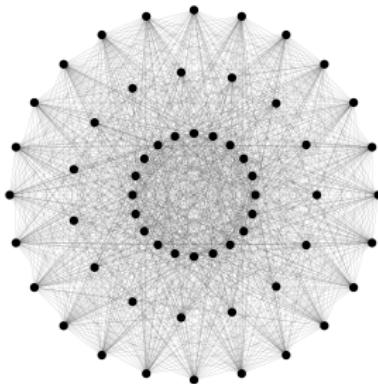


# Group Generation Through the Generations

Scott Harper

University of Bristol



St Andrews University Mathematics Society

3rd March 2017

So what do you do?

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the art of measuring symmetry

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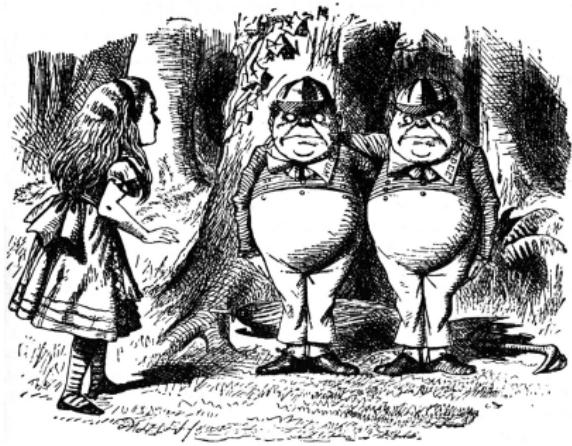
*'Perhaps Looking-Glass milk isn't  
good to drink?'*



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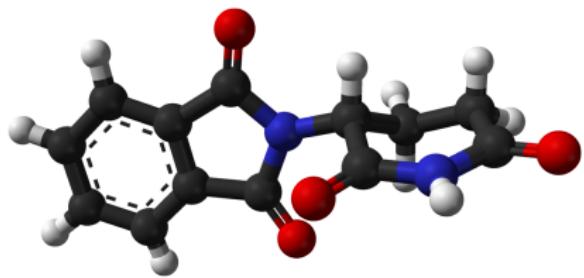


*Alice knew which was which  
in a moment.*

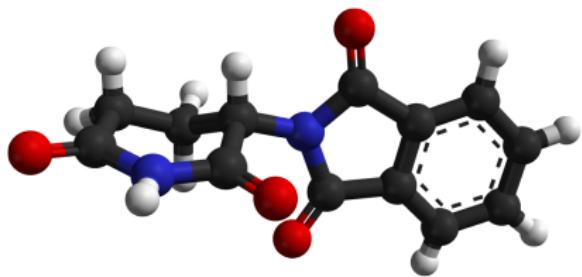
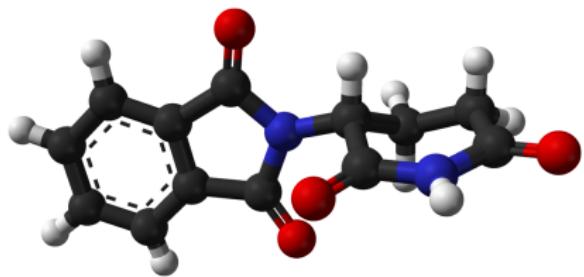


thalidomide

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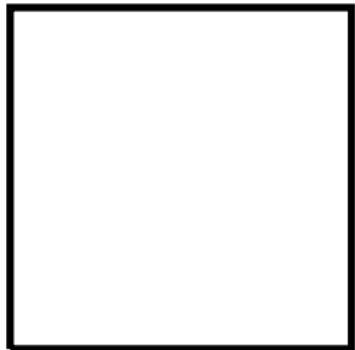


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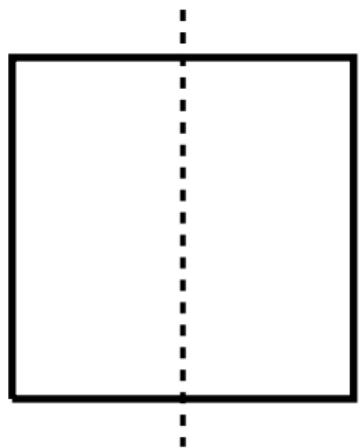


# Examples of Groups

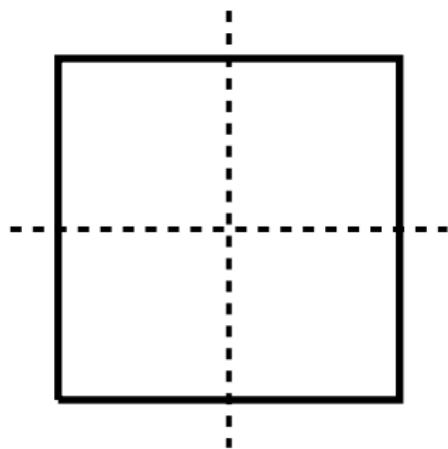
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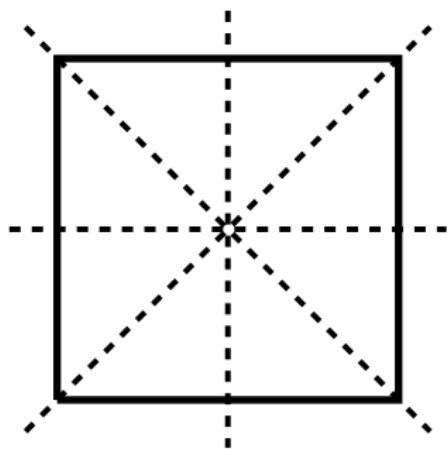
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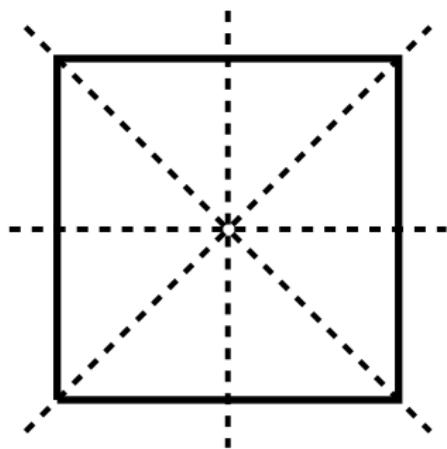
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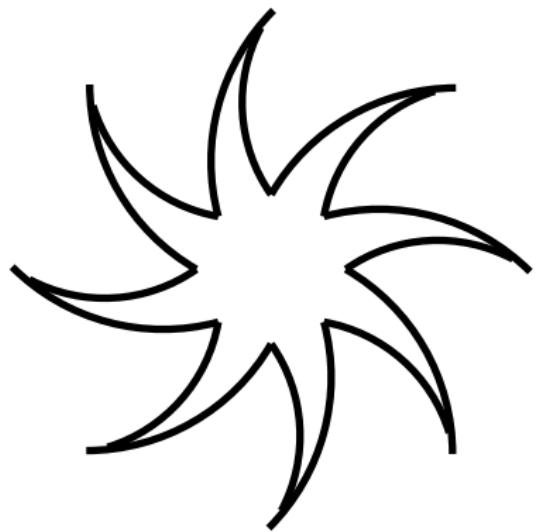
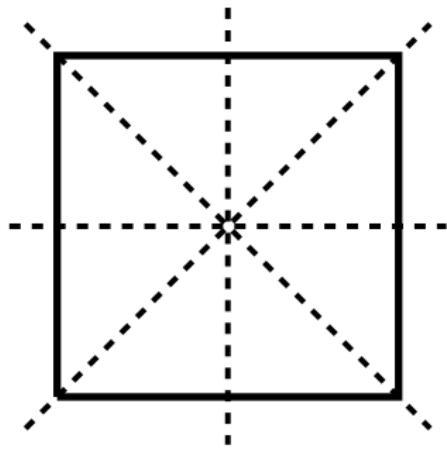


