THE COMPLEX POLYNOMIALS

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CERTIFICATE

This is to certify that the work contained in this report entitled **"The Complex Polynomials"** submitted by **Harosh Kumar (Roll Number: 202123018)** to the Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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ABSTRACT

In the analytic theory of polynomials, the main problem is to find the location of zeros of polynomials over the field of real numbers or over the field of complex numbers. Firstly, the results determining the number of zeros of real polynomial in an interval of the real line are presented. The main objective of this report is to present various results on the zeros of polynomials and the number of zeros of polynomials in the disks.

Contents

Chapter 1

Preliminaries

Let $\mathbb{C}$ denote the set of complex numbers/ complex plane.

Definition 1.0.1. *Let $f(z)$ be a complex function which is analytic in an open set D of $\mathbb{C}$. Let $z_0 \in D$. Then the Taylor series of f about the point z_0 is given by*

$$f(z) = f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\frac{f'''(z_0)}{3!}(z-z_0)^3+\cdots \quad \textit{for} \quad |z-z_0| < R$$

for some $R > 0$. This can be written as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad |z-z_0| < R\ . \tag{1.1}$$

Remark 1.0.1. *When $z_0 = 0$ in (1.1), the Taylor series of f is also called Maclaurin series.*

Example 1.0.1. *The Maclaurin series of $f(z) = e^z$ is given by*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \textit{for} \qquad z \in \mathbb{C}\ .$$

Definition 1.0.2. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1}z^{n-1} + a_n z^n$ be a* *polynomial of degree n. If all the coefficients of $p(z)$ are real numbers, then it is called a real polynomial. If the all the coefficients of $p(z)$ are complex numbers, then it is called a complex polynomial. A point $z_0 \in \mathbb{C}$ is said to be a zero of $p(z)$ if $p(z_0) = 0$.*

Definition 1.0.3. *For a given polynomial $p(z)$ of degree $n \geq 1$, if a zero (say z_0) of $p(z)$ is found then we can find the lower degree polynomial $q(z)$ after deleting the linear factor of $(z - z_0)$ and this lower degree polynomial $q(z)$ is called a* ***deflected polynomial****. That is, $p(z) = (z - z_0)q(z)$ where the degree of $q(z)$ is $(n - 1)$.*

Theorem 1.0.1. ***(Fundamental Theorem of Algebra)***
Every polynomial equation $p(z) = 0$, where the degree of $p(z)$ is n, has at most n roots, or solutions, in the field $\mathbb{R}$ of real numbers and exactly n roots exist, in the field $\mathbb{C}$ of complex numbers.

Definition 1.0.4. *Let $\{a_0,\ a_1,\ \ldots,\ a_n\}$ be a finite sequence of real numbers. We shall say that a* ***sign change*** *occurs between a_k and a_m when $0 \leq k < m \leq n$ if*

$$a_k a_m < 0$$

and if either $m = k + 1$ or $m > k + 1$ and

$$a_l = 0, \qquad k < l < m\ .$$

The total number of sign changes that occurs between the elements of a sequence is called the ***number of sign changes*** *of the sequence or* ***number of variations of sign*** *in the sequence.*

Example 1.0.2. *The sequence* $\{1,\ 0,\ 2,\ -3,\ 0,\ 0,\ 1,\ 2,\ -1,\ 0,\ 2\}$ *has three sign changes.*

Definition 1.0.5. *Let* $p(x) = 0$ *be a polynomial equation of degree* n *with real coefficients. Let* V_a *denote the number of sign changes in the sequence*

$$p(x),\ p'(x),\ p''(x),\ \ldots,\ p^{(n)}(x)\ ,$$

when $x = a$*, where* a *is any real number.*

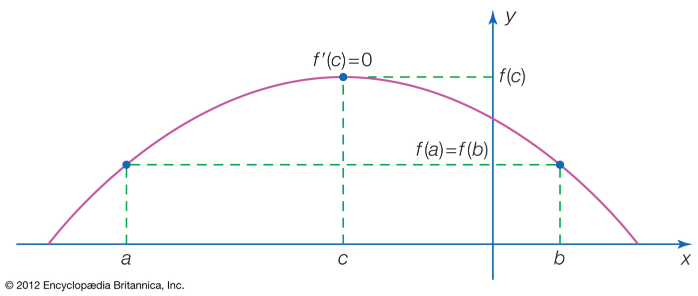
Theorem 1.0.2. ***(The Budan-Fourier Theorem)***

Let $p(x)$ *be a polynomial equation of degree* n *with real coefficients. Let* a *and* b *be any real numbers,* $a < b$*. Then the number of roots of* $p(x) = 0$ *in the interval* $(a,\ b]$ *(counting multiplicity) is* $V_a - V_b - 2k$*, where* k *is a positive integer or zero.*

For example, if $s = 3$, then there are either 3 or 1 real roots. On the other hand, if $s = 4$, then there are either 4, 2, or no real roots.

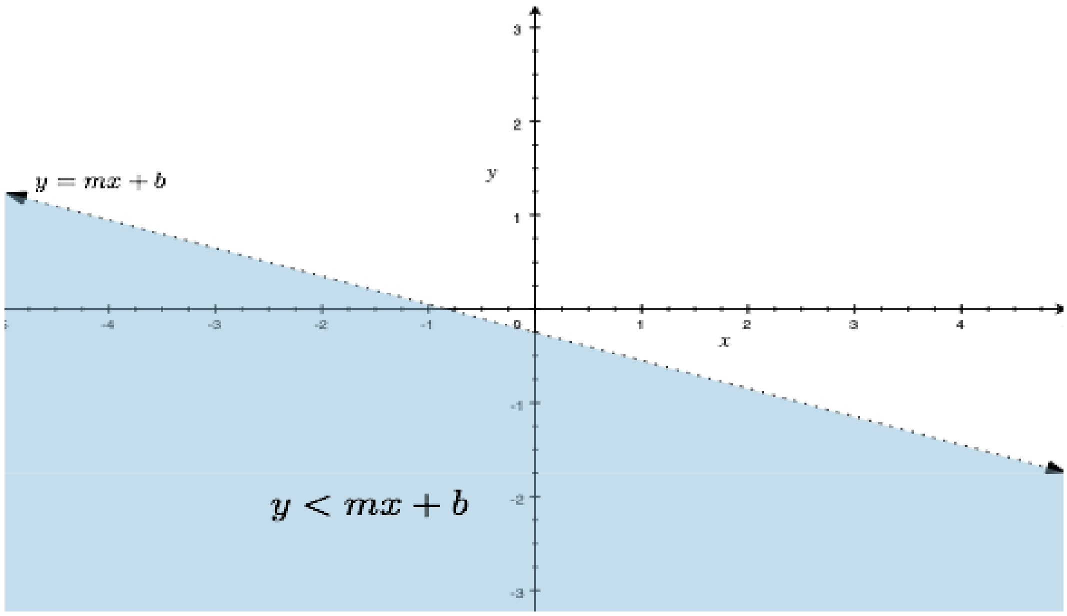
Theorem 1.0.3. ***(Rolle's Theorem)***

If a real-valued function f *is continuous on a closed interval* $[a, b]$*, differentiable on the open interval* (a, b)*, and* $f(a) = f(b)$*, then there exists at least one* c *in the open interval* (a, b) *such that* $f'(c) = 0$*.*



Definition 1.0.6. ***(Half Plane)***

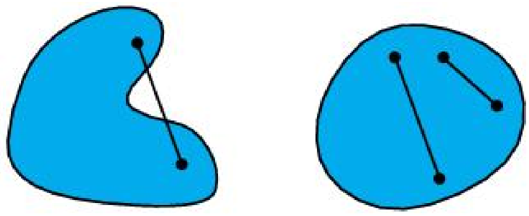
*A **half-plane** is a planar region consisting of all points on one side of an infinite straight line, and no points on the other side. If the points on the line are included, then it is called an closed half-plane. If the points on the line are not included, then it is called an open half-plane.*



The line divides the Cartesian plane into two half-planes, The set of points in the Cartesian plane which lie above the line form the upper half-plane, and the set of points which lie below the line form the lower half-plane.

