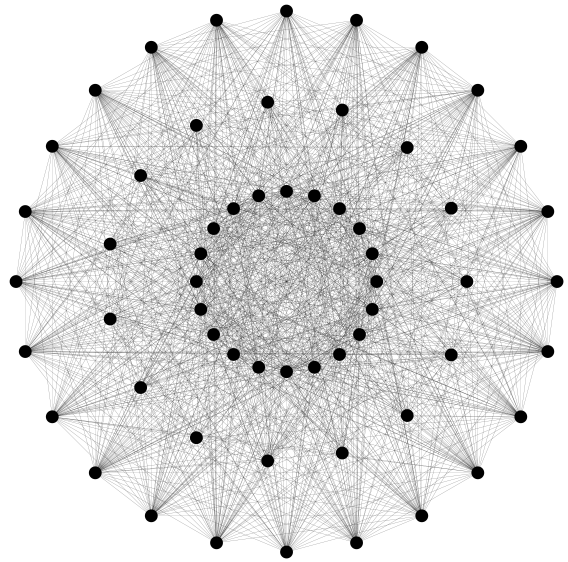
# Group Generation Through the Generations

Scott Harper

University of Bristol



St Andrews University Mathematics Society

3rd March 2017

# So what do you do?

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the art of measuring symmetry

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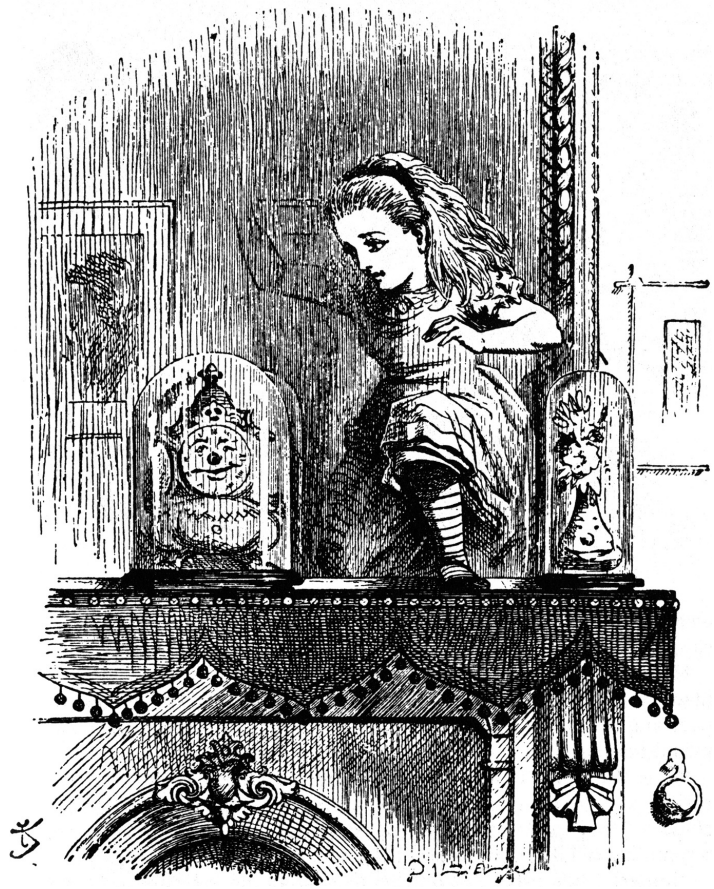

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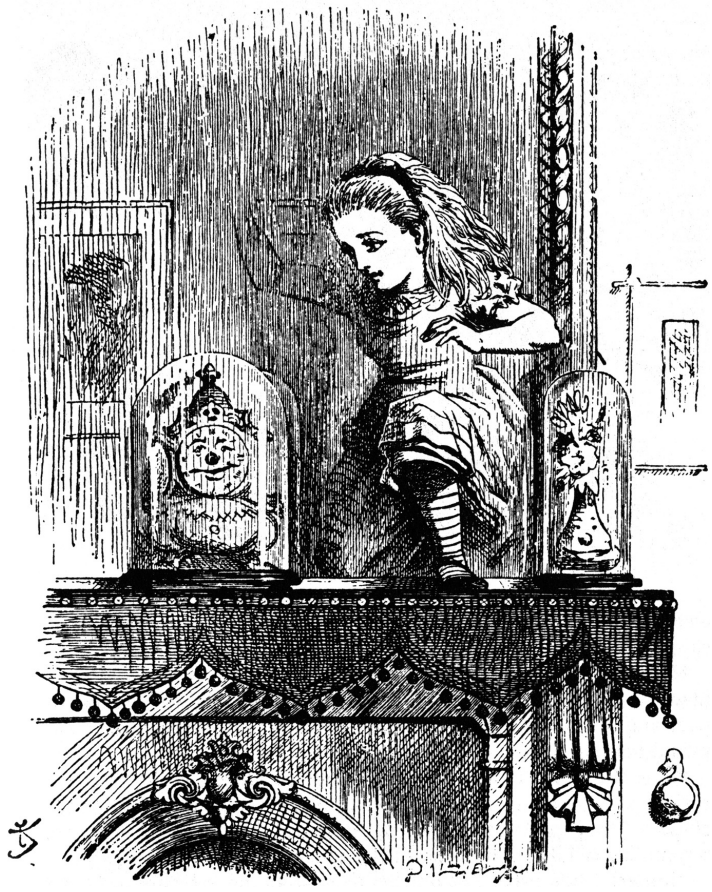
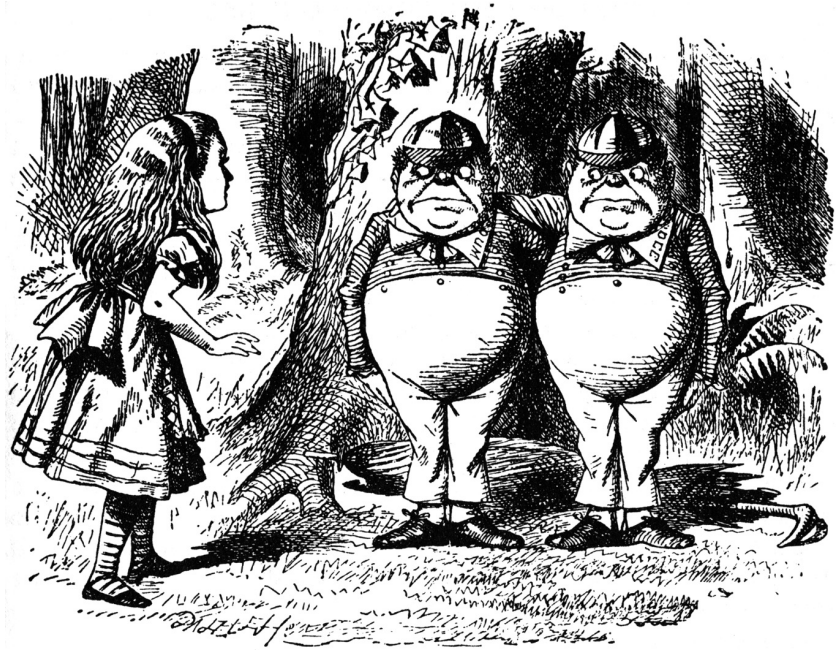


*Alice knew which was which in a moment.*



thalidomide

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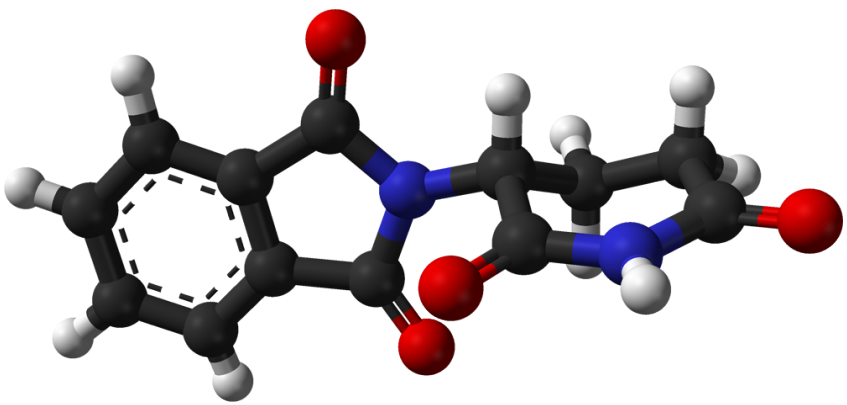

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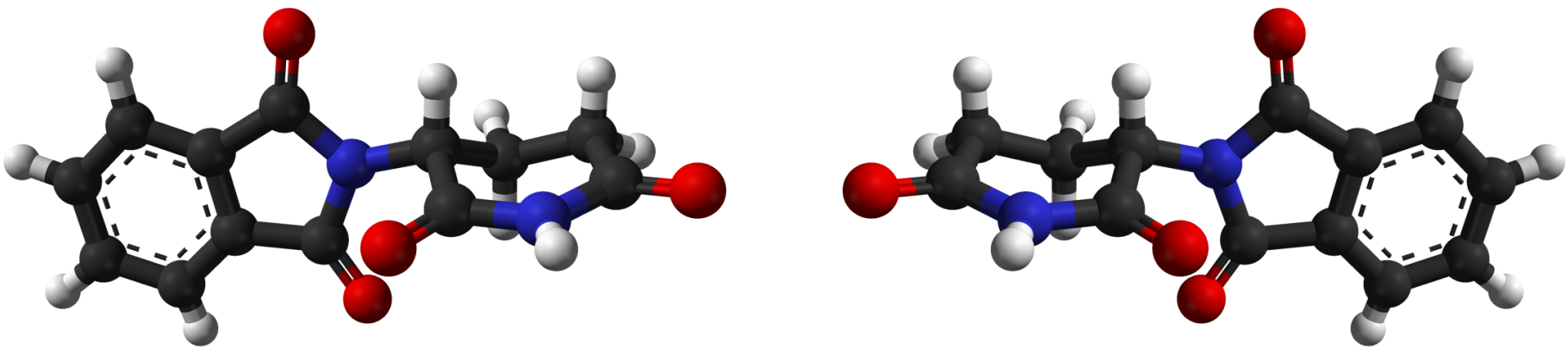

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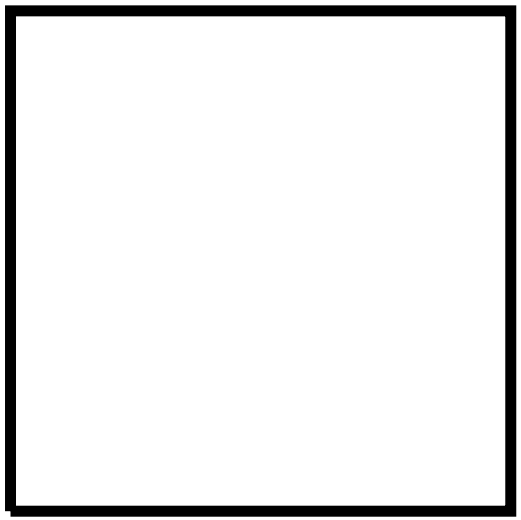

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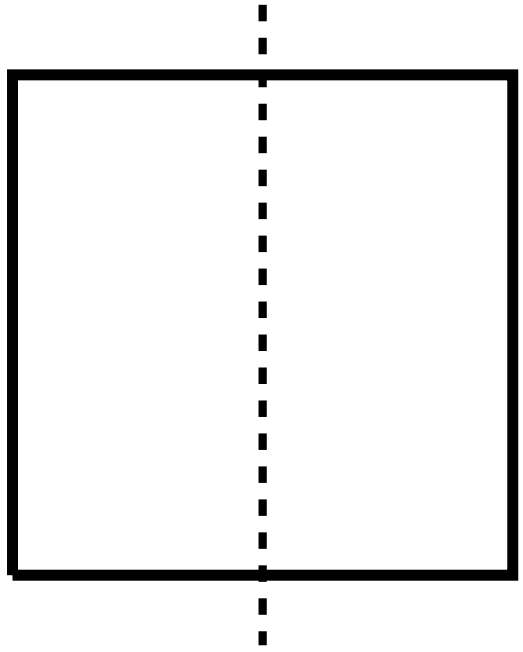

