asset space with managed portfolios, we can use the Markowitz solution to find the optimal strategy for a mean-variance investor. The optimal strategy is a combination of managed portfolios but it is trivial to recover the corresponding investment in the basis assets at each point in time given the values of the conditioning

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[1] Hansen and Richard (1987) introduced this idea to develop tests of conditional asset pricing models. Bansal and Harvey (1996) use conditional portfolios in performance evaluation.

variables. We show that the weight invested in each basis asset at each point in time is a simple linear function of the state variables. Our approach can thus be seen as parameterizing the portfolio policy as a function of the state variables and then maximizing the investor's utility by choosing optimally the coefficients of this function.

The advantage of framing the dynamic portfolio problem in a static context is that all the refinements developed over the years for the Markowitz model are at our disposal. These include the use of portfolio constraints, shrinkage estimation, and the combination of the investor's prior beliefs with the information contained in the history of returns.

In general, our approach relies on sample moments of the long-horizon returns of the expanded set of assets. However, if the log returns on the basis assets and the log state variables follow a vector auto-regression (VAR) with normally distributed innovations, as is typically assumed in the line of research initiated by Campbell and Viceira (1999),[2] the long-horizon moments of returns can all be expressed in terms of the parameters of the VAR. In this special but popular case, we use our approach to obtain approximate closed-form solutions for the finite-horizon dynamic portfolio choice which complement the approximate closed-form solutions for the infinite-horizon case with intermediate consumption derived by Campbell and Viceira.

Our approach is similar in spirit to that of Cox and Huang (1989) and its empirical implementation by Aït-Sahalia and Brandt (2004). They solve the dynamic portfolio problem in two steps. The investor first chooses the optimal portfolio of Arrow-Debreu securities and then figures out how to replicate this portfolio by dynamically trading the basis assets or derivatives on the basis assets. In contrast, we solve the portfolio problem in one step as the optimal choice across simple dynamic trading strategies. Note also that the Cox-Huang approach requires financial markets to be complete, for only then all Arrow-Debreu securities can be replicated, whereas we do not need to assume market completeness since the investor only chooses among feasible strategies.

Our paper also relates to Ferson and Siegel (2001). They assume that the conditional mean vector and covariance matrix of asset returns are *known* functions of the state variables and then derive the optimal portfolio weights by maximizing a mean-variance utility function (in an unconditional sense similar to ours). They show that the resulting portfolio weights are also functions of the state variables since they depend

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[2]See also the survey by Campbell and Viceira (2002).

on the conditional means and conditional variances and covariances of asset returns. In contrast, we model the portfolio weights directly as functions of the state variables and find the coefficients of these functions that maximize the investor's utility. Our portfolio weights implicitly take into account the impact of the state variables on both the means and the variances and covariances of asset returns since all of these moments affect the portfolio's expected return and risk, and thereby the investor's expected utility. Therefore, our method can be interpreted as an approximation of the solution offered by Ferson and Siegel. For instance, by postulating that the optimal portfolio weights are linear in the state variables, we implicitly constrain the forms of the mean vector and the covariance matrix of returns as functions of the state variables.

The two methods are quite different when applied in practice. To use Ferson and Siegel's approach, we need to estimate conditional means, variances, and covariances of returns as functions of the state variables. While conditional mean functions can easily be estimated by regressing returns on the state variables, it is notoriously difficult to estimate a conditional covariance matrix as a function of state variables in a manner that guarantees positive positive semi-definiteness at all times. In contrast, estimating the portfolio weight function in our approach does not require imposing any sort of nonlinear constraints. Furthermore, our approach has the advantage of being much more parsimonious. Suppose we are interested in forming optimal portfolios of $N$ assets. With Ferson and Siegel's approach, we have to estimate $N$ functions of the state variables for the expected return vector and $N(N+1)/2$ functions for the covariance matrix. With our approach, we only need to estimate $N$ functions for the optimal portfolio weights. The gains in computation and estimation precision are evident.

The paper proceeds as follows. We first describe our approach in Sections 2.1 and 2.2. We then illustrate the mechanics of our approach through a simple example in Section 2.3 and examine its accuracy in Section 2.4. Section 3 deals with the special case in which the log returns and log state variables follow a Gaussian VAR and Section 4 discusses briefly how several refinements of the static Markowitz approach can be directly applied to our approach. We illustrate our approach through an empirical application in Section 5 and conclude in Section 6.

# 2 The Method

We solve a conditional portfolio choice problem with parameterized portfolio weights of the form $x_t = \theta z_t$, where $z_t$ denotes a vector of state variables and $\theta$ is a matrix of coefficients. This conditional portfolio choice problem is mathematically *equivalent* to solving an unconditional problem within an augmented asset space that includes naively managed zero-investment portfolios with excess returns of the form $z_t$ times the excess return of each basis asset. We first establish this idea in the context of a single-period problem and then extend the approach to the multiperiod case. Finally, we illustrate both cases in a simple example and examine the accuracy of the solutions in a numerical experiment.

## 2.1 Single-Period Problem

Suppose we want to find the optimal portfolio policy under a quadratic criterium on the excess returns of the portfolio:

$$\max \mathrm{E}_t \left[ r_{t+1}^p - \frac{\gamma}{2} (r_{t+1}^p)^2 \right], \tag{1}$$

where $\gamma$ is a positive constant and $r_{t+1}^p = R_{t+1}^p - R_t^f$ is the excess return on the portfolio from $t$ to $t+1$.[3] Throughout the paper we use capital letters to denote gross returns and lower-case letters to denote excess returns. We date all variables with a subscript that corresponds to the time at which the variable is known. For example, returns of risky assets from time $t$ to time $t+1$ are denoted $R_{t+1}$. The risk-free rate for the same period is denoted $R_t^f$, since that is known at the beginning of the return period.

Denote the vector of portfolio weights on the risky assets at time $t$ by $x_t$. The above

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[3]This criterium function arises from quadratic utility over wealth $u(W_{t+1}) = W_{t+1} - \frac{a}{2} W_{t+1}^2$. In this case, the relative risk aversion coefficient is given by $\kappa = \frac{a W_t}{1 - a W_t}$, so we can write $a W_t = \frac{\kappa}{1+\kappa}$. Now reconsider the utility function:

$$
\begin{aligned}
u(W_{t+1}) &= W_{t+1} - \frac{a}{2} (W_{t+1})^2 = W_t R_{t+1}^p - \frac{a}{2} (W_t R_{t+1}^p)^2 \\
&= W_t \left[ R_{t+1}^p - \frac{a}{2} W_t (R_{t+1}^p)^2 \right] = W_t \left[ R_{t+1}^p - \frac{\kappa}{2(1+\kappa)} (R_{t+1}^p)^2 \right],
\end{aligned}
$$

where $R_{t+1}^p$ denotes the gross return on the investor's portfolio. For a given (constant) initial wealth $W_t$, maximizing the expectation of the function above is equivalent to problem (1) with $\gamma = \frac{\kappa}{1+\kappa}$.

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optimization problem then becomes:

$$\max_{x_t} \mathrm{E}_t \left[ x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top r_{t+1} r_{t+1}^\top x_t \right], \tag{2}$$

where $r_{t+1} = R_{t+1} - R_t^f$ is the vector of excess returns on the $N$ risky assets. By formulating the problem in terms of excess returns, we are implicitly assuming that the remainder of the value of the portfolio is invested in the risk-free asset with return $R_t^f$.

When the returns are iid and the portfolio weights are constant through time, $x_t = x$, we can replace the conditional expectation with an unconditional one and solve for the weights:

$$x = \frac{1}{\gamma} \mathrm{E} \left[ r_{t+1} r_{t+1}^\top \right]^{-1} \mathrm{E} \left[ r_{t+1} \right]. \tag{3}$$

This is the well-known Markowitz solution, which can be implemented in practice by replacing the population moments by sample averages:

$$x = \frac{1}{\gamma} \left[ \sum_{t=1}^{T-1} r_{t+1} r_{t+1}^\top \right]^{-1} \left[ \sum_{t=1}^{T-1} r_{t+1} \right]. \tag{4}$$

(Notice that the $1/T$ terms in the sample averages cancel.)

Consider now the more realistic case of non-iid returns and assume that the optimal portfolio policies are linear in a vector of $K$ state variables (the first of which we will generally take to be a constant):

$$x_t = \theta z_t, \tag{5}$$

where $\theta$ is an $N \times K$ matrix. Our problem is then:

$$\max_{\theta} \mathrm{E}_t \left[ (\theta z_t)^\top r_{t+1} - \frac{\gamma}{2} (\theta z_t)^\top r_{t+1} r_{t+1}^\top (\theta z_t) \right]. \tag{6}$$

We can use the following result from linear algebra:

$$(\theta z_t)^\top r_{t+1} = z_t^\top \theta^\top r_{t+1} = \mathrm{vec}(\theta)^\top (z_t \otimes r_{t+1}), \tag{7}$$

where $\mathrm{vec}(\theta)$ piles up the columns of matrix $\theta$ into a vector and $\otimes$ is the Kronecker

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product of two matrices, and write:

$$\tilde{x} = \text{vec}(\theta) \tag{8}$$

$$\tilde{r}_{t+1} = z_t \otimes r_{t+1} . \tag{9}$$

Our problem can now be written as:

$$\max_{\tilde{x}} \text{E}_t \left[ \tilde{x}^\top \tilde{r}_{t+1} - \frac{\gamma}{2} \tilde{x}^\top \tilde{r}_{t+1} \tilde{r}_{t+1}^\top \tilde{x} \right] . \tag{10}$$

Since the same $\tilde{x}$ maximizes the conditional expected utility at all dates $t$, it also maximizes the unconditional expected utility:

$$\max_{\tilde{x}} \text{E} \left[ \tilde{x}^\top \tilde{r}_{t+1} - \frac{\gamma}{2} \tilde{x}^\top \tilde{r}_{t+1} \tilde{r}_{t+1}^\top \tilde{x} \right] , \tag{11}$$

which corresponds simply to the problem of finding the unconditional portfolio weights $\tilde{x}$ for the expanded set of $(N \times K)$ assets with returns $\tilde{r}_{t+1}$. The expanded set of assets can be interpreted as managed portfolios, each of which invests in a single basis asset an amount proportional to the value of one of the state variables. We term these "conditional portfolios."

