Errata

original publication

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1. (Error) p.323 (and repeated on p.341): There is a fifth exception to the bound $\mu(G) \leqslant 3$ for simple groups G. This is an error in [11, Theorem 5], which, in turn, is due to an error in [24, p.767]. The additional exception is $G = \mathrm{P}\Omega_8^+(5)$ which satisfies $\mu(G) = 5$.

   Thanks to Saul Freedman for pointing out this error.

2. (Typo) p.325, Proof of Lemma 2.2: "$(g, s\sigma_1) = (t\sigma_2, h)^a$" should be "$(g, s\sigma_1)^a = (t\sigma_2, h)$". (What is written is true, but it does not lead to what follows.)

3. (Error) p.327, Proof of Theorem 2.8: The technical details in the paragraph leading up to equation (2.2) are a little muddled, but the conclusion in (2.2) is correct. The two sentences "For $s, x \in X$ we can $\ldots = (s\sigma_1, t\sigma_2)^x$." should be replaced with the following.

   "Let $x \in X$ and write $(s\sigma_1, t\sigma_2)^x = (s'\sigma_1, t'\sigma_2)$. A valid choice for $(t')_2$ is $x^{-1}t_2$ since
   $$[(x^{-1}t_2)^{-1}, \sigma_2^{-1}] = (t_2\sigma_2 t_2^{-1})^x \sigma_2^{-1} = ([t_2^{-1}, \sigma_2^{-1}]\sigma_2)^x \sigma_2^{-1} = (t\sigma_2)^x \sigma_2^{-1} = t'.$$
   Therefore,
   $$(Y^{\circ}_{(t'\sigma_2)}(s'\widetilde{\sigma}_1))^{(t')_2} = (Y^{\circ}_{(t\sigma_2)^x}(s\widetilde{\sigma}_1)^x)^{x^{-1}t_2} = (Y^{\circ}_{t\sigma_2} s\widetilde{\sigma}_1)^{t_2}.$$
   In particular, we have $n_{(t',s')} = n_{(t,s)}$."

4. (Typo) p.327, Proof of Theorem 2.8: In the sentence immediately following equation (2.2), "$[s\sigma_1, t\sigma_2]$" should be "$[s\sigma_1, t\sigma_2] = 1$".

5. (Typo/Improve) p.329, Example 2.9: "$g = \varphi^i$" should have been introduced after φ was defined. Additionally, for consistency, "$g = \varphi^i$" should be "$g = \widetilde{\varphi}^i$", and all instances of φ in part (ii) should also be $\widetilde{\varphi}$.

6. (Typo) p.329, Proposition 2.11: The usual convention that $q = p^f$ is missing. Alternatively, change "$p = 2$" to "$q = 2^f$".

7. (Typo) p.330, Proof of Lemma 2.12: The Shintani maps are defined on conjugacy classes, not elements, so there are two typos:
   (a) In the first display, the final term "$\pi(F(x\sigma_2))$" should be "$\pi(F(x\sigma_2^{X_{\sigma_1}}))$".
   (b) In the second display, the final term "$\pi(F^{-1}(y\sigma_1))$" should be "$\pi(F^{-1}(y\sigma_1^{X_{\sigma_2}}))$".

8. (Improve) p.330, Example 2.15: The proof of [9, Lemma 5.3] that is alluded to here has a problem with it, so there is all the more reason to take this alternative approach. We discuss this at length in point 17 below.

9. (Error/Typo) p.331, Lemma 2.16(ii): "$F(x\widetilde{\sigma}_2)$" should be "$F(x\widetilde{\sigma}_2)^d$". (This error is not repeated in the proof.)

10. (Typo) p.331, Proof of Lemma 2.16: There are several typos:

    (a) In the third sentence, "$a^{-1}a^{\sigma_1^{-1}}\widetilde{\sigma}_2$" should be "$a^{-1}a^{\sigma_1^{-1}}\widetilde{\sigma}_1$".
    (b) In the fifth sentence, "$|F(x\widetilde{\sigma}_2)^d|$" should be "$e|F(x\widetilde{\sigma})^d|$".
    (c) In the sixth sentence, "$de|F(x\widetilde{\sigma})^d|$" should be "$de|F(x\widetilde{\sigma}_2)^d|$".

    Thanks to Daniele Nemmi for spotting these typos.