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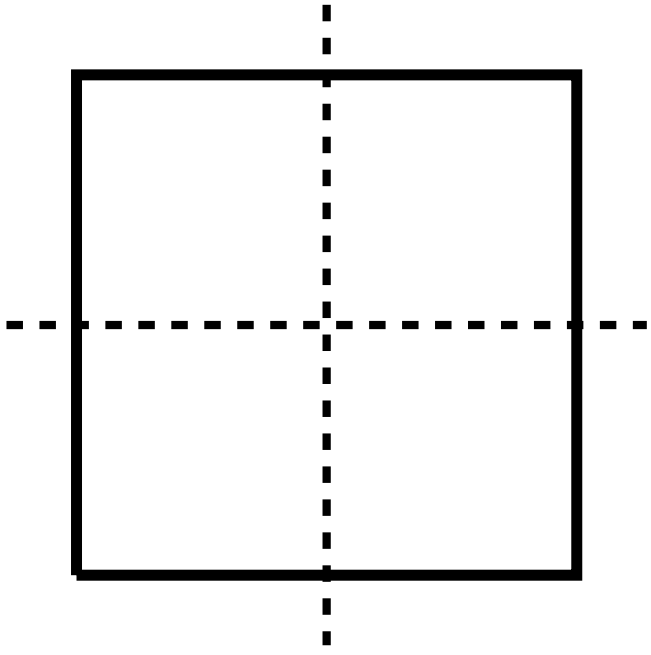

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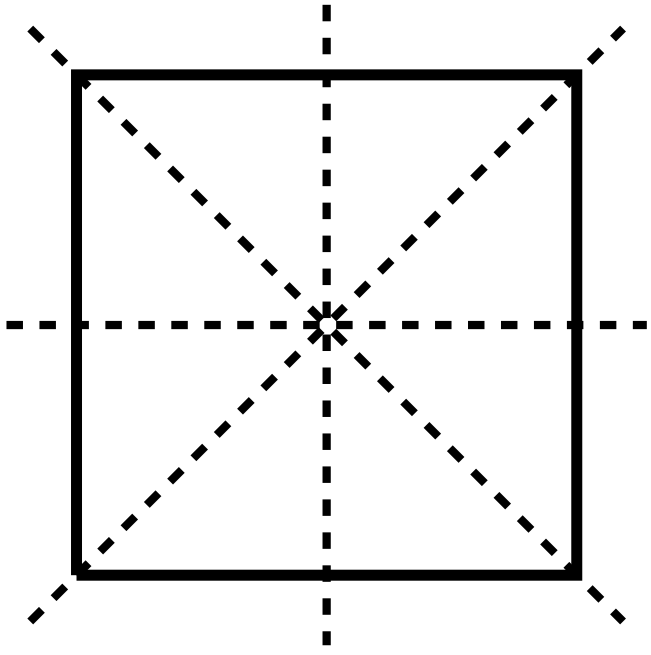

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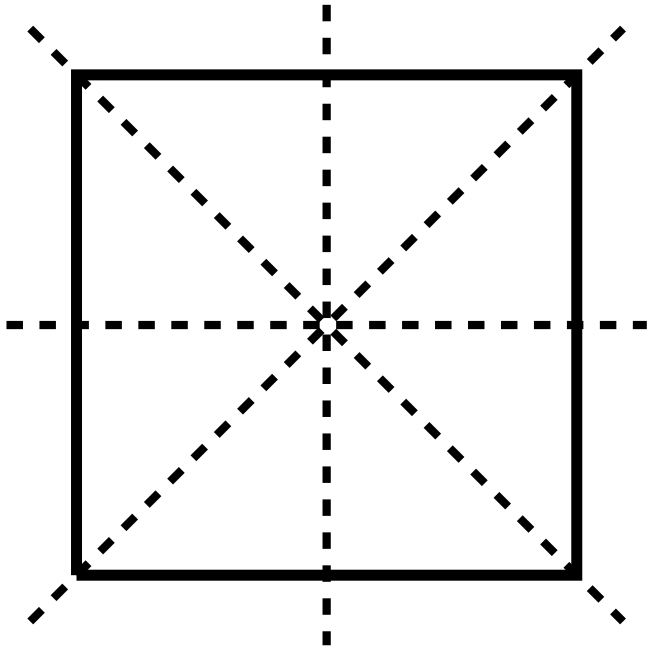

the dihedral group $D_{\square}$

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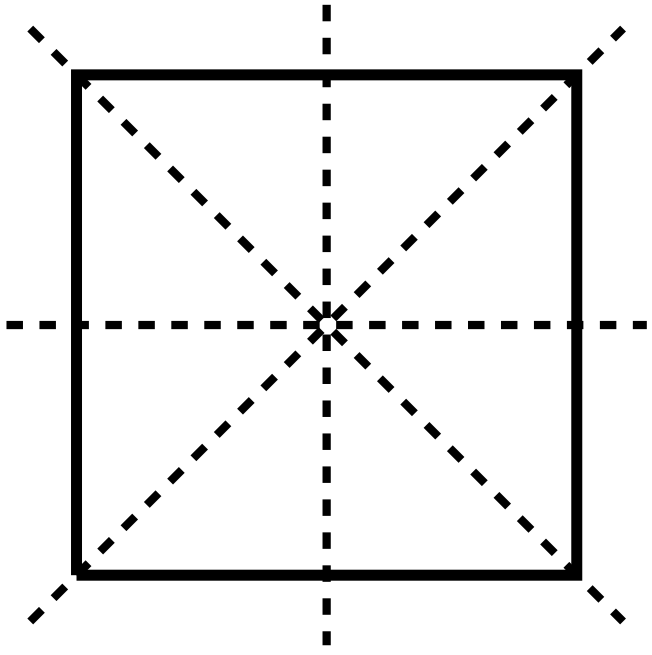

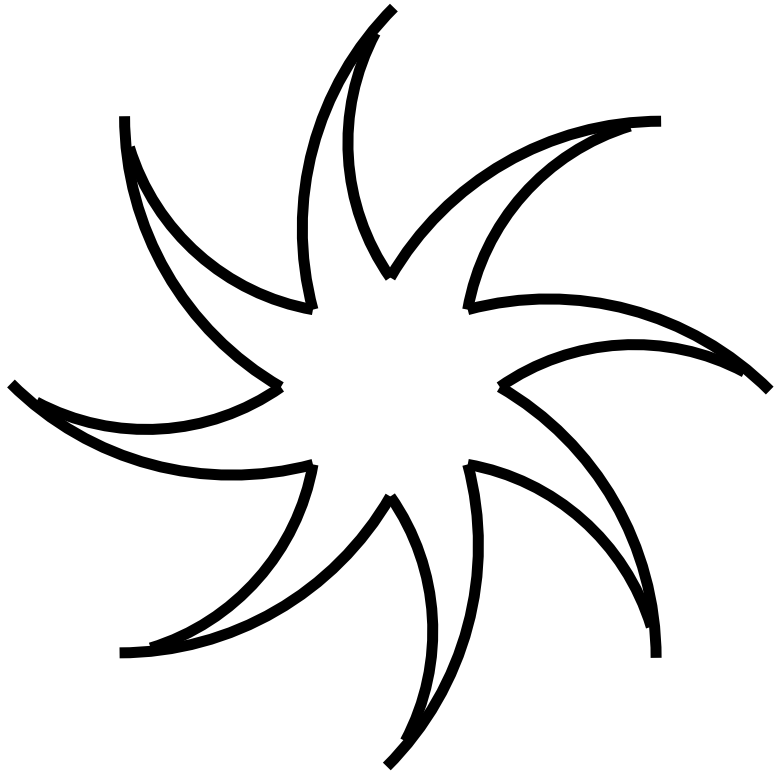

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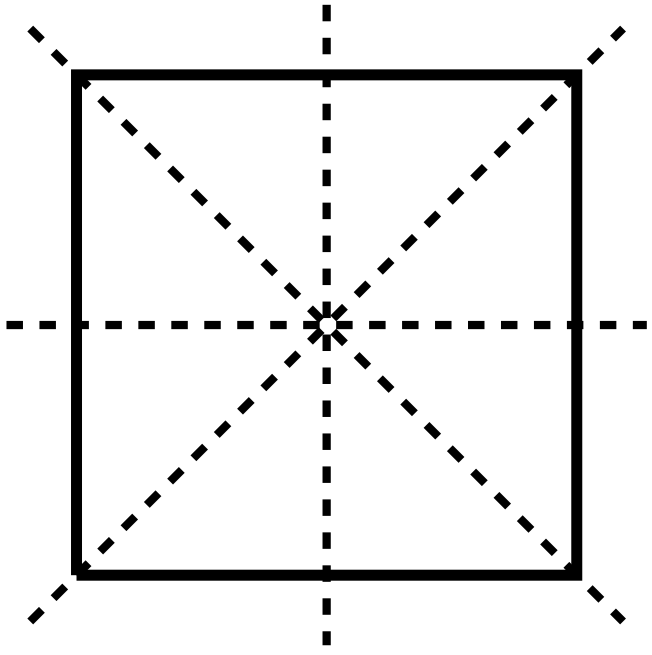

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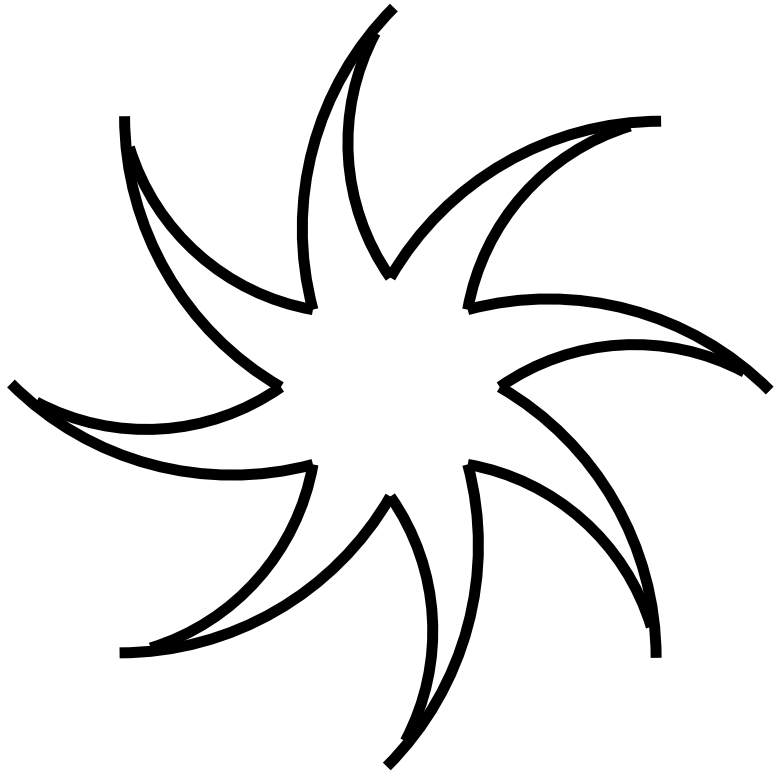

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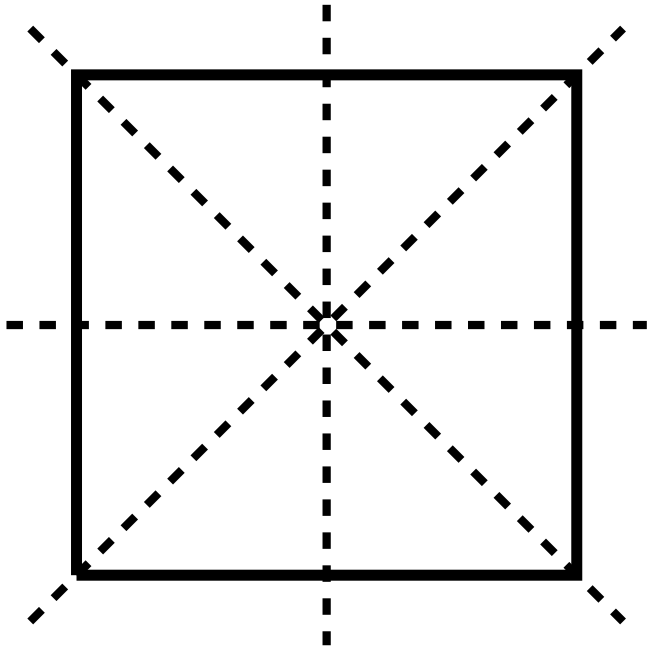