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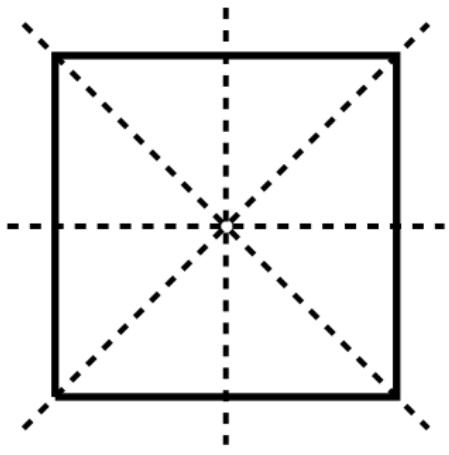
the dihedral group  $D_{\square}$

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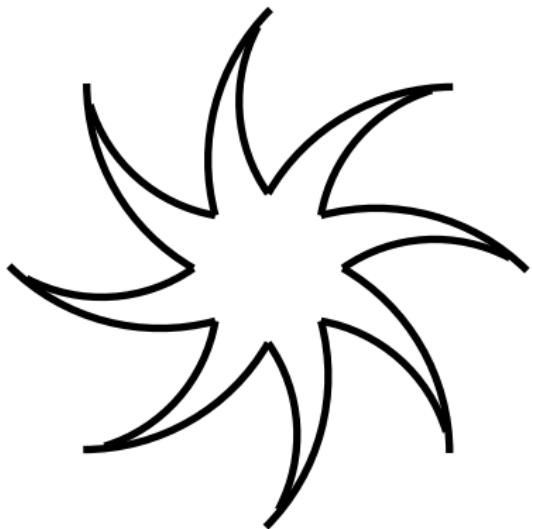


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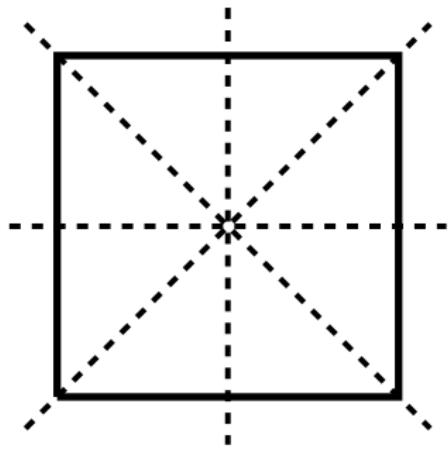


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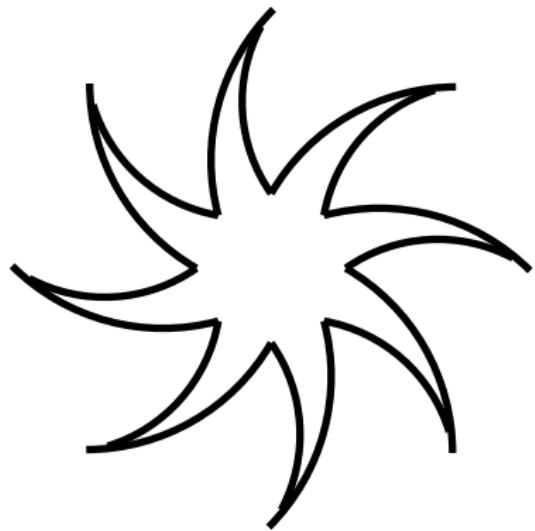


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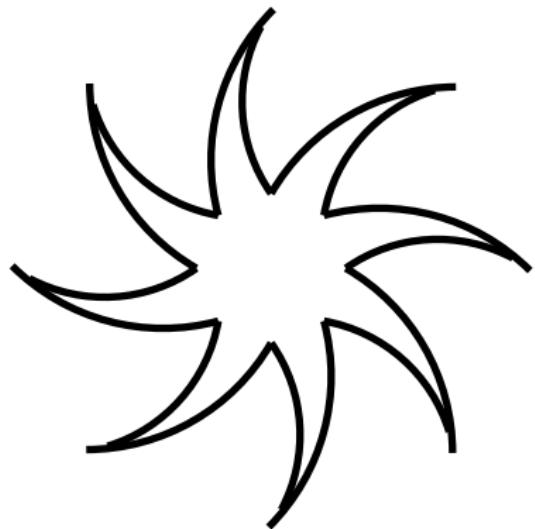
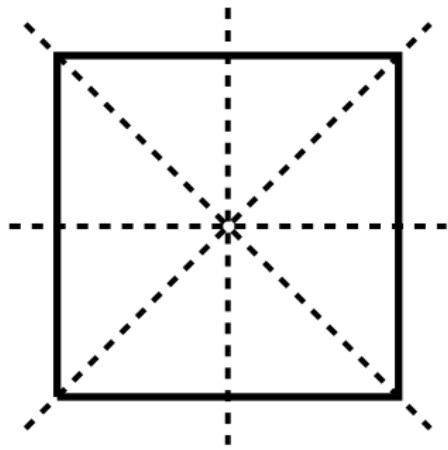


