Linearity: It adds up ...

... but the converse doesn't hold.

Matt McDevitt

Scott Harper

SUMS Lunchtime Talk

21st April 2015

Linearity: It adds up ...

... but the converse doesn't hold.

Matt McDevitt

Scott Harper

SUMS Lunchtime Talk

21st April 2015

Non-Constructive Proofs

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ We are done! $\sqrt{2}^{\sqrt{2}}$ is rational.

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ We are done! $\sqrt{2}^{\sqrt{2}}$ is rational.

Case 2: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ We are done! $\sqrt{2}^{\sqrt{2}}$ is rational.

Case 2: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ We are done! $\sqrt{2}^{\sqrt{2}}$ is rational.

Case 2: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}}$

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ We are done! $\sqrt{2}^{\sqrt{2}}$ is rational.

Case 2: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2}$

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ We are done! $\sqrt{2}^{\sqrt{2}}$ is rational.

Case 2: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2 \in \mathbb{Q}$.

Non-Constructive Proofs

Theorem

There exist irrational numbers a and b such that a^b is rational.

Proof

Consider $\sqrt{2}^{\sqrt{2}}$.

Case 1: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ We are done! $\sqrt{2}^{\sqrt{2}}$ is rational.

Case 2: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2 \in \mathbb{Q}$. ■

Real Numbers and Rational Numbers

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$

(ii) $x + 0 = 0 + x = x$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$

(ii) $x + 0 = 0 + x = x$

(iii) $x + (-x) = 0 = (-x) + x$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x+y)+z = x+(y+z)$

(ii) $x+0 = 0+x = x$

(iii) $x+(-x) = 0 = (-x)+x$

(iv) $x+y = y+x$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$

(ii) $x + 0 = 0 + x = x$

(iii) $x + (-x) = 0 = (-x) + x$

(iv) $x + y = y + x$

(v) $(qr)x = q(rx)$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$

(ii) $x + 0 = 0 + x = x$

(iii) $x + (-x) = 0 = (-x) + x$

(iv) $x + y = y + x$

(v) $(qr)x = q(rx)$

(vi) $1x = x$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$

(ii) $x + 0 = 0 + x = x$

(iii) $x + (-x) = 0 = (-x) + x$

(iv) $x + y = y + x$

(v) $(qr)x = q(rx)$

(vi) $1x = x$

(vii) $q(x + y) = qx + qr$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$
(ii) $x + 0 = 0 + x = x$
(iii) $x + (-x) = 0 = (-x) + x$
(iv) $x + y = y + x$
(v) $(qr)x = q(rx)$
(vi) $1x = x$
(vii) $q(x + y) = qx + qr$
(viii) $(q + r)x = qx + rx.$

Real Numbers and Rational Numbers

Let $x, y, z \in \mathbb{R}$ and let $q, r \in \mathbb{Q}$.

(i) $(x + y) + z = x + (y + z)$

(ii) $x + 0 = 0 + x = x$

(iii) $x + (-x) = 0 = (-x) + x$

(iv) $x + y = y + x$

(v) $(qr)x = q(rx)$

(vi) $1x = x$

(vii) $q(x + y) = qx + qr$

(viii) $(q + r)x = qx + rx.$

Theorem

$\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Bases

Let V be a vector space over a field F.

Bases

Let V be a vector space over a field F.

Definition (Familiar)

A set $\mathcal{B} = \{x_1, \ldots, x_n\} \subseteq V$ is a basis for V if:

Bases

Let V be a vector space over a field F.

Definition (Familiar)

A set $\mathcal{B} = \{x_1, \ldots, x_n\} \subseteq V$ is a basis for V if:

Spanning For every $x \in V$ there exist $q_1, \ldots, q_n \in F$ such that $x = q_1 x_1 + \cdots + q_n x_n$.

Bases

Let V be a vector space over a field F.

Definition (Familiar)

A set $\mathcal{B} = \{x_1, \ldots, x_n\} \subseteq V$ is a basis for V if:

Spanning For every $x \in V$ there exist $q_1, \ldots, q_n \in F$ such that
$$x = q_1 x_1 + \cdots + q_n x_n.$$

Linear Independence If $q_1 x_1 + \cdots + q_n x_n = 0$ then $q_1 = \cdots = q_n = 0$.