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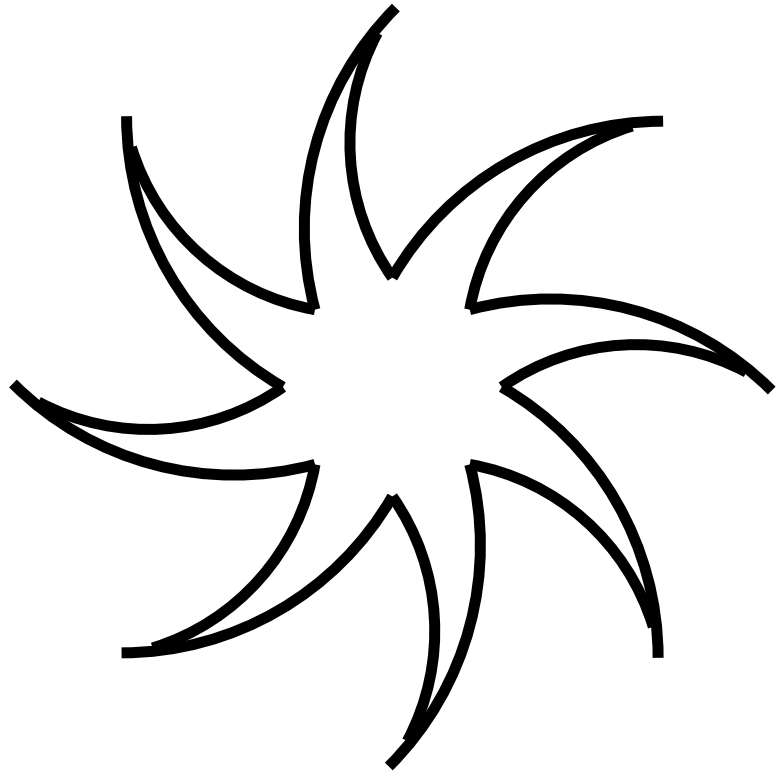

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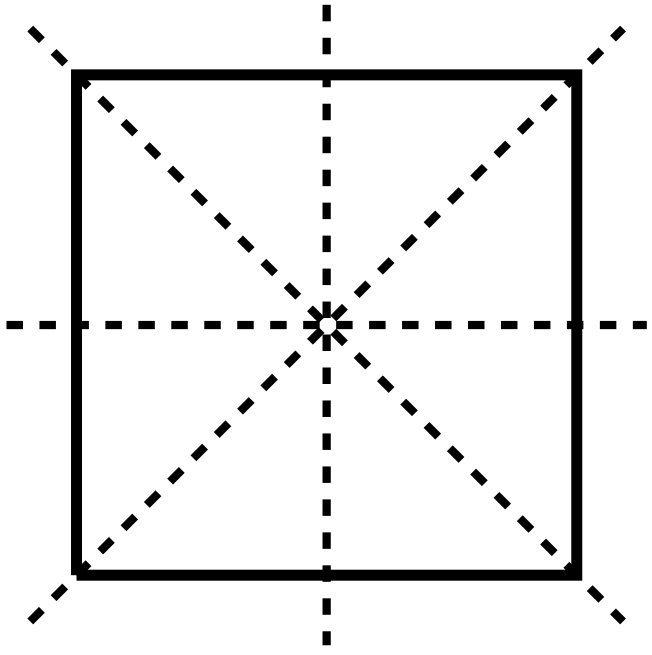

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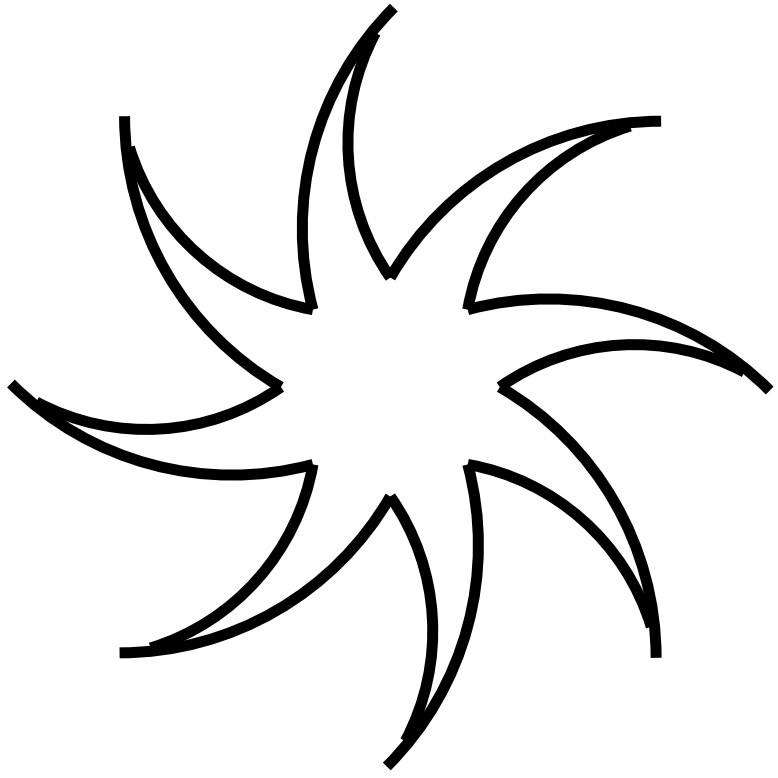

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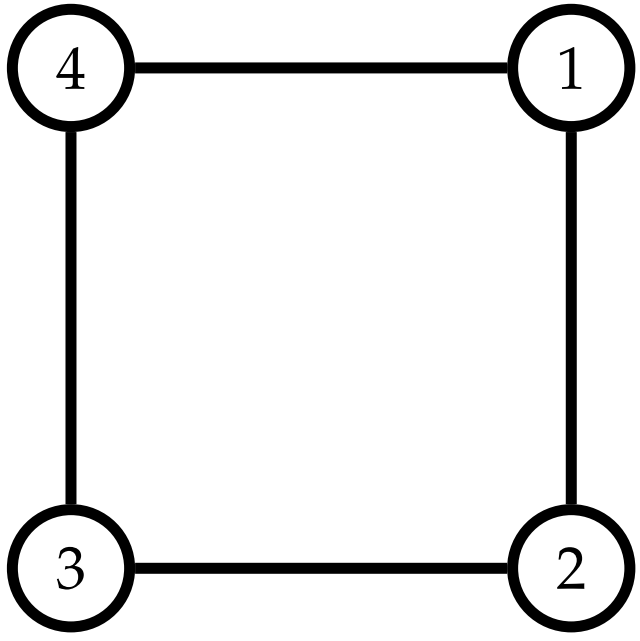


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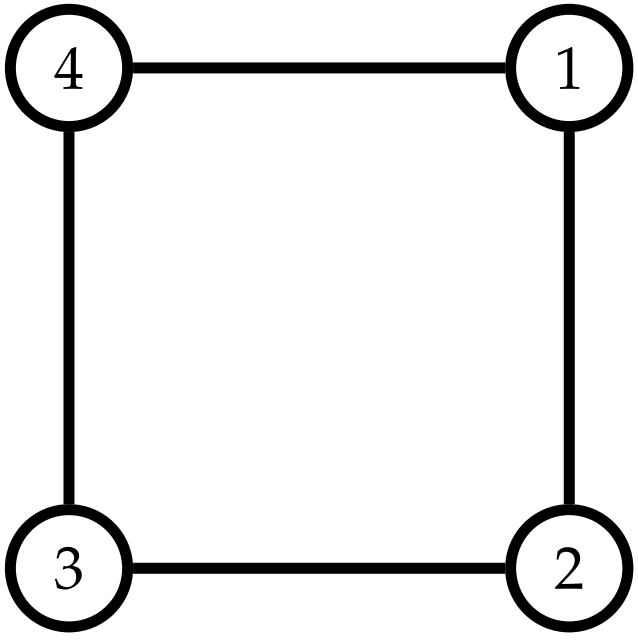


Rotation by 90° clockwise

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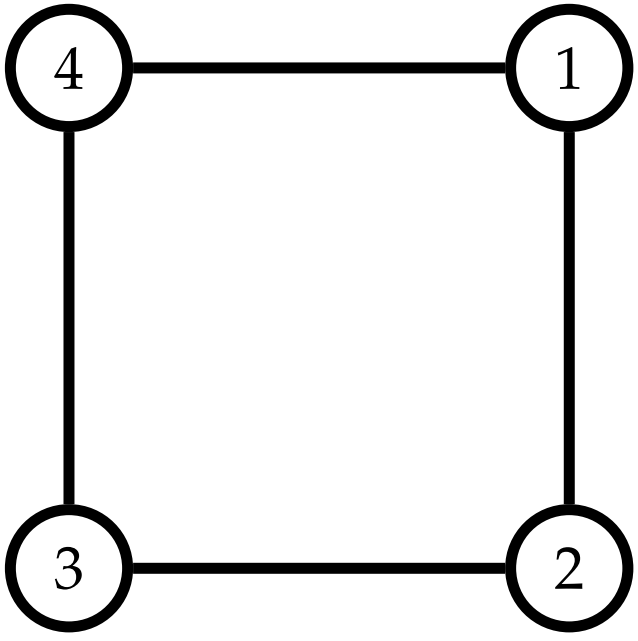


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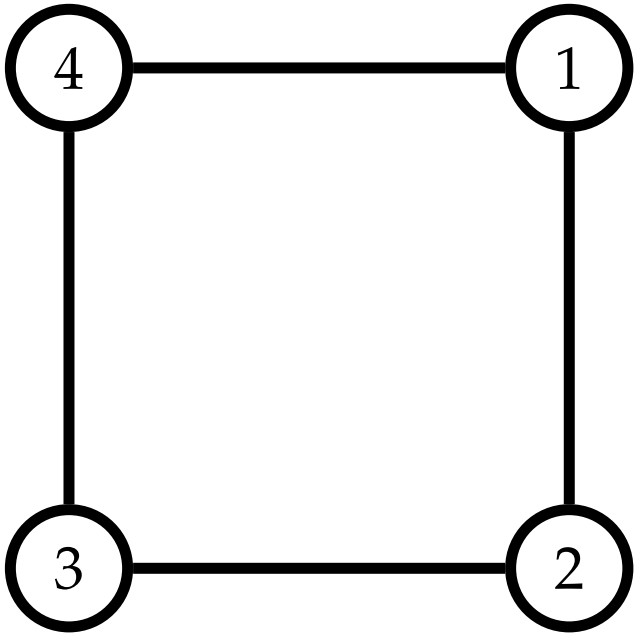


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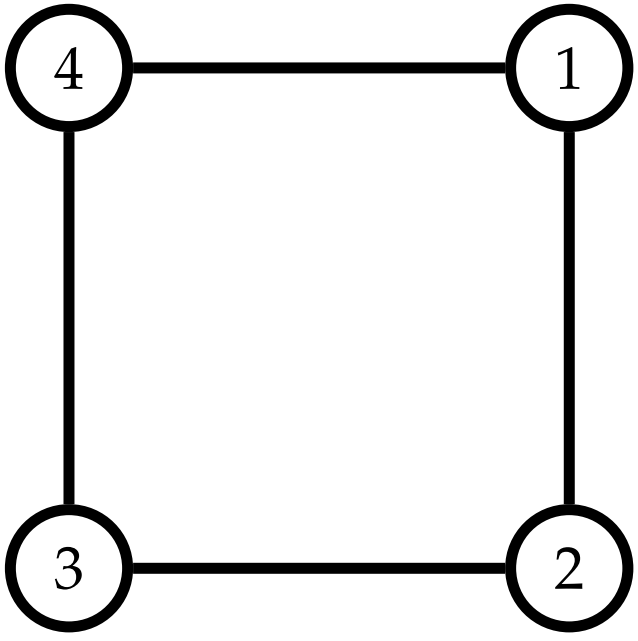


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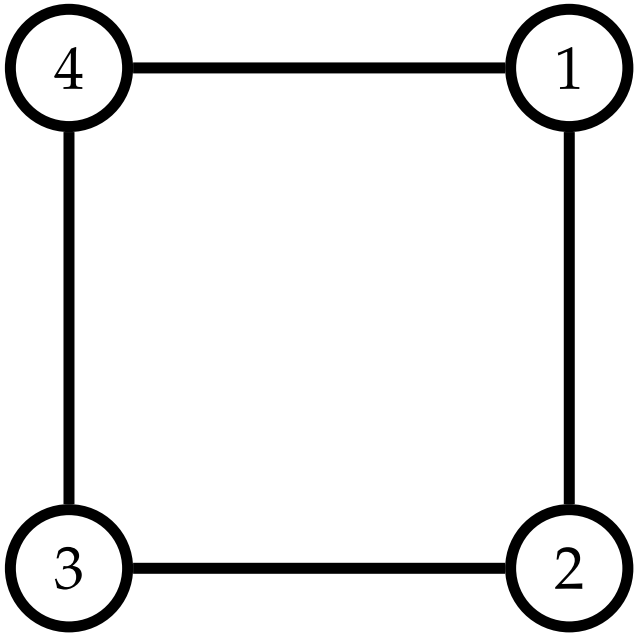