Definition 1.0.7. ***(Convex set)***

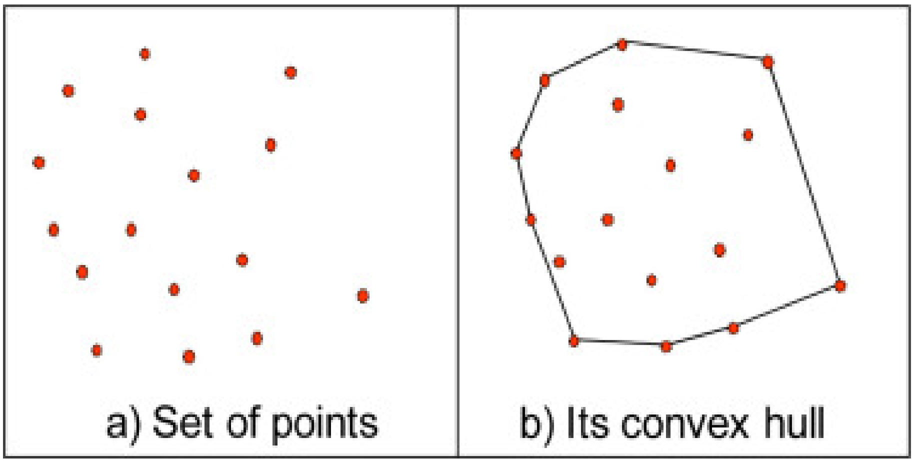
*Let S be a subset of $\mathbb{R}^2$. Then, S is said to be **convex** if every straight line segment L joining any two points of S lies entirely inside the set S.*



Not Convex Convex

Definition 1.0.8. ***(Convex hull)***

*Let S be a subset of $\mathbb{R}^2$. The **convex hull** of S is the smallest convex set that contain S. So, the convex hull of S is the intersection of all convex sets containing S.*



Example 1.0.3. *The intersection of closed half-planes is an example of a closed convex hull.*

Definition 1.0.9. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a complex function in D. Let $z_0 \in D$. A point z_0 is called a **zero of order** (or **multiplicity**) m for the function f if*

$$f(z_0) = 0 \qquad \textit{and}$$

$$f^{(k)}(z_0) = 0 \quad \textit{for} \quad k = 1,\ 2,\ \cdots (m-1) \qquad \textit{and} \qquad f^{(m)}(z_0) \neq 0\ .$$

Theorem 1.0.4. *If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is analytic and not constant, $z_0 \in D$, and $f(z_0) = 0$ then there is an $R > 0$ such that the open disk $B(z_0, R) \subset D$ and $f(z) \neq 0$ for $0 < |z - z_0| < R$. That is, the zeros of an analytic function f are isolated.*

Definition 1.0.10. *If $f(z)$ is not analytic at z_0 and $f(z)$ is analytic in the punctured/ deleted neighborhood $N_r^*(z_0) = \{z \in \mathbb{C} \ : \ 0 < |z - z_0| < r\}$ for some $r > 0$ then the point z_0 is called an **isolated singular point** of $f(z)$.*

Definition 1.0.11. *Let $f(z)$ have an isolated singularity at $z = z_0$. Then, the point z_0 is a **pole of order** n of f if and only if z_0 is a zero of order n for the function $1/f$.*

Definition 1.0.12. *If γ is a closed contour in $\mathbb{C}$ then for a point 'a' not on γ,*

$$n(\gamma;\, a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

*is called the **index of γ with respect to the point** a. It is also called the **winding number of γ around** a. The other notation is Ind$(\gamma;\, a)$.*

Example 1.0.4.

If $\gamma : z(t) = z_0 + re^{it}$ for $t \in [0,\, 2\pi]$, then $n(\gamma;\, z_0) = 1$.

If $\gamma : z(t) = 5 + 4e^{it}$ for $t \in [0,\, 6\pi]$, then $n(\gamma;\, 5) = 3$, $n(\gamma;\, 8) = 3$ and $n(\gamma;\, 10) = 0$.

If γ is a closed contour in $\mathbb{C}$ and z_0 is a point in $\mathbb{C}$ that is not enclosed by the contour γ, then $n(\gamma;\, z_0) = 0$.

If γ is a negatively oriented (clockwise direction) closed contour in $\mathbb{C}$ and it is enclosing a point z_0 in $\mathbb{C}$, then $n(\gamma;\, z_0)$ will be a negative integer.

If γ is a positively oriented (anticlockwise direction) closed contour in $\mathbb{C}$ and it is enclosing a point z_0 in $\mathbb{C}$, then $n(\gamma; z_0)$ will be a positive integer.

Theorem 1.0.5 (**Argument Principle**).

Let C be a positively oriented simple closed contour. Let $f(z)$ be analytic and nonzero at each point of C. Further f is analytic inside C except for a finite number of poles inside C. Suppose that, inside C, the function $f(z)$ has zeros $z_1, z_2, \cdots, z_k$ with multiplicities $m_1, m_2, \cdots, m_k$ and poles $w_1, w_2, \cdots, w_\ell$ with multiplicities $n_1, n_2, \cdots, n_\ell$ respectively. If $\Delta_C \arg(f(z))$ denotes the change in $\arg(f(z))$ as z traverses the curve C once in the counterclockwise sense, then

$$\begin{aligned} \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi}\Delta_C \arg f(z) = \sum_{i=1}^{k} m_i - \sum_{j=1}^{l} n_j = \\ \textit{Winding Number of } f(C) \textit{ about the origin} &= \textit{Number of zeros} - \textit{Number of poles inside } C \end{aligned}$$

The following theorem is closely related to the argument principle and helps us to find number of zeros of an analytic function inside a simple closed contour.

Theorem 1.0.6 (**Rouche's Theorem**).

Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour C. Suppose that

$$|g(z)| < |f(z)| \qquad \textit{for each point } z \textit{ on the contour } C\ .$$

Then, the functions $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting multiplicities, inside the contour C.

Chapter 2

Zeros of Polynomials and Disks Containing Zeros

2.1 Horner Algorithm

A given polynomial $p(z)$ can be represented in many different ways like Taylor's form or it may be in factored form. Each representation suggest a special algorithm for computing the value of the polynomial and possibly its derivatives. If, we wish to represent a polynomial $p(z)$ in Taylor's series then

$$p(z) = u_0 z^n + u_1 z^{n-1} + u_2 z^{n-2} + \ldots + u_n.$$

To compute $p(z)$ for a given z, we need to compute $z^2, \ldots, z^n$ and then multiply by appropriate coefficients u_k. Now we shall discuss a procedure that permits the calculation of $p(z)$ by n multiplication and the computation of all Taylor coefficients by $\frac{1}{2}n(n-1)$ multiplication called **Horner algorithm**.

Now let's move on to obtain Horner algorithm which is based on Horner's rule. Rewrite

$$p(z) = u_0 z^n + u_1 z^{n-1} + u_2 z^{n-2} + \ldots + u_n$$

as

$$p(z) = (((...(((u_0 z + u_1)z + u_2)z + ... + u_{n-1})z + u_n \ .$$

We see that $p(z)$ can be obtained in n multiplication. Expressing the procedure analytically, we compute numbers v_0, v_1, v_2, …, v_n by the recurrence relation

$$\begin{aligned} v_0 &= u_0 \ , \\ v_k &= u_k + z v_{k-1}, \qquad k = 1,\ 2,\ \ldots,\ n \ . \end{aligned}$$

Then, we have $v_n = p(z)$.