11. (Extra) p.331, Lemma 2.18: This is generalised in [Section 2.3, S. Harper, *Totally deranged elements of almost simple groups and invariable generating sets*, J. Algebra **109** (2024), e12935, `https://doi.org/10.1112/jlms.12935`]. This generalisation answers the questions raised in Example 4.6.

12. (Typo) p.331, Proof of Lemma 2.18: In the first sentence, "$(x\widetilde{\sigma})^g$" should be "$(x\widetilde{\sigma})^{g^{-1}}$".

13. (Typo) p.333: "$G_2(2) \cong \mathrm{P\Gamma L}_3(3)$" should be "$G_2(2) \cong \mathrm{P\Gamma U}_3(3)$". (This observation is never used in the paper.)

14. (Error) p.336, Table 2, rows 1, 2 and 5: The conditions on j are confused (and k has not even been defined). It is easier to just give a more general condition on j and allow sets S to arise multiple times. In rows 1 and 2, the description of S could be improved, and in row 5, the description of S needs to be corrected. Accordingly, the final two columns of these three rows should be changed to the following.

"row 1	$\{\delta^j, \delta^{-j}, \dots, \delta^{jp^{i-1}}, \delta^{-jp^{i-1}}\}\langle\delta^{p^i-1}\rangle$	$0 \leqslant j \leqslant p^i - 1$
row 2	$\{\delta^j, \delta^{-j}, \dots, \delta^{jp^{i-1}}, \delta^{-jp^{i-1}}\}\langle\delta^{p^i+1}\rangle$	$0 \leqslant j \leqslant p^i + 1$
row 5	$\{\delta^j, \delta^{-j}, \dots, \delta^{jp^{i-1}}, \delta^{-jp^{i-1}}\}\langle\delta^{p^i+1}\rangle$	$0 \leqslant j \leqslant p^i + 1$"

15. (Typo) p.336, Theorem 3.2:
$$\text{“}\ddot{F}((Txg)^{\mathrm{Outdiag}(T)}) = (T_0x_0g_0)^{\mathrm{Outdiag}(T_0)}\text{”}$$
    should be
$$\text{“}\ddot{F}((Txg)^{\mathrm{Out}(T)}) = (T_0x_0g_0)^{\mathrm{Out}(T_0)}\text{”}.$$
    (This error is not repeated in the proof. The corrected version is the only thing that could have been meant based on the immediately preceding definition.)

16. (Improve) p.338, Proof of Theorem 3.2, case $T = \mathrm{PSL}_n^{\varepsilon}(q)$: It would have been useful to note that for a group G and a positive integer d, we write $\frac{1}{d}G$ for the unique index d subgroup of G if it exists.

17. (Error) p.338, Proof of Theorem 3.2, case $T = \mathrm{PSL}_n^{\varepsilon}(q)$ and $g_0 = 1$: This is the one case in the proof where Corollary 2.14 was not applied (though a special case was established using Corollary 2.14 in Example 2.15). Corollary 2.14 does apply to this case, but the argument is more technical since $\mathrm{Aut}(T)$ is more complicated (see the proof below). Because of this, and because the result had been known in this case for a while, the original proof from [9] (and a variation in [29]) was sketched instead. However, unfortunately, it is not clear that the proofs of [9, Lemmas 4.2 and 5.3] and [29, Lemma 6.4.2] work, owing to the fact that they overlook a subtlety.

    For the sake of exposition, when explaining this oversight, we will focus on the case where $\varepsilon = \varepsilon_0 = +$ (this corresponds to [9, Lemma 4.2]), but the same problem occurs in all cases. We will also exploit the harmless abuse of notation of identifying $F(xg)$ with a representative of the $\mathrm{PGL}_n(q_0)$-class it defines. The critical subtlety concerns the implicit deduction

$$F(xg) = a^{-1}(xg)^{-e}a \text{ for some } a \in \mathrm{PGL}_n(\overline{\mathbb{F}}_p) \implies \det(F(xg)) = \det((xg)^{-e}).$$