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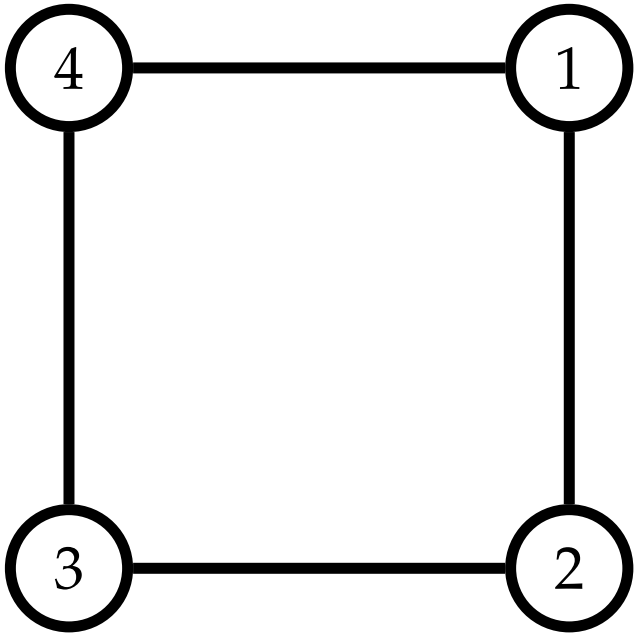


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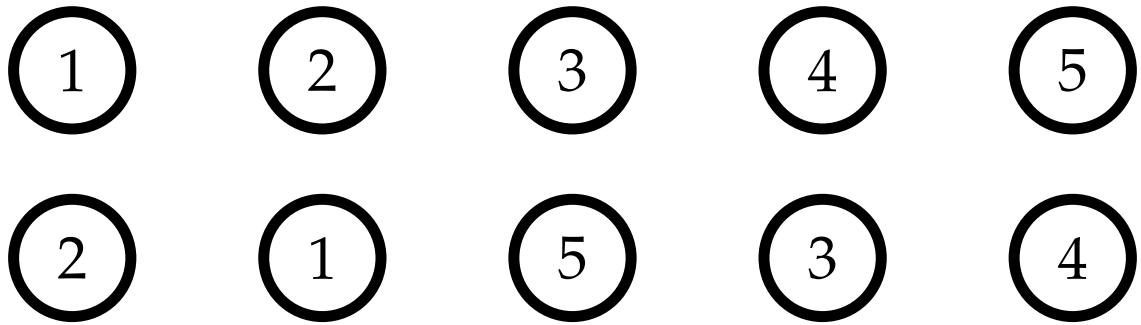

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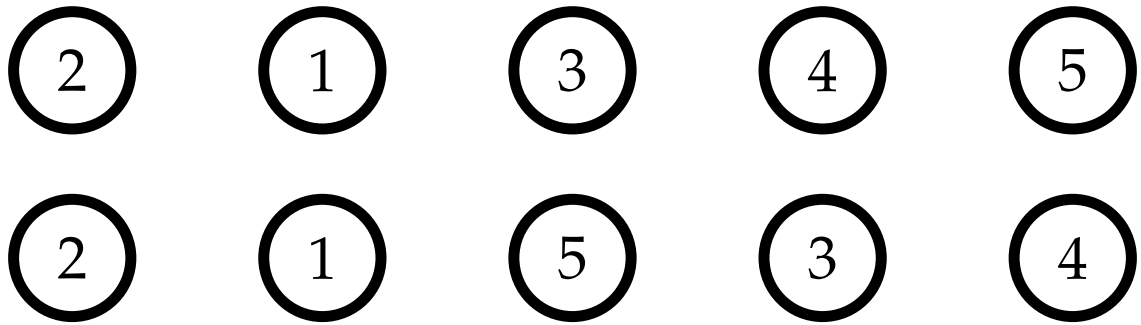

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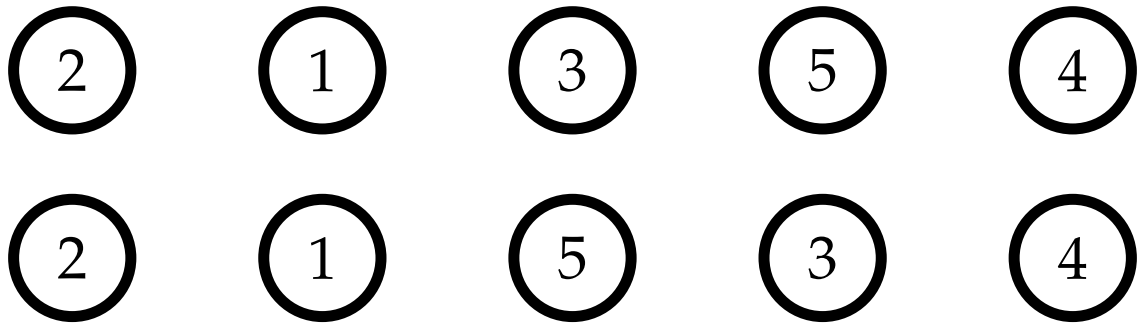

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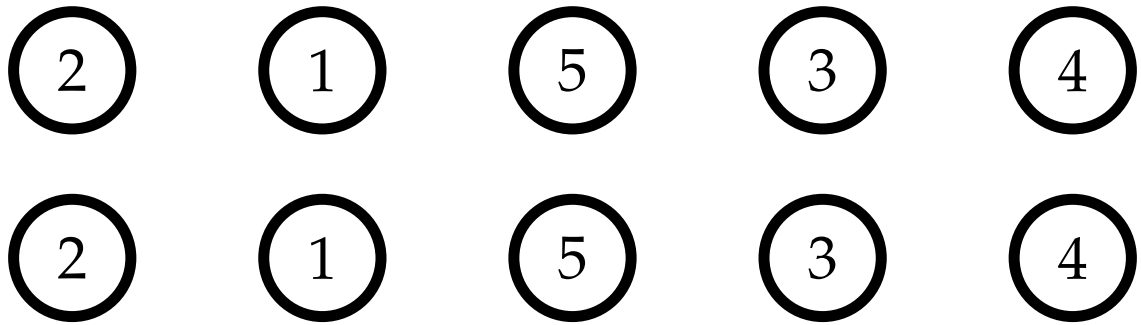

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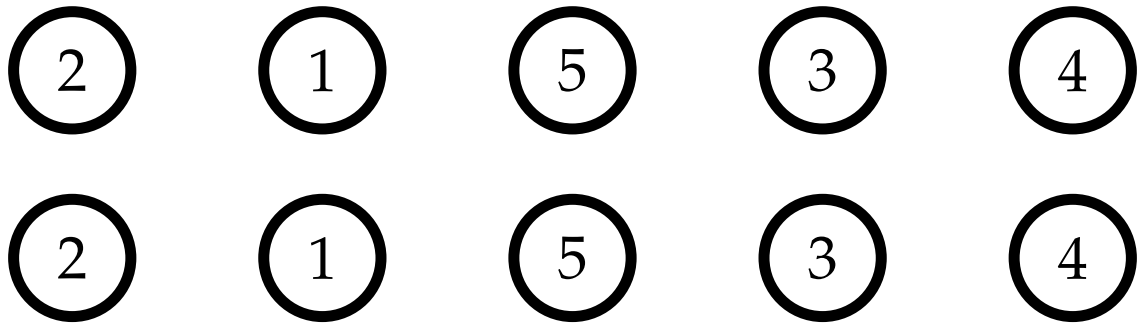


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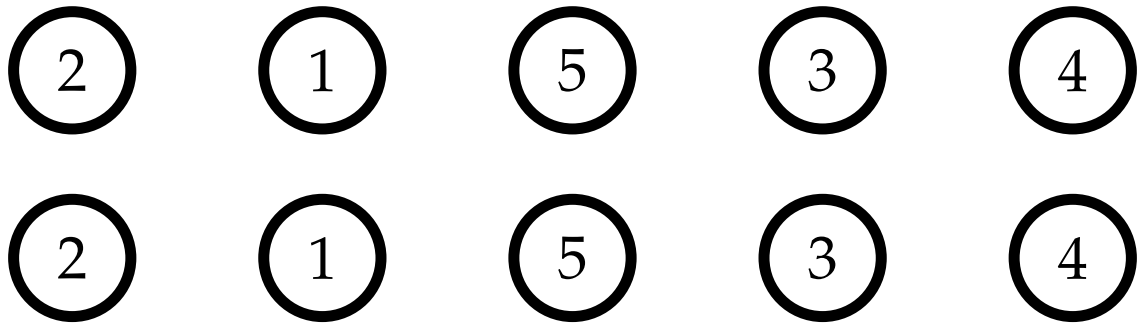


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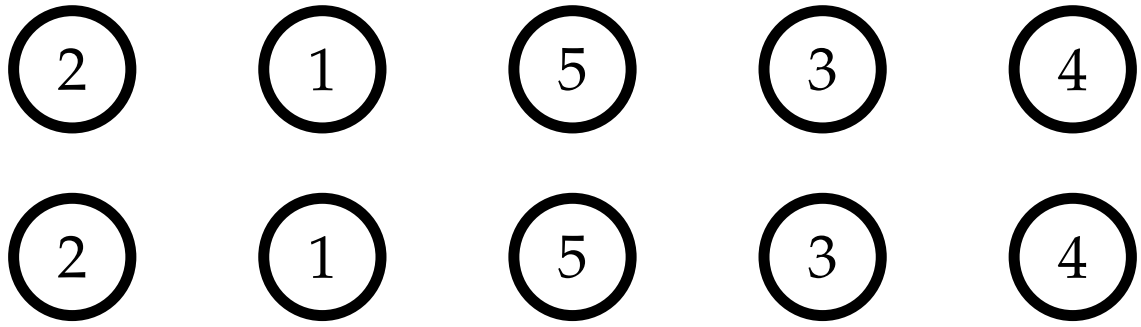


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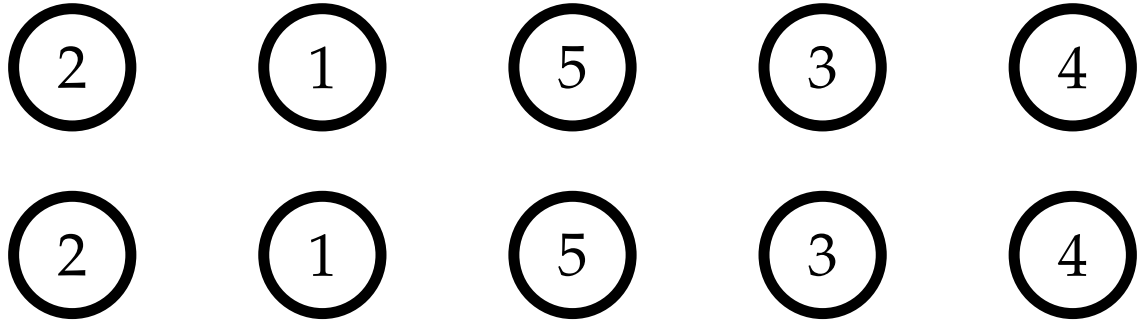


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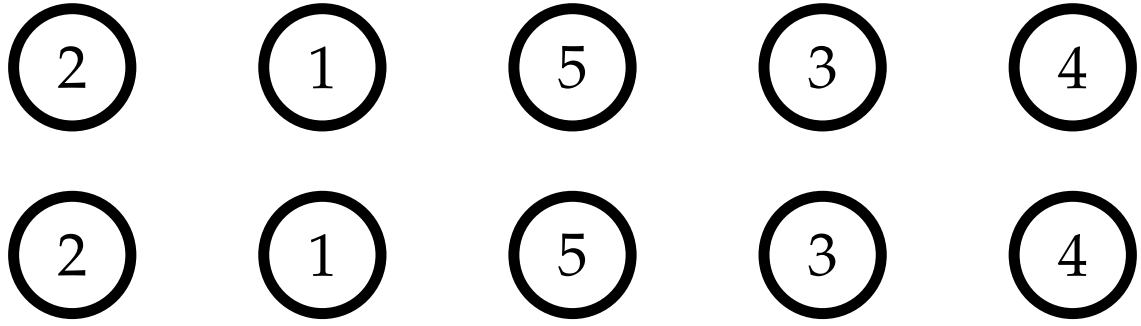


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For example, $A_5$ is generated by $(1\,2\,3)$ and $(1\,2\,3\,4\,5)$.