More generally, $v_k = p_k(z)$, $\quad k = 0,\ 1,\ 2,\ \ldots,\ n$, where

$$p_k(z) = u_0 \ z^k + u_1 \ z^{k-1} + u_2 \ z^{k-2} + ... + u_k.$$

Now, we apply the same algorithm to the numbers $v_k^0 := v_k$, we get

$$\begin{aligned} v_0^1 &= v_0^0 \\ v_k^1 &= v_k^0 + z \ v_{k-1}^1 \qquad k = 1,\ 2,\ \ldots,\ n \end{aligned}$$

Now, by the above result, if $k = 0,\ 1,\ 2,\ \ldots,\ n$,

$$\begin{aligned} v_k^1 &= v_0^0 \ z^k + v_1^0 \ z^{k-1} + \cdots + v_k^0 \\ &= u_0 \ z^k + (u_0 \ z + u_1) \ z^{k-1} + (u_0 \ z^2 + u_1 \ z + u_2) \ z^{k-2} + \cdots \\ &\quad \cdots + (u_0 \ z^k + u_1 \ z^{k-1} + u_2 \ z^{k-2} + \cdots + u_k) \\ &= (k+1) u_0 z^k + k u_1 z^{k-1} + ... + u_k \\ &= p'_{k+1}(z) \end{aligned}$$

Thus in particular, $v_{n-1}^1 = p'(z)$.

By repeating this process and finding v_k^n which amounts to the following recurrence relation, called ***Horner Algorithm***:

$$
\begin{aligned}
v_k^{-1} &:= u_k, \qquad k = 0,\ 1,\ \ldots,\ n\ ; \\
v_0^{m} &:= u_0, \qquad m = 0,\ 1,\ \ldots,\ n\ ; \\
v_k^{m} &:= v_k^{m-1} + z\, v_{k-1}^{m}, \qquad m = 0,\ 1,\ \ldots,\ n; \qquad k = 1,\ 2, \ldots,\ n-m\ .
\end{aligned}
$$

Theorem 2.1.1. *Let $p(z) = u_0 z^n + u_1 z^{n-1} + u_2 z^{n-2} + ... + u_n$ be the polynomial, and let*

$$w_m \ = \ \frac{1}{m!}\, p^{(m)}(z)\ , \qquad m = 0,\ 1,\ 2,\ \ldots,\ n\ .$$

If the coefficients v_k^m are constructed by the Horner Algorithm, then for $m = 0$, 1, 2, …, n,

$$v_{n-m}^{m} \ = \ w_m$$

Furthermore, if x is any complex number

$$v_0^m x^{n-m-1} + v_1^m x^{n-m-2} + ... + v_{n-m-1}^m = w_{m+1} + w_{m+2}(x-z) + ... + w_n (x-z)^{n-m-1}.$$

Proof. The proof is by induction with respect to m. We verify the assertion for $m = 0$. Here the recurrence relation,

$u_0 = v_0^0$, $u_k = v_k^0 - z v_{k-1}^0$ $\qquad\qquad$ $k = 1$, 2, …, n

Now

$$p(x) = u_0 x^n + u_1 x^{n-1} + u_2 x^{n-2} + ... + u_n.$$

$$= (v_0^0 x^{n-1} + v_1^0 x^{n-2} + ... + v_{n-1}^0)(x-z) + v_n^0$$

Assuming, $\qquad z \neq x$

and

$$v_0^0 x^{n-1} + v_1^0 x^{n-2} + ... + v_{n-1}^0 = \frac{p(x) - w_0}{x - z} \tag{2.1}$$

Now using Taylor's expansion of p at z $p(x) = w_0 + w_1(x - z) + ... + w_n(x - z)^n$

So from Equation 2.1

$$v_0^0 x^{n-1} + v_1^0 x^{n-2} + ... + v_{n-1}^0 = w_1 + w_2(x - z) + ... + w_n(x - z)^{n-1}$$

which proves for $m = 0$.

Now, let it is proved for $m - 1$. Then

$$v_0^{m-1} x^{n-m} + v_1^{m-1} x^{n-m-1} + ... + v_{n-m}^{m-1} = w_m + w_{m+1}(x - z) + ... + w_n(w - z)^{n-m}. \tag{2.2}$$

Now left side expression can be expand by relation $v_k^m = v_k^{m-1} + z v_{k-1}^m$ and

$$v_0^m = u_0 \qquad\qquad m = 0,\ 1,\ 2,\ \ldots,\ n$$

So, $v_0^{m-1} = u_0 = v_0^m$ So,

$$v_0^m x^{n-m} + v_1^m x^{n-m-1} - z v_0^m x^{n-m-1} + v_2^m x^{n-m-2} - z v_1^m x^{n-m-2} + ... + v_{n-m}^m - z v_{n-m-1}^m$$

Now by combining the terms containing $v_0, v_1, ..., v_{n-m-1}, v_{n-m}$ we will get

$$(v_0^m x^{n-m-1} + v_1^m x^{n-m-2} + ... + v_{n-m-1}^m)(x - z) + v_{n-m}^m$$

Now from equation (2.2) above expression equals,

$$w_m + w_{m+1}(x - z) + ... + w_n(w - z)^{n-m}.$$

Letting $x = z$ thus yields $v_{n-m} = w_m$ and assuming that $z \neq x$ and solving for

$$(v_0^m x^{n-m-1} + v_1^m x^{n-m-2} + ... + v_{n-m-1}^m)$$

We get

$v_0^m x^{n-m-1} + v_1^m x^{n-m-2} + ... + v_{n-m-1}^m = w_{m+1} + w_{m+2}(x-z) + ... + w_n(w-z)^{n-m-1}$

This proves the theorem.

$\square$

Corollary 2.1.1.

$$v_0^m x^{n-m-1} + v_1^m x^{n-m-2} + ... + v_{n-m-1}^m = \frac{p(x) - w_0 - w_1(x-z) - ... - w_m(x-z)^m}{(x-z)^{m+1}}$$

And hence for particularly $m = 1$,

$$\frac{p(x)}{(x-z)} = v_0^0 x^{n-1} + v_1^0 x^{n-2} + ... + v_{n-1}^0.$$

We can prove the corollary using the expression $(v_0^m x^{n-m-1} + v_1^m x^{n-m-2} + ... + v_{n-m-1}^m)(x-z) + v_{n-m}^m = w_m + w_{m+1}(x-z) + ... + w_n(w-z)^{n-m}$ which have got above.

The coefficients v_k^m constructed by the Horner algorithm are arranged in the following two dimensional array, called the **Horner algorithm**.

u_0	u_1	u_2	u_3
v_0^0	v_1^0	v_2^0	v_3^0
v_0^1	v_1^1	v_2^1	
v_0^2	v_1^2		
v_0^3			

The circled entries are the Taylor coefficients of $p(z)$ at z.

Example 2.1.1. *Compute the Taylor expansion of the polynomial*

$$p(z) = z^4 - 4z^3 + 3z^2 - 2z + 5$$

at $z = 2$.

The Horner scheme's results are tabulated below:

1	-4	3	-2	5
1	-2	-1	-4	-3
1	0	-1	-6	
1	2	3		
1	4			
1				

The circled entries are the Taylor coefficients of $p(z)$ at 2.

Thus, the desired expansion is

$$p(2 + h) = h^4 + 4h^3 + 3h^2 - 6h - 3 \ .$$

It only requires $\frac{1}{2}n(n-1)$ multiplication because multiplication requires only for $v_k^m = v_k^{m-1} + zv_{k-1}^m$ and $p(2) = 0^4 + 4 * 0^3 + 3 * 0^2 - 6 * 0 - 3 = -3$.

Example 2.1.2. *The polynomial* $p(z) = z^4 - 2z^3 - 7z^2 + 8z + 12$ *has a zero at* $z = -2$*. Forming the first row of the Horner scheme with* $z = -2$ *yields*

1	-2	-7	8	12
1	-4	1	6	0

Thus, the deflected polynomial is

$$\frac{p(z)}{z+2} = z^3 - 4z^2 + z + 6.$$

2.2 Number of Zeros of Real Polynomial in a Real Interval

The Theorem of Fourier-Budan yields an upper bound for the number of zeros of a real polynomial in a real interval. An exact determination of this number is possible by means of the Cauchy Index. This is an integer that can be associated with any real rational function r and any interval whose end points are not poles of r.