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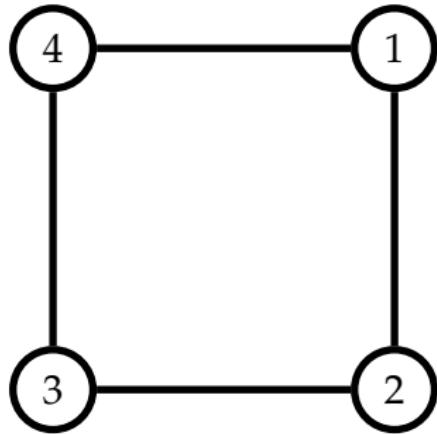


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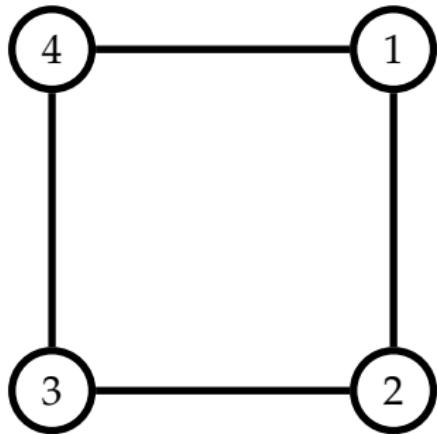
generated by **two** elements

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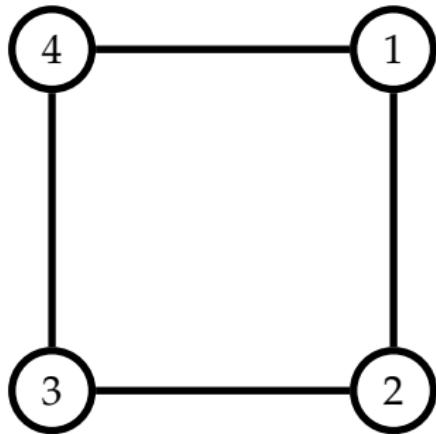


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Rotation by  $90^\circ$  clockwise

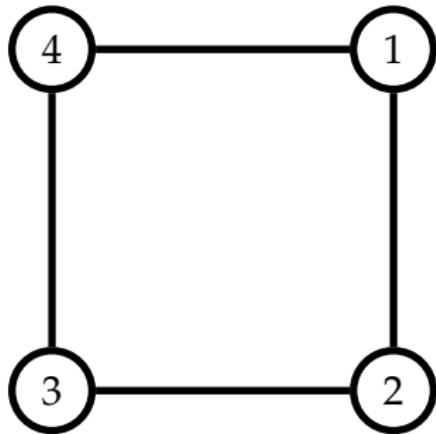
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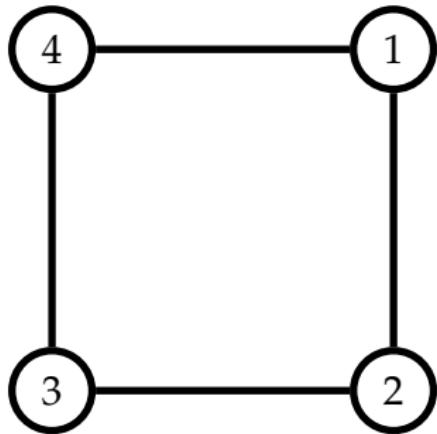
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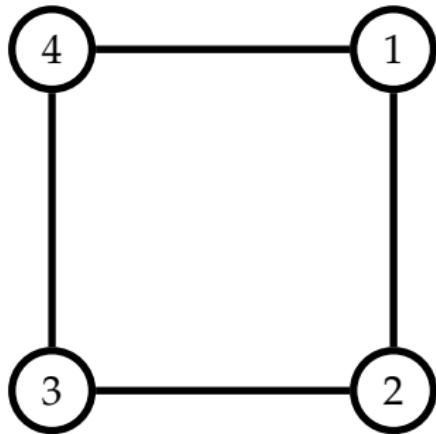


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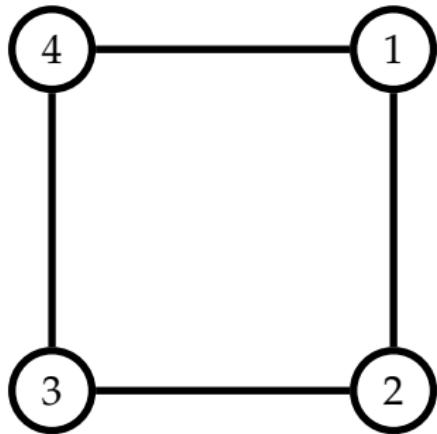
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For example,  $A_5$  is generated by  $(1\ 2\ 3)$  and  $(1\ 2\ 3\ 4\ 5)$ .

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**Theorem** (Abel; 1802) There is **no** formula for the quintic.

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**Theorem** For  $n \geq 5$ , the alternating group  $A_n$  is simple.

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“Please request publicly that Jacobi or Gauss give their opinions, not on the truth but on the importance, of these theorems.

After that I hope people will be found who profit by sorting out all this mess.

I embrace you with affection.  
É. GALOIS, 29 May 1832”

## Generation II: Netto's Conjecture

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If we arbitrarily select two or more substitutions of  $n$  elements, it is to be regarded as extremely probable that the group of lowest order which contains these is the symmetric group, or at least the alternating group.