It follows that the optimal $\tilde{x}$ is:

$$\begin{aligned} \tilde{x} &= \frac{1}{\gamma} \text{E} \left[ \tilde{r}_{t+1} \tilde{r}_{t+1}^\top \right]^{-1} \text{E} \left[ \tilde{r}_{t+1} \right] \\ &= \frac{1}{\gamma} \text{E} \left[ (z_t z_t^\top) \otimes (r_{t+1} r_{t+1}^\top) \right]^{-1} \text{E} \left[ z_t \otimes r_{t+1} \right] , \end{aligned} \tag{12}$$

which we can again implement in practice by replacing the population moments by sample averages:

$$\tilde{x} = \frac{1}{\gamma} \left[ \sum_{t=0}^{T} (z_t z_t^\top) \otimes (r_{t+1} r_{t+1}^\top) \right]^{-1} \left[ \sum_{t=0}^{T} z_t \otimes r_{t+1} \right] . \tag{13}$$

From this solution we can trivially recover the weight invested in each of the basis assets by adding the corresponding products of elements of $\tilde{x}$ and $z_t$.

Note that the solution (13) depends only on the data and hence does not require

any assumptions about the distribution of returns besides stationarity. In particular, it does not require any assumptions about how the distribution of returns depends on the state variables. The state variables can predict time-variation in the first, second, and, if we consider more general utility functions, even higher-order moments of returns. As Brandt (1999) and Aït-Sahalia and Brandt (2001) emphasize, the advantage of focusing directly on the portfolio weights is that we bypass the estimation of the conditional return distribution. This intermediate estimation step typically involves ad-hoc distributional assumptions and inevitably misspecified models for the conditional moments of returns. In contrast, estimating the conditional portfolio weights in a single step is robust to misspecification of the conditional return distribution. It can also result in more precise estimates if the dependence of the optimal portfolio weights on the state variables is less noisy than the dependence of the return moments on the state variables.

At this point, it is instructive to compare our approach to Ferson and Siegel (2001). They assume that the conditional expected returns and conditional variances and covariances of asset returns are known functions of the state variables:

$$r_{t+1} = \mu(z_t) + \epsilon_{t+1}, \tag{14}$$

where the conditional covariance matrix of $\epsilon_{t+1}$ is $\Sigma(z_t)$. Ferson and Siegel then derive the mean-variance optimal portfolio weights as a function of the state variables:

$$x(z_t) = \pi(\mu(z_t) - R^f \iota)^\top \Lambda(z_t), \tag{15}$$

where

$$\Lambda(z_t) = \left[ (\mu(z_t) - R^f \iota)(\mu(z_t) - R^f \iota)^\top + \Sigma(z_t) \right]^{-1}, \tag{16}$$

$\iota$ is a vector of ones, and $\pi$ is a constant.

Our approach of modeling the portfolio weights as a function of the state variables can be seen as an approximation of the solution provided by Ferson and Siegel. For instance, postulating that the portfolio weights are linear in the state variables:

$$x(z_t) = \theta z_t \tag{17}$$

implicitly constrains the functional forms of $\mu(z)$ and $\Sigma(z)$ in equations (15) and (16).

Ferson and Siegel show that when the returns are homoskedastic, the optimal portfolio weights are approximately linear in the expected returns for an extended range of the state variables around their unconditional mean. Therefore, if the expected returns are linear in the state variables, the portfolio weights will also be linear in the state variables. Of course, homoskedastic returns with linear means is only one of many return models that deliver approximately linear portfolio weights. Also, our approach can easily accommodate non-linear portfolio weights by simply including non-linear transformations of the state variables in the portfolio weight functions.

Applying Ferson and Siegel's approach in practice raises a number of issues. While the conditional mean functions $\mu(z_t)$ can easily be estimated by regressing excess returns $r_{t+1}$ on the state variables $z_t$, it is notoriously difficult to estimate a conditional covariance matrix $\Sigma(z_t)$ as a function of the state variables in a manner that guarantees positive semi-definiteness at all times. Estimating the portfolio weight function in our approach does not require imposing any sort of nonlinear constraints. Furthermore, our approach has the advantage of being much more parsimonious. Suppose that we are interested in forming optimal portfolios of $N$ assets. With Ferson and Siegel's approach, we have to estimate $N$ functions of the state variables for the expected return vector and $N(N+1)/2$ functions for the covariance matrix. With our approach, we only need to estimate $N$ functions for the optimal portfolio weights. The gains in computation and estimation precision are evident.

Since we express the portfolio problem in an estimation context, we can use standard sampling theory to compute standard errors for the portfolio weights and then test hypotheses about them. Specifically, following Britten-Jones (1999), we can interpret the solution (13) as being proportional (with constant of proportionality $1/\gamma$) to the coefficients of a standard least-squares regression of a vector of ones on the excess returns $\tilde{r}_{t+1}$. This allows us to compute standard errors for $\tilde{x}$ from the standard errors of the regression coefficients. These standard errors can be used to test, for example, whether some state variable is a significant determinant of the portfolio policy. Using our notation, the covariance matrix of the vector $\tilde{x}$ is:

$$\frac{1}{\gamma^2} \frac{1}{T - N \times K} (\iota_T - \tilde{r}\tilde{x})^\top (\iota - \tilde{r}\tilde{x})(\tilde{r}^\top \tilde{r})^{-1} \tag{18}$$

where $\iota_T$ denotes a $T \times 1$ vector of ones and $\tilde{r}$ is a $T \times K$ matrix with the time series of returns of the $K$ managed portfolios.

As we already mentioned, the assumption that the optimal portfolio weights are linear functions of the state variables is innocuous because $z_t$ can include non-linear transformations of a set of more basic state variables $y_t$. This means that the linear portfolio weights can be interpreted as a more general portfolio policy function $x_t = g(y_t)$ for any $g(\cdot)$ that can be spanned by a polynomial expansion in the more basic state variables $y_t$. In other words, our approach can in principle accommodate very general dependence of the optimal portfolio weights on the state variables.

In practice, we need to choose a finite set of state variables and possible non-linear transformations of these state variables to use in the portfolio policy. From a statistical perspective, variable selection for modeling portfolio weights is no different from variable selection for modeling returns. Variables can be chosen on the basis of individual $t$ tests and joint $F$ tests computed with the covariance matrix of the portfolio weights in equation (18), or on the basis of out-of-sample performance. From an economic perspective, however, there are distinct advantages in focusing directly on the optimal portfolio weights. As Aït-Sahalia and Brandt (2001) demonstrate, it is more natural in an asset allocation framework to choose variables that predict optimal portfolio weights than it is to choose variables that predict moments of return. In particular, a variable may be a statistically important predictor of both means and variances, but be useless for determining optimal portfolio weights because the variation in the moments offset each other (e.g., the corresponding conditional Sharpe ratio is constant).

Finally, we can extend our approach to allow some or all of the state variables to be asset-specific. In a companion paper, Brandt, Santa-Clara, and Valkanov (2003), we study optimal stock portfolios by parameterizing the weight invested in each stock as a function of the company's characteristics, including its book-to-market ratio, market capitalization, and return over the past year. Importantly, the parameters of the weight function are constrained to be the same for all stocks, which makes the problem highly tractable and computationally efficient. The resulting optimal portfolios (of this very large set of assets) do not suffer from exploding weights (as mean-variance efficient portfolios often do) and have outstanding performance both in and out of sample.

## 2.2 Multiperiod Problem

The idea of augmenting the asset space with naively managed portfolios extends to the multiperiod case. Consider an investor who maximizes the following two-period mean-variance objective:

$$\max \mathrm{E}_t \left[ r^p_{t \to t+2} - \frac{\gamma}{2} (r^p_{t \to t+2})^2 \right], \tag{19}$$

where $r^p_{t \to t+2}$ denotes the excess return of a two-period investment strategy:

$$
\begin{aligned}
r^p_{t \to t+2} &= \left( R^f_t + x_t^\top r_{t+1} \right) \left( R^f_{t+1} + x_{t+1}^\top r_{t+2} \right) - R^f_t R^f_{t+1} \\
&= x_t^\top (R^f_{t+1} r_{t+1}) + x_{t+1}^\top (R^f_t r_{t+2}) + (x_t^\top r_{t+1})(x_{t+1}^\top r_{t+2}).
\end{aligned}
\tag{20}
$$

The first line of this expression shows why we call $r^p_{t \to t+2}$ a two-period excess return. The investor borrows a dollar at date $t$ and allocates it to the risky and risk-free assets according to the first-period portfolio weights $x_t$. After the first period, at date $t+1$, the one-dollar investment results in $(R^f_t + x_t^\top r_{t+1})$ dollars, which the investor then allocates again to the risky and risk-free assets according to the second-period portfolio weights $x_{t+1}$. Finally, at date $t+2$, the investor has $(R^f_t + x_t^\top r_{t+1})(R^f_{t+1} + x_{t+1}^\top r_{t+2})$ dollars but must pay $R^f_t R^f_{t+1}$ dollars for the principal and interest of the one-dollar loan. The remainder is the two-period excess return.

The second line of equation (20) decomposes the two-period excess return into three terms. The first two terms have a natural interpretation as the excess return of investing in the risk-free rate in the first (second) period and in the risky asset in the second (first) period.[4] Notice that the portfolio weights on these two intertemporal portfolios are the same as the weights on the risky asset in the first and second periods, respectively. The third term in this expression captures the effect of compounding.