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$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

**Quartic** Similar to cubic but even more involved!

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**Cubic** (Renaissance Italy) If $ax^3 + bx^2 + cx + d = 0$ with $a \neq 0$ then

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

**Quartic** Similar to cubic but even more involved!

**Theorem** (Abel; 1802) There is no formula for the quintic.

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Repeatedly divide by big divisors

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**Theorem** For $n \geq 5$, the alternating group $A_n$ is simple.

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"Please request publicly that Jacobi or Gauss give their opinions, not on the truth but on the importance, of these theorems.

After that I hope people will be found who profit by sorting out all this mess.

I embrace you with affection.

É. GALOIS, 29 May 1832"

# Generation II: Netto's Conjecture

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If we arbitrarily select two or more substitutions of $n$ elements, it is to be regarded as extremely probable that the group of lowest order which contains these is the symmetric group, or at least the alternating group. In the case of two substitutions the probability in favor of the symmetric group may be taken as about $\frac{3}{4}$, and in favor of the alternating, but not symmetric, group as about $\frac{1}{4}$. In order that any given substitutions may generate a group which is only a part of the $n!$ possible substitutions, very special relations are necessary, and it is highly improbable that arbitrarily chosen substitutions $s_i = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1} & x_{i_2} & \dots & x_{i_n} \end{pmatrix}$ should satisfy these conditions. The exception most likely to occur would be that all the given substitutions were severally equivalent to an even number of transpositions and would consequently generate the alternating group.

E. Netto, *The theory of substitutions and its application to algebra*, Trans. F. N. Cole, Ann Arbor, Michigan, (1892)

Let $P(G)$ be the probability that two random elements generate $G$.

**Netto's Conjecture** $P(A_n) \to 1$ as $n \to \infty$.

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Let $p$ be prime. Then

$$P(C_p) = \frac{p^2 - 1}{p^2}$$

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## Numerical evidence

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|
| $P(A_n)$ | 0.633 | 0.588 | 0.726 | 0.738 | 0.848 | 0.875 |

GAP computations by N. Menezes, M. Quick and C. M. Roney-Dougal

# Generation III: Dixon's Conjecture

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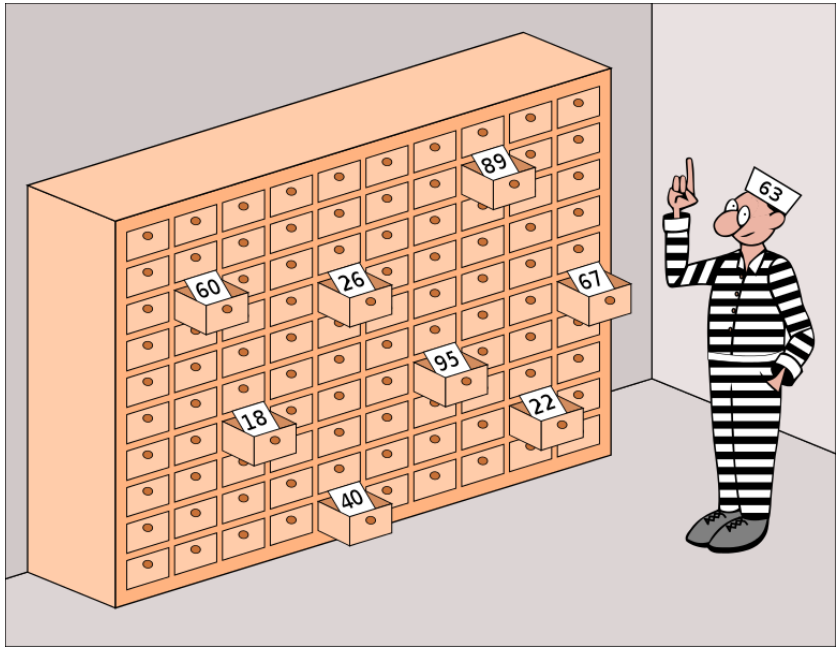
A final chance is given to 100 prisoners, numbered 1 to 100. A room contains 100 closed drawers which contain the numbers 1 to 100. One by one, the prisoners enter the room and choose 50 drawers to open. The drawers are closed again. If every prisoner finds his number, all prisoners are pardoned. However, if just one prisoner does not find his number, all prisoners remain imprisoned. Before any prisoner enters the room, they may decide on a strategy but they may not communicate once the first prisoner enters. What is the best strategy?

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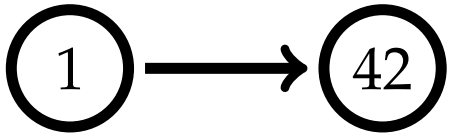
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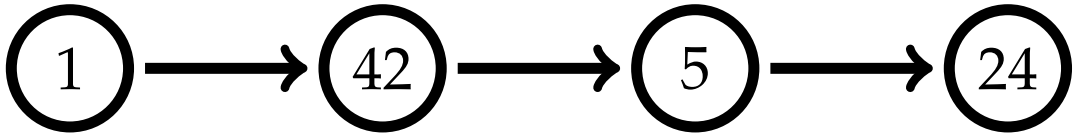
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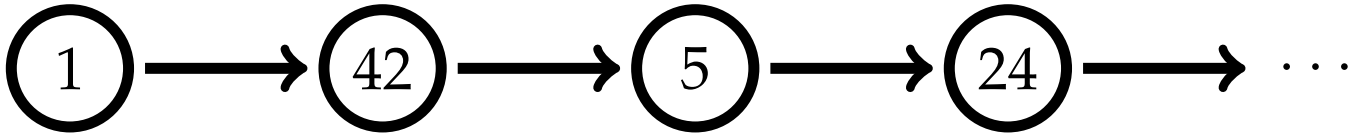
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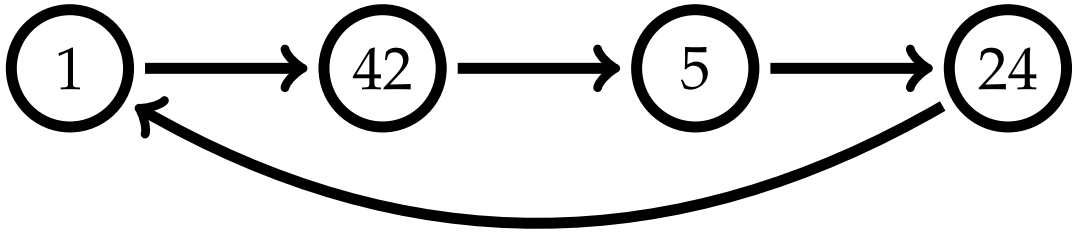
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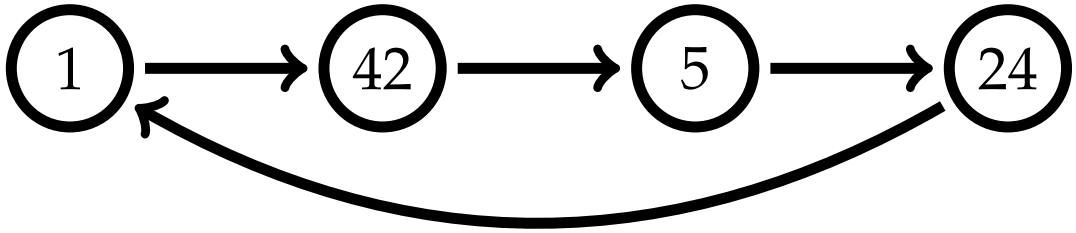
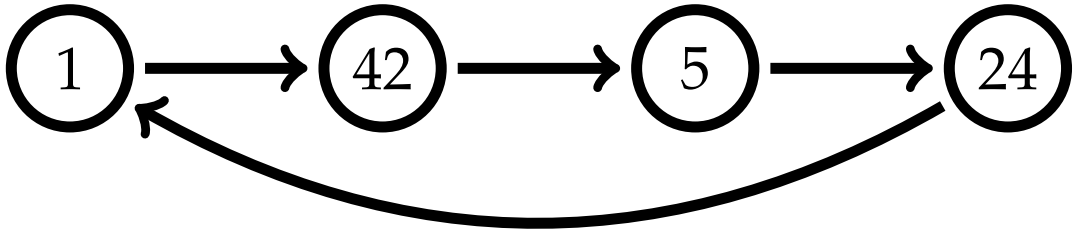



All prisoners succeed if there is no cycle of length longer than 50.

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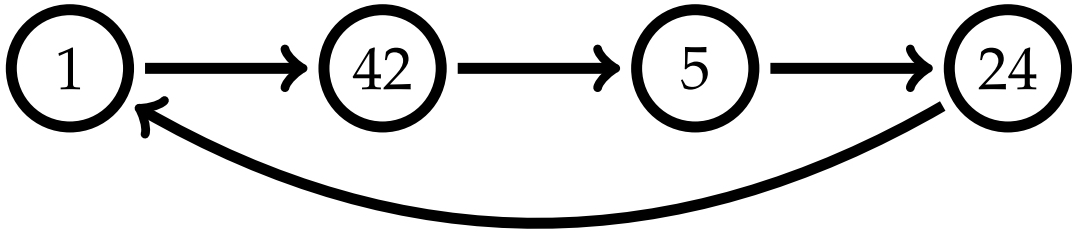


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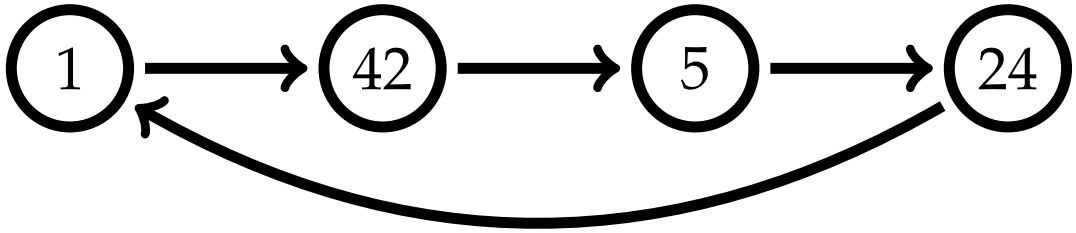


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**Exercise** For $\ell > n$, the probability that a permutation of $2n$ points has a cycle of length $\ell$ is $1/\ell$.

The probability that a permutation of 100 points has no cycle of length longer than 50 is

$$1 - \left( \frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{100} \right) \approx 0.312$$

which is the probability of success.

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What if there were other finite simple groups?

# Generation IV: An Enormous Theorem

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1965 Janko finds three more sporadic groups

# A Simple Ballad

What are the orders of all simple groups?
I speak of the honest ones, not of the loops.
It seems that old Burnside their orders has guessed:
Except of the cyclic ones, even the rest.

Groups made up of permutes will produce more:
For $A_n$ is simple, if $n$ exceeds 4.
Then, there was Sir Matthew who came into view
Exhibiting groups of an order quite new.

Still others have come on to study this thing.
Of Artin and Chevalley now we shall sing.
With matrices finite they made quite a list.
The question is: Could there be others they've missed?

Suzuki and Ree then maintained it's the case
These methods had not reached the end of the chase.
They wrote down some matrices, just four by four,
that made up a simple group. Why not make more?

And then came up the opus of Thompson and Feit
Which shed on the problem remarkable light.
A group, when the order won't factor by two,
Is cyclic or solvable. That's what true.

Suzuki and Ree had caused eyebrows to raise,
But the theoreticians they just couldn't face.
Their groups were not new: if you added a twist,
You'd get them from old ones with a flick of the wrist.

Still, some hardy souls felt a thorn in their side.
For the five groups of Mathieu all reason defied:
Not $A_n$, not twisted, and not Chevalley.
They called them sporadic and filed them away.

Are Mathieu groups creatures of heaven or hell?
Zvonimir Janko determined to tell.
He found out what nobody wanted to know:
The masters had missed 1 7 5 5 6 0.

The floodgates were open! New groups were the rage!
(And twelve or more sprouted, to greet the new age.)
By Janko and Conway and Fischer and Held,
McLaughlin, Suzuki, and Higman, and Sims.