In the case of two substitutions the probability in favor of the symmetric group may be taken as about  $\frac{3}{4}$ , and in favor of the alternating, but not symmetric, group as about  $\frac{1}{4}$ . In order that any given substitutions may generate a group which is only a part of the  $n!$  possible substitutions, very special relations are necessary, and it is highly improbable that arbitrarily chosen substitutions  $s_i = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1} & x_{i_2} & \dots & x_{i_n} \end{pmatrix}$  should satisfy these conditions. The exception most likely to occur would be that all the given substitutions were severally equivalent to an even number of transpositions and would consequently generate the alternating group.

E. Netto, *The theory of substitutions and its application to algebra*,  
Trans. F. N. Cole, Ann Arbor, Michigan, (1892)

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Cyclic groups of prime order

Let  $p$  be prime. Then

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and so  $P(C_p) \rightarrow 1$  as  $p \rightarrow \infty$ .

Numerical evidence

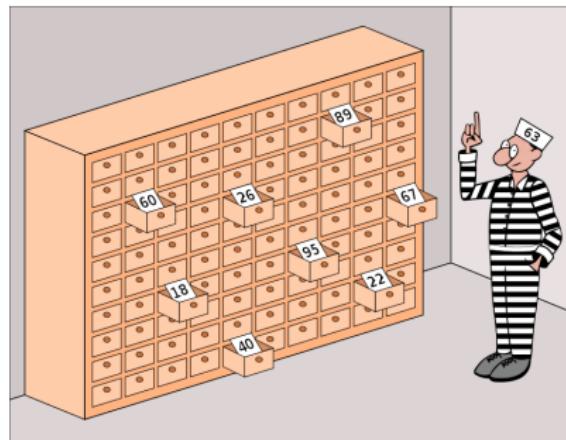
$n$	5	6	7	8	9	10
$P(A_n)$	0.633	0.588	0.726	0.738	0.848	0.875

GAP computations by N. Menezes, M. Quick and C. M. Roney-Dougal

## Generation III: Dixon's Conjecture

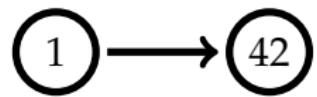
## Generation III: Dixon's Conjecture

A final chance is given to 100 prisoners, numbered 1 to 100. A room contains 100 closed drawers which contain the numbers 1 to 100. One by one, the prisoners enter the room and choose 50 drawers to open. The drawers are closed again. If every prisoner finds his number, all prisoners are pardoned. However, if just one prisoner does not find his number, all prisoners remain imprisoned. Before any prisoner enters the room, they may decide on a strategy but they may not communicate once the first prisoner enters. What is the best strategy?

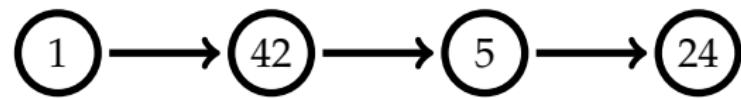




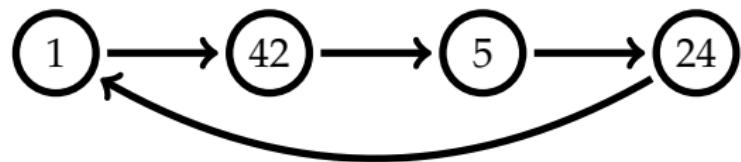
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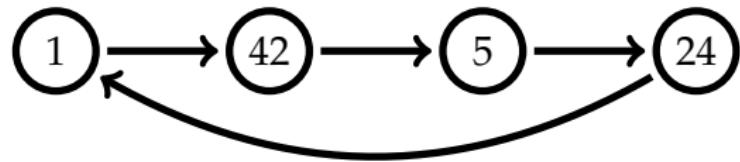




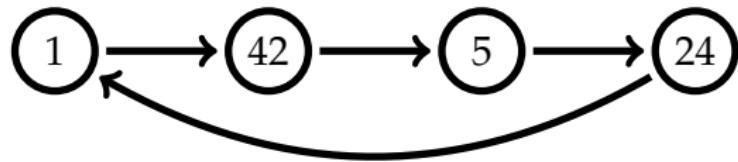




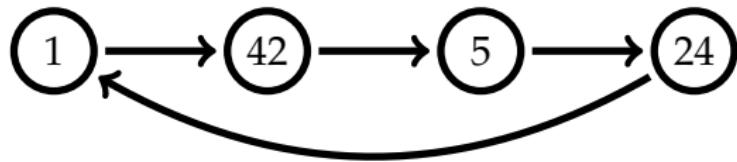




All prisoners succeed if there is no cycle of length longer than 50.

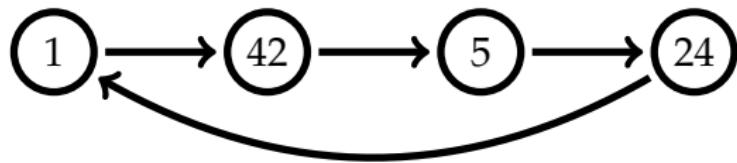


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**Exercise** For  $\ell > n$ , the probability that a permutation of  $2n$  points has a cycle of length  $\ell$  is  $1/\ell$ .



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**Exercise** For  $\ell > n$ , the probability that a permutation of  $2n$  points has a cycle of length  $\ell$  is  $1/\ell$ .

The probability that a permutation of 100 points has no cycle of length longer than 50 is

$$1 - \left( \frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{100} \right) \approx 0.312$$

which is the probability of success.

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What if there were other finite simple groups?

## Generation IV: An Enormous Theorem

Which finite groups are simple?

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## A Simple Ballad

What are the orders of all simple groups?  
I speak of the honest ones, not of the loops.  
It seems that old Burnside their orders has guessed:  
Except of the cyclic ones, even the rest.

Groups made up of permutes will produce more:  
For  $A_n$  is simple, if  $n$  exceeds 4.  
Then, there was Sir Matthew who came into view  
Exhibiting groups of an order quite new.