(In [9], the Shintani map is defined slightly differently, so we have $F(xg) = a^{-1}(xg)^{e}a$, but this difference does not affect the main point of this discussion). This deduction is not justified. For example, if p is odd, then $\mathrm{PGL}_2(p)$ has two conjugacy classes of involutions, referred to as t_1 a t_1', exactly one of which is contained in $\mathrm{PSL}_2(q_0)$, but t_1 and t_1' are conjugate as elements of $\mathrm{PGL}_2(q_0^2)$ (see [22, Table 4.5.1], for example, for all these claims). Therefore, when p is odd and $n = 2$, one could have $F(g\sigma) \in \mathrm{PSL}_2(q_0)$ and $(xg)^{-e} \in \mathrm{PGL}_2(q_0) \setminus \mathrm{PSL}_2(q_0)$, while still having the relationship $a^{-1}(xg)^{-e}a$ for some $a \in \mathrm{PGL}_n(\overline{\mathbb{F}}_p)$. For matrices $A, B \in \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, of course $\det(A^{-1}BA) = \det(B)$. The problem is that $F(xg) \in \mathrm{PGL}_n(q_0)$, so $\det(F(xg))$ is only defined up to multiplication by an nth power in $\mathbb{F}_{q_0}^{\times}$, and $a \in \mathrm{PGL}_n(\overline{\mathbb{F}}_p)$, so $\det(a)$ is only defined up to multiplication by an nth power in $\overline{\mathbb{F}}_p^{\times}$, but all elements of $\overline{\mathbb{F}}_p^{\times}$ are nth powers.

In light of the above, below we give a proof of the result in this case using Corollary 2.14. (To be inkeeping with the spirit of the paper, this is the proof that should really have been given all along.) This proof should replace the two sentences "Let $x \in \mathrm{PGL}_n^{\varepsilon}(q)$. Then ... and [29, Lemma 6.4.2])."

"Let (X, σ_1, σ_2) be the Shintani setup of (T, g). Note that $X = \mathrm{PSL}_n$, and, since $g_0 = 1$, there is a Steinberg endomorphism σ of X such that $\sigma_1 = \sigma^e$ and $\sigma_2 = \sigma$ where $q = q_0^e$. Write $Z = Z(\mathrm{GL}_n)$. Let d be a divisor of $(n, q_0 - \varepsilon_0)$. Write

$$Z_d = \{z^d \mid z \in Z\} \quad \text{and} \quad X_d = \mathrm{SL}_n/(\mathrm{SL}_n \cap Z_d),$$

and observe that

$$(X_d)_{\sigma^e} = d.\mathrm{PSL}_n^{\varepsilon}(q).((n, q-\varepsilon)/d) \quad \text{and} \quad (X_d)_{\sigma} = d.\mathrm{PSL}_n^{\varepsilon_0}(q_0).((n, q_0-\varepsilon_0)/d).$$

Note that, $X_1 = X$ and $Z_1 = Z$. Let $\pi_d \colon X_d \to X$ be the isogeny with kernel Z/Z_d, so

$$\pi_d((X_d)_{\sigma^e}) = \tfrac{1}{d}\mathrm{PGL}_n^{\varepsilon}(q) \quad \text{and} \quad \pi_d((X_d)_{\sigma}) = \tfrac{1}{d}\mathrm{PGL}_n^{\varepsilon_0}(q_0).$$

By Corollary 2.14, the Shintani map

$$F \colon \{(x\sigma)^{\mathrm{PGL}_n^{\varepsilon}(q)} \mid x \in \mathrm{PGL}_n^{\varepsilon}(q)\} \to \{x_0^{\mathrm{PGL}_n^{\varepsilon_0}(q_0)} \mid x_0 \in \mathrm{PGL}_n^{\varepsilon_0}(q_0)\}$$

restricts to

$$\{(x\sigma)^{\mathrm{PGL}_n^{\varepsilon}(q)} \mid x \in \tfrac{1}{d}\mathrm{PGL}_n^{\varepsilon}(q)\} \to \{x_0^{\mathrm{PGL}_n^{\varepsilon_0}(q_0)} \mid x_0 \in \tfrac{1}{d}\mathrm{PGL}_n^{\varepsilon_0}(q_0)\}.$$