Bases

Let V be a vector space over a field F.

Definition (Familiar)

A set $\mathcal{B} = \{x_1, \ldots, x_n\} \subseteq V$ is a basis for V if:

Spanning For every $x \in V$ there exist $q_1, \ldots, q_n \in F$ such that
$$x = q_1 x_1 + \cdots + q_n x_n.$$

Linear Independence If $q_1 x_1 + \cdots + q_n x_n = 0$ then $q_1 = \cdots = q_n = 0$.

Definition (New)

A set $\mathcal{B} \subseteq V$ is a basis for V if:

Let V be a vector space over a field F.

Definition (Familiar)

A set $\mathcal{B} = \{x_1, \ldots, x_n\} \subseteq V$ is a basis for V if:

Spanning For every $x \in V$ there exist $q_1, \ldots, q_n \in F$ such that $x = q_1 x_1 + \cdots + q_n x_n$.

Linear Independence If $q_1 x_1 + \cdots + q_n x_n = 0$ then $q_1 = \cdots = q_n = 0$.

Definition (New)

A set $\mathcal{B} \subseteq V$ is a basis for V if:

Spanning For each $x \in V$ there exist $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ such that $x = q_1 x_1 + \cdots + q_n x_n$,

Bases

Let V be a vector space over a field F.

Definition (Familiar)

A set $\mathcal{B} = \{x_1, \dots, x_n\} \subseteq V$ is a basis for V if:

- **Spanning** For every $x \in V$ there exist $q_1, \dots, q_n \in F$ such that $x = q_1 x_1 + \cdots + q_n x_n$.
- **Linear Independence** If $q_1 x_1 + \cdots + q_n x_n = 0$ then $q_1 = \cdots = q_n = 0$.

Definition (New)

A set $\mathcal{B} \subseteq V$ is a basis for V if:

- **Spanning** For each $x \in V$ there exist $x_1, \dots, x_n \in \mathcal{B}$ and $q_1, \dots, q_n \in F$ such that $x = q_1 x_1 + \cdots + q_n x_n$,
- **Linear Independence** For any $x_1, \dots, x_m \in \mathcal{B}$, if $q_1 x_1 + \cdots + q_m x_m = 0$ then $q_1 = \cdots = q_m = 0$.

Why do we care?

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$.

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Define $f : \mathcal{B} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1 & \text{if x=b} \\ x & \text{otherwise} \end{cases}$$

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Define $f : \mathcal{B} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1 & \text{if x=b} \\ x & \text{otherwise} \end{cases}$$

For each $x \in \mathbb{R}$ write $x = q_1 b_1 + \cdots q_n b_n$

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Define $f : \mathcal{B} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1 & \text{if x=b} \\ x & \text{otherwise} \end{cases}$$

For each $x \in \mathbb{R}$ write $x = q_1 b_1 + \cdots q_n b_n$ and define

$$f^*(x) = q_1 f(b_1) + \cdots + q_n f(b_n).$$

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Define $f : \mathcal{B} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1 & \text{if x=b} \\ x & \text{otherwise} \end{cases}$$

For each $x \in \mathbb{R}$ write $x = q_1 b_1 + \cdots q_n b_n$ and define

$$f^*(x) = q_1 f(b_1) + \cdots + q_n f(b_n).$$

Claim: f^* is non-linear.

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Define $f : \mathcal{B} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1 & \text{if x=b} \\ x & \text{otherwise} \end{cases}$$

For each $x \in \mathbb{R}$ write $x = q_1 b_1 + \cdots q_n b_n$ and define

$$f^*(x) = q_1 f(b_1) + \cdots + q_n f(b_n).$$

Claim: f^* is non-linear. f^* is additive.

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Define $f : \mathcal{B} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1 & \text{if x=b} \\ x & \text{otherwise} \end{cases}$$

For each $x \in \mathbb{R}$ write $x = q_1 b_1 + \cdots q_n b_n$ and define

$$f^*(x) = q_1 f(b_1) + \cdots + q_n f(b_n).$$

Claim: f^* is non-linear. f^* is additive.

Theorem

Every vector space has a basis.

Why do we care?

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Fix $b \in \mathcal{B}$.