Still others have come on to study this thing.  
Of Artin and Chevalley now we shall sing.  
With matrices finite they made quite a list.  
The question is: Could there be others they've missed?

Suzuki and Ree then maintained it's the case  
These methods had not reached the end of the chase.  
They wrote down some matrices, just four by four,  
that made up a simple group. Why not make more?

And then came up the opus of Thompson and Feit  
Which shed on the problem remarkable light.  
A group, when the order won't factor by two,  
Is cyclic or solvable. That's what true.

Suzuki and Ree had caused eyebrows to raise,  
But the theoreticians they just couldn't face.  
Their groups were not new: if you added a twist,  
You'd get them from old ones with a flick of the wrist.

Still, some hardy souls felt a thorn in their side.  
For the five groups of Mathieu all reason defied:  
Not  $A_n$ , not twisted, and not Chevalley.  
They called them sporadic and filed them away.

Are Mathieu groups creatures of heaven or hell?  
Zvonimir Janko determined to tell.  
He found out what nobody wanted to know:  
The masters had missed 1 7 5 5 6 0.

The floodgates were open! New groups were the rage!  
(And twelve or more sprouted, to greet the new age.)  
By Janko and Conway and Fischer and Held,  
McLaughlin, Suzuki, and Higman, and Sims.

No doubt you noted the last lines don't rhyme.  
Well, that is, quite simply, a sign of the time.  
There's chaos, not order, among simple groups,  
And maybe we'd better go back to the loops.

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### Theorem (The Classification of Finite Simple Groups)

Any finite simple group is isomorphic to one of the following

- a cyclic group of prime order
- an alternating group of degree at least five
- a group of Lie type
- one of 26 sporadic groups

# The Periodic Table Of Finite Simple Groups

$0, C_1, Z_1$	Dynkin Diagrams of Simple Lie Algebras												$C_2$							
1													2							
1													3							
$A_1(4), A_1(5)$	$A_1(2)$	$A_1(7)$	$B_n$	$D_n$	$E_{6,7,8}$	$F_4$	$G_2$	$^3A_6(4)$	$B_2(3)$	$C_3(3)$	$D_4(2)$	$^2D_4(2^2)$	$^2A_2(9)$	$C_3$						
$A_5$			$B_n$	$D_n$	$E_{6,7,8}$	$F_4$	$G_2$	$^3A_6(4)$	$25920$	$458351680$	$174182400$	$197466720$	$6048$	$3$						
60		168																		
$A_1(9), B_2(2)^*$	$^2G_2(5)'$	$A_1(8)$	$C_n$	$E_{6,7,8}$	$F_4$	$G_2$	$^3A_6(4)$	$B_2(4)$	$C_3(5)$	$D_4(3)$	$^2D_4(3^2)$	$^2A_2(16)$	$C_5$	$5$						
$A_6$			$C_n$	$E_{6,7,8}$				$25920$	$228501$	$458351680$	$458217981400$	$1015196616520$	$62400$	$$						
360		504																		
$A_7$	$A_1(11)$	$E_6(2)$	$E_7(2)$	$E_8(2)$	$F_4(2)$	$G_2(3)$	$^3D_4(2^3)$	$^2E_6(2^2)$	$^2B_2(2^3)$	Tits*	$^2F_4(2)'$	$^2G_2(3^3)$	$B_3(2)$	$C_4(3)$	$D_5(2)$	$^2D_5(2^2)$	$^2A_2(25)$	$C_7$		
2520	460	216441975322 85527570000	747751402 85527570000	33011268 85527570000	4245096	211341312	76481479680 779483499200	29120	17971200	10073446472	14511520	63784756 654489466	2349925548000 25415379558400	126000					7	
$A_1(2)$	$A_8$	$A_1(13)$	$E_6(3)$	$E_7(3)$	$E_8(3)$	$F_4(3)$	$G_2(4)$	$^3D_4(3^3)$	$^2E_6(3^2)$	$^2B_2(2^5)$	$^2F_4(2^5)$	$^2G_2(3^5)$	$B_2(5)$	$C_3(7)$	$D_4(5)$	$^2D_4(4^2)$	$^2A_3(9)$	$C_{11}$		
20160	1092	1217172815644236 671164471600	1217172815644236 671164471600	5734430792466 671164471600	251596100	203030531564912	330531564912	32537600	264905302469 264905302469	499325657 499325657	4686000	273407218 664195300	8911539000 8911539000	1975440000 1975440000	126000					11
$A_9$	$A_1(17)$	$E_6(4)$	$E_7(4)$	$E_8(4)$	$F_4(4)$	$G_2(5)$	$^3D_4(4^3)$	$^2E_6(4^2)$	$^2B_2(2^7)$	$^2F_4(2^5)$	$^2G_2(3^7)$	$B_2(7)$	$C_3(9)$	$D_5(3)$	$^2D_4(5^2)$	$^2A_2(64)$	$C_{13}$			
181440	2488	1217172815644236 671164471600	1217172815644236 671164471600	3918570715200 671164471600	58593000000	642798400	47402356 34095583688	323917173000 55234932652	239191912644 55234932652	138297600 499304000	54425731402 54425731402	1269312796 1269312796	37680285250 37680285250	891305339200 891305339200	5515776 5515776					13
$A_{10}$	$A_n(q)$	$E_6(q)$	$E_7(q)$	$E_8(q)$	$F_4(q)$	$G_2(q)$	$^3D_4(q^3)$	$^2E_6(q^2)$	$^2B_2(2^{n+1})$	$^2F_4(2^{n+1})$	$^2G_2(3^{n+1})$	$B_{10}(q)$	$C_{10}(q)$	$D_{10}(q)$	$^2D_{10}(q^2)$	$^2A_{10}(q^2)$	$Z_p$			
$\frac{q!}{2}$		$\frac{(q^n-1)(q^{n-1}-1)\dots(q-1)}{2}$	$\frac{(q^n-1)(q^{n-1}-1)\dots(q-1)}{2}$	$\frac{(q^{2n}-1)(q^{2n-1}-1)\dots(q-1)}{2}$	$\frac{(q^n-1)(q^{n-1}-1)\dots(q-1)}{2}$	$\frac{(q^n-1)(q^{n-1}-1)\dots(q-1)}{2}$	$\frac{(q^{2n}-1)(q^{2n-1}-1)\dots(q-1)}{2}$	$\frac{(q^n-1)(q^{n-1}-1)\dots(q-1)}{2}$	$C_p$											
																	$p$			

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group\*
- Sporadic Groups
- Cyclic Groups

\*The Ree group  $G_2(q^2)$  is not a group of Lie type, but is the 2nd centralizer subgroup of  $Sp(6, q)$ . It is usually given by its Lie type.

