The Art of Measuring Symmetry

Scott Harper

What is symmetry?

Have you ever marvelled at the stunning symmetry of a butterfly or the Taj Mahal? Or felt the urge to rearrange things that have been laid out asymmetrically? Or been frustrated by a photograph that is taken slightly off-centre? As humans we are attracted to symmetry, and we encounter it every day in nature, art and architecture.

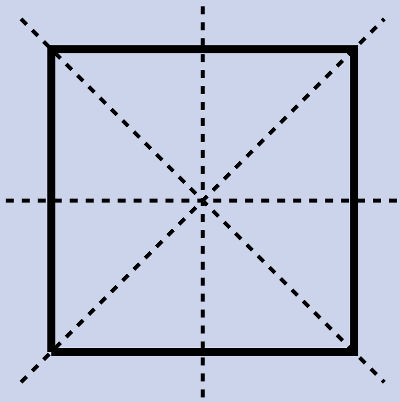
But what exactly is symmetry? What is it about a butterfly that makes it symmetrical? You might say that it is the same on both sides. Or you might say that it looks the same if you reflect it in a mirror. That second perspective gets to the heart of what is symmetry is all about: we should not think of shapes as simply having symmetry but rather having symmetries.

A symmetry of a shape is something you can do to the shape after which the shape looks the same.

Reflecting a butterfly in the vertical line is a symmetry, because the butterfly looks the same afterwards.

Measuring symmetry

Like a butterfly, the square can be reflected in the vertical line down its centre, but reflecting the square in the horizontal or diagonal lines shown also leaves the square arranged exactly as it was.

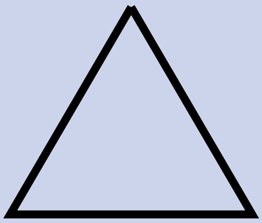


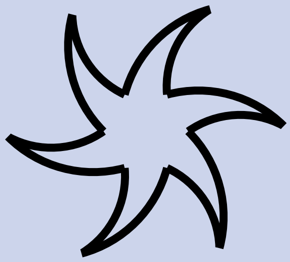
However, reflection is not the only thing we can do; rotating the square (about its centre) by 90° clockwise is also a symmetry of the square. Moreover, doing this twice (i.e. rotating by 180°) is also a symmetry as is doing this thrice (i.e. rotating by 270°).

What about rotating by a full 360°? This amounts to not doing anything at all, so should we count this as a symmetry? Yes, because it satisfies our definition: square definitely looks the same after doing nothing. We call this "do nothing" symmetry the *identity symmetry*, and this is the one symmetry that every shape obviously has.

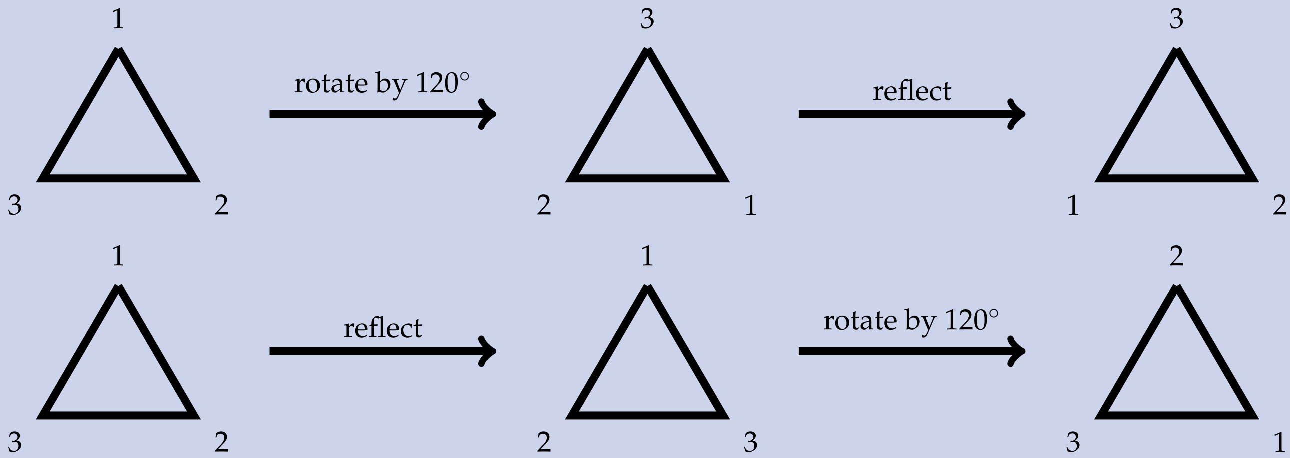
A square, therefore, has eight symmetries: four reflections (vertical, horizontal, two diagonal) and four rotations (by 0° – the identity, by 90°, by 180°, by 270°). In contrast, the butterfly has only two symmetries: the identity and the reflection. In this sense, a square is more symmetrical than a butterfly, because a square has eight symmetries but a butterfly only has two. This is our first step in measuring symmetry.