Comparing the first two terms to the third, we see that the latter is two orders of magnitude smaller than the former. The return $(x_t^\top r_{t+1})(x_{t+1}^\top r_{t+2})$ is a product of two single-period *excess* returns, which means that its units are typically of the order of 1/100th of a percent per year. The returns on the first two portfolios, in contrast, are

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[4]To see that $x_t^\top (R^f_{t+1} r_{t+1})$ is a two-period excess return from investing in risky assets in the first period and the risk-free asset in the second period, just follow the argument above with $x_{t+1} = 0$. Investing the first-period proceeds of $(R^f_t + x_t^\top r_{t+1})$ in the risk-free asset in the second period yields $(R^f_t + x_t^\top r_{t+1})R^f_{t+1}$. After paying back $R^f_t R^f_{t+1}$, the investor is left with an excess return of $x_t^\top (R^f_t r_{t+1})$.

products of a gross return ($R_t^f$ or $R_{t+1}^f$) and an excess return ($r_{t+1}$ or $r_{t+2}$), so their units are likely to be percent per year.

Given that the compounding term is orders of magnitude smaller than the two intertemporal portfolios, we will for now ignore it. (We discuss the effect of ignoring the compounding term below.) The two-period portfolio choice is then simply a choice between two intertemporal portfolios, one that holds the risky asset in the first period only and the other that holds the risky asset in the second period only. We term these "timing portfolios." We can solve the dynamic problem as a simple static choice between these two managed portfolios. In particular, for the two-period case, the sample analogue of the optimal portfolio weights are given by:

$$\tilde{x} = \frac{1}{\gamma} \left[ \sum_{t=1}^{T-2} \tilde{r}_{t\to t+2} \tilde{r}_{t\to t+2}^{\top} \right]^{-1} \left[ \sum_{t=1}^{T-2} \tilde{r}_{t\to t+2} \right], \tag{21}$$

where $\tilde{r}_{t\to t+2} = [R_{t+1}^f r_{t+1}, R_t^f r_{t+2}]$. The first set of elements of $\tilde{x}$ (corresponding to the returns $R_{t+1}^f r_{t+1}$) then represents the fraction of wealth invested in the risky assets in the first period and the second set of elements (corresponding to $R_t^f r_{t+2}$) are for the risky assets in the second period.

In a general $H$-period problem, we proceed in exactly the same fashion. We construct a set of timing portfolios:

$$\tilde{r}_{t\to t+H} = \left\{ \prod_{\substack{i=0 \\ i\neq j}}^{H-1} R_{t+i}^f r_{t+j+1} \right\}_{j=0}^{H-1}, \tag{22}$$

where each term represents a portfolio that invests in risky assets in period $t + j$ and in the risk-free rate in all other periods $t + i$, with $i \neq j$. The sample analogue of the optimal portfolio weights are then again given by the static solution:

$$\tilde{x} = \frac{1}{\gamma} \left[ \sum_{t=1}^{T-H} \tilde{r}_{t\to t+H} \tilde{r}_{t\to t+H}^{\top} \right]^{-1} \left[ \sum_{t=1}^{T-H} \tilde{r}_{t\to t+H} \right]. \tag{23}$$

It is important to realize that, in contrast to a long-horizon buy-and-hold problem, the random components of the timing portfolios are *non-overlapping*. We thereby

avoid the usual statistical problems associated with overlapping long-horizon returns. Notice, however, that as the length of the horizon $H$ increases, we loose observations for computing the mean and covariance matrix of $\tilde{r}_{t \to t+H}$, which may compromise the statistical precision of the solution.

We can naturally combine the ideas of conditional portfolios and timing portfolios. For this, we replace the risky returns $r_{t+j+1}$ in equation (22) with the conditional portfolio returns $z_{t+j} \otimes r_{t+j+1}$. The resulting optimal portfolio weights $\tilde{x}$ from equation (23) then provide the optimal allocations to the conditional portfolios at each time $t + j$.

The obvious appeal of our approach is its simplicity. Naturally, this simplicity comes with drawbacks. First, by ignoring the compounding terms, our approach no longer provides the exact solution to the multiperiod problem. Writing out the return on an $H$-period dynamic portfolio strategy analogous to the two-period case in equation (20) shows that the multiperiod portfolio returns are only spanned when we include the compounding terms in the static portfolio problem. Unfortunately, the presence of the compounding terms imposes a set of non-linear constraints on the static portfolio weights. The portfolio weights on the compounding terms are constrained to be products of the portfolio weights on the timing portfolios. Due to the non-linearity of these constraints, solving the static constrained problem with compounding terms for a large number of assets and/or a large number of rebalancing periods is not much simpler than solving the corresponding dynamic problem using numeric optimization techniques. Our suggestion is to ignore the compounding terms on the grounds that they are orders of magnitude smaller than the timing portfolio returns. However, in ignoring the compounding terms, our solution is at best a good approximation of the solution to the multiperiod problem. The quality of the approximation is naturally specific to each application. Intuitively, it depends on the growth rate of wealth per period and on the number of periods considered. We further examine this issue in Section 2.4.

The second drawback of our approach is that it can be quite data-intensive for problems with very long horizons. For example, suppose we want to solve a ten-year portfolio choice problem with quarterly rebalancing using a 60-year post-war data sample of quarterly returns and state variable realizations. Since each timing-portfolio involves a ten-year return, we would only have six independent observations to compute the moments of the timing-portfolio returns and hence the optimal portfolio weights. The

obvious way to overcome this data issue is to impose a statistical model for the returns and state variables that allows us to compute the long-horizon moments analytically (or by simulation) from the parameters of the statistical model. Specifically, if the log returns on the basis assets and the log state variables follow a VAR with normally distributed innovations, the long-horizon moments can be expressed in terms of the parameters of the VAR. This use of a statistical model allows us to solve dynamic portfolio choice problems with arbitrarily long horizons using only a finite data sample. We elaborate on this idea in Section 3.

## 2.3   Illustrative Example

To illustrate more concretely the mechanics of our approach, consider a time series of only six observations (for simplicity) of excess returns of two risky assets, stocks denoted by $s$ and bonds denoted by $b$:

$$
\begin{bmatrix}
r_1^s & r_1^b \\
r_2^s & r_2^b \\
\dots & \dots \\
r_6^s & r_6^b
\end{bmatrix}
\tag{24}
$$

The optimal static portfolio in equation (4) directly gives us the weight $x^s$ invested in the stock and the weight $x^b$ invested in the bond (with the remainder invested in the risk-free asset). The solution takes into account the sample covariance matrix of asset returns and the vector of sample mean excess returns.

Suppose now that there is one conditioning variable, such as the dividend yield or the spread between long and short Treasury yields, which affects the conditional distribution of returns. We observe a time series of this state variable:

$$
\begin{bmatrix}
z_0 \\
z_1 \\
\dots \\
z_5
\end{bmatrix},
\tag{25}
$$

where the dating reflects the fact that $z$ is known at the beginning of each return period. We take into account the information in the conditioning variable by estimating

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a portfolio policy that depends on it. For this, we expand the matrix of returns (24) in the following manner:

$$\begin{bmatrix} r_1^s & r_1^b & z_0 r_1^s & z_0 r_1^b \\ r_2^s & r_2^b & z_1 r_2^s & z_1 r_2^b \\ \cdots & \cdots & \cdots & \cdots \\ r_6^s & r_6^b & z_5 r_6^s & z_5 r_6^b \end{bmatrix} \tag{26}$$

and compute the optimal static portfolio of this expanded set of assets. This static solution gives us a vector of four portfolio weights $\tilde{x}$ corresponding to each of the basis assets and managed portfolios in the matrix above. We find the weight invested in stocks at each time by using the first and third elements of $\tilde{x}$, $x_t^s = \tilde{x}_1 + \tilde{x}_3 z_t$. Similarly, the weight invested in the bond at each time is $x_t^b = \tilde{x}_2 + \tilde{x}_4 z_t$. Note that when we use the Markowitz solution (4) on the matrix of returns of the expanded asset set (26), the covariance matrix and vector of means takes into account the covariances among returns *and* between returns and lagged state variables. The latter covariances capture the impact of predictability of returns on the optimal portfolio policy.

Consider now a two-period portfolio choice problem. We construct the matrix of returns of the timing portfolios as described in equation (22):

$$\begin{bmatrix} r_1^s R_1^f & R_0^f r_2^s & r_1^b R_1^f & R_0^f r_2^b \\ r_3^s R_3^f & R_2^f r_4^s & r_3^b R_3^f & R_2^f r_4^b \\ r_5^s R_5^f & R_4^f r_6^s & r_5^b R_5^f & R_4^f r_6^b \end{bmatrix} . \tag{27}$$

This matrix contains two-period non-overlapping returns of four trading strategies. The corresponding optimal portfolio vector $\tilde{x}$ gives us the weights on "stocks in period 1," "stocks in period 2," "bonds in period 1," and "bonds in period 2." The covariance matrix and vector of means which show up in the static portfolio solution account for the contemporaneous covariances of returns as well as the one-period serial covariances of returns. The latter covariances induce hedging demands.

Finally, we can consider a two-period problem with the conditioning variable. The

returns of the expanded asset set are:

$$
\begin{bmatrix}
r_1^s R_1^f & R_0^f r_2^s & r_1^b R_1^f & R_0^f r_2^b & z_0 r_1^s R_1^f & R_0^f z_1 r_2^s & z_0 r_1^b R_1^f & R_0^f z_1 r_2^b \\
r_3^s R_3^f & R_2^f r_4^s & r_3^b R_3^f & R_2^f r_4^b & z_2 r_3^s R_3^f & R_2^f z_3 r_4^s & z_2 r_3^b R_3^f & R_2^f z_3 r_4^b \\
r_5^s R_5^f & R_4^f r_6^s & r_5^b R_5^f & R_4^f r_6^b & z_4 r_5^s R_5^f & R_4^f z_5 r_6^s & z_4 r_5^b R_5^f & R_4^f z_5 r_6^b
\end{bmatrix}.
\tag{28}
$$

The optimal portfolio of these eight assets now includes the weight on "stocks in period 1, conditional on the level of $z$," for example. The portfolio solution takes into account the covariances between returns and state variables over subsequent periods.