Define $f : \mathcal{B} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x + 1 & \text{if x=b} \\ x & \text{otherwise} \end{cases}$$

For each $x \in \mathbb{R}$ write $x = q_1 b_1 + \cdots q_n b_n$ and define

$$f^*(x) = q_1 f(b_1) + \cdots + q_n f(b_n).$$

Claim: f^* is non-linear. f^* is additive.

Theorem

Every vector space has a basis.

Corollary

There exists an additive function $f : \mathbb{R} \to \mathbb{R}$ which is non-linear.

Every vector space has a basis

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1, 0, 0)\}$

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Let $\mathcal{B}$ be a maximal linearly independent set. Does $\mathcal{B}$ span V?

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Let $\mathcal{B}$ be a maximal linearly independent set. Does $\mathcal{B}$ span V?

Certainly $\mathcal{B}$ spans $\mathcal{B}$, so let $x \notin \mathcal{B}$.

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Let $\mathcal{B}$ be a maximal linearly independent set. Does $\mathcal{B}$ span V?

Certainly $\mathcal{B}$ spans $\mathcal{B}$, so let $x \notin \mathcal{B}$.

Since $\mathcal{B}$ is a maximal l.i. set $\mathcal{B} \cup \{x\}$ is not l.i.

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Let $\mathcal{B}$ be a maximal linearly independent set. Does $\mathcal{B}$ span V?

Certainly $\mathcal{B}$ spans $\mathcal{B}$, so let $x \notin \mathcal{B}$.

Since $\mathcal{B}$ is a maximal l.i. set $\mathcal{B} \cup \{x\}$ is not l.i.

There are $x_1, \ldots, x_n \in \mathcal{B} \cup \{x\}$ and $q_1, \ldots, q_n \in F$ s.t.
$q_1 x_1 + \cdots + q_n x_n = 0$.

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Let $\mathcal{B}$ be a maximal linearly independent set. Does $\mathcal{B}$ span V?

Certainly $\mathcal{B}$ spans $\mathcal{B}$, so let $x \notin \mathcal{B}$.

Since $\mathcal{B}$ is a maximal l.i. set $\mathcal{B} \cup \{x\}$ is not l.i.

There are $x_1, \ldots, x_n \in \mathcal{B} \cup \{x\}$ and $q_1, \ldots, q_n \in F$ s.t.
$q_1 x_1 + \cdots + q_n x_n = 0$.

Since $\mathcal{B}$ is l.i. assume that $x_1 = x$.

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Let $\mathcal{B}$ be a maximal linearly independent set. Does $\mathcal{B}$ span V?

Certainly $\mathcal{B}$ spans $\mathcal{B}$, so let $x \notin \mathcal{B}$.

Since $\mathcal{B}$ is a maximal l.i. set $\mathcal{B} \cup \{x\}$ is not l.i.

There are $x_1, \ldots, x_n \in \mathcal{B} \cup \{x\}$ and $q_1, \ldots, q_n \in F$ s.t.
$q_1 x_1 + \cdots + q_n x_n = 0$.

Since $\mathcal{B}$ is l.i. assume that $x_1 = x$.

So $x = -\frac{q_2}{q_1} x_2 - \cdots - \frac{q_n}{q_1} x_n$.

Every vector space has a basis

Example 1 ($\mathbb{R}^3$)

$\mathcal{B}_1 = \{(1,0,0)\}$ $\mathcal{B}_2 = \{(1,0,0),(0,1,0)\}$ $\mathcal{B}_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Claim 1

A maximal linearly independent set is a basis.

Proof

Let $\mathcal{B}$ be a maximal linearly independent set. Does $\mathcal{B}$ span V?

Certainly $\mathcal{B}$ spans $\mathcal{B}$, so let $x \notin \mathcal{B}$.

Since $\mathcal{B}$ is a maximal l.i. set $\mathcal{B} \cup \{x\}$ is not l.i.

There are $x_1, \dots, x_n \in \mathcal{B} \cup \{x\}$ and $q_1, \dots, q_n \in F$ s.t. $q_1 x_1 + \cdots + q_n x_n = 0$.

Since $\mathcal{B}$ is l.i. assume that $x_1 = x$.