The groups starting on the second row are the classical groups. The sporadic second group is related to the family of binary groups.

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Alternates*	Symbol	M <sub>11</sub>	M <sub>12</sub>	M <sub>22</sub>	M <sub>23</sub>	M <sub>24</sub>	J(1), J(11)	HJ	HJM	J <sub>4</sub>	HS	McL	J <sub>3</sub> , HJM, HTM	Ru	
Order <sup>†</sup>		7920	93040	443520	10200960	244823040	175360	604800	50232960	96775371046	977562000	44352000	896125000	4350307200	145152144000

\*\*The sporadic groups and families alternate names in the upper left or other names by which they are known. The symbol in the upper right indicates which these are used to indicate isomorphisms. All such symbols are used in the table except the family  $(G, G')$ .

†With the following exceptions:

$R(q)$

$A = A(q)$

$A \cong A(q)$

$S = S(q)$

$S \cong S(q)$

$O = O(q)$

$O \cong O(q)$

$N = N(q)$

$N \cong N(q)$

$L = L(q)$

$L \cong L(q)$

$E = E(q)$

$E \cong E(q)$

$H = H(q)$

$H \cong H(q)$

$I = I(q)$

$I \cong I(q)$

$P = P(q)$

$P \cong P(q)$

$R = R(q)$

$R \cong R(q)$

$T = T(q)$

$T \cong T(q)$

$U = U(q)$

$U \cong U(q)$

$V = V(q)$

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$W = W(q)$

$W \cong W(q)$

$X = X(q)$

$X \cong X(q)$

$Y = Y(q)$

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$Z = Z(q)$

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A theorem of Menezes, Quick and Roney-Dougal (2013)