## 2.4  Importance of the Compounding Terms

Our approach to the multiperiod portfolio problem relies critically on the presumption that the compounding terms (i.e., the cross-products of the excess returns in different time periods) are negligible relative to the returns on the timing portfolios. We now examine to what extent and under which circumstances this is valid.

We apply our method to the following model for monthly excess stock and bond returns (the basis assets) and the term spread (the state variable):

$$
\begin{bmatrix}
\ln(1 + r_{t+1}^s) \\
\ln(1 + r_{t+1}^b) \\
z_{t+1}
\end{bmatrix}
=
\begin{bmatrix}
0.0059 \\
0.0007 \\
-0.0028
\end{bmatrix}
+
\begin{bmatrix}
0.0060 \\
0.0035 \\
0.9597
\end{bmatrix}
\times z_t
+
\begin{bmatrix}
\epsilon_{t+1}^s \\
\epsilon_{t+1}^b \\
\epsilon_{t+1}^z
\end{bmatrix},
\tag{29}
$$

with

$$
\begin{bmatrix}
\epsilon_{t+1}^s \\
\epsilon_{t+1}^b \\
\epsilon_{t+1}^z
\end{bmatrix}
\sim \mathrm{MVN}
\begin{bmatrix}
0,
\begin{bmatrix}
0.0018 & 0.0002 & -0.0005 \\
0.0002 & 0.0006 & 0.0007 \\
-0.0005 & 0.0007 & 0.0802
\end{bmatrix}
\end{bmatrix}.
\tag{30}
$$

The choice of state variable is based on our empirical results in Section 5, where we identify the term spread as an important return predictor (other important predictors include the dividend yield and detrended short-term interest rate). The functional form of the model follows the literature on portfolio choice under predictability and is also related to our setup in Section 3. The parameter values are OLS estimates based on monthly data from January 1945 through December 2000.

To assess the importance of the compounding terms in the solution of the multiperiod

portfolio problem, we compare portfolio policies that ignore the compounding terms (using our simplified approach based on the timing portfolios) with policies that incorporate the compounding terms (obtained through numeric optimization). We label these solutions "approximate" and "exact," respectively.[5] Intuitively, there are two factors that affect the role of the compounding terms, the rebalancing frequency and the portfolio horizon. The less frequently the portfolio is rebalanced, the larger are the magnitudes of the excess returns per period, and therefore the larger are the magnitudes of the compounding terms. The longer the horizon, the more compounding terms there are in the expanded budget constraint. Hence, we study multiperiod portfolio problems with rebalancing frequencies ranging from monthly to annual and horizons ranging from one to five years.

The results of our experiments are displayed in Table 1. The table describes the multiperiod returns from the approximate and exact portfolio policies for an investor with quadratic utility and $\gamma = 4$ (the value we use in our empirical application). Panel A presents the results for unconditional portfolio policies and panel B the results for conditional policies in which the stock and bond returns each period are scaled by the state variable. All summary statistics (mean, standard deviation, Sharpe ratio, and certainty equivalent return) are annualized. Finally, the last row in each panel reports the average absolute difference between the approximate and exact allocations to (scaled) stock and bond in all periods.

Broadly reviewing the results across horizons, rebalancing frequencies, and panels, it is clear that, consistent with our intuition, the compounding terms are relatively unimportant in the multiperiod portfolio problem. For example, the largest increase in the annualized Sharpe ratio from taking into account the compounding terms in the unconditional case is 0.0026, which corresponds to an increase in the annualized certainty equivalent return of only three basis points. The largest average absolute difference in the unconditional portfolio weights is 4.26 percent, which is less than 1/10th of the magnitude of the typical allocation to stocks and far smaller than the standard errors

---

[5]The exact solution is obtained by numerically maximizing the expected utility of terminal wealth with respect to the portfolio weights in every period. For a given set of portfolio weights, the moments of the multiperiod portfolio returns are evaluated using 1,000,000 data points simulated from the model (29). To keep the comparison as fair as possible and to abstract from sampling error, we use the same simulations to evaluate the moments of the timing portfolios for the approximate solution.

on the unconditional portfolio weights in our empirical application. The results are similar for the conditional case. The certainty equivalent gains from taking into account the compounding terms are less than 12 basis points per year and the average absolute differences in the portfolio weights are less than five percent.

Analyzing the results more closely reveals some intuitive patterns. The importance of the compounding terms increases with the horizon (holding constant the rebalancing frequency) as well as with the rebalancing frequency (holding constant the horizon). The compounding terms are more important for the conditional polices because these are associated with a higher expected growth rate of wealth and a larger number of compounding terms due to the inclusion of the scaled returns.

We conclude from this experiment that our approach of solving the multiperiod portfolio problem with timing portfolios, which ignore the compounding of excess returns over time, results in little economic loss. This is particularly true for problems with relatively short horizons and infrequent rebalancing. This small economic loss is more than compensated by the computational gains arising from the simplicity of our approach, especially when compared to the usual numeric solutions of multiperiod portfolio problems.

## 3    Optimal Portfolio Weights Implied by a VAR

As mentioned above, our approach can be data intensive for solving portfolio problems with very long horizons. However, this issue can be overcome by using a statistical model for the returns and state variables. For example, consider a problem with a single risky asset and one conditioning variable and assume that the log (gross) return and log conditioning variable evolve jointly according to the following restricted VAR with normally distributed innovations:

$$
\begin{bmatrix} \ln R_{t+1} \\ \ln z_{t+1} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \ln z_t + \epsilon_{t+1},
\tag{31}
$$

where $\epsilon_{t+1} \sim \mathrm{N}[0, \Omega]$. We also assume for simplicity that the risk-free rate is constant.

The dynamics of returns in equation (31) imply the following expanded VAR:

$$\ln Y_{t+1} = A + B \ln Y_t + \nu_{t+1}, \tag{32}$$

where $\ln Y_{t+1} = [\ln R_{t+1}, \ln z_{t+1}, \ln z_t, \ln z_t + \ln R_{t+1}]^\top$ and $\eta_{t+1} \sim N[0, \Gamma]$ with

$$A = \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b_1+1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \Gamma = \begin{bmatrix} \omega_{11} & \omega_{12} & 0 & \omega_{11} \\ \omega_{12} & \omega_{22} & 0 & \omega_{12} \\ 0 & 0 & 0 & 0 \\ \omega_{11} & \omega_{12} & 0 & \omega_{11} \end{bmatrix}, \tag{33}$$

where $\omega_{ij}$ are the elements of the covariance matrix $\Omega$. The first two unconditional moments of this expanded VAR are given by:

$$\mu \equiv \mathrm{E}[\ln Y_{t+1}] = (I - B)^{-1} A$$
$$\mathrm{vec}(\Sigma) \equiv \mathrm{vec}(\mathrm{Var}[\ln Y_{t+1}]) = (I - B \otimes B)\mathrm{vec}(\Gamma). \tag{34}$$

We use this expanded VAR to solve for the moments of returns involved in our solution to the dynamic portfolio choice problem.

## 3.1   Single-Period Problem

Consider first the single-period portfolio problem. Following equation (9), we construct excess returns on the managed portfolios:

$$\tilde{r}_{t+1} = [R_{t+1} - R^f, \; z_t(R_{t+1} - R^f)]^\top \tag{35}$$

From the extended VAR (34), these returns can be written as:

$$\tilde{r}_{t+1} = \Lambda Y_{t+1} + \lambda, \tag{36}$$

where:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -R^f & 1 \end{bmatrix} \quad \text{and} \quad \lambda = \begin{bmatrix} -R^f \\ 0 \end{bmatrix}. \tag{37}$$

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The optimal single-period portfolio choice for the expanded asset space in equation (13) depends on the first two moments of these returns, which are given by:

$$E[\tilde{r}_{t+1}] = \Lambda E[Y_{t+1}] + \lambda \quad \text{and} \quad \text{Var}[\tilde{r}_{t+1}] = \Lambda \text{Var}[Y_{t+1}]\Lambda^\top, \tag{38}$$

where, from the joint log-normality of $Y_{t+1}$ and the unconditional moments of the VAR:

$$E[Y_{t+1}] = \exp\left\{E[\ln Y_{t+1}] + \frac{1}{2}\text{Diag}\left[\text{Var}[\ln Y_{t+1}]\right]\right\}$$
$$\text{Var}[Y_{t+1}] = \left(\exp\left\{\text{Var}[\ln Y_{t+1}]\right\} - 1\right)E[Y_{t+1}]E[Y_{t+1}]^\top. \tag{39}$$

The moments in equation (38), and hence the optimal portfolio weights, can therefore be evaluated using the unconditional moments of the VAR in equation (34).

