$$\mathrm{Sgn}(v) = \frac{|x|}{x} \qquad |v|\, v$$

$$m \overbrace{\frac{d^2x}{dt^2}}^{\dot{v}} = -mg - kx - \frac{1}{2} C_D \rho_f A\, \mathrm{Sgn}(v) V^2 + \rho g \tau$$

$$\boxed{\dot{v} = \overbrace{\left[g\left(-1 + \frac{\rho \tau}{m}\right)\right]}^{\alpha} - \overbrace{\frac{K}{m}}^{\beta} x - \overbrace{\frac{1}{2} C_D \rho_f A}^{\gamma} \mathrm{Sgn}(v)\, V^2}$$

$$\dot{v} = \alpha - \beta x - \gamma\, \mathrm{Sgn}(v)\, V^2 \qquad \mathrm{Sgn}(v) = \frac{|v|}{v}$$

$$\dot{v} = \alpha - \beta x - \gamma |v|\, v$$

$$\dot{x} = v$$

$$\tilde{x} = \frac{\alpha}{\beta} + x$$

$$\dot{\tilde{v}} = \cancel{\alpha} - \beta\left(\tilde{x} - \cancel{\frac{\alpha}{\beta}}\right) - \gamma |v|\, v$$

$$\dot{\tilde{v}} = \dot{v}$$

$$\dot{\tilde{v}} = -\beta \tilde{x} - \gamma |\tilde{v}|\, \tilde{v} \qquad \tilde{v} = v$$

$$\begin{matrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{matrix} = \begin{pmatrix} \tilde{v} \\ -\beta \tilde{x} - \gamma |\tilde{v}|\, \tilde{v} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\gamma |\tilde{v}| \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix}}$$

$$\dot{u} = \Omega(\tilde{v})\, u$$

$$\boxed{f(x, v, t) = \Omega(v)\, u(\tilde{x}, \tilde{v}, t)}$$

Método numérico

Sea un sistema con condiciones iniciales

$$u_0 = \begin{pmatrix} \tilde{x}_0 \\ \tilde{v}_0 \end{pmatrix} \qquad t_0,\ t_f$$

definimo un incremento de tiempo h, teniento en cuenta un número de pasos N

$$\boxed{\delta t = \frac{t_f - t_0}{N} \qquad t_{n+1} = t_n + h}$$

Tal que para calcular la solución aproximado

$$\boxed{U_{n+1} = U_n + A(U_n, t_n)\,\delta t}$$

Siendo A el algoritmo a usar

A es la función de incremento

Sea

$\dot{u} = f(u, t)$ la función que se desea solución

Con el método de Euler, A es escrito:

$$\boxed{A(U_n, t_n) = f(U_n, t_n)}$$

Con el método midpoint es escrito:

$$\boxed{A(U_n, \delta t) = f\left(U_n + \frac{\delta t}{2} f(U_n, t_n), t_n + \frac{\delta t}{2}\right)}$$

Con el método de Heum:

$$\boxed{A(U_n, \delta t) = \frac{1}{2}\left[f(U_n, t_n) + f\left(U_n + \delta t\, f(U_n, t_n), t_{n+1}\right)\right]}$$

Con el método RK4:

$$\boxed{\begin{aligned} K_1 &= f(U_n, t_n) \\ K_2 &= f\left(U_n + \frac{\delta t}{2} K_1, t_n + \frac{1}{2}\delta t\right) \\ K_3 &= f\left(U_n + \frac{\delta t}{2} K_2, t_n + \frac{1}{2}\delta t\right) \\ K_4 &= f(U_n + \delta t K_3, t_n + \delta t) \end{aligned}}$$

$$\boxed{A(U_n, t_n) = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)}$$

Soluciones analíticas

$$\boxed{\dot{V} = \overbrace{\left[g\left(-1 + \frac{\rho \tau}{m}\right)\right]}^{\alpha} - \overbrace{\frac{K}{m}}^{\beta} X - \overbrace{\frac{1}{2} C_D \rho_f A \operatorname{Sgn}(V)}^{\gamma} V^2}$$