2.2.1 Cauchy Indices

Let r be a rational functional and let ξ be a real pole of r. Consider each one sided limit of r at ξ. For the function r, the Cauchy index at the pole ξ is

$$I_\xi\, r = \begin{cases} +1 & \text{if} \quad \lim_{x\to\xi-} r(x) = -\infty, \quad \lim_{x\to\xi+} r(x) = \infty \\ -1 & \text{if} \quad \lim_{x\to\xi-} r(x) = \infty, \quad \lim_{x\to\xi+} r(x) = -\infty \\ 0 & \text{Otherwise.} \end{cases}$$

Now let α and β be two real numbers that are not poles of r, $\alpha < \beta$. Then the **Cauchy index of r for the interval** $[\alpha,\beta]$ is defined as the sum of the Cauchy indices of r for all poles ξ such that $\alpha < \xi < \beta$. It will be denoted by

$$I_{\alpha}^{\beta}\, r \qquad \text{or} \qquad I_{\alpha}^{\beta}\, r(x)$$

Frequently, this number is referred to as the number of times r jumps from -∞ to $+\infty$ minus the number of times it jumps from $+\infty$ to -∞. For infinite interval the Cauchy index is defined by

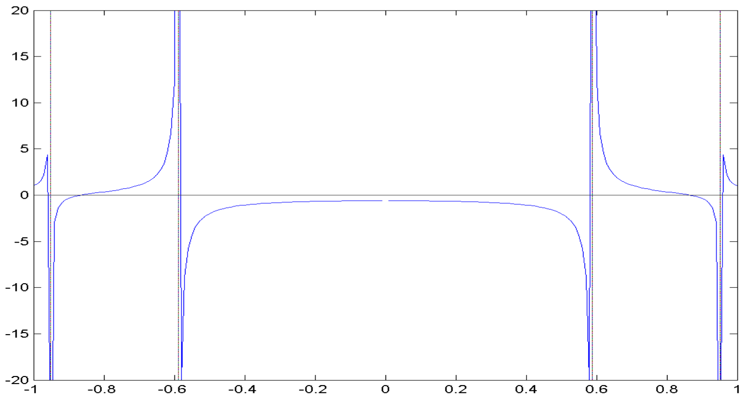
$$I_{\alpha}^{\infty} r = \lim_{\beta \to \infty} I_{\alpha}^{\beta} r\ .$$

Since a rational function has only a finite number of poles, $I_{\alpha}^{\beta}\, r$ is constant for β sufficiently large and the limit always exist.

Example 2.2.1. *Consider the rational function:*

$$r(x) = \frac{4x^3 - 3x}{16x^5 - 20x^3 + 5x} = \frac{p(x)}{q(x)}\ .$$

The function $r(x)$ has poles at $x_1 = 0.9511, x_2 = 0.5878, x_3 = 0, x_4 = -0.5878$ and $x_5 = -0.9511$.



We can see in the above figure that $I_{x_1}r = I_{x_2}r = 1$ *and* $I_{x_4}r = I_{x_5}r = -1$. *For the pole in zero, we have* $I_{x_3}r = 0$ *since the left and right limits are equal. So we conclude that* $I_{-1}^{1}r = 0$ *in [-1,1].*

2.2.2 Number of Zeros in an Interval

Let p be a polynomial with complex coefficients and distinct zeros w_k, $k = 1, 2, \ldots, n$ and let m_k be the multiplicity of w_k then

$$p(z) = A(z-w_1)^{m_1}(z-w_2)^{m_2}....(z-w_n)^{m_n}$$

$$p'(z) = (Am_1(z-w_1)^{m_1-1}(z-w_2)^{m_2}....(z-w_n)^{m_n})+$$

$$(Am_2(z-w_1)^{m_1}(z-w_2)^{m_2-1}....(z-w_n)^{m_n}) + ...$$

$$+Am_n(z-w_1)^{m_1}(z-w_2)^{m_2}....(z-w_n)^{m_n-1}$$

Now,

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{m_k}{z-w_k}$$

If p is a real polynomial and $\xi_1, \xi_2, \ldots, \xi_j$ are its distinct real zeros then

$$\frac{p'(x)}{p(x)} = \sum_{i=1}^{j} \frac{m_i}{x-\xi_i} + r(x)\ ,$$

where m_i is again the multiplicity of ξ_i and r is a real rational function without real poles.

Theorem 2.2.1. *Let* p *be a polynomial with real coefficients. If* $[\alpha,\ \beta]$ *is any interval such that* $p(\alpha)p(\beta) \neq 0$, *then the number of distinct zeros of* p *in* $[\alpha,\ \beta]$

equals the Cauchy index

$$I_{\alpha}^{\beta}\frac{p'}{p}\ .$$

2.3 Disks Containing Specified Number of Zeros

Let $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. Let z_0 be a complex number and let $r > 0$. Then it is quite interesting to construct a disk $|z - z_0| < r$ that contains a specified number of zeros of p.

2.3.1 Disks Containing No Zeros

Theorem 2.3.1. *Let $z_0 \in \mathbb{C}$. Let*

$$p(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots + b_n(z - z_0)^n \qquad \textit{for all} \quad z \in \mathbb{C}\ ,$$

where $b_0 \neq 0$ and $n \geq 1$. Assume that $b_n = 1$.

Let $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ be positive numbers such that $\lambda_1 + \lambda_2 + ... + \lambda_n \leq 1$ and let

$$r = \min_{1\leq m\leq n,\ b_m\neq 0} \lambda_m^{\frac{1}{m}} \left|\frac{b_0}{b_m}\right|^{\frac{1}{m}}$$

Then the open disk $|z - z_0| < r$ contains no zero of p.

Proof.

Given that $z_0 \in \mathbb{C}$ such that $p(z_0) \neq 0$.

Now,

$$p(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots + b_n(z - z_0)^n \qquad \text{for all} \quad z \in \mathbb{C}\ ,$$

where

$$b_m = \frac{1}{m!}\, p^{(m)}(z_0), \qquad \text{for} \qquad m = 0,\ 1,\ 2,\ \ldots,\ n\ .$$

$$\begin{aligned} |p(z)| &= \left| \sum_{m=0}^{n} b_m (z - z_0)^m \right| > |b_0| - \sum_{m=1}^{n} |b_m| r^m \\ &= |b_0| \left\{ 1 - \sum_{m=1}^{n} \left| \frac{b_m}{b_0} \right| r^m \right\} \end{aligned}$$

Whenever $b_m \neq 0$ in the m-th term, we replace r^m with $\lambda_m \left| \dfrac{b_0}{b_m} \right|$, which is not smaller.

This yields,

$$\begin{aligned} |p(z)| &> |b_0| \left\{ 1 - \sum_{b_m \neq 0} \lambda_m \right\} \\ &\geq |b_0| \left\{ 1 - \sum_{m=1}^{n} \lambda_m \right\} \geq 0 \end{aligned}$$

Hence $p(z) \neq 0$.

That is, there is no zeros of p in the open disk $|z - z_0| < r$. □

Corollary 2.3.1. *Let* $z_0 \in \mathbb{C}$. *Let*

$$p(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots + b_n(z - z_0)^n \qquad \textit{for all} \quad z \in \mathbb{C} \,,$$

where $b_0 \neq 0$ *and* $n \geq 1$. *Assume that* $b_n = 1$.

Let

$$r = r(z_0) = \min_{1 \leq m \leq n} \left| \frac{b_0}{b_m} \right|^{\frac{1}{m}} .$$

Then, the closed disk $|z - z_0| \leq \frac{r}{2}$ *contains no zero of the polynomial* p.