So $x = -\frac{q_2}{q_1} x_2 - \cdots - \frac{q_n}{q_1} x_n$. So $\mathcal{B}$ spans V. ∎

Every vector space has a basis

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

$\mathcal{B}_1 = \{1\}$,

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{1, X\}$,

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{1, X\}$, $\mathcal{B}_3 = \{1, X, X^2\}$,

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{1, X\}$, $\mathcal{B}_3 = \{1, X, X^2\}$, ...

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{1, X\}$, $\mathcal{B}_3 = \{1, X, X^2\}$, ...

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n = \{1, X, X^2, X^3, \dots\}$$

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{1, X\}$, $\mathcal{B}_3 = \{1, X, X^2\}$, ...

$$\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n = \{1, X, X^2, X^3, \dots\}$$

Claim $2 - \varepsilon$

Let $X = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \dots\}$ be a collection of l.i. sets such that $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \cdots$. Then $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is l.i.

Every vector space has a basis

Example 2 ($\mathbb{R}[X]$)

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{1, X\}$, $\mathcal{B}_3 = \{1, X, X^2\}$, ...

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n = \{1, X, X^2, X^3, \dots\}$$

Claim 2 $- \varepsilon$

Let $X = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \dots\}$ be a collection of l.i. sets such that $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \cdots$. Then $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is l.i.

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ *be a collection of l.i. sets such that for any sets* $\mathcal{B}_i, \mathcal{B}_j \in X$*, either* $\mathcal{B}_i \subseteq \mathcal{B}_j$ *or* $\mathcal{B}_j \subseteq \mathcal{B}_i$*. Then* $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ *is l.i.*

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ *be a collection of l.i. sets such that for any sets* $\mathcal{B}_i, \mathcal{B}_j \in X$, *either* $\mathcal{B}_i \subseteq \mathcal{B}_j$ *or* $\mathcal{B}_j \subseteq \mathcal{B}_i$. *Then* $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ *is l.i.*

Proof

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Proof

Let $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ be such that $q_1 x_1 + \cdots + q_n x_n = 0$.

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Proof

Let $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ be such that $q_1 x_1 + \cdots + q_n x_n = 0$.

For i we know $x_i \in \mathcal{B}_{j_i}$.

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Proof

Let $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ be such that $q_1x_1 + \cdots + q_nx_n = 0$.

For i we know $x_i \in \mathcal{B}_{j_i}$.

Either $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$ or $\mathcal{B}_{j_2} \subseteq \mathcal{B}_{j_1}$.

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Proof

Let $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ be such that $q_1 x_1 + \cdots + q_n x_n = 0$.

For i we know $x_i \in \mathcal{B}_{j_i}$.

Either $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$ or $\mathcal{B}_{j_2} \subseteq \mathcal{B}_{j_1}$. Assume that $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$.

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Proof

Let $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ be such that $q_1 x_1 + \cdots + q_n x_n = 0$.

For i we know $x_i \in \mathcal{B}_{j_i}$.

Either $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$ or $\mathcal{B}_{j_2} \subseteq \mathcal{B}_{j_1}$. Assume that $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$.

Then $x_1, x_2 \in \mathcal{B}_{j_2}$.

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Proof

Let $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ be such that $q_1 x_1 + \cdots + q_n x_n = 0$.

For i we know $x_i \in \mathcal{B}_{j_i}$.

Either $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$ or $\mathcal{B}_{j_2} \subseteq \mathcal{B}_{j_1}$. Assume that $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$.

Then $x_1, x_2 \in \mathcal{B}_{j_2}$.

Continuing this gives $x_1, \ldots, x_n \in \mathcal{B}_{j_n}$.

Every vector space has a basis

Claim 2

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a collection of l.i. sets such that for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is l.i.

Proof

Let $x_1, \ldots, x_n \in \mathcal{B}$ and $q_1, \ldots, q_n \in F$ be such that $q_1 x_1 + \cdots + q_n x_n = 0$.

For i we know $x_i \in \mathcal{B}_{j_i}$.

Either $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$ or $\mathcal{B}_{j_2} \subseteq \mathcal{B}_{j_1}$. Assume that $\mathcal{B}_{j_1} \subseteq \mathcal{B}_{j_2}$.

Then $x_1, x_2 \in \mathcal{B}_{j_2}$.

Continuing this gives $x_1, \ldots, x_n \in \mathcal{B}_{j_n}$.