## 3.2 Multiperiod Portfolio Choice

Consider next a two-period dynamic problem. The excess returns of the conditional and timing portfolios are:

$$\tilde{r}_{t\to t+2} = [\underbrace{(R_{t+1} - R^f)R^f, \ z_t(R_{t+1} - R^f)R^f}_{\substack{\text{stocks in period 1,} \\ \text{conditional on } z}}, \ \underbrace{R^f(R_{t+2} - R^f), \ R^f z_{t+1}(R_{t+2} - R^f)}_{\substack{\text{stocks in period 2,} \\ \text{conditional on } z}}]$$

$$= R^f\left(\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}\begin{bmatrix} Y_{t+1} \\ Y_{t+2} \end{bmatrix} + \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}\right)^\top. \tag{40}$$

The corresponding first and second moments are:

$$E[\tilde{r}_{t\to t+2}] = R^f\left(\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}E\begin{bmatrix} Y_{t+1} \\ Y_{t+2} \end{bmatrix} + \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}\right)$$
$$\text{Var}[\tilde{r}_{t\to t+2}] = (R^f)^2\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}\text{Var}\begin{bmatrix} Y_{t+1} \\ Y_{t+2} \end{bmatrix}\begin{bmatrix} \Lambda^\top & 0 \\ 0 & \Lambda^\top \end{bmatrix}, \tag{41}$$

where:

$$
\mathrm{E}\begin{bmatrix} Y_{t+1} \\ Y_{t+2} \end{bmatrix} = \exp\left\{ \mathrm{E}\begin{bmatrix} \ln Y_{t+1} \\ \ln Y_{t+2} \end{bmatrix} + \frac{1}{2}\mathrm{Diag}\begin{bmatrix} \mathrm{Var}\begin{bmatrix} \ln Y_{t+1} \\ \ln Y_{t+2} \end{bmatrix} \end{bmatrix} \right\}
$$

$$
\mathrm{Var}\begin{bmatrix} Y_{t+1} \\ Y_{t+2} \end{bmatrix} = \left( \exp\left\{ \mathrm{Var}\begin{bmatrix} \ln Y_{t+1} \\ \ln Y_{t+2} \end{bmatrix} \right\} - 1 \right) \mathrm{E}\begin{bmatrix} Y_{t+1} \\ Y_{t+2} \end{bmatrix} \mathrm{E}\begin{bmatrix} Y_{t+1} \\ Y_{t+2} \end{bmatrix}^{\top}
$$

$$(42)$$

and from the unconditional moments of the VAR:

$$
\mathrm{E}\begin{bmatrix} \ln Y_{t+1} \\ \ln Y_{t+2} \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \end{bmatrix}
$$

$$
\mathrm{Var}\begin{bmatrix} \ln Y_{t+1} \\ \ln Y_{t+2} \end{bmatrix} = \begin{bmatrix} \Sigma & B\Sigma \\ B\Sigma & \Sigma \end{bmatrix}.
$$

$$(43)$$

Finally, consider an $N$-period dynamic problem. Using basic matrix algebra, the excess returns on the conditional and timing portfolios can be written as:

$$
\tilde{r}_{t \to t+N} = \left(R^f\right)^{N-1}\left( \left(I_N \otimes \Lambda\right)\begin{bmatrix} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{bmatrix} + \left(\iota_N \otimes \lambda\right) \right),
$$

$$(44)$$

where $I_N$ and $\iota_N$ denote an $N$-dimensional identity matrix and vector of ones, respectively. The corresponding first and second moments are:

$$
\mathrm{E}[\tilde{r}_{t \to t+N}] = \left(R^f\right)^{N-1}\left( \left(I_N \otimes \Lambda\right)\mathrm{E}\begin{bmatrix} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{bmatrix} + \left(\iota_N \otimes \lambda\right) \right)
$$

$$
\mathrm{Var}[\tilde{r}_{t \to t+N}] = \left(R^f\right)^{2(N-1)}\left(I_N \otimes \Lambda\right)\mathrm{Var}\begin{bmatrix} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{bmatrix}\left(I_N \otimes \Lambda^{\top}\right),
$$

$$(45)$$

where:

$$
\mathrm{E}\left[\begin{array}{c} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{array}\right] = \exp\left\{\mathrm{E}\left[\begin{array}{c} \ln Y_{t+1} \\ \cdots \\ \ln Y_{t+Nfs} \end{array}\right] + \frac{1}{2}\mathrm{Diag}\left[\mathrm{Var}\left[\begin{array}{c} \ln Y_{t+1} \\ \cdots \\ \ln Y_{t+N} \end{array}\right]\right]\right\}
$$

$$
\mathrm{Var}\left[\begin{array}{c} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{array}\right] = \left(\exp\left\{\mathrm{Var}\left[\begin{array}{c} \ln Y_{t+1} \\ \cdots \\ \ln Y_{t+N} \end{array}\right]\right\} - 1\right)\mathrm{E}\left[\begin{array}{c} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{array}\right]\mathrm{E}\left[\begin{array}{c} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{array}\right]^{\top} \tag{46}
$$

and:

$$
\mathrm{E}\left[\begin{array}{c} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{array}\right] = \iota_N \otimes \mu
$$

$$
\mathrm{Var}\left[\begin{array}{c} Y_{t+1} \\ \cdots \\ Y_{t+N} \end{array}\right] = \left[\begin{array}{ccccc} B^0 & B^1 & B^2 & \cdots & B^{N-1} \\ B^1 & B^0 & B^1 & \cdots & B^{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B^{N-1} & B^{N-2} & B^{N-3} & \cdots & B^0 \end{array}\right] \otimes \Sigma. \tag{47}
$$

To summarize, the optimal portfolio weights for the $N$-period dynamic problem with conditional and timing portfolios, which depend on the first and second moments of the managed portfolio returns, can be evaluated analytically using the coefficient matrix $B$ and the unconditional moments $\mu$ and $\Sigma$ of the VAR (which in turn depend on $A$, $B$, and $\Gamma$). Since we can estimate the VAR with a relatively modest time-series of returns and state variable realizations, we can solve dynamic portfolio choice problems with arbitrarily long horizons using finite data samples in this VAR context. Of course, this comes at the cost of having to impose strong structure on the dynamics of returns.

# 4    Extensions and Refinements

Our approach can be extended and refined along a number of dimensions. In this section, we show how to generalize the investor's utility function and how to compute robust portfolio weights for a large numbers of assets using techniques developed originally for the static Markowitz approach.

## 4.1 Objective Functions

The mean-variance objective function can be extended to an arbitrary utility function $u(W_{t+1})$. In that case, we solve the problem:

$$\max_{\theta} \mathrm{E}_t \left[ u(R_t^f + (\theta z_t)^\top r_{t+1}) \right],$$ (48)

or the corresponding first-order conditions, using numeric optimization methods. While high-dimensional numeric solutions are non-trivial, our approach benefits from being static and unconstrained (since we ignore the compounding terms). Furthermore, there exists by now an extensive literature on effective and fast algorithms for solving high-dimensional optimization problems that can applied to our framework.[6]

The quadratic objective function (1) can alternatively be interpreted as a second-order approximation of a more general utility function, such as power or more general HARA preferences. To increase the precision of this approximation, Brandt, Goyal, Santa-Clara, and Stroud (2003) propose a fourth-order expansion that includes adjustments for the skewness and kurtosis of returns and their effects on expected utility. Specifically, the expansion of expected utility around the current wealth growing at the risk-free rate is:

$$\mathrm{E}_t \left[ u(W_{t+1}) \right] \approx \mathrm{E}_t \left[ u(W_t R_t^f) + u'(W_t R_t^f)(W_t\, x_t{}^\top r_{t+1}) + \frac{1}{2} u''(W_t R_t^f)(W_t\, x_t{}^\top r_{t+1})^2 \right. $$
$$\left. + \frac{1}{6} u'''(W_t R_t^f)(W_t\, x_t{}^\top r_{t+1})^3 + \frac{1}{24} u''''(W_t R_t^f)(W_t\, x_t{}^\top r_{t+1})^4 \right].$$ (49)

In this case, the FOCs define an *implicit* solution for the optimal weights in terms of the joint moments of the derivatives of the utility function and returns:

$$x_t \approx - \left\{ \mathrm{E}_t \left[ u''(W_t R_t^f)(r_{t+1} r_{t+1}{}^\top) \right] W_t^2 \right\}^{-1} \times \left\{ \mathrm{E}_t \left[ u'(W_t R_t^f)(r_{t+1}) \right] W_t \right.$$
$$\left. + \frac{1}{2}\mathrm{E}_t \left[ u'''(W_t R_t^f) \left( x_t{}^\top r_{t+1} \right)^2 r_{t+1} \right] W_t^3 + \frac{1}{6}\mathrm{E}_t \left[ u''''(W_t R_t^f) \left( x_t{}^\top r_{t+1} \right)^3 r_{t+1} \right] W_t^4 \right\}.$$ (50)

This implicit expression for the optimal weights is easy to solve in practice. Start with

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[6]These algorithms including variants of the Newton method (e.g., Conn, Gould, and Toint, 1988; Moré and Toraldo, 1989), the quasi-Newton or BFGS method (e.g., Byrd, Lu, Nocedal, and Zhu, 1995), and the sequential quadratic programming approach (e.g., Gill, Murray, Saunders, 2002).

an initial "guess" for the optimal weights (such as equal weights in each asset), denoted $x_t(0)$. Then, enter this guess on the right-hand side of equation (50) and obtain a new solution for the optimal weights on the left-hand-side, denoted $x_t(1)$. After a few iterations $n$, the guess $x_t(n)$ is very close to the solution $x_t(n+1)$ and we can take this value to be the solution of equation (50). Brandt, Goyal, Santa-Clara, and Stroud show that this expansion is highly accurate for investment horizons up to one year, even when returns are far from normally distributed. It is straightforward to use this expansion approach in our extended asset space approach.

We can also consider performance benchmarks in the objective function. Frequently, money managers are evaluated on their performance relative to a benchmark index portfolio over a given period. Such problems can easily be solved within our approach. Simply use returns of the basis assets in excess of the benchmark index (instead of in excess of the risk-free interest rate), $R_t - R_t^i \iota$, in the portfolio optimization. The objective function defined on these excess returns thus defines a gain from beating the benchmark index with low tracking error. The optimal portfolio weights can be interpreted as deviations from the benchmark, usually termed "active" weights.