**Theorem** For simple groups  $G$ ,  $P(G) \geq \frac{53}{90}$ .

# Generation V: Our Generation('s) Work

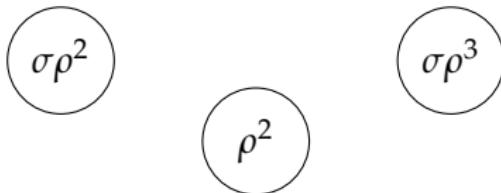
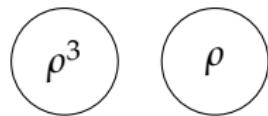
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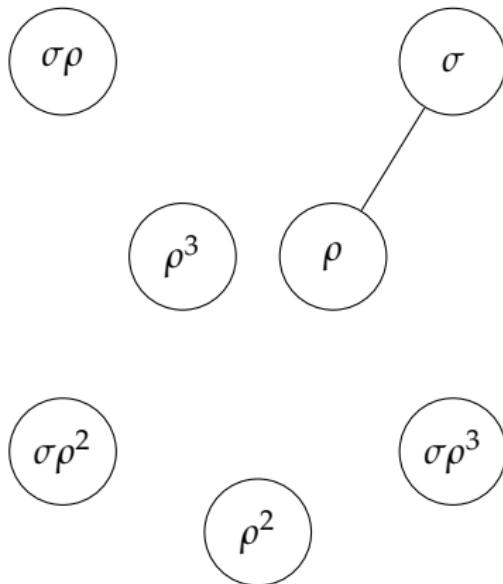
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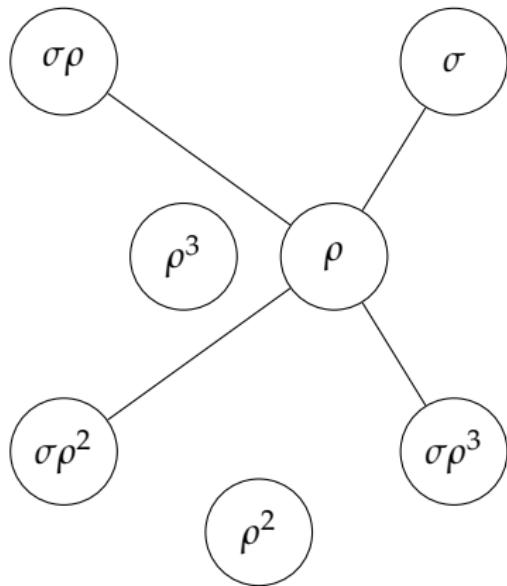
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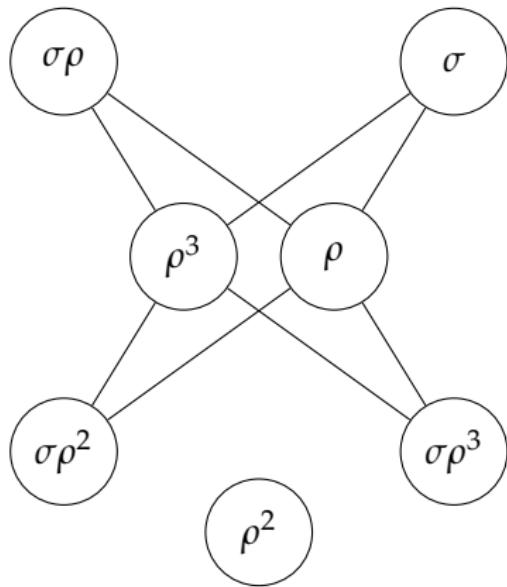
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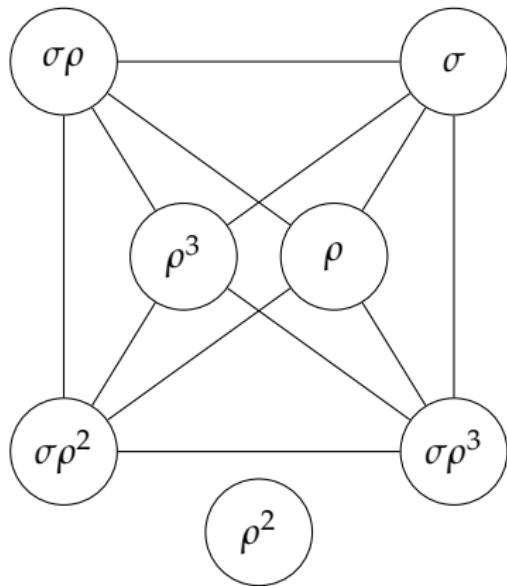
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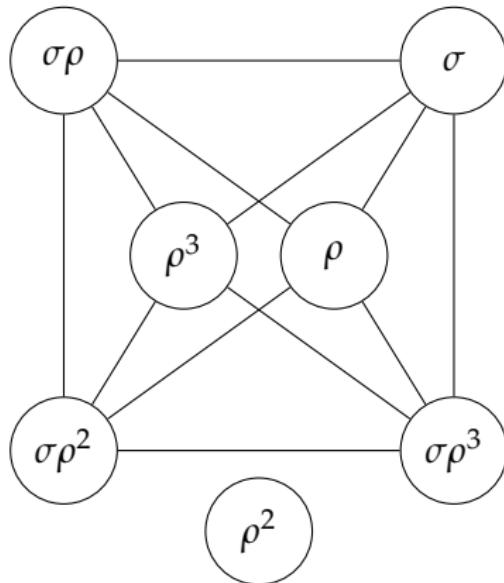
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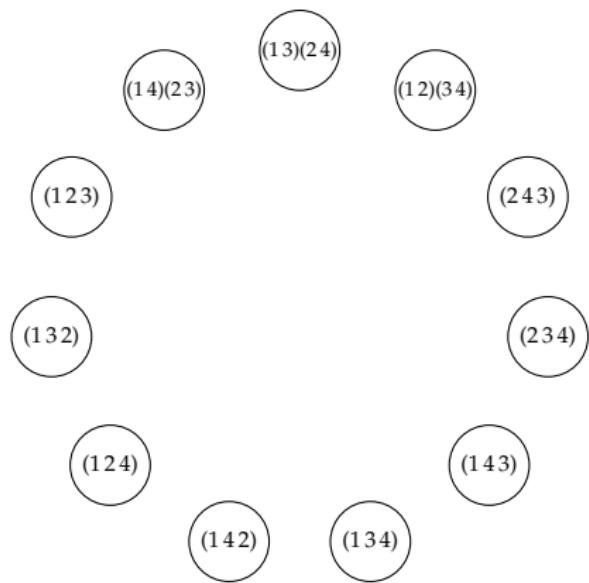
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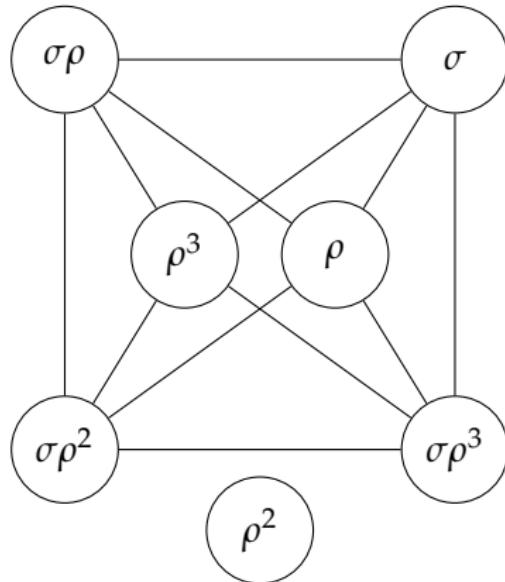
Alternating group  $A_4$