No doubt you noted the last lines don't rhyme.
Well, that is, quite simply, a sign of the time.
There's chaos, not order, among simple groups,
And maybe we'd better go back to the loops.

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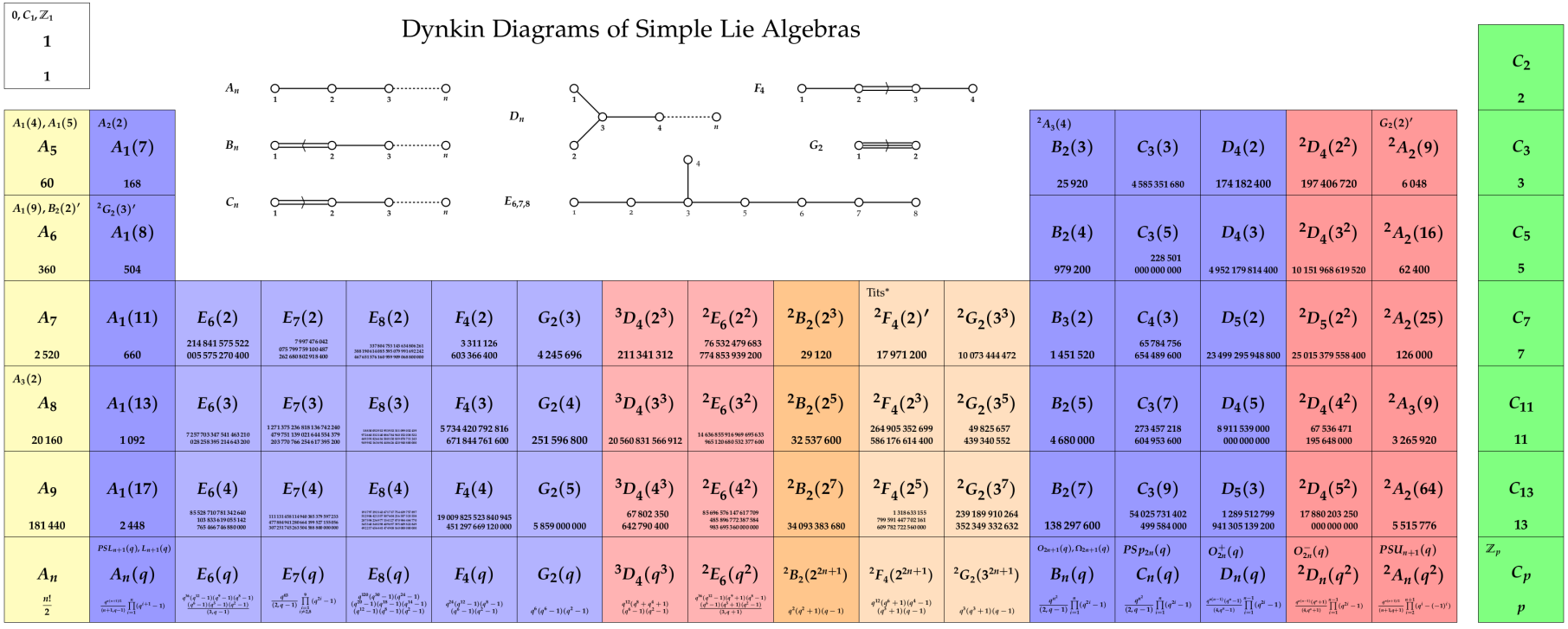
**Theorem** (The Classification of Finite Simple Groups)

Any finite simple group is isomorphic to one of the following

- a cyclic group of prime order
- an alternating group of degree at least five
- a group of Lie type
- one of 26 sporadic groups

# The Periodic Table Of Finite Simple Groups




https://irandrus.files.wordpress.com/2012/06/periodic-table-of-groups.pdf

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A theorem of Menezes, Quick and Roney-Dougal (2013)

**Theorem** For simple groups $G$, $P(G) \geq \frac{53}{90}$.

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Consider the graph whose vertices are the non-identity elements of $G$ such that two vertices $g$ and $h$ are adjacent if and only if $\langle g, h \rangle = G$.

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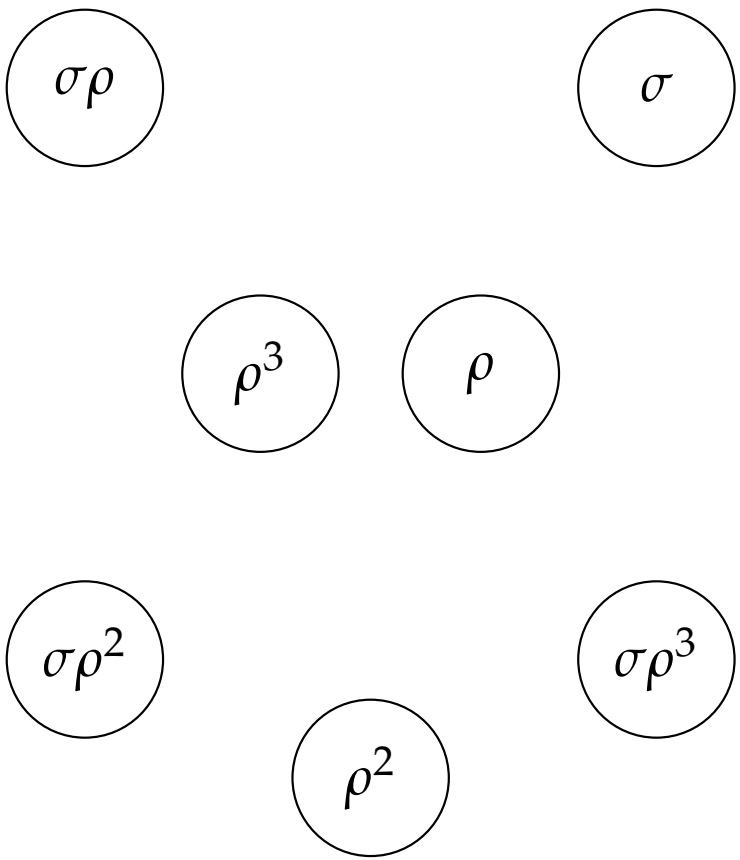

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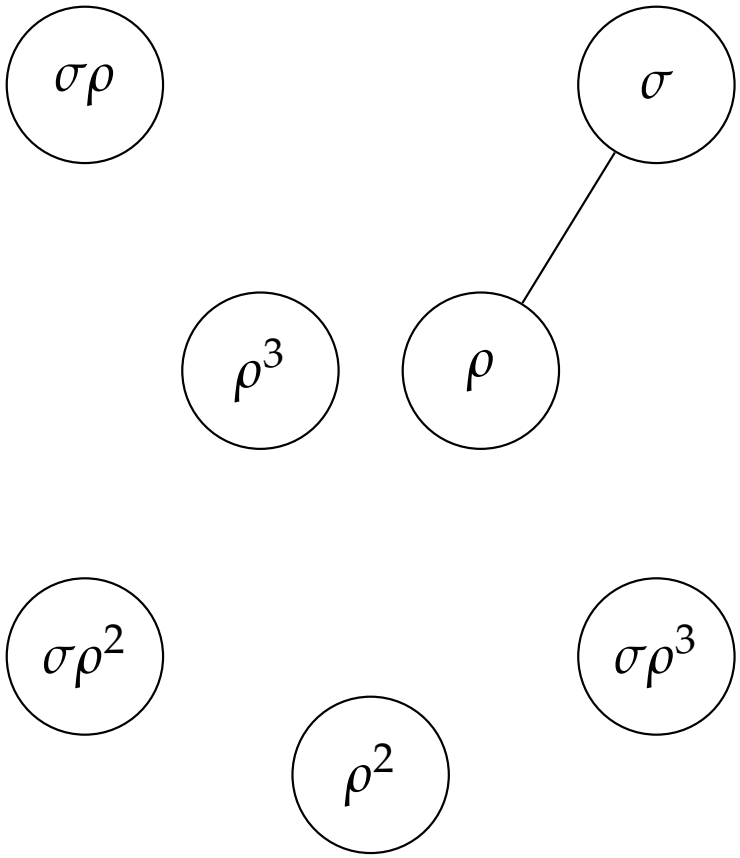

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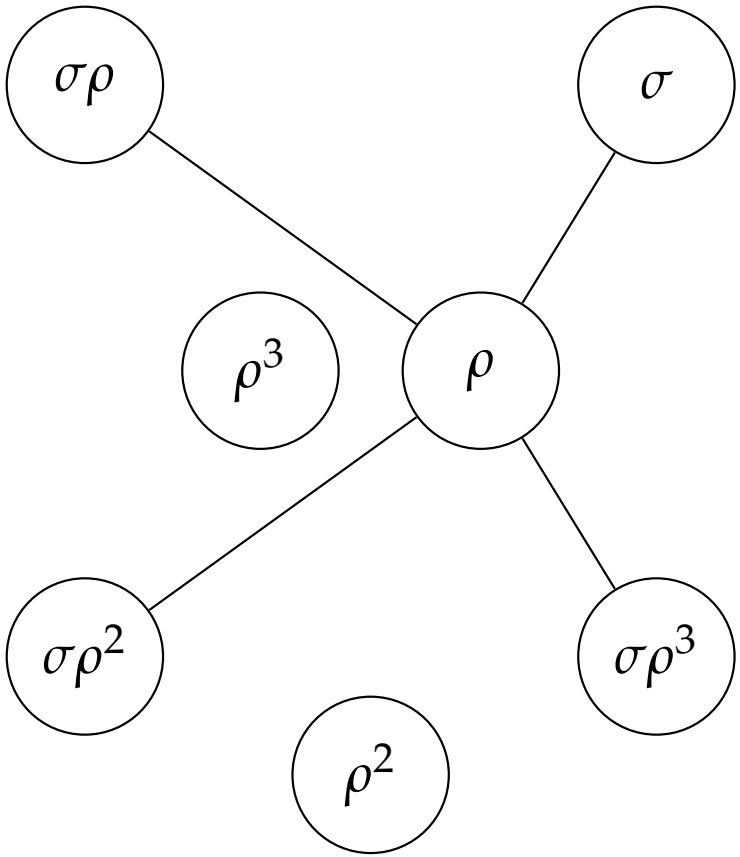

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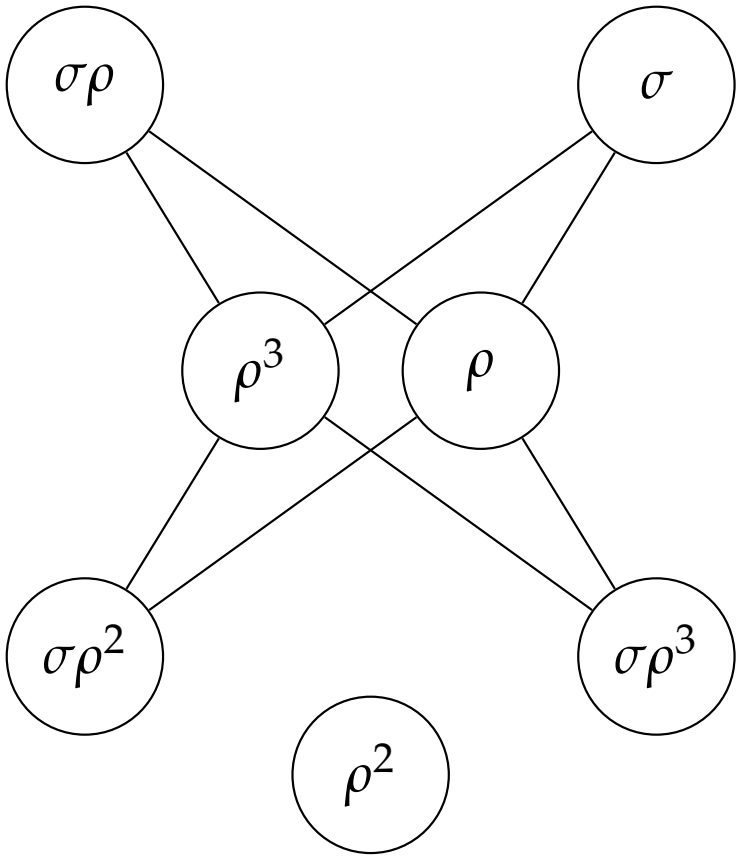