Cuando $\gamma = 0$

$$\dot{v} = \alpha - \beta X$$

Cambio de variable

$$\tilde{X} = \frac{\alpha}{\beta} - X \qquad X = \frac{\alpha}{\beta} - \tilde{X}$$

$$\dot{\tilde{X}} = -V$$

$$\ddot{\tilde{X}} = \dot{\tilde{V}} = -\dot{V}$$

$$-\dot{\tilde{V}} = \cancel{\alpha} - \beta\left(\frac{\cancel{\alpha}}{\beta} - \tilde{X}\right)$$

$$\dot{\tilde{V}} = -\beta \tilde{X}$$

$X(0) = X_0$

$V(0) = V_0$

$$\frac{d^2 X}{dt^2} + \beta \tilde{X} = 0$$

$$W^2 = \beta$$

$$\hat{X} = A \cos(\omega t + \varnothing) + B \operatorname{Sen}(\omega t + \varnothing)$$

$$\hat{X} = \frac{\alpha}{\beta} - X$$

$$X(t) = \frac{\alpha}{\beta} - A \cos(\omega t + \varnothing) - B \operatorname{Sen}(\omega t + \varnothing)$$

$$\varnothing =$$

$$X(0) = X_0 = \frac{\alpha}{\beta} - A \qquad A = \frac{\alpha}{\beta} - X_0$$

$$\dot{x} = \frac{A \operatorname{Sen} \omega t}{\omega} - \frac{B \cos \omega t}{\omega}$$

$$V(0) = V_0 = -\frac{B}{\omega_0} \qquad V_0 = -\frac{B}{\omega}$$

$$B = -V_0 \omega$$

$$X(t) = \frac{\alpha}{\beta} - \left[\frac{\alpha}{\beta} - x_0\right] \cos(\omega t) + V_0 \omega \operatorname{Sen}(\omega t)$$

$$\boxed{X(t) = \frac{\alpha}{\beta}\left[1 - \cos \omega t\right] + X_0 \cos(\omega t) + V_0 \omega \operatorname{Sen}(\omega t)}$$

$$\boxed{V(t) = \frac{\alpha}{\omega} \operatorname{Sen} \omega t + V_0 \omega^2 \cos \omega t - X_0 \omega \operatorname{Sen} \omega t}$$

$$\boxed{\begin{aligned} \omega^2 &= \frac{K}{m} \\ \alpha &= -g + \frac{\rho g V}{m} \\ \beta &= \frac{K}{m} = W^2 \end{aligned}}$$

Cuando $\alpha = \beta = 0$

$$\dot{v} = -\gamma v^2$$

Si $V_0 < 0$ La solución es:

$$\boxed{\begin{aligned} x(t) &= x_0 - \frac{1}{\gamma}\ln[1 - \gamma V_0 t] \\ V(t) &= \frac{V_0}{1 - \gamma V_0 t} \end{aligned}}$$

$[\gamma] = m^{-1}$

Sí $V_0 > 0$

$$\boxed{\begin{aligned} x(t) &= x_0 + \frac{1}{\gamma}\ln[1 + \gamma V_0 t] \\ V(t) &= \frac{V_0}{1 + \gamma V_0 t} \end{aligned}}$$

Sí $V_0 = 0$

$$\boxed{\begin{aligned} x(t) &= x_0 \\ V(t) &= 0 \end{aligned}}$$

De forma general podemos expresar la solución de la siguiente manera:

$$\boxed{\begin{aligned} X(t) &= X_0 + \frac{1}{\gamma}\,\mathrm{Sgn}(V_0)\ln\left[1+\gamma|V_0|t\right] \\ V(t) &= \frac{V_0}{1+\gamma\,\mathrm{Sgn}(V_0)V_0 t} \qquad \mathrm{Sgn}(V_0) = \begin{cases} x>0 & 1 \\ x<0 & -1 \\ x=0 & 0 \end{cases} \end{aligned}}$$