Finally, we can expand the mean-variance objective to penalize covariance with the return of a particular portfolio such as the market, $r^m$.[7] In this case, the objective is:

$$\mathrm{E}_t\left[ r_{t+1}^p - \frac{\gamma}{2}(r_{t+1}^p)^2 - \lambda r_{t+1}^p r_{t+1}^m \right], \tag{51}$$

with some positive penalty constant $\lambda$. The solution in the unconditional case is (replacing sample moments for the population moments):

$$x = \frac{1}{\gamma}\left[ \mathrm{E}r_{t+1}r_{t+1}^\top \right]^{-1}\left[ \sum_{t=1}^{T-1}(1 - \lambda r_{t+1}^m)r_{t+1} \right], \tag{52}$$

which can trivially be extended to the conditional and multiperiod problems.

## 4.2  Constraints, Shrinkage, and Prior Views

A benefit of framing the dynamic portfolio problem in a static context is that we have at our disposal all of the refinements of the Markowitz approach developed over the

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[7]Or, similarly, penalize covariance with consumption growth.

past decades. These include the use of portfolio constraints to avoid extreme positions (e.g., Frost and Savarino, 1988; Jagannathan and Ma, 2003), the use of shrinkage to improve the estimates of the means (e.g., Jobson and Korkie, 1981) as well as of the covariance matrix (e.g., Ledoit, 1995), and the combination of the investor's prior from an alternative data source or the belief in a pricing model with the information contained in returns (e.g., Treynor and Black, 1973; Black and Litterman, 1992; Pastor and Stambaugh, 2000).

For the last approach, which is particularly useful in practice, a natural prior is that the market is in equilibrium. In that case the market portfolio is the tangency portfolio. Suppose that the estimated portfolio weight on asset $i$ is of the form $x_t^i = a + bz_t$, and assume that $z$ has been standardized to have mean zero. Using the equilibrium prior, we would shrink $a$ towards the market capitalization weight of the asset and $b$ towards zero. The shrinkage weights can be determined from the standard errors of the estimates of $a$ and $b$, coupled with a prior on the efficiency of the market.

# 5    Application

There is substantial evidence that economic variables related to the business cycle help forecast stock and bond returns. For instance, Campbell (1991), Campbell and Shiller (1988), Fama (1990), Fama and French (1988,1989), Hodrick (1992), and Keim and Stambaugh (1986) report evidence that stock market returns are predictable by the dividend-price ratio, short-term interest rate, term spread, and credit spread. Fama and French (1989) show that the same variables also predict bond returns. We use these four conditioning variables in a simple application of our method to the dynamic portfolio choice between stocks, bonds, and cash. This application is similar to Brennan, Schwartz, and Lagnado (1997) and Campbell, Chan, and Viceira (2003).

We take the stock to be the CRSP value-weighted market index, the bond to be the index of long-term Treasuries constructed by Ibbotson Associates, and cash to be the three-month Treasury bill, also obtained from Ibbotson Associates. The dividend-price ratio (D/P) is calculated as the difference between the log of the last twelve month dividends and the log of the current price index of the CRSP value-weighted index. The relative Treasury bill (Tbill) stochastically detrends the raw series by taking the

difference between the Treasury bill rate and its twelve-month moving average. The term spread (Term) is the difference between the yields on 10-year and 1-year government bonds. The default spread (Default) is calculated as the difference between the yield on BAA- and AAA-rated corporate bonds. The interest rate data is obtained from the DRI/Citibase database. We standardize the three conditioning variables to ease the interpretation of the coefficients of the portfolio policy. The sample period is January 1945 through December 2000.

Table 2 reports the results for both unconditional and conditional portfolio policies at monthly, quarterly, and annual holding periods. The investor is assumed to have quadratic utility. We set $\gamma = 4$ in all cases, as this leads to an unconditional asset allocation that holds a small but positive amount in cash, which is roughly similar to portfolios typically recommended by financial consultants.

There are some differences in the unconditional portfolio weights across the three holding periods. With monthly rebalancing, the weight in equities is 51.15 percent, whereas it is only 37.13 and 41.66 percent at the quarterly and annual frequencies, respectively. This pattern is due to differences in the joint distribution of stock and bond excess returns over the different holding periods. In particular, there is a small amount of positive serial correlation in returns at the monthly frequency that turns negative at the quarterly and annual frequencies. This makes the volatility of stock and bond returns proportionately lower at the monthly frequency.

The conditional policies are quite sensitive to the state variables. For the monthly conditional policy, the coefficients of the bond weight on Term, Default, and D/P, as well as the coefficient of the stock weight on Default and D/P are all significant at the 95 percent level. Furthermore, the average weight held in stocks by the conditional policy is 81.21 percent, which significantly exceeds the corresponding unconditional weight of 51.15 percent. The reason is that the predictability in the first and second moments of returns allows the investor to be more aggressive on average since the exposure can be reduced in bad times. The average weight on bonds of the conditional policy is actually negative, $-36.24$ percent, compared to the unconditional weight of 47.09 percent. An $F$-test of the hypothesis that all coefficients on the state variables are equal to zero has a $p$-value of zero. Finally, the (annualized) Sharpe ratio of the conditional policy is 0.92, which is twice that of the unconditional policy of 0.46. Overall, it is clear that

the conditional return distribution is very different on average than the unconditional return distribution.

The results are less pronounced for the longer holding periods. At the quarterly frequency, for example, only the coefficients of the bond weight on Default and of the stock weight on Tbill are significant. However, the hypothesis that all coefficients on the state variables are zero is still rejected with a $p$-value of zero. More importantly, the Sharpe ratio of the conditional policy is still one and a half times that of the unconditional policy, 0.64 versus 0.41. The results for the annual policy are qualitatively similar, with an increase in Sharpe ratio from 0.44 to 0.54. Judging by the relative increase in the Sharpe ratio, the conditioning information becomes less important as the holding period increases. On the one hand, this is sensible, as a conditional strategy turns into an unconditional one in the limit as the horizon increases. On the other hand, this result is at odds with the general notion that returns become more predictable at longer horizon. The reason for why this long-run predictability is not as noticeable in our results is that for our sample period the interest rate variables are the most important predictors and those variables mainly forecast returns at short horizons (e.g., Fama and French, 1989).

Figure 1 displays the time series of portfolio weights of the conditional policies. For comparison, the figure also shows the unconditional portfolio weights. Overall, the shorter the holding period, the more extreme positions the policies take at times. It is striking that the conditional policies can be substantially different at different frequencies.

As mentioned earlier, by focussing directly on the portfolio weights we capture time-variation in the entire return distribution as opposed to just the expected returns. To get a sense of the importance of this aspect of our approach, we compare the conditional policies to more traditional strategies based only on predictive return regressions. Specifically, we regress the excess stock and bond returns on the state variables and then use the corresponding one-period ahead forecasts of the returns together with the unconditional covariance matrix to form portfolio weights. In this way, the strategy only takes into account the predictability of expected returns and ignores the impact of the state variables on variances and covariances. Table 3 compares the two approaches, and Figure 2 plots the time series of portfolio weights on the stock.

The advantage of our approach is most striking at the quarterly frequency. Both strategies generate an average premium of about nine percent per year, but our conditional strategy has a volatility that is less than half that of the regression-based strategy, 14.5 versus 32.6 percent, resulting in a Sharpe ratio that is more than twice as large. In fact, the investor would be willing to pay an annual fee of more than 16 percent to obtain the improved performance associated with exploiting the joint time-variation of the entire return distribution, as opposed to exploiting just the time-variation of the mean returns. Although the differences between the strategies are less dramatic at the monthly and annual frequencies, the conclusion holds nevertheless. The fee the investor is willing to pay for using the conditional strategy is about two percent for monthly returns and more than four percent for annual returns. It is interesting to note that as the holding period increases, the benefit of our approach shifts from generating a substantially higher expected return at a slightly higher level of risk to generating a slightly lower expected return at a substantially lower level of risk.

We now turn our attention to multiperiod strategies. Table 4 reports the portfolio weights of the multiperiod portfolio policy for a one-year horizon with monthly or quarterly rebalancing. For simplicity, we report only the unconditional strategy and the conditional strategy with a single state variable, the detrended T-bill rate. The table reports the estimated portfolio weights for month one, four, eight, and twelve as well as for all four quarters of the twelve-month or four-quarter problems, respectively.

With monthly rebalancing, the weight on stocks decreases and the weight on bonds increases as the end of the horizon approaches. This horizon pattern is roughly the same for the unconditional and conditional policies, which means that it is generated by the serial-covariance structure of the returns on the basis assets. With quarterly rebalancing, the unconditional and average conditional (the constant term in the conditional policy) stock holdings are similar to each other and to the results with monthly rebalancing. The unconditional and average conditional bond holdings, in contrast, are very different from each other. In the unconditional policy, the bond holding increases from $-4$ to 56 percent as the end of the horizon approaches, while in the conditional policy the average bond holding decreases from $-17$ to $-27$ percent. This difference in the horizon patterns can only be attributed to the serial-covariance structure of the conditional portfolio returns, which illustrates the importance of augmenting the asset space in this

multiperiod problem.

Finally, notice that the Sharpe ratios of the multiperiod policies are higher than in the corresponding single-period case. Consistent with the prediction of intertemporal portfolio theory, both the mean and the volatility of the portfolio returns are lower. Given the serial-covariance structure in the data, the investor sacrifices mean return for intertemporal diversification.

# 6   Conclusion

We presented a simple approach for dynamic portfolio selection. The solution extends the Markowitz approach to the choice between managed portfolios: conditional portfolios that invest in each asset a weight proportional to some conditioning variable, and timing portfolios that invest in each asset in a single period. The intuition underlying our approach is that the static choice among these mechanically managed portfolios is equivalent to a dynamic strategy in the basis assets. Our hope is that, by making dynamic portfolio selection no more difficult to implement than the static Markowitz approach, it will finally leave the confines of the ivory tower and make its way into the day-to-day practice of the investment industry.

# References

Aït-Sahalia, Yacine, and Michael W. Brandt, 2001, Variable selection for portfolio choice, *Journal of Finance* 56, 1297–1351.

——— , 2004, Portfolio and consumption choice with option-implied state prices, Working Paper, Princeton University.