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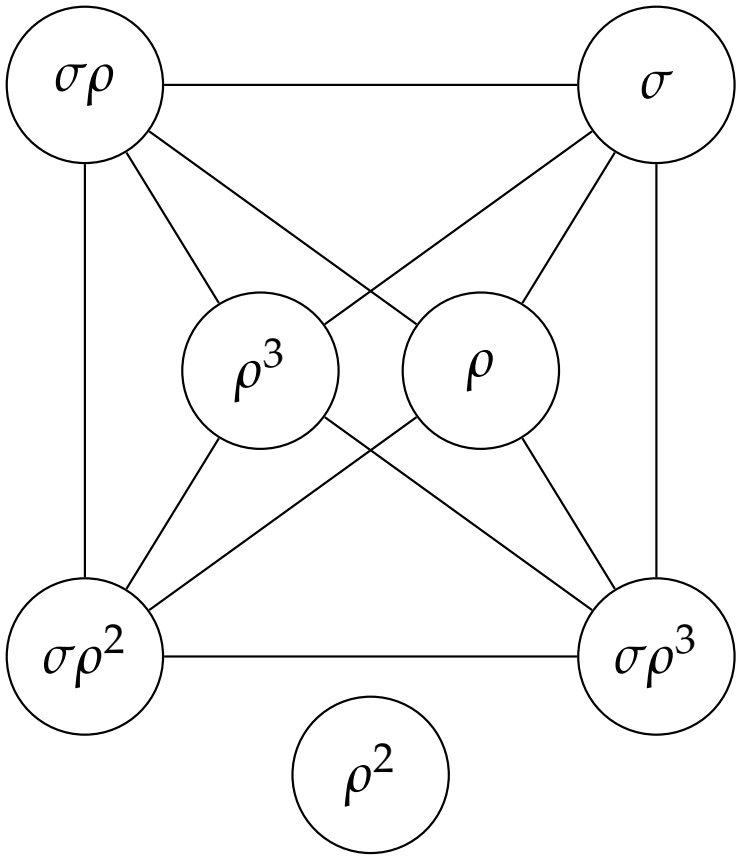

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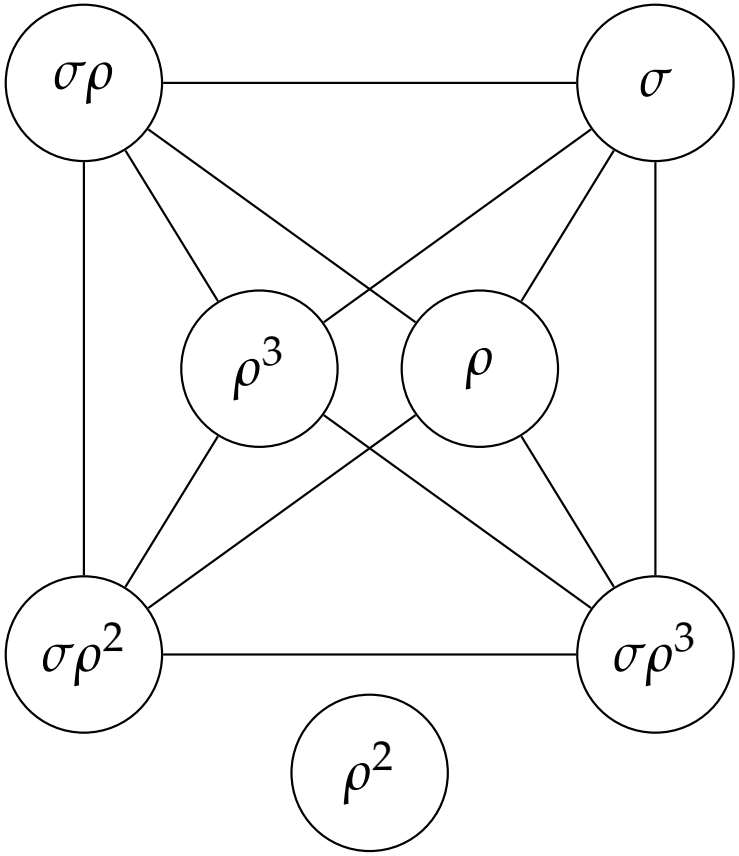
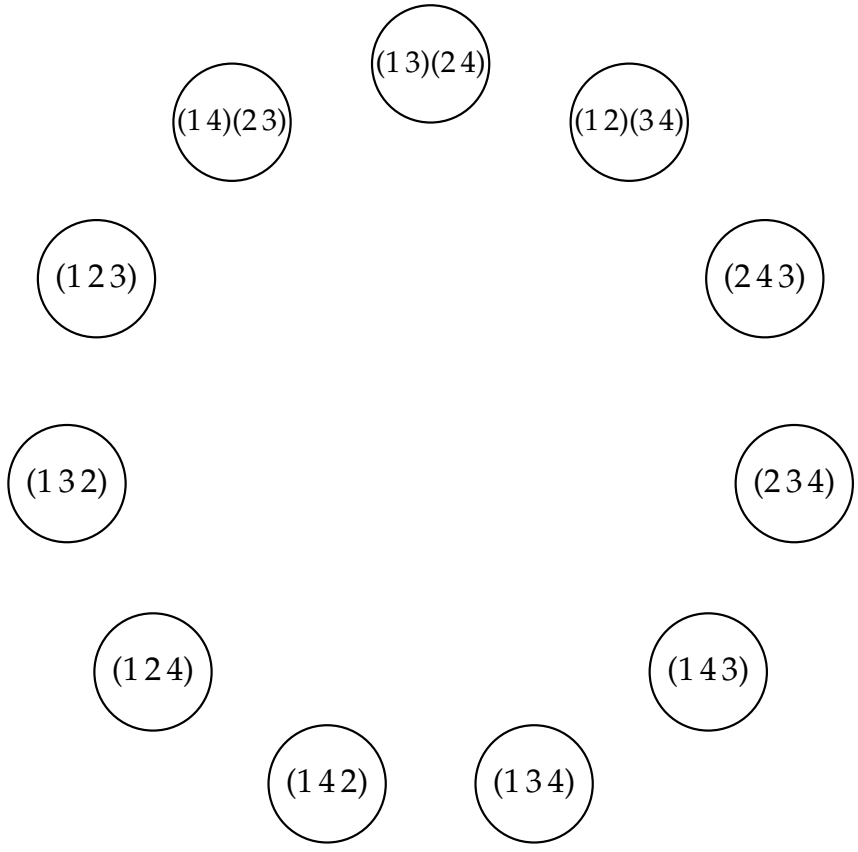



## Alternating group $A_4$



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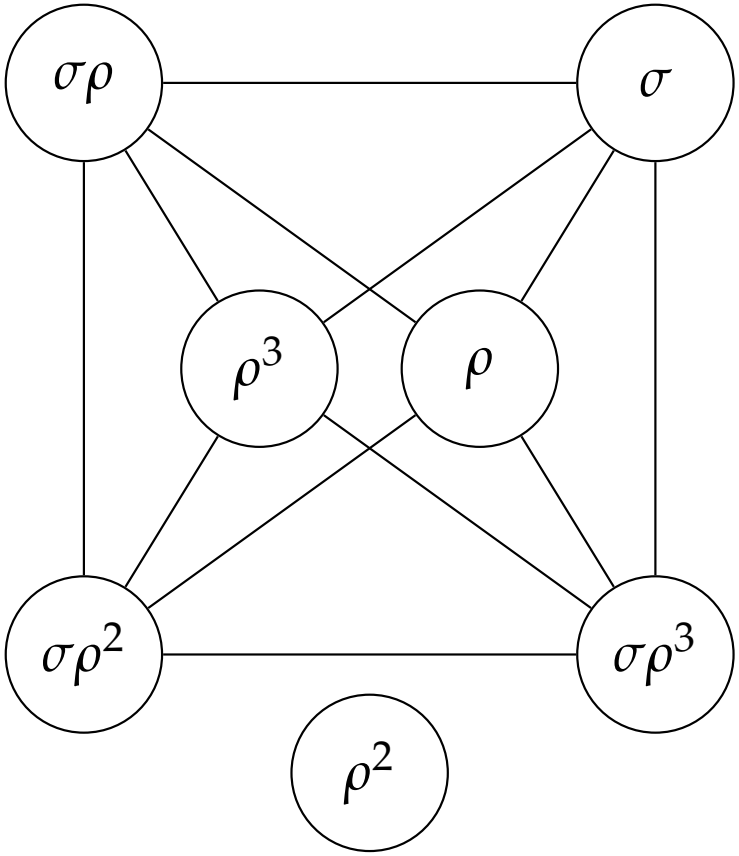


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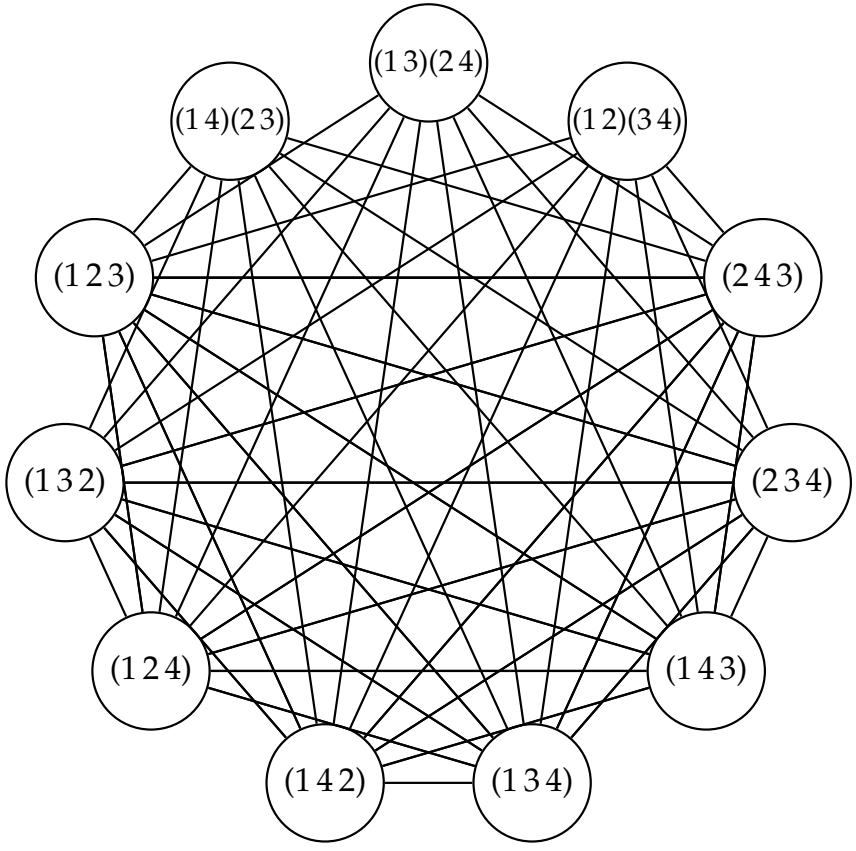

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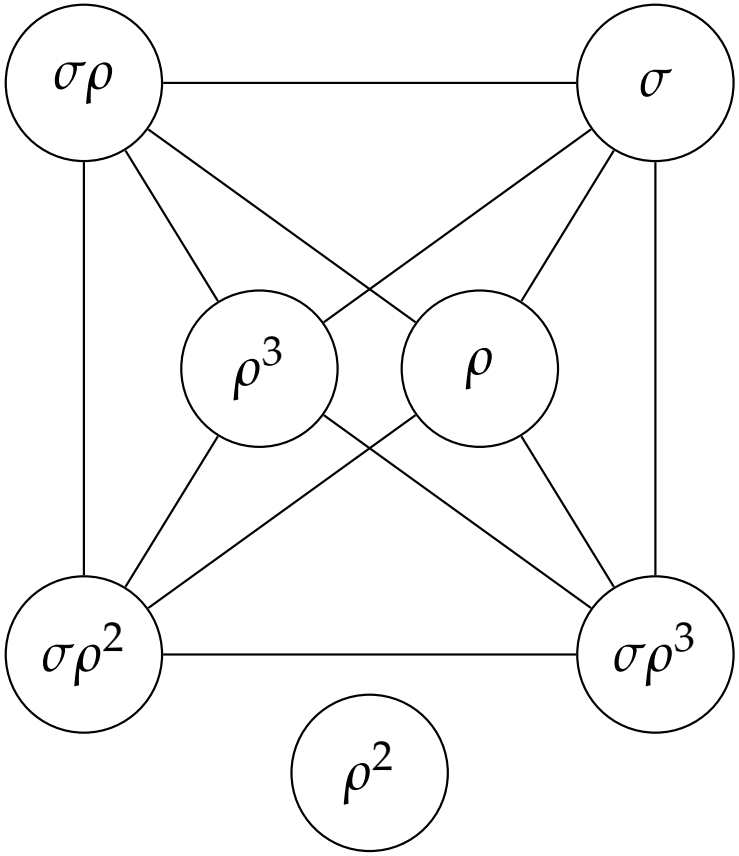