Proof. In Theorem 2.3.1, let us take

$$\lambda_m = \frac{1}{2^m} \qquad \text{for} \qquad m = 1,\ 2,\ \ldots,\ n \,.$$

Then, set

$$\rho = \min_{1\le m\le n,\, b_m\neq 0} \lambda_m^{\frac{1}{m}} \left|\frac{b_0}{b_m}\right|^{\frac{1}{m}} = \left(\frac{1}{2}\right) \min_{1\le m\le n} \left|\frac{b_0}{b_m}\right|^{\frac{1}{m}} .$$

Observe that $\rho = r/2$.

By Theorem 2.3.1, the disk $|z-z_0| < \rho = \frac{r}{2}$ does not contain any zeros of p. $\square$

Theorem 2.3.2. *Let $z_0 \in \mathbb{C}$. Let*

$$p(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots + b_n(z-z_0)^n \qquad \textit{for all} \quad z \in \mathbb{C} ,$$

where $b_0 \neq 0$ and $n \geq 1$. Assume that $b_n = 1$.

Let r_1 denote the positive solution of $|b_0| = |b_1|r + |b_1|r^2 + \cdots + r^n$. Then, no zero of p is contained in the open disk $|z-z_0| < r_1$.

Proof. Let

$$\lambda_m \;=\; \left|\frac{b_m}{b_0}\right| r_1^m, \qquad m = 1,\ 2,\ \ldots,\ n .$$

Then

$$\lambda_1 + \lambda_2 + ... + \lambda_n = \left|\frac{b_1}{b_0}\right| r_1 + \left|\frac{b_2}{b_0}\right| r_1^2 + \cdots + \left|\frac{b_{n-1}}{b_0}\right| r_1^{n-1} + r_1^n = 1 .$$

Now applying Theorem 2.3.1 with these λ_is, we get

$$\rho = \min_{1\le m\le n,\, b_m\neq 0} \lambda_m^{\frac{1}{m}} \left|\frac{b_0}{b_m}\right|^{\frac{1}{m}} = \min_{1\le m\le n,\, b_m\neq 0} \left(\left|\frac{b_m}{b_0}\right| r_1^m\right)^{\frac{1}{m}} \left|\frac{b_0}{b_m}\right|^{\frac{1}{m}} = r_1 .$$

By Theorem 2.3.1, the disk $|z-z_0| < \rho = r_1$ does not contain any zeros of p.

This completes the proof of the theorem. $\square$

Theorem 2.3.3. *Let $z_0 \in \mathbb{C}$. Let*

$$p(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots + b_n(z-z_0)^n \qquad \textit{for all} \quad z \in \mathbb{C} ,$$

where $b_0 \neq 0$ *and* $n \geq 1$*. Assume that* $b_n = 1$*.*

Let

$$\beta(z_0) \;=\; \min_{1 \leq m \leq n} \left[\binom{n}{m} \left| \frac{b_0}{b_m} \right| \right]^{\frac{1}{m}} .$$

Then, no zero of p *is contained in the disk* $|z - z_0| \leq \frac{1}{ne}\, \beta(z_0)$*.*

Proof. For $0 < \gamma < 1$, suppose $\lambda_m = \binom{n}{m} \gamma^m (1-\gamma)^{n-m}$.

Then, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.

Now applying Theorem 2.3.1 with these λ_is, we get

$$\rho = \min_{1 \leq m \leq n,\, b_m \neq 0} \lambda_m^{\frac{1}{m}} \left| \frac{b_0}{b_m} \right|^{\frac{1}{m}} = \min_{1 \leq m \leq n,\, b_m \neq 0} \left\{ \binom{n}{m} \gamma^m (1-\gamma)^{n-m} \right\}^{\frac{1}{m}} \left| \frac{b_0}{b_m} \right|^{\frac{1}{m}}$$

$$= \frac{\gamma}{1-\gamma} \min_{1 \leq m \leq n} (1-\gamma)^{\frac{n}{m}} \left[\binom{n}{m} \left| \frac{b_0}{b_m} \right| \right]^{\frac{1}{m}} \;\geq\; \gamma (1-\gamma)^{n-1} \beta(z_0)$$

For $\gamma = 1/n$, we have

$$\gamma (1-\gamma)^{n-1} \geq \frac{1}{n} \left(1 - \frac{1}{n} \right)^{n-1} > \frac{1}{n\, e} .$$

Thus,

$$\rho > \frac{1}{n\, e} \beta(z_0) .$$

By Theorem 2.3.1, the open disk $|z - z_0| < \rho$ does not contain any zeros of p.

Therefore, the closed disk $|z - z_0| \leq \frac{1}{n\, e} \beta(z_0)$ does not contain any zeros of p.

This completes the proof. $\square$

2.3.2 Disks containing at least m zero ($m \geq 1$)

Let $z_0 \in \mathbb{C}$. Let

$$p(z) = b_0 + b_1 (z - z_0) + b_2 (z - z_0)^2 + \cdots + b_n (z - z_0)^n \qquad \text{for all} \quad z \in \mathbb{C} ,$$

where $b_0 \neq 0$ and $n \geq 1$. Assume that $b_n = 1$.

Let m be an integer with $1 \leq m \leq n$. If $b_0 = b_1 = \cdots = b_{m-1} = 0$ then p has a zero of multiplicity m at z_0. Thus if $|b_0|, \ldots, |b_{m-1}|$ are small then p can be expected to have m zeros close to z_0.

Theorem 2.3.4. *Let $z_0 \in \mathbb{C}$. Let*

$$p(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots + b_n(z - z_0)^n \qquad \textit{for all} \quad z \in \mathbb{C}\,,$$

where $b_0 \neq 0$ and $n \geq 1$. Assume that $b_n = 1$. Let r_m denote the unique non-negative solution of the following equation in r,

$$r^n = \binom{n-1}{m-1}|b_0| + \binom{n-2}{m-2}|b_1|r + \cdots + \binom{n-m}{0}|b_{m-1}|r^{m-1}\,.$$

Then at least m zeros of p are contained in the closed disk $|z - z_0| \leq r_m$.

Theorem 2.3.5. *Let $z_0 \in \mathbb{C}$. Let*

$$p(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots + b_n(z - z_0)^n \qquad \textit{for all} \quad z \in \mathbb{C}\,,$$

where $b_0 \neq 0$ and $n \geq 1$. Assume that $b_n = 1$.
Let $b_m \neq 0$ and let r_m^* denote the unique non-negative solution of the following equation in r,

$$|b_m|r^m = \binom{n}{m}|b_0| + \binom{n-1}{m-1}|b_1|r + \cdots + \binom{n-m+1}{1}|b_{m-1}|r^{m-1}\,.$$

Then at least m zeros of p are contained in the closed disk $|z - z_0| \leq r_m^*$.

2.3.3 Disks Containing All Zeros

Definition 2.3.1. *Let* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n$ *be a given polynomial of degree* $n \geq 1$*. An* **inclusion radius** *for the given polynomial* $p(z)$ *is a number* σ *such that all the zeros of* $p(z)$ *lie in the closed disk* $|z| \leq \sigma$*.*
A **proper inclusion radius** *for the given polynomial* $p(z)$ *is a number* σ *such that all the zeros of* $p(z)$ *lie in the open disk* $|z| < \sigma$*.*

Theorem 2.3.6. *Let*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n$$

be a given polynomial of degree $n \geq 1$*. Assume that* $a_0 \neq 0$*. Let* $w_1, \ldots, w_n$ *be the zeros of* $p(z)$*. Consider the polynomial with the coefficients taken in reverse order*

$$q(z) = 1 + a_{n-1} z + a_{n-2} z^2 + \cdots + a_1 z^{n-1} + a_0 z^n \ .$$

Then $q(z)$ *has zeros* $\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_n}$*.*

Proof. Let w be a zero of $p(z)$. Since $a_0 \neq 0$, the zero w of $p(z)$ must be a non-zero complex number.
We will show that $1/w$ is a zero of $q(z)$.
Since w is a zero of $p(z)$, we have

$$\sum_{k=0}^{n} a_k w^k \ = \ 0 \ ,$$

where $a_n = 1$.