Consider two more shapes: a triangle and a twisted six-pointed star. The triangle has three reflections (in the lines through each corner) and three rotations (by 0°, 120° and 240°). However, the twisted star does not have any reflectional symmetries (for if we were to reflect it, the points would twist anticlockwise instead of clockwise), but it can be rotated by six angles: 0°, 60°, 120°, 180°, 240° and 300°. Both of these shapes have six symmetries, so neither is more symmetrical than the other. Does this mean that they have the same symmetry?





Number the corners of the triangle 1, 2 and 3 to keep track of how the triangle moves when we carry out each symmetry. Compare what happens if we first rotate the triangle by 120° then reflect the triangle in the vertical line, with what results from carrying out the same two symmetries in the opposite order.



We see that the order that we carry out symmetries in matters when we are dealing with the triangle. However, you can check that the order does not matter for the twisted star, as every symmetry is just some number of rotations by 60°. This means that, although these two shapes have the same number of symmetries, for these two shapes, the symmetries interact with each other in different ways.

The collection of symmetries of a shape is called its *symmetry group*. The number of symmetries in this group is gives a rough measure of the symmetry of the shape, but, as we have just seen, only by studying how the symmetries combine (when one is followed by another) can we fully understand the symmetry of the shape.

Group theory

Symmetry plays a fundamental role in numerous subjects, and in these settings it is often useful to be able to measure how symmetrical something is. Symmetry groups allow mathematicians to do exactly this, and this branch of mathematics is known as *group theory*. Galileo famously said that the laws of nature are written in the language of mathematics; indeed, the Standard Model, fundamental to modern physics, is written in the language of group theory. We will also see an application of symmetry in medicine in the next section.

However, for as long as humans have been using numbers for the practical purpose of measuring things, we have also been asking interesting questions about the numbers themselves, and some of these questions continue to occupy pure mathematicians today. (For example, the Ancient Greeks knew that there are infinitely many prime numbers, but we still do not know whether there are infinitely many pairs of prime numbers that differ by just two.) In the same way, as well applying group theory to other subjects, mathematicians investigate natural questions about groups themselves and in the final section we'll see one question that I recently answered.

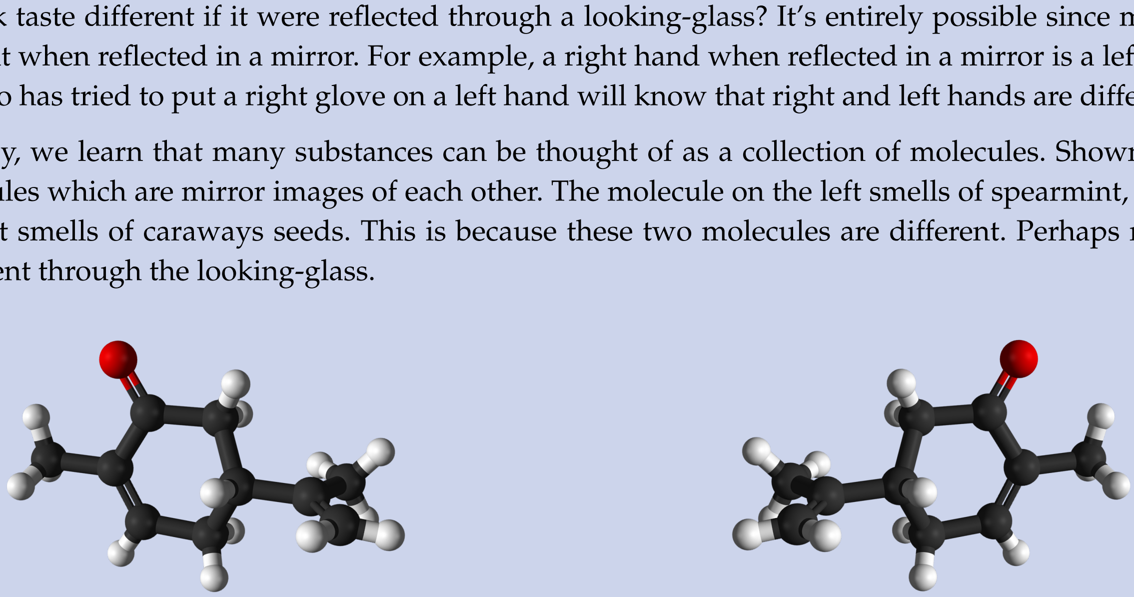
Through the Looking-Glass

Lewis Carroll's *Through the Looking-Glass* begins with Alice pondering

> "'Perhaps Looking-Glass milk isn't good to drink?'"

Would milk taste different if it were reflected through a looking-glass? It's entirely possible since many things are different when reflected in a mirror. For example, a right hand when reflected in a mirror is a left hand, and anyone who has tried to put a right glove on a left hand will know that right and left hands are different.

In chemistry, we learn that many substances can be thought of as a collection of molecules. Shown below are two molecules which are mirror images of each other. The molecule on the left smells of spearmint, but the one on the right smells of caraways seeds. This is because these two molecules are different. Perhaps milk would taste different through the looking-glass.