Since this holds for all divisors d of $(n, q_0 - \varepsilon_0)$, we deduce that

$$\ddot{F}((\Delta(d)\ddot{\sigma})^{\mathrm{Out}(T)}) = \Delta_0(d)^{\mathrm{Out}(T_0)},$$

where we write

$$\Delta(d) = \{h \in \langle \ddot{\delta} \rangle \mid |h| = |\ddot{\delta}^d|\} \quad \text{and} \quad \Delta_0(d) = \{h \in \langle \ddot{\delta}_0 \rangle \mid |h| = |\ddot{\delta}_0^d|\}.$$

Inspecting Table 2, we see that if $\ddot{x}_0 \in \langle \ddot{\delta}_0 \rangle$, then $\ddot{x}_0^{\mathrm{Out}(T_0)} = \ddot{x}_0^{\langle \ddot{\varphi}, \ddot{\gamma} \rangle}$, and if $\ddot{x} \in \langle \ddot{\delta} \rangle$, then $(\ddot{x}\ddot{\sigma})^{\mathrm{Out}(T)} = (\langle \ddot{\delta}^{q_0 - \varepsilon_0} \rangle \ddot{x}\ddot{\sigma})^{\langle \ddot{\varphi}, \ddot{\gamma} \rangle}$. Therefore, it suffices to show that F commutes with the action of $\langle \phi, \gamma \rangle$ on $\mathrm{Inndiag}(T)\sigma$ and $\mathrm{Inndiag}(T_0)$. To this end, let $\psi \in \langle \phi, \gamma \rangle$. Then ψ is an automorphism of X as an abstract group, and $[\psi, \sigma] = 1$. Therefore,

$$F((x\sigma)^{X_{\sigma^e}}) = y^{X_\sigma} \iff (x\sigma, \sigma^e)^X = (\sigma, y\sigma^e)^X \iff ((x\sigma)^\psi, (\sigma^e)^\psi)^{X\psi} = (\sigma^\psi, (y\sigma^e)^\psi)^{X\psi}$$
$$\iff ((x\sigma)^\psi, \sigma^e)^X = (\sigma, y^\psi \sigma^e)^X \iff F(((x\sigma)^\psi)^{X_{\sigma^e}}) = (y^\psi)^{X_\sigma},$$

as required. ”

Thanks to Daniele Nemmi for pointing out this oversight.

18. (Typo) p.338, Proof of Theorem 3.2, case $T = \mathrm{PSL}_n^\varepsilon(q)$ and $g_0 = \gamma$: Both instances of “$\delta\langle\ddot{\delta}^2\rangle\ddot{g}$” should be “$\ddot{\delta}\langle\ddot{\delta}^2\rangle\ddot{g}$”, and both instances of “$\delta_0\langle\ddot{\delta}_0^2\rangle\ddot{g}_0$” should be “$\ddot{\delta}_0\langle\ddot{\delta}_0^2\rangle\ddot{g}_0$”.

19. (Error) pp.339–340, Remark 3.5 and Theorem 3.6: In part (i), the generic $\mathcal{S}$-type subgroups should be included in the description of (I) and not (IV). Accordingly, Theorem 3.6 is not wrong, but the categories (I)–(IV) do not correspond exactly with the categories (I)–(IV) in Theorem 3.4. In part (ii), the subgroups in [34, Theorem 1(c) and (d)] should be included in the description of (I).

20. (Improve) p.341: The sentence introducing $\mu(G)$ and $\mu^*(G)$ should conclude with “, so $\mu^*(G)$ is only defined when G^* is nonempty.”

21. (Error) pp.341–342, Proposition 4.1: The following two errors are present in part (iii):
    (a) For $G = \mathrm{M}_{12}.2$, Table 4 correctly reports that $\mu^*(G) = 3$ witnessed by `12B`, but we actually have $\mu(G) = 2$ witnessed by `10A`.
    (b) The case $G = \mathrm{Aut}(A_6)$, has been overlooked entirely. Here $\mu^*(G)$ is not defined since G^* is empty, and $\mu(G) = 2$ witnessed by `6AB`.

    Accordingly, the statement of Proposition 4.1 should be corrected to:

    “**Proposition 4.1.** Let G be an almost simple group whose socle is alternating, sporadic or ${}^2F_4(2)'$. If $G = \mathrm{Aut}(A_6)$, then $\mu^*(G)$ is not defined, and $\mu(G) = 2$ witnessed by an element of order 6. Otherwise, $\mu(G) \leqslant \mu^*(G) \leqslant 3$, and, moreover, if G is not simple, then the following hold.