Multiplying both side by w^{-n}, we get

$$\sum_{k=0}^{n} a_k w^{k-n} \;=\; 0 \;.$$

Changing the summation index as $i = n - k$, we get

$$\sum_{i=0}^{n} a_{n-i} w^{-i} = 0 \;.$$

This gives that

$$a_n \left(\frac{1}{w}\right)^0 + a_{n-1}\left(\frac{1}{w}\right)^1 + a_{n-2}\left(\frac{1}{w}\right)^2 + \cdots + a_1\left(\frac{1}{w}\right)^{n-1} + a_0\left(\frac{1}{w}\right)^n = 0 \;.$$

That is,

$$q\left(\frac{1}{w}\right) = 0 \;.$$

This completes the proof of the theorem. □

Note: In view of Theorem 2.3.6,

- $\sigma > 0$ is an inclusion radius for $p(z)$ if and only if the open disk $|z| < 1/\sigma$ is free of zeros of $q(z)$.
- $\sigma > 0$ is a proper inclusion radius for $p(z)$ if and only if the closed disk $|z| \leq 1/\sigma$ is free of zeros of $q(z)$.

Theorem 2.3.7. *Let* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n$, *where* $n \geq 1$ *and* $a_0 \neq 0$. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be positive numbers such that* $\lambda_1 + \lambda_2 + \cdots + \lambda_n \leq 1$. *Let*

$$\sigma \;=\; \max_{1\leq m\leq n} \lambda_m^{\frac{-1}{m}} \, |a_{n-m}|^{\frac{1}{m}} \;.$$

Then σ *is an inclusion radius for* p.

Proof.

Consider the polynomial $q(z)$ with the coefficients taken in reverse order

$$q(z) = 1 + a_{n-1}z + a_{n-2}z^2 + \cdots + a_1 z^{n-1} + a_0 z^n \ .$$

Note that $\sigma > 0$ is an inclusion radius for $p(z)$ if and only if the open disk $|z| < 1/\sigma$ is free of zeros of $q(z)$.

Set

$$\rho = \frac{1}{\sigma} = \frac{1}{\max_{1\le m\le n} \lambda_m^{\frac{-1}{m}} \ |a_{n-m}|^{\frac{1}{m}}} = \min_{1\le m\le n} \lambda_m^{\frac{1}{m}} \left| \frac{1}{a_{n-m}} \right|^{\frac{1}{m}} \ .$$

By Theorem 2.3.1, the open disk $|z| < \rho$ does not contain any zeros of q.

Therefore, $|z| \le \sigma$ contains all zeros of p.

This completes the proof. $\square$

Theorem 2.3.8. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1}z^{n-1} + z^n$, where $n \ge 1$ and $a_0 \ne 0$. Let*

$$\sigma = 2 \max_{1\le m\le n} |a_{n-m}|^{1/m} \ .$$

Then σ is a proper inclusion radius for p.

Proof.

Consider the polynomial $q(z)$ with the coefficients taken in reverse order

$$q(z) = 1 + a_{n-1}z + a_{n-2}z^2 + \cdots + a_1 z^{n-1} + a_0 z^n \ .$$

Note that $\sigma > 0$ is a proper inclusion radius for $p(z)$ if and only if the closed disk $|z| \le 1/\sigma$ is free of zeros of $q(z)$.

Set

$$\rho = \frac{1}{\sigma} = \frac{1}{2\max_{1\le m\le n} |a_{n-m}|^{1/m}} = \frac{1}{2} \min_{1\le m\le n} \left| \frac{1}{a_{n-m}} \right|^{1/m} \ .$$

By Corollary 2.3.1, the closed disk $|z| \le \frac{1}{\sigma}$ contains no zero of the polynomial q.

Therefore, $|z| < \sigma$ contains all zeros of p.

This completes the proof. $\square$

Theorem 2.3.9. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n$, where $n \ge 1$ and $a_0 \neq 0$. Let σ_1 denote the positive solution of the following equation in σ*

$$\sigma^n \;=\; |a_0| + |a_1|\sigma + \cdots + |a_{n-1}|\sigma^{n-1} \; .$$

Then σ is an inclusion radius of p.

Proof.

Consider the polynomial $q(z)$ with the coefficients taken in reverse order

$$q(z) = 1 + a_{n-1} z + a_{n-2} z^2 + \cdots + a_1 z^{n-1} + a_0 z^n \; .$$

Note that $\sigma > 0$ is an inclusion radius for $p(z)$ if and only if the open disk $|z| < 1/\sigma$ is free of zeros of $q(z)$.

Since σ_1 is the positive solution of the equation

$$|a_0| + |a_1|\sigma + \cdots + |a_{n-1}|\sigma^{n-1} - \sigma^n \;=\; 0 \; ,$$

it follows that $1/\sigma_1$ is the positive solution of the equation

$$|a_0|\sigma^n + |a_1|\sigma^{n-1} + \cdots + |a_{n-1}|\sigma - 1 = 0 \; .$$

By Theorem 2.3.2, it follows that no zero of q is contained in the open disk $|z| < \frac{1}{\sigma_1}$.

Therefore, $|z| \le \sigma_1$ contains all zeros of p.

This completes the proof. $\square$

In 2022, Prasanna Kumar and Ritu Dhankar proved the following result.

Theorem 2.3.10. *Let* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ *be a polynomial of degree* n*. Consider*

$$T(z) = \frac{d}{dz}\left(z^n p\left(\frac{1}{z}\right)\right) = n a_0 z^{n-1} + (n-1) a_1 z^{n-2} + \cdots + 2a_{n-2} z + a_{n-1}$$

and

$$S(z) = z^{n-1}\,\frac{d}{dz}\left(p\left(\frac{1}{z}\right)\right) = (-1)\left[a_1 z^{n-3} + 2a_2 z^{n-4} + \cdots + (n-1)a_{n-1} z^{-1} + n a_n z^{-2}\right] .$$

If

$$|T(z)| < |S(z)| \qquad \textit{in} \qquad |z| \leq 1$$

then $p(z)$ *has all its zeros in* $|z| < 1$*.*

Proof. We have

$$|T(z)| < |S(z)| \qquad \text{in} \qquad |z| \leq 1 .$$

So, $S(z) \neq 0$ in $|z| \leq 1$.
But then $z^{n-1} S\left(\frac{1}{z}\right)$ has all its zero in $|z| < 1$.
Since for every complex number z on $|z| = 1$ the complex number $\frac{1}{z}$ is also on $|z| = 1$. It follows from $|T(z)| < |S(z)|$ in $|z| \leq 1$ that

$$\left|T\left(\frac{1}{z}\right)\right| < \left|S\left(\frac{1}{z}\right)\right| \qquad \text{on} \qquad |z| = 1 .$$

Further it implies that

$$\left|z^{n-1}\, T\left(\frac{1}{z}\right)\right| < \left|z^n\, S\left(\frac{1}{z}\right)\right| \qquad \text{on} \qquad |z| = 1 .$$

Note that the polynomial $z^n S\left(\frac{1}{z}\right)$ has all its zeros in $|z| < 1$.

By using Rouche's theorem (Theorem 1.0.6), it follows that the polynomial $z^{n-1}T\left(\frac{1}{z}\right) + z^n S\left(\frac{1}{z}\right)$ has all its zeros in $|z| < 1$.

A simple calculation shows that

$$z^{n-1}T\left(\frac{1}{z}\right) + z^n S\left(\frac{1}{z}\right) = n\,p(z) - z\,p'(z) + z\,p'(z) = n\,p(z)\ .$$

Therefore $p(z)$ has all its zeros in $|z| < 1$.