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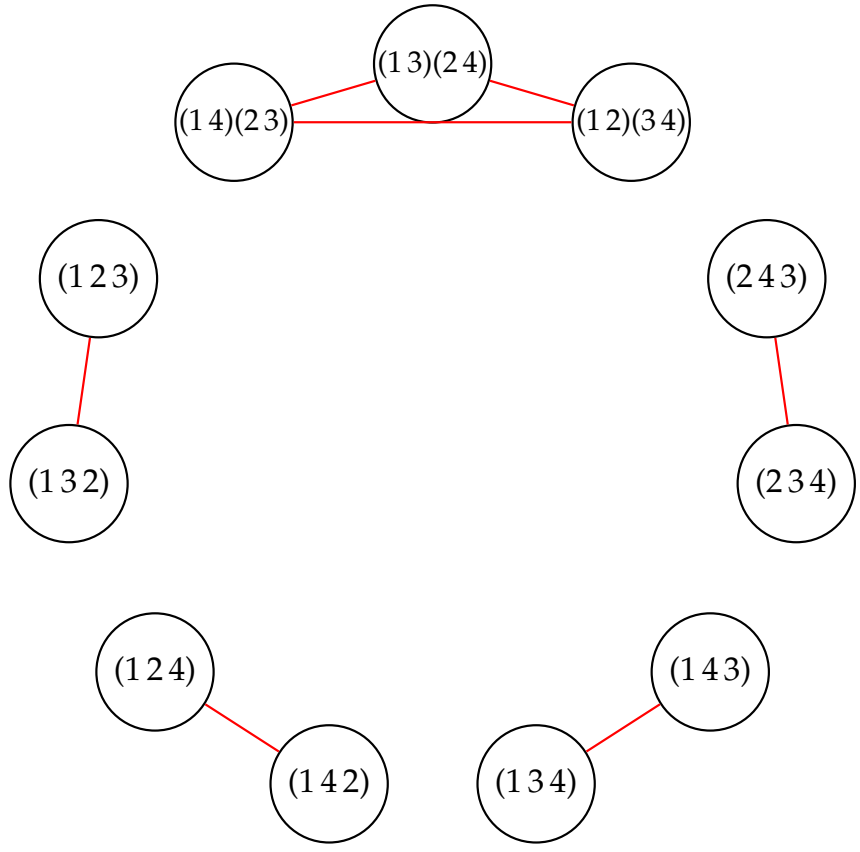

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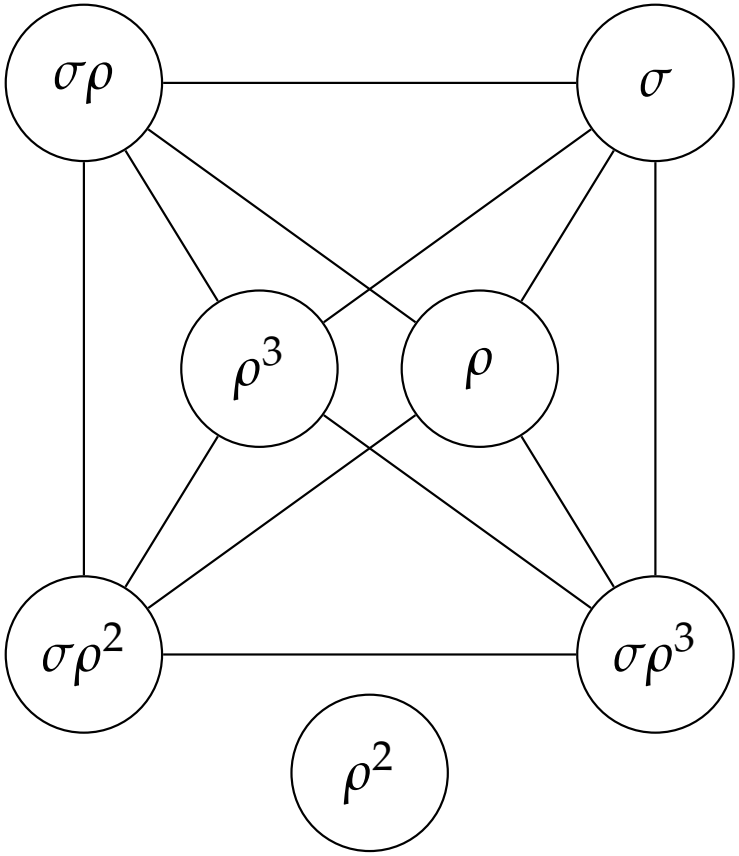


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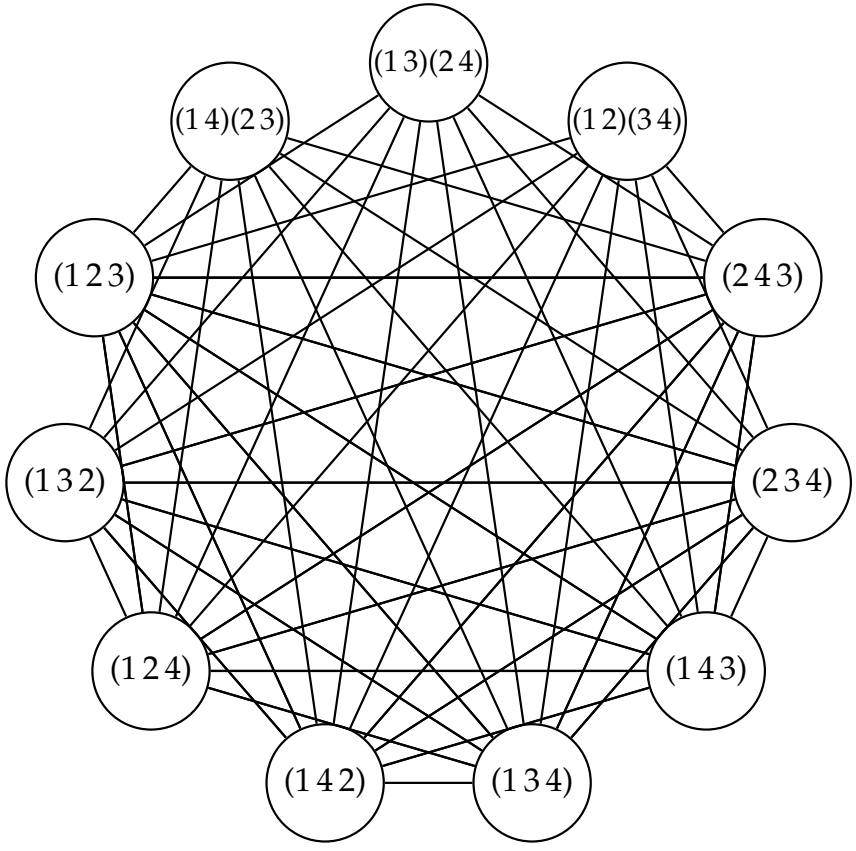

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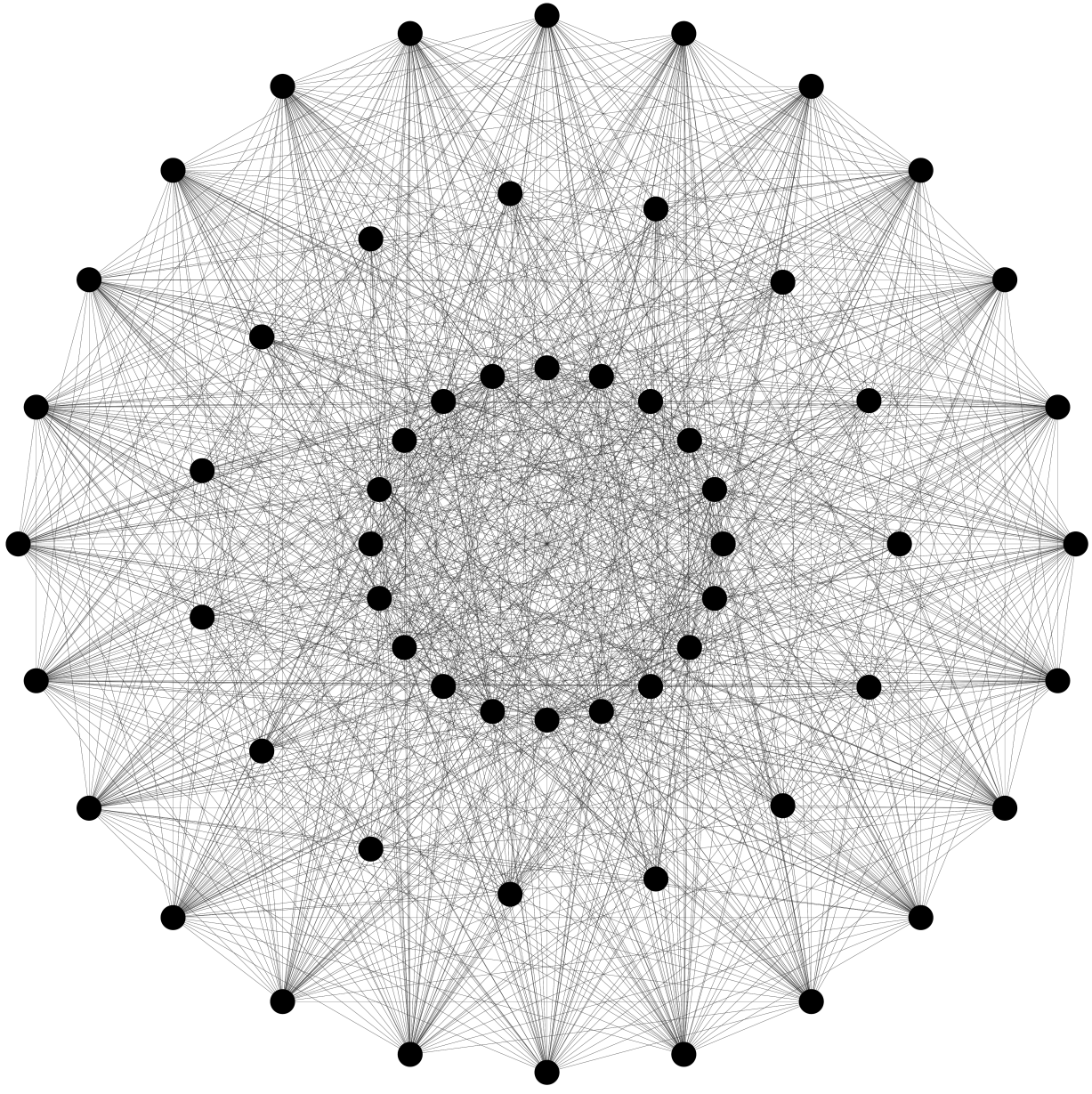

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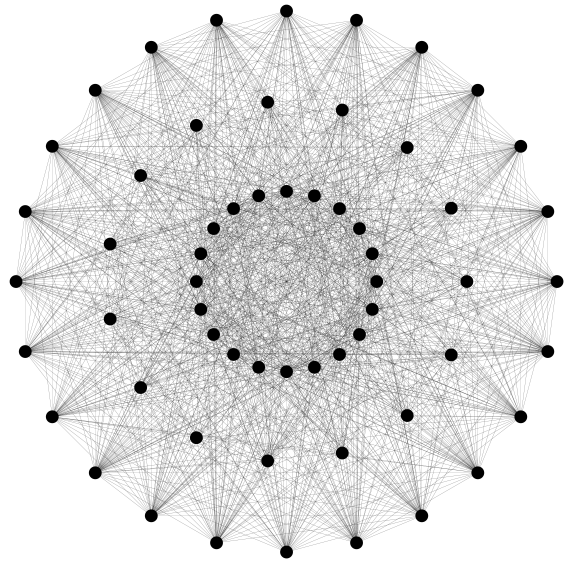
**Conjecture** For finite groups, every non-identity element is contained in a generating pair if and only if every proper quotient is cyclic.

# Group Generation Through the Generations

Scott Harper

University of Bristol



St Andrews University Mathematics Society

3rd March 2017