Bansal, Ravi, and Campbell R. Harvey, 1996, Performance evaluation in the presence of dynamic trading strategies, Working Paper, Duke University.

Black, Fisher, and Robert Litterman, 1992, Global portfolio optimization, *Financial Analysts Journal* 48, 28–43.

Brandt, Michael W., 1999, Estimating portfolio and consumption choice: A conditional method of moments approach, *Journal of Finance* 54, 1609–1646.

——— , Amit Goyal, Pedro Santa-Clara, and Jonathan R. Stroud, 2003, A simulation approach to dynamic portfolio choice with an application to learning about return predictability, Working Paper, Duke University.

Brandt, Michael W., Pedro Santa-Clara, and Rossen Valkanov, 2003, Equity portfolios with parametric weights: Exploiting size, book to market, and momentum, Working Paper, Duke University.

Brennan, Michael J., Eduardo S. Schwartz, and Ronald Lagnado, 1997, Strategic asset allocation, *Journal of Economic Dynamics and Control* 21, 1377–1403.

Britten-Jones, Mark, 1999, The sampling error in estimates of mean-variance efficient portfolio weights, *Journal of Finance* 54, 655–671.

Byrd, Richard H., Peihuang Lu, Jorge Nocedal, and Ciyou Zhu, 1995, A limited memory algorithm for bound constrained optimization, *SIAM Journal on Scientific Computing* 16, 1190–1208.

Campbell, John Y., 1991, A variance decomposition for stock returns, *Economic Journal* 101, 157–179.

———— , Yeung Lewis Chan, and Luis M. Viceira, 2003, A multivariate model of strategic asset allocation, .

Campbell, John Y., and Robert J. Shiller, 1988, Stock prices, earnings, and expected dividends, *Journal of Finance* 43, 661–676.

Campbell, John Y., and Luis M. Viceira, 1999, Consumption and portfolio decisions when expected returns are time varying, *Quarterly Journal of Economics* 114, 433–495.

———— , 2002, *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors* (Oxford University Press: New York, NY).

Conn, Andy R., Nick I.M. Gould, and Philippe Toit, 1988, Global convergence for a class of trust-region algorithms for optimization with simple bounds, *SIAM Journal of Numerical Analysis* 25, 433–460.

Cox, John C., and Chi-Fu Huang, 1989, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33–83.

Fama, Eugene F., 1990, Stock returns, expected returns, and real activity, *Journal of Finance* 45, 1089–1108.

———— , and Kenneth R. French, 1988, Permanent and temporary components of stock prices, *Journal of Political Economy* 96, 246–273.

———— , 1989, Business conditions and expected returns on stocks and bonds, *Journal of Financial Economics* 25, 23–49.

Frost, Peter A., and James E. Savarino, 1988, For better performance: Constrain portfolio weights, *Journal of Portfolio Management* 15, 29–34.

Gill, Philip E., Walter Murray, and Michael A. Saunders, 2002, SNOPT: an SQP algorithm for large-scale constrained optimization, *SIAM Journal of Optimization* 12, 979–1006.

Hansen, Lars P., and Scott F. Richard, 1987, The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models, *Econometrica* 55, 587–613.

Hodrick, Robert J., 1992, Dividend yields and expected stock returns: Alternative procedures for inference and measurement, *Review of Financial Studies* 5, 257–286.

Jagannathan, Ravi, and Tongshu Ma, 2003, Risk reduction in large portfolios: Why imposing the wrong constraints helps, *Journal of Finance* 58, 1651–1683.

Jobson, J. David, and Bob Korkie, 1981, Putting Markowitz theory to work, *Journal of Portfolio Management* 7, 70–74.

Keim, Donald B., and Robert F. Stambaugh, 1986, Predicting returns in the stock and bond markets, *Journal of Financial Economics* 17, 357–390.

Ledoit, Olivier, 1995, A well-conditioned estimator for large dimensional covariance matrices, Working Paper, UCLA.

Moré, Jorge J., and Gerardo Toraldo, 1989, Algorithms for bound constrained quadratic programming problems, *Numerical Mathematics* 55, 377–400.

Pastor, Lubos, and Robert F. Stambaugh, 2000, Comparing asset pricing models: An investment perspective, *Journal of Financial Economics* 56, 335–381.

Treynor, Jack L., and Fisher Black, 1973, How to use security analysis to improve portfolio selection, *Journal of Business* 46, 66–86.

# Table 1: Approximation Error in Multiperiod Portfolio Policies

This table describes approximate and exact portfolio policies for an investor with quadratic utility and $\gamma = 4$. The approximate solution is based on timing portfolios which ignore the compounding of excess returns over time. The exact solution takes compounding into account. Panel A is for unconditional portfolio policies involving stocks and bonds. Panel B is for conditional portfolio policies in which the stock and bond returns are also scaled by the term spread. Each panel reports the annualized mean, standard deviation, Sharpe ratio, and certainty equivalent of the multiperiod portfolio returns. It also reports the average absolute difference between the approximate versus exact allocations.

## Panel A: Unconditional Policies

| | 1 Year Horizon | | | | 2 Year Horizon | | | | 5 Year Horizon | | | |
| | Monthly | | Quarterly | | Quarterly | | Semi-Annual | | Semi-Annual | | Annual | |
| | Approx. | Exact | Approx. | Exact | Approx. | Exact | Approx. | Exact | Approx. | Exact | Approx. | Exact |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mean | 0.0452 | 0.0427 | 0.0446 | 0.0425 | 0.0401 | 0.0367 | 0.0393 | 0.0365 | 0.0359 | 0.0311 | 0.0350 | 0.0308 |
| Std. Dev. | 0.0918 | 0.0867 | 0.0909 | 0.0867 | 0.0857 | 0.0783 | 0.0846 | 0.0783 | 0.0839 | 0.0720 | 0.0826 | 0.0722 |
| Sharpe Ratio | 0.4924 | 0.4925 | 0.4906 | 0.4902 | 0.4673 | 0.4684 | 0.4647 | 0.4657 | 0.4281 | 0.4317 | 0.4242 | 0.4267 |
| Cert. Equiv. | 0.0215 | 0.0216 | 0.0214 | 0.0214 | 0.0185 | 0.0187 | 0.0184 | 0.0185 | 0.0155 | 0.0159 | 0.0154 | 0.0157 |
| Avg \|Δ Weight\| | 0.0296 | | 0.0246 | | 0.0383 | | 0.0331 | | 0.0426 | | 0.0369 | |

## Panel B: Conditional Policies

| | 1 Year Horizon | | | | 2 Year Horizon | | | | 5 Year Horizon | | | |
| | Monthly | | Quarterly | | Quarterly | | Semi-Annual | | Semi-Annual | | Annual | |
| | Approx. | Exact | Approx. | Exact | Approx. | Exact | Approx. | Exact | Approx. | Exact | Approx. | Exact |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mean | 0.0777 | 0.0668 | 0.0738 | 0.0655 | 0.0637 | 0.0540 | 0.0603 | 0.0527 | 0.0627 | 0.0496 | 0.0586 | 0.0479 |
| Std. Dev. | 0.1225 | 0.1041 | 0.1179 | 0.1040 | 0.1122 | 0.0939 | 0.1083 | 0.0940 | 0.1295 | 0.0995 | 0.1251 | 0.1002 |
| Sharpe Ratio | 0.6343 | 0.6417 | 0.6260 | 0.6298 | 0.5672 | 0.5745 | 0.5565 | 0.5608 | 0.4840 | 0.4987 | 0.4680 | 0.4775 |
| Cert. Equiv. | 0.0332 | 0.0344 | 0.0329 | 0.0336 | 0.0267 | 0.0279 | 0.0264 | 0.0271 | 0.0235 | 0.0257 | 0.0233 | 0.0247 |
| Avg \|Δ Weight\| | 0.0312 | | 0.0267 | | 0.0395 | | 0.0346 | | 0.0431 | | 0.0370 | |

## Table 2: Single-Period Portfolio Policies

This table shows estimates of the single-period portfolio policy. Standard errors for the coefficients of the portfolio policies in parenthesis. The $p$-value refers to an $F$-test of the hypothesis that all the coefficients on the state variables other than the constant are jointly zero. The last three rows present statistics of the returns generated by the portfolio policies. Data from January of 1945 to December of 2000.

| State Variable | | Monthly | | Quarterly | | Annual | |
|---|---|---|---|---|---|---|---|
| | | Unconditional | Conditional | Unconditional | Conditional | Unconditional | Conditional |
| Stock | Const | 0.764 (0.185) | 0.873 (0.186) | 0.770 (0.179) | 0.649 (0.185) | 0.572 (0.139) | 0.591 (0.177) |
| | Term | | -0.166 (0.184) | | 0.304 (0.223) | | 0.249 (0.203) |
| | Default | | -0.504 (0.173) | | -0.333 (0.183) | | 0.046 (0.155) |
| | D/P | | 0.796 (0.306) | | 0.126 (0.190) | | 0.037 (0.149) |
| | TBill | | -0.485 (0.459) | | -0.434 (0.235) | | -0.172 (0.242) |
| Bond | Const | -0.005 (0.323) | 0.291 (0.174) | -0.040 (0.319) | -0.084 (0.389) | -0.074 (0.255) | -0.696 (0.323) |
| | Term | | -0.096 (0.173) | | 0.840 (0.336) | | 0.352 (0.278) |
| | Default | | -0.465 (0.404) | | 0.144 (0.351) | | 0.337 (0.291) |
| | D/P | | 0.672 (0.310) | | -0.076 (0.458) | | -0.264 (0.373) |
| | TBill | | 0.773 (0.247) | | 0.601 (0.346) | | -0.152 (0.397) |
| $p$-value | | | 0.000 | | 0.000 | | 0.000 |
| Mean Excess Return | | 0.063 | 0.157 | 0.065 | 0.126 | 0.049 | 0.095 |
| Std. Dev. Return | | 0.111 | 0.178 | 0.109 | 0.146 | 0.087 | 0.101 |
| Sharpe Ratio | | 0.589 | 0.882 | 0.602 | 0.864 | 0.571 | 0.938 |

## Table 3: Traditional versus Optimal Conditional Policies

This table shows estimates of the traditional approach to tactical asset allocation. In this approach, conditional expected returns are obtained from an in-sample regression of returns on the state variables and the Markowitz solution is applied to these conditional expected returns together with the unconditional covariance matrix. Panel A displays the estimated regressions of stock and bond returns on the conditioning variables, each estimated at monthly, quarterly, and annual frequency. Panel B summarizes the traditional portfolio policy (Trdnl) and, for comparison, the full conditional policy (Cndtnl) that takes into account the impact of the conditioning variables both on expected returns and their covariance matrix. The first two rows present the time-series average of the weights on stocks and bonds of the two policies. The next three lines offer statistics for the time series of portfolio returns. The last row shows the yearly fee that a mean-variance investor would be willing to pay to be able to use the full conditional policy instead of using traditional approach.

