1. **Logistic Regression: Training stability**
2. dataset A converges very quickly, but dataset B doesn’t seem to converge ever.
3. The short answer is that Logistic Regression will not converge when data is perfectly separable.

* First part of the proof below will show that if the data is perfectly separated, the gradient of loss function w.r.t. can never be 0, which means the log-likelihood loss can never reach to a maximum;

The probability of a point is:

For simplicity, assume only has one feature, is just a scaler. A correctly classified point should satisfy ; Otherwise the sample is misclassified.

The MLE is the solution of . To further simply the notation let’s assume , the log-likelihood loss can be written as:

Take the derivative w.r.t. :

Since ,, we have for all . This explains why there is no convergence. The plot of should look something like the following. Note that when , we should have to make , which is the lower part of the diagram; when , we should have to make , which is the upper part of the diagram.

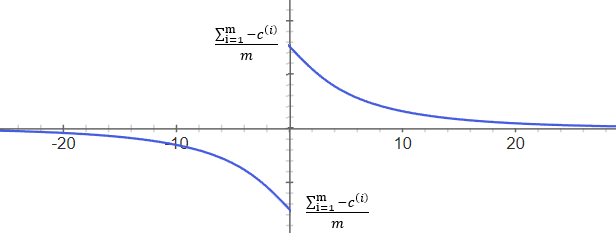


Figure 1, Gradient w.r.t.

* Now consider there is a misclassified point. Just one misclassified will make the algorithm converge. A misclassified point means in the assumption made above, , this will always make . Now we need to show that in this case will have a solution. Suppose the jth point is misclassified:

The left part of the above are all the correctly classified points which will take the value in () when , or () when , it will have the similar plot as Figure 1; The right side of the above will take the value () when , ; () when . The below Figure 2 shows their relationship. It can be seen that no matter whether is positive or negative, there will always be an intersection between and .

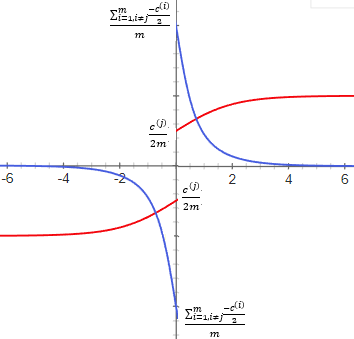


Figure 2, Gradient for both correct (blue)   
 and misclassified(red) samples

Note that there is a possibility that even though there is a misclassified point, never have a solution, like the following, there is no intersection between red and blue plot:

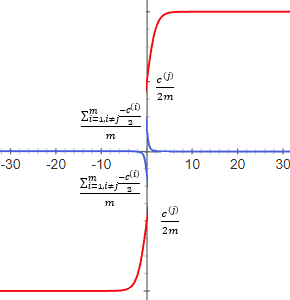


Figure 3, Not possible

It turns out that this is impossible so long as the data points are generated IID from the same distribution; there is always a solution for in this case.

1. For each of these possible modifications, state whether it would lead to Convergence.

* Using a different constant learning rate.   
  Answer:   
  No. The problem is with the parameter being scaled infinitely to get a larger probability to get a maximum log-likelihood. Change of learning rate would not solve this problem.
* Decreasing the learning rate over time.  
  Answer:  
  No. the answer is same as above.
* Linear scaling of the input features.  
  Answer:  
  No. It has the same effect as to scale the parameters which doesn’t help.
* Adding a regularization term to the loss function.   
  Answer:  
  Yes. L2 regularization will penalize large hence it will prevent the parameters to scale infinitely when it tries to maximize the log-likelihood.
* Adding zero-mean Gaussian noise to the training data or labels: will have misclassified points.  
  Answer:  
  Yes. It will add misclassified samples to the dataset which will make logistic regression converge.

1. In SVM, is going to be normalized and cannot be infinitely increased when trying to maximize the log-likelihood loss; strict separation issue would no longer be a problem.
2. **Model Calibration**
3. Prove calibration condition holds true for Logistic regression over range (a, b) = (0, 1). Firstly, the log-likelihood of all points is given by

The maximum likelihood parameter is just the solution given by . Note that

with . Use the fact that we include a bias term: , by observing the first component of and , we have:

1. Perfect calibration doesn’t mean perfect accuracy. If for any the property in the question holds true, consider two train examples with index i and j, they have different probability range and . By switching two samples’ probability ranges, the perfect calibration still holds true, but the two training examples will have different probability ranges separately.

Conversely if the model achieves perfect accuracy, it is perfectly calibrated. This can be explained by clapping to the probability of every single training sample, the perfect calibration condition holds true always.

1. What effect including L2 regularization in the logistic regression objective has on model calibration.

L2 regularization penalizes large value of , which makes not able to grow infinitely to get a larger probability, which means the calibration range is smaller than (0,1).

1. **Bayesian Interpretation of Regularization**
2. Show that if we assume that .

**PROOF:**

The posterior distribution of parameter is:

The first equality is based on Bayesian rule with a condition, the second equality is the marginal probability over conditioned on x. The denominator of the second is independent on the parameter, so we have:

Which is exactly what we need to prove.

1. Show that MAP estimation with zero-mean Gaussian prior is equivalent to applying L2 regularization with MLE.

**PROOF:**

From above we have the MAP estimation given by:

Change the max log-likelihood estimation to min negative log-likelihood estimation, and use the fact the is a zero mean Gaussian:

From the last equality we have:

1. Closed form expression for with Gaussian prior and Gaussian distributed dataset

* The Likelihood function is:
* The negative log-likelihood loss can then be derived from above:

The MAP estimation is just the solution at the stationary point. Take derivative w.r.t. and set it to :

Vectorize the above sum, ,, :

So the closed form .

1. Show that in the case of Laplace prior is equivalent to the solution of a linear regression with L1 regularization.

PROOF. Similar to c), the Likelihood function is given by

The negative log-likelihood loss is then:

Scale the loss doesn’t change anything to the problem, so the loss can just be written as:

Where

1. **Constructing** **kernels.**Apply Mercier’s theorem, to show a matrix is a kernel matrix, just need to show that the matrix is symmetric Positive Semi-Definite. (For the ease of readability, And are the same thing in this answer.)
   1. :

It’s a kernel, it is obvious that if K1 and K2 are both symmetric and positive semi-definitive.

Frist, K is symmetric because, from:

Second, K is positive semi-definitive, because both K1 and K2 are PSD, K=K1+K2 is also a PSD matrix:

* 1. :   
     Not necessarily a kernel. Same as (a) it is symmetric, but not always positive definitive.   
     For example, , ( is identity matrix), they both PSDs, but is negative definite.
  2. :   
     It is a kernel only when a >= 0. If a<0, , then is negative definite.
  3. :   
     Similar to c), It is a kernel only when a <= 0. For example, a = 1, , we have
  4. :   
     It is a kernel since because,

Summarize the above 3 equations, which means K is symmetric;

since both K1 and K2 are PSD, for any given vector x, we have , it implies that:

* 1. :  
     It is a kernel. From class we know that for a feature mapping , is a kernel. In this case we simply can set , it’s just a 1d mapping.
  2. :  
     It is a kernel. Since K3 is a Kernel, for a feature mapping .
  3. :  
     It is a kernel. First, is still a kernel; is still a kernel when , is a kernel if ; combine them together we have that is still a kernel.

1. Kernelizing the Perceptron
   1. How you would apply the “kernel trick” to the perceptron
   2. Implement your approach by completing
   3. Which kernel performs badly and why does it fail?
2. Spam classification