    (i) ...
    (ii) ...
    (iii) Otherwise, G appears in Table 4 with $\mu^*(G) = k$ witnessed by s, and $\mu(G) = \mu^*(G)$ unless $G = \mathrm{M}_{12}.2$, in which case $\mu(G) = 2$ witnessed by `10A`.”,

    and the caption of Table 4 should be corrected to:

    “**Table 4**
    The data for Proposition 4.1(iii), where $\mu^*(G) = |\mathcal{M}(G, s)| = k$.”

22. (Improve) p.341–342, Proof of Proposition 4.1(iii): The code available in [12] was intended to be applied to simple groups, and, as such, does not behave as expected for almost simple groups. (This is what led to the error in point 21(a).) The relevant GAP and MAGMA code is now available at [S. Harper, *maximal-overgroups*, `https://github.com/harper-scott/maximal-overgroups`, 2025].

23. (Improve) p.343, Proposition 4.5: This statement passes one of Serre's tests on how to write mathematics badly. In the final sentence, following “then”, one should insert “s can be chosen such that”.

24. (Improve) p.346, Corollary 5.5: The quantifiers in this statement are unclear. The final sentence would be clearer if it was replaced with the following. "Assume that there exists $1 < d < n/2$ such that every element of the coset Tg stabilises a k-space for some $1 \leqslant k < d$. Then $s(\langle T, g\rangle) \leqslant |K|^{4d}$."

25. (Improve) p.346, Corollary 5.5: It might have been worth giving the proof of this corollary since there is one subtlety. We give the proof below.

    "*Proof.* Fix a subset $S = \{x_1, \ldots, x_s\} \subseteq T$ satisfying the conditions in the statement of Proposition 5.4, and fix an element $x_{s+1} \in T$. Note that $s + 1 \leqslant |K|^{4d}$. Let $y \in \langle T, g\rangle$. We claim that there exists $1 \leqslant i \leqslant s + 1$ such that $\langle x_i, y\rangle \neq \langle T, g\rangle$. First assume that $\langle T, y\rangle$ is a proper subgroup of $\langle T, g\rangle$. Then $\langle x_{s+1}, y\rangle \neq \langle T, g\rangle$ since $\langle T, x_{s+1}, y\rangle = \langle T, y\rangle$ is a proper subgroup of $\langle T, g\rangle$. Now assume that $\langle T, y\rangle = \langle T, g\rangle$. Then a power of y is contained in Tg, so, by hypothesis, y stabilises a k-space U for some $1 \leqslant k < d$. Hence, by construction, there exists $1 \leqslant i \leqslant s$ such that x_i stabilises U, so $\langle x_i, y\rangle \neq \langle T, g\rangle$ since $\langle x_i, y\rangle$ stabilises U. This proves the claim, and we conclude that $s(\langle T, g\rangle) < s + 1 \leqslant |K|^{4d}$, as required. $\square$"

26. (Error/Typo) p.347, Table 6: "$\mathrm{PSL}_{2m+1}(q)$" should be "$\mathrm{PSL}_{2m}(q)$", and "$\mathrm{PSU}_{2m+1}(q)$" should be "$\mathrm{PSU}_{2m}(q)$". (The error is not repeated in the proof of Theorem 5.8, see Case 3 on p.349.)

27. (Improve) p.348, Proof of Theorem 5.8, Case 2: It would be useful to point out that for a positive integer N, we write N_2 for the largest power of 2 dividing N.

28. (Error) p.351, Proof of Theorem 1: In the first paragraph, "$T = \mathrm{PSp}_4(q)$ with a graph-field automorphism g" should be "$T = \mathrm{PSp}_4(q)$ with a graph-field automorphism g or $T = \mathrm{P}\Omega_8^+(q)$ with a triality graph or graph-field automorphism g".

29. (Improve) p.353, Proof of Theorem 1, Case 2: As in point 27, in the first paragraph, for a positive integer N, we write N_2 for the largest power of 2 dividing N.

Thanks to David Craven for the helpful categorisation of issues into (Extra), (Improve), (Typo) and (Error). I do not employ these terms as formally as he does, but the errata of his papers could be consulted to see how he defines these categories.

Please get in touch with me at `s.harper.3@bham.ac.uk` if you discover any other issues with this paper.

SH, 16 June 2025