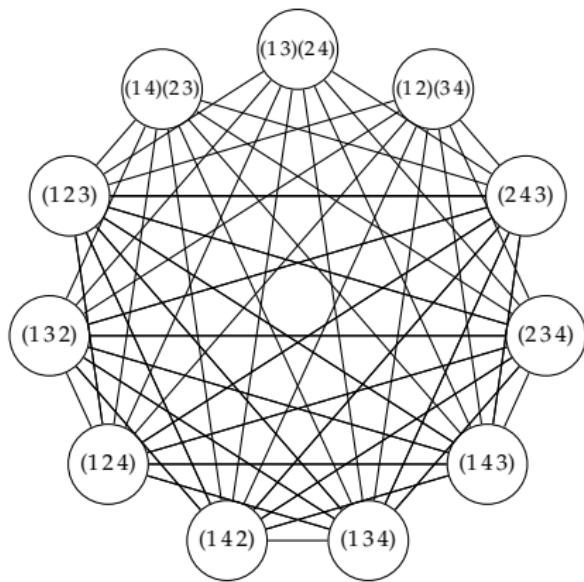
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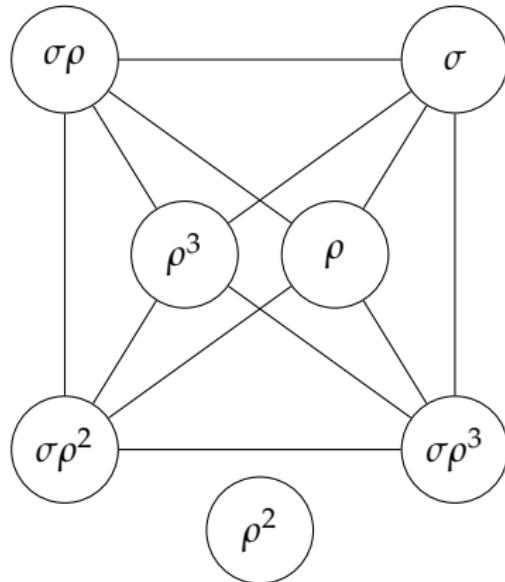
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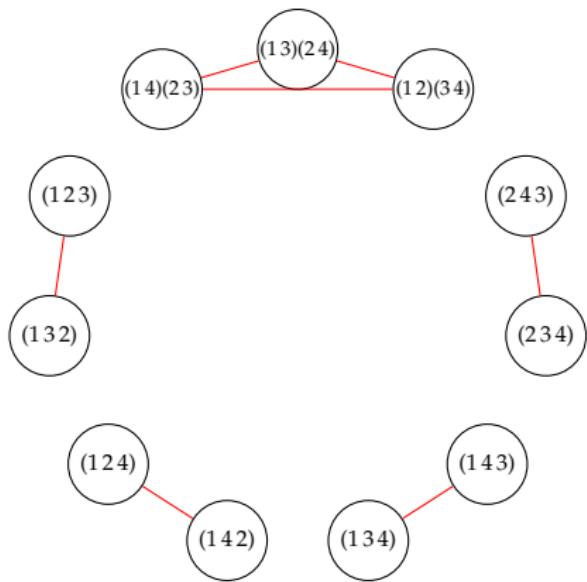
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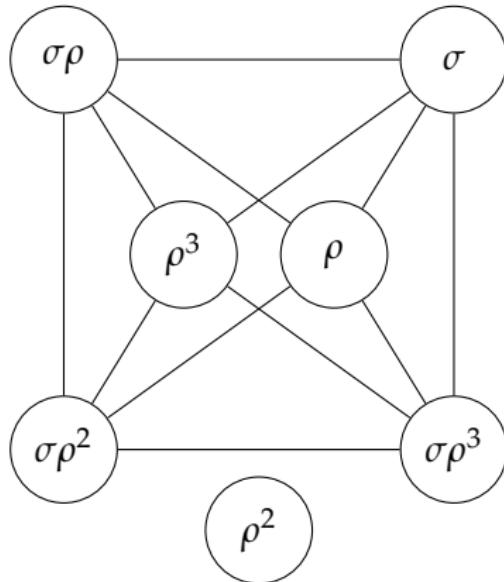
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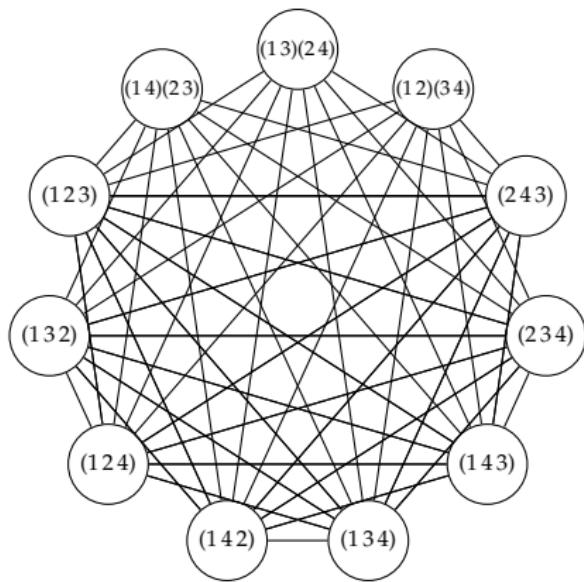
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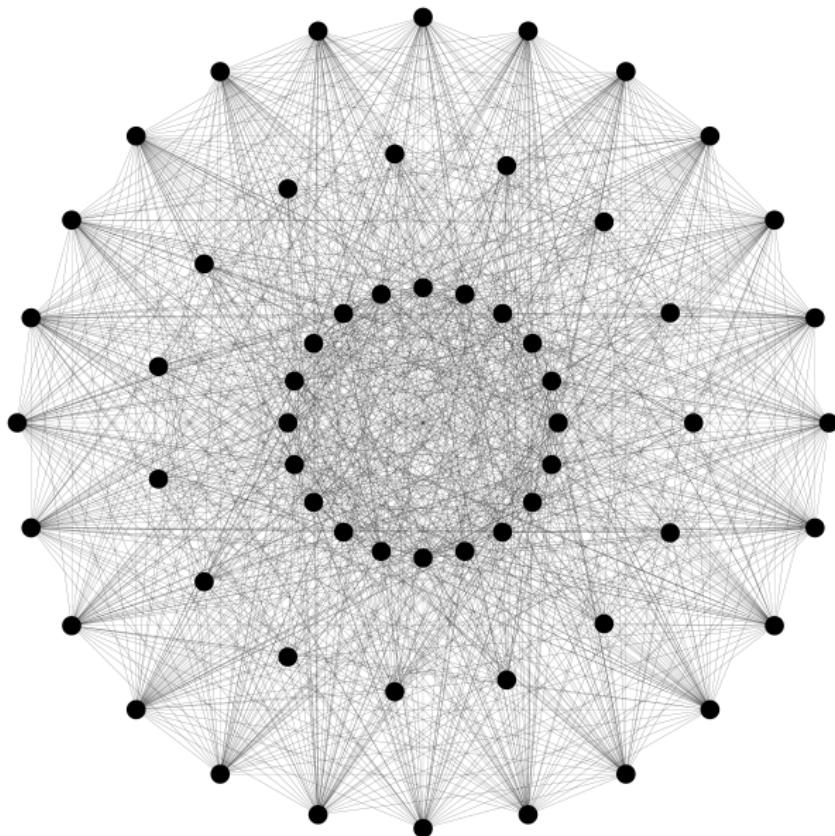
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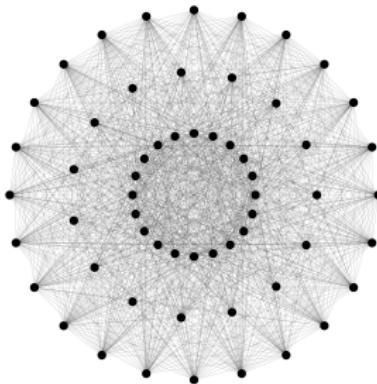
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**Conjecture** For finite groups, every non-identity element is contained in a generating pair if and only if every proper quotient is cyclic.

# Group Generation Through the Generations

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University of Bristol



St Andrews University Mathematics Society

3rd March 2017