In the late 1950s, a drug known as thalidomide was taken by pregnant women to reduce morning sickness. But tragically thousands of babies were born with birth defects. What went wrong? One thalidomide molecule does reduces morning sickness with no side affects. However, like the molecule above, thalidomide has a mirror image which is different from it, and it is this mirror image that causes the birth defect. The crucial problem is that when you manufacture thalidomide you end up with a mixture of both of the mirror images. Therefore, in order to safely make drugs we need to make sure we understand when two molecules are the same up to rotational symmetry, and this is exactly the sort of question which the tools of group theory allow us to answer.

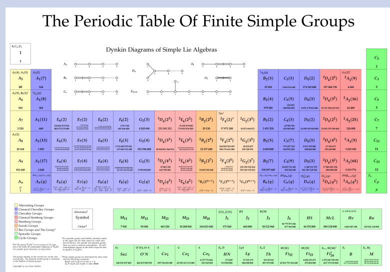
A periodic table of symmetry

Some shapes can have infinitely many symmetries. For instance, a circle can be reflected in any line through its centre. From now on, we will only consider groups with a finite number of symmetries.

Just as lego constructions can be broken into lego bricks, as whole numbers can be broken into prime numbers and as molecules can be broken into atoms, groups can be broken into smaller indivisible groups called *simple groups*. For example, the group of six symmetries of the triangle can be broken into two simple groups: one containing just a reflection and the identity (like the symmetries of a butterfly), and one containing three rotations (like the symmetries of a twisted three-pointed star).

If all your lego bricks are blue, then anything you build from those bricks will also be blue, and you know this without having to build all possible constructions and checking them. In group theory there is a similar philosophy: we can deduce facts about all groups from knowledge of these building-block simple groups.

Therefore, it is important that we know what all of these simple groups are. One of the greatest achievements of twentieth century mathematics, was the effort of hundreds of mathematicians across the world, writing thousands of pages of mathematics, to obtain the *Classification of Finite Simple Groups*. This is a list of several families of groups that they proved were simple and a proof that any simple group is on their list. If we continue with the analogy that the simple groups are akin to atoms, then the Classification of Finite Simple Groups is akin to the Periodic Table of Elements.



The impossible equation

Mathematicians, together with scientists and social scientists, use *equations* to model the world, so being able to solve equations has been a key feature of mathematics throughout the ages. At school, you will likely have come across quadratic equations such as $x^2 + 5x + 6 = 0$, and you might remember that you can solve these equations using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In renaissance Italy, mathematicians challenged each other to high-profile mathematical duels, setting each other a series of equations to solve in an allotted period of time. These equations were not quadratic, but cubic, such as $x^3 - 6x^2 + 11x - 6 = 0$. This pressure led to the discovery of a formula to solve these sorts of equations and, later, quartic equations (where x^4 also appears). What about quintic equations like $x^5 + 20x + 16 = 0$?

Despite lots of effort nobody could find a formula to solve these equations. This is because it is impossible! There is, and there provably can never be, a formula, just involving basic operations such as addition, division and taking roots, that you can use to solve any quintic equation.

The first mathematician to truly understand why was Évariste Galois, in the 1830s. He realised that whether an equation could be solved using a formula depended on the symmetry of the equation, and he essentially invented group theory in order to explain this! You can solve an equation using a formula only if the symmetry group of the equation (whatever that means) is built up from particular simple groups, and this is not true for all quintic equations. Tragically, Galois died following a duel at the age of twenty.

The generation game

Consider once again the symmetries of a triangle. Although the triangle has six symmetries we can obtain any of these symmetries by taking the rotation by 120° and the reflection in the vertical line and combining these in various ways. For example, we can obtain a rotation by 240° by doing the rotation by 120° twice. Or we can do a reflection in the line through the bottom-right corner in two steps by first rotating by 120° then reflecting in the vertical line. We say that the symmetry group of the triangle is *generated* by these two symmetries.

In fact, for any symmetry except the identity there is a second symmetry such that these two symmetries generate the whole group. This is stronger than what we said earlier: before we just said that there are two symmetries that generate the symmetry group, but now we are saying that if you choose any symmetry other than the identity (for example, you might choose the reflection in the line through the bottom-left corner), then there is a second symmetry (in this case rotation by 240° works) that together with the first generates the symmetry group.

Which symmetry groups have the property that for any symmetry other than the identity there is a second symmetry such that these two symmetries generate the whole symmetry group? The symmetry group of the triangle has the property and the symmetry group of the square does not. More generally, the symmetry group of a regular polygon has this property if and only if the polygon has a prime number of sides. For symmetry groups in general, Tim Burness, Robert Guralnick and I answered this question fully. We did this by turning the question into a question about just the simple groups and then studying the simple groups very carefully.

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Our paper *The spread of a finite group*, T.C. Burness, S. Harper & R.M. Guralnick, Annals of Mathematics **193** (2021), 619–687.

Further reading *Groups and symmetry*, M.A. Armstrong, Springer, 2010.