This completes the proof. $\square$

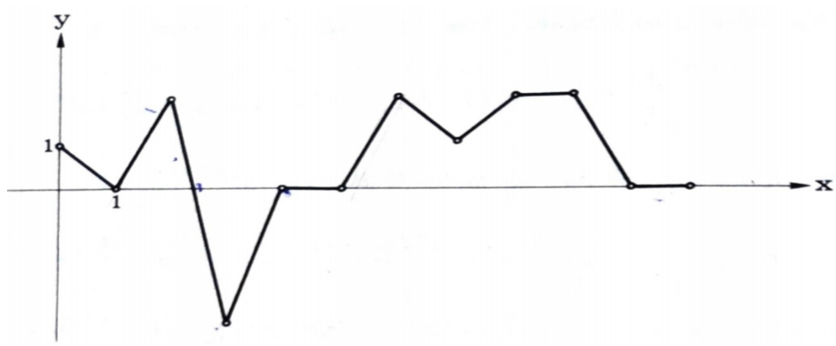
Chapter 3

Geometry of Zeros

3.1 The Rule of Descartes

As very important application of sign change of sequence of numbers in finding numbers of positive zeros of a polynomial known as Descartes rule, is fundamental in subsequent sections.

The sequence $\{1, 0, 2, -3, 0, 0, 2, 1, 2, 2, 0, 0\}$ has two sign changes. To interpret the number of sign change graphically we consider the piecewise linear function f defined on $[0, n]$ whose graph passes through the points (k, u_k) and is a linear between. the number of sign changes equals the number of times the graph crosses the x-axis.



Note: The number of sign changes is not altered if sequence is multiplied by a constant $c \neq 0$ or by a sequence of constant sign.

Lemma 3.1. *Let $0 \leq k < m \leq n$, $u_k u_m \neq 0$ The number of sign changes of the sequence $\{u_k, u_{k+1}, \ldots, u_m\}$ is even if $u_k u_m > 0$ and odd if $u_k u_m < 0$.*

Along with the given sequence $\{u_k\}$ we consider the sequence $\{v_k\}$ of its difference

$$v_k \; = \; u_k \; - \; u_{k+1}, \qquad k = 0,\ 1,\ 2,\ \ldots,\ (n+1)\ ,$$

where $u_{-1} = u_{n+1} = 0$
Then the number of sign change of the sequence $\{v_k\}$ exceeds the number of sign change of the given sequence $\{u_k\}$ by an odd positive number.

Theorem 3.1.1 (Descartes' rule of signs).
Let $p(x) = u_0 + u_1 x + ... + u_n x^n$ be a real polynomial which is not a zero polynomial. Let s denote the number of sign changes in the sequence $\{u_k\}$ of its coefficients and let r denote the number of real positive zeros (counted with multiplicity). Then $s - r$ is even and non-negative.

OR

Number of positive real zeros of polynomial can't exceed the number of sign changes in the sequence of its coefficients.

Proof. We prove the theorem by induction with respect to r.
If $r = 0$, the factor representation shows that the first nonzero coefficient has the same sign as u_n hence by Lemma 3.1, the number of sign changes is even and the theorem is true for $r = 0$.

Assume that the theorem is true for some $r \geq 0$.

If the real polynomial p has $r+1$ positive zeros, then we can write it in the form

$$p(x) = (x - \xi)\, p_1(x) \ ,$$

where $\xi \geq 0$ and $p_1(x) = u_0 + u_1 x + ... + u_n x^n$ has r positive zeros.

By induction hypothesis the sequence $\{u_n\}$ has $r + 2k$ sign changes, where k is suitable non-negative integer.

Now the coefficients of p are

$$v_m \ = \ u_{m-1} - \xi\, u_m \qquad m = 0,\ 1,\ 2,\ \ldots,\ n+1\ ,$$

where $u_{-1} = u_{n+1} = 0$.

So, the number s of sign changes of the sequence v_m is the same as the number of sign changes in

$$\left\{-\xi^{m-1}\, v_m\right\} \ = \ \left\{\xi^m u_m \ - \ \xi^{m-1} u_{m-1}\right\}\ .$$

The sequence in the right side is the sequence of differences of two sequence $\{\xi^m u_m\}$ which has the same number of sign changes as $\{u_m\}$, namely $r + 2k$.

Therefore, it follows that

$$\begin{aligned} s \ &= \ r + 2k + \text{an odd positive number} \\ &= \ r + 1 + \text{an even positive number} \end{aligned}$$

The assertion of the theorem thus holds for polynomials with $r+1$ positive zeros.

This completes the induction step. □

Example 3.1.1. *The polynomial* $p(x) = x^4 - 2x^3 - 3x^2 + 4x - 5$ *has three sign changes. By Theorem 3.1.1, this polynomial* p *has maximum 3 positive zeros.*

Corollary 3.1.1. *The number of negative zeros of an equation* $f(x) = 0$ *can't exceed the number of sign changes of* $f(-x) = 0$.

Proof.

Let $g(x) = f(-x)$.

Then, the number of positive zeros of $g(x) = 0$ can't exceeds the number of sign changes of $g(x) = 0$. This implies that the number of negative zeros of $g(-x) = 0$ can't exceed the number of sign changes of $g(x) = 0$.

Consequently, the number of negative zeros of $f(x) = 0$ can't exceed the number of sign changes of $f(-x) = 0$.

This completes the proof.

□

3.2 Location of Zeros

Gauss-Lucas theorem gives a geometric relation between the roots of a polynomial p and the roots of its derivative p'.

Theorem 3.2.1. ***(Gauss-Lucas theorem)***

If p *is a polynomial with complex coefficients, all zeros of* p' *belongs to the convex hull of the set of zero of* p*. That is, any closed half-plane that contains all zeros of* p *also contains all zeros of* p'*.*

Proof. Let the degree of p be n and let w_1, w_2, ..., w_n be the zeros of p. Given any closed half-plane H, there exist a complex number $c \neq 0$ and a real number

α such that $z \in H$ iff $Re(cz) \geq \alpha$ (Because it is an infinite plane).

Therefore, if all zeros are in H, then

$$\text{Re}(c\, w_m) \geq \alpha\ , \qquad m = 1,\ 2,\ \ldots,\ m\ .$$

Let V be convex hull and z not be in V. We wish to show that $p'(z) \neq 0$. Since $z \notin V$, there will be a half-plane H (say) such that $z \notin H$. So, $\text{Re}(cz) < \alpha$, we have $\text{Re}(cz) - \alpha < 0$ and let $\alpha = \text{Re}(cw_m)$ So,

$$Re(c(z - w_m)) < 0\ , \qquad m = 1,\ 2,\ \ldots,\ n\ .$$

Hence,

$$\text{Re}\frac{1}{c(z - w_m)} < 0$$

$$\implies \qquad \text{Re}\ \frac{\bar{c}}{|c|(z - w_m)} < 0$$

$$\implies \text{Re}\left(\frac{\bar{c}}{(z - w_m)}\right) < 0\ , \qquad m = 1,\ 2,\ \ldots, n$$