### Panel A: Regression Estimates

|  | Coefficient | Monthly | Quarterly | Annual |
|---|---|---|---|---|
| Stock | Cnst | 0.0813 (0.0220) | 0.0896 (0.0265) | 0.0828 (0.0236) |
|  | Term | -0.0618 (0.0233) | -0.0423 (0.0281) | -0.0258 (0.0264) |
|  | Default | 0.0355 (0.0254) | 0.0735 (0.0301) | 0.0057 (0.0286) |
|  | D/P | -0.0950 (0.0259) | -0.0431 (0.0315) | 0.0177 (0.0286) |
|  | Tbill | -0.0619 (0.0224) | -0.0374 (0.0274) | -0.0332 (0.0245) |
|  | $R^2$ | 0.0266 | 0.0339 | 0.0500 |
| Bond | Cnst | 0.0170 (0.0089) | 0.0170 (0.0098) | 0.0164 (0.0098) |
|  | Term | -0.0144 (0.0094) | 0.0044 (0.0104) | 0.0053 (0.0110) |
|  | Default | 0.0328 (0.0103) | 0.0248 (0.0111) | 0.0155 (0.0119) |
|  | D/P | -0.0178 (0.0104) | -0.0040 (0.0116) | -0.0044 (0.0119) |
|  | Tbill | 0.0004 (0.0091) | 0.0090 (0.0102) | -0.0213 (0.0102) |
|  | $R^2$ | 0.0126 | 0.0216 | 0.1182 |

### Panel B: Portfolio Policies

|  | Monthly | | Quarterly | | Annual | |
|---|---|---|---|---|---|---|
|  | Trdnl | Cndtnl | Trdnl | Cndtnl | Trdnl | Cndtnl |
| Mean Weight Stock | 0.5196 | 0.8121 | 0.3853 | 0.5580 | 0.4923 | 0.4562 |
| Mean Weight Bond | 0.4783 | -0.3624 | 0.4251 | 0.6356 | 0.3929 | 0.3208 |
| Mean Excess Return | 0.1615 | 0.1985 | 0.0940 | 0.0921 | 0.0884 | 0.0574 |
| Std. Dev. Return | 0.1969 | 0.2154 | 0.3257 | 0.1449 | 0.2098 | 0.1059 |
| Sharpe Ratio | 0.8204 | 0.9215 | 0.2885 | 0.6361 | 0.4213 | 0.5423 |
| Equalization Fee | 0.0195 |  | 0.1679 |  | 0.0428 |  |

### Table 4: Multiperiod Portfolio Policies

This table shows estimates of the multiperiod portfolio policy with a one-year horizon and monthly or quarterly rebalancing. Standard errors for the coefficients of the portfolio policies in parenthesis. The $p$-value refers to an $F$-test of the hypothesis that all the coefficients on the state variables other than the constant are jointly zero. The last three rows present statistics of the returns generated by the portfolio policies.

| Asset | Month/ Quarter | State Variable | Monthly Unconditional | Monthly Conditional | Quarterly Unconditional | Quarterly Conditional |
|-------|------|------|------|------|------|------|
| Stock | 1/1 | Cnst | 0.4899 (0.1429) | 0.5046 (0.1437) | 0.4495 (0.1188) | 0.4219 (0.1207) |
|       |     | Tbill |                 | -0.2568 (0.1428) |                | -0.0598 (0.3368) |
|       | 4/2 | Cnst | 0.4563 (0.1445) | 0.4998 (0.1449) | 0.3720 (0.1171) | 0.4469 (0.1174) |
|       |     | Tbill |                 | -0.2264 (0.1451) |                | 0.0923 (0.3427) |
|       | 8/3 | Cnst | 0.4354 (0.1453) | 0.4174 (0.1453) | 0.3942 (0.1172) | 0.4129 (0.1177) |
|       |     | Tbill |                 | -0.1549 (0.1442) |                | 0.3406 (0.3388) |
|       | 12/4 | Cnst | 0.2788 (0.1437) | 0.2206 (0.1436) | 0.2477 (0.1198) | 0.2074 (0.1214) |
|       |     | Tbill |                 | -0.3654 (0.1439) |                | 0.6009 (0.3275) |
| Bond | 1/1 | Cnst | -0.2249 (0.3589) | -0.2174 (0.3622) | -0.0367 (0.3237) | -0.1729 (0.1114) |
|       |     | Tbill |                 | 0.0685 (0.2059) |                | 0.0537 (0.1946) |
|       | 4/2 | Cnst | 0.0575 (0.3578) | 0.0688 (0.3608) | 0.1104 (0.3236) | -0.1597 (0.1119) |
|       |     | Tbill |                 | 0.0528 (0.2059) |                | 0.0553 (0.2145) |
|       | 8/3 | Cnst | 0.5084 (0.3561) | 0.5267 (0.3579) | 0.4375 (0.3234) | -0.1322 (0.1121) |
|       |     | Tbill |                 | 0.0176 (0.2060) |                | 0.0194 (0.2153) |
|       | 12/4 | Cnst | 0.6943 (0.3559) | 0.6646 (0.3613) | 0.5633 (0.3188) | -0.2662 (0.1151) |
|       |     | Tbill |                 | 0.0343 (0.2063) |                | 0.1320 (0.1962) |
| $p$-value |   |   | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Mean Excess Return |   |   | 0.0424 | 0.0570 | 0.0407 | 0.0505 |
| Std. Dev. Return |   |   | 0.0882 | 0.0972 | 0.0858 | 0.0914 |
| Sharpe Ratio |   |   | 0.4812 | 0.5860 | 0.4740 | 0.5524 |

## Figure 1: Portfolio Weights of Conditional and Unconditional Policies

This figure displays the time series of conditional portfolio weights. The solid line corresponds to the portfolio weight on the stock and the dash-dotted line corresponds to the portfolio weight on the bond. The constant portfolio weights from the unconditional policy are depicted as straight lines.
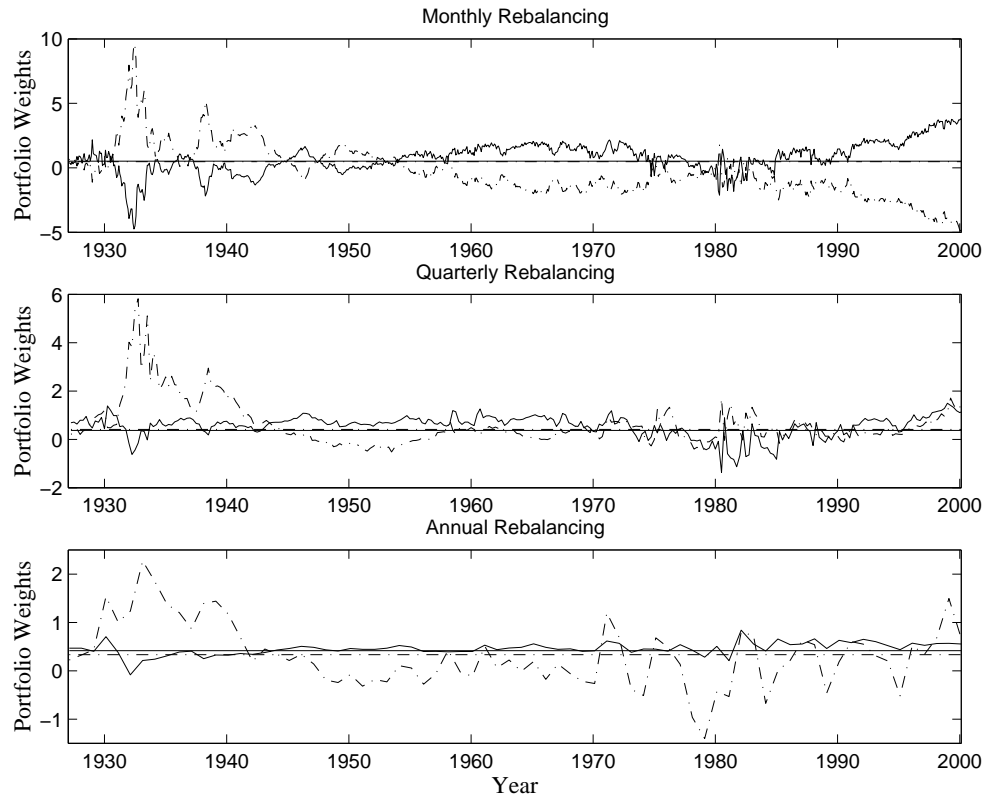
## Figure 2: Portfolio Weights of Conditional and Regression-Based Policies

This figure displays the time series of the portfolio weight on the stock obtained from the conditional approach as solid line and from the regression-based approach as dashed line. In the regression-based approach, conditional expected returns are computed from an in-sample regression of returns on the state variables, and the Markowitz solution is applied to these conditional expected returns together with the unconditional covariance matrix.