and

$$\sum_{m=1}^{n} \frac{1}{z - w_m} < 0\ .$$

Therefore, it follows that

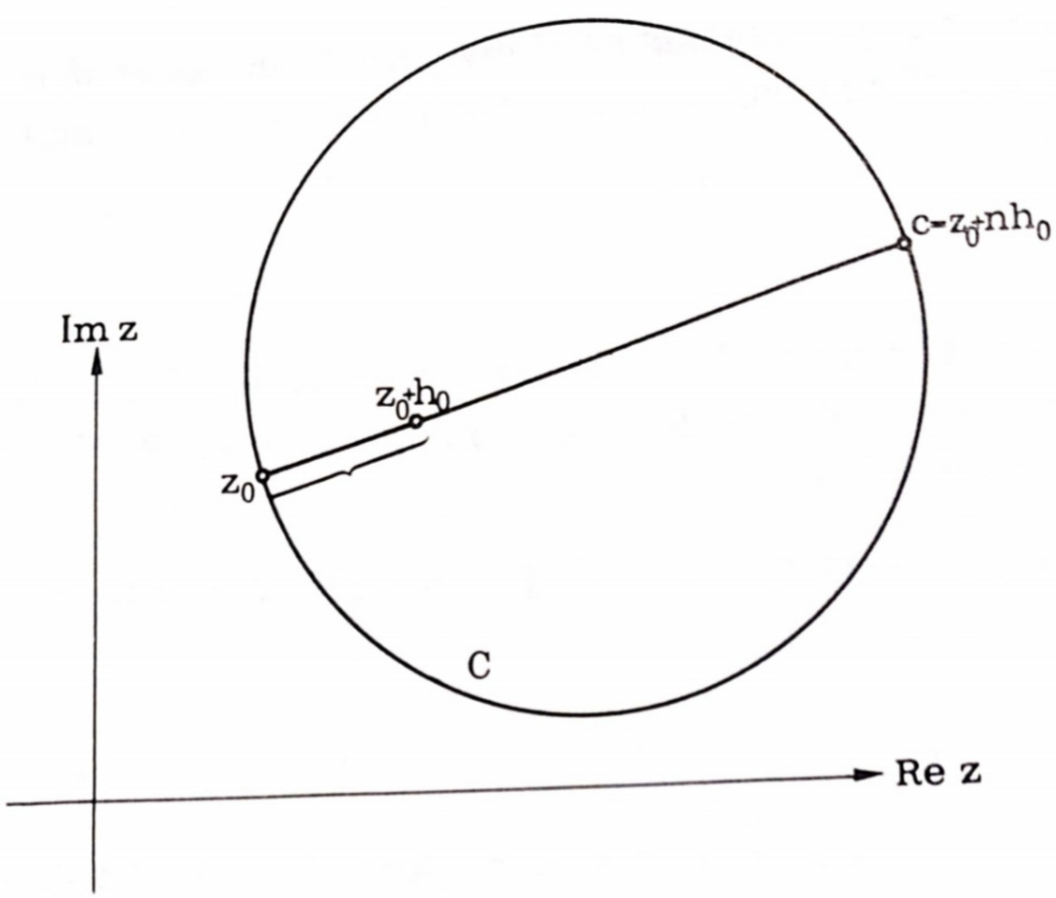
$$\sum_{m=1}^{n} \frac{1}{z - w_m} \neq 0$$

$$\implies \frac{p'(z)}{p(z)} \neq 0$$

$$\implies p'(z) \neq 0.$$

□

Theorem 3.2.2. *Let p be a polynomial of degree n and let z_0 be a point such that $p'(z_0) \neq 0$. Let h_0 denotes the newton correction at z_0, i.e $h_0 = \frac{p(z_0)}{p'(z_0)}$. Then any circular region with the points z_0 and $z_0 + nh_0$ on its boundary contains at least one zero of p.*



In particular, we may choose for c a circular disk with a straight line segment from z_0 to $z_0 + nh_0$ as its diameter.

Proof. Let p be a polynomial of degree n and let z_0 be a finite point such that $p(z_0)p'(z_0) \neq 0$.

The newton correction $h_0 = -\frac{p(z_0)}{p'(z)}$ can then be formed and is different from zero. The next point in the newton sequence of approximation would be $z_1 = z_0 + h_0$.

In this way we get $c = z_0 + nh_0$ from newton's method for roots of multiplicity 1. Now due to applying newton method, we are going near and near to zero iteratively. So, contain atleast one zero. $\square$

3.3 Schur and Cohn Algorithm

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n. We define $p^*(z)$ the reciprocal polynomail of p by

$$p^*(z) = \overline{a_0}\ z^n + \overline{a_1}\ z^{n-1} + \cdots + \overline{a_n}\ .$$

The receprocal polynomial of a polynomial of degree n is always consider to be a polynomial of degree n.

For $z \neq 0$,

$$p^*(z) = z^n\ \overline{p\Big(\frac{1}{\overline{z}}\Big)}\ .$$

from this it follows immediately that $|z| = 1$ implies $|p^*(z)| = |p(z)|$

Thus any zero of modulus 1 of p is a zero of p^*.

In fact, if p is a polynomial of degree $n \geq 0, p \neq 0$ with zeros $w_1,\ w_2,\ \ldots,$ w_n (including ∞, being listed according to their multiplicity), the receprocal polynomial p^* is a polynomial of degree n with the zero $\frac{1}{\overline{w_k}}$, $k = 1, 2, ..., n$, where, $\frac{1}{\infty} = 0, \frac{1}{0} = \infty$.

Because, for $z \neq 0$,

$$p^*(z) = z^n\ \overline{p\Big(\frac{1}{\overline{z}}\Big)}\ ,$$

it is easy to see that if w with $w \neq 0$ and $w \neq \infty$, is a zero of $p(z) = 0$, then $p^*(1/\overline{w}) = 0$ and hence $\frac{1}{\overline{w}}$ is a zero of $p^*(z)$.

Definition 3.3.1 (Schur Transform).

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ *be a polynomial of degree* n*. Then the* **Schur transform** Tp *of the polynomial* p *is defined by*

$$\begin{aligned} Tp(z) &= \overline{a_0}\, p(z) - a_n\, p^*(z) \\ &= \sum_{k=0}^{n-1} (\overline{a_0}\, a_k - a_n\, \overline{a_{n-k}})\, z^k \end{aligned}$$

The Schur transform of a polynomial of degree 0 *is the zero polynomial.*

Now,

$$Tp(0) = \bar{a_0} a_0 - a_n \bar{a_n} = |a_0|^2 - |a_n|^2$$

is always a real number.

We define the iterated schur transform $T^2 p, T^3, ..., T^n p$ by

$$T^k p \;=\; T(T^{k-1} p), \qquad k = 2,\ 3,\ \ldots,\ n\ ,$$

where $T^{k-1}p$ is to be regarded as a polynomial of degree $n - k + 1$ even if its leading coefficient is zero.

Now, we set

$$\gamma_k = T^k p(0), \qquad k = 1, 2, ..., n$$

Theorem 3.3.1. *Let* $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ *be a polynomial of degree* $n \geq 1$*. All zeros of* p *lie outside the closed unit disk* $|z| \leq 1$ *iff* $\gamma_k > 0$ *for* $k = 1,\ 2,\ \ldots,\ n$*.*

Proof. We will prove forward part by induction on k.

For $k = 1$:

Suppose p has no zero in $|z| \leq 1$. So, By vieta's formulla $|a_0| > |a_n|$. hence $\gamma_1 > 0$.

Also, we have for $|z| = 1$

$$|\bar{a}_0 p(z)| > |a_n p^*(z)|$$

Hence, by Rouche's theorem to the unit circle $Tp = \bar{a}_0 p - a_n p^*$ has no zero on and inside the unit disk. so it holds for $k = 1$

Now, suppose it holds for $n = k$ and let

$$T^k p(z) = a_0^k + a_1^k z + ... + a_{n-k}^k z^{n-k}$$

and assume that all zeros of $T^k p$ lie outside the unit disk. Then according to vieta's $|a_0^k| > |a_{n-k}^k|$. hence

$$\gamma_{k+1} = |a_0^k|^2 - |a_{n-k}^k|^2 > 0$$

. So, by Again by using Rouche's theorem, all zeros of

$$T^{k+1} p = \overline{a_0^k} T^k p - a_{n-k}^k (T^k p)^*$$

lie outside $|z| \leq 1$ Hence, $\gamma_k > 0$

Conversely,

Let $\gamma_k > 0, k = 1, 2, ..., n$ This means that $|a_0^k| > |a_{n-k}^k|, k = 0, 1, 2, ..., n-1$. Hence it follows as, as above, that

$$T^{k+1} p = \overline{a_0^k} T^k p - a_{n-k}^k (T^k p)^*$$

has an equal number of zero in $|z| \leq 1$ as $T^k p$. Since $T^n p = \gamma_n$ has no zero, it follows that $T^0 p = p$ likewise has no zeros in $|z| \leq 1$.

$\square$

The above algorithm is called **Schur-Cohn algorithm.** This algorithm can always be used to decide whether a given polynomial is free of zero in the clossed unit disk.

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