Since $\mathcal{B}_{j_n}$ is linearly independent, $q_1 = \cdots = q_n = 0$. $\blacksquare$

Diversion: Set Theory

Diversion: Set Theory

Banach-Tarksi Paradox

The unit sphere can be partitioned into finitely many parts which can be reassembled (following some isometries) to form two copies of the unit sphere.

Diversion: Set Theory

Banach-Tarksi Paradox

The unit sphere can be partitioned into finitely many parts which can be reassembled (following some isometries) to form two copies of the unit sphere.

For any set $\{A_i \mid i \in I\}$ of non-empty sets, there is a set $X = \{a_i \mid i \in I\}$ such that $a_i \in A_i$ for all $i \in I$.

Diversion: Set Theory

Banach-Tarksi Paradox

The unit sphere can be partitioned into finitely many parts which can be reassembled (following some isometries) to form two copies of the unit sphere.

Axiom of Choice

For any set $\{A_i \mid i \in I\}$ of non-empty sets, there is a set $X = \{a_i \mid i \in I\}$ such that $a_i \in A_i$ for all $i \in I$.

Diversion: Set Theory

Banach-Tarksi Paradox

The unit sphere can be partitioned into finitely many parts which can be reassembled (following some isometries) to form two copies of the unit sphere.

Axiom of Choice

For any set $\{A_i \mid i \in I\}$ of non-empty sets, there is a set $X = \{a_i \mid i \in I\}$ such that $a_i \in A_i$ for all $i \in I$.

(Source: https://xkcd.com/804/)

Diversion: Set Theory

Banach-Tarksi Paradox

The unit sphere can be partitioned into finitely many parts which can be reassembled (following some isometries) to form two copies of the unit sphere.

Axiom of Choice

For any set $\{A_i \mid i \in I\}$ of non-empty sets, there is a set $X = \{a_i \mid i \in I\}$ such that $a_i \in A_i$ for all $i \in I$.

Zorn's Lemma

Let X be a partially ordered set such that every totally ordered subset has an upper bound in X. Then X has a maximal element.

(Source: `https://xkcd.com/804/`)

Diversion: Set Theory

Diversion: Set Theory

Zorn's Lemma

Let X be a partially ordered set such that every totally ordered subset has an upper bound in X. Then X has a maximal element.

Diversion: Set Theory

Zorn's Lemma

Let X be a partially ordered set such that every totally ordered subset has an upper bound in X. Then X has a maximal element.

Definition

A partial order on a set X is a relation $\leq$ such that for all $x, y, z \in X$:

(i) $x \leq x$;

(ii) If $x \leq y$ and $y \leq x$ then $x = y$;

(iii) If $x \leq y$ and $y \leq z$ then $x \leq z$.

Diversion: Set Theory

Zorn's Lemma

Let X be a partially ordered set such that every totally ordered subset has an upper bound in X. Then X has a maximal element.

Definition

A partial order on a set X is a relation $\leq$ such that for all $x, y, z \in X$:

(i) $x \leq x$;

(ii) If $x \leq y$ and $y \leq x$ then $x = y$;

(iii) If $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition

A total order on a set X is a partial order $\leq$ such that for all $x, y \in X$,

(iv) $x \leq y$ or $y \leq x$.

Every vector space has a basis

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$. For all $i \in I$, $\mathcal{B}_i \subseteq \mathcal{B}$.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$. For all $i \in I$, $\mathcal{B}_i \subseteq \mathcal{B}$. So $\mathcal{B}$ is an upper bound for X.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$. For all $i \in I$, $\mathcal{B}_i \subseteq \mathcal{B}$. So $\mathcal{B}$ is an upper bound for X.

Claim 2: $\mathcal{B}$ is in $\mathcal{L}(V)$.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$. For all $i \in I$, $\mathcal{B}_i \subseteq \mathcal{B}$. So $\mathcal{B}$ is an upper bound for X.

Claim 2: $\mathcal{B}$ is in $\mathcal{L}(V)$.

Zorn's Lemma: Since every totally ordered subset of $\mathcal{L}(V)$ has an upper bound in $\mathcal{L}(V)$, $\mathcal{L}(V)$ has a maximal element.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$. For all $i \in I$, $\mathcal{B}_i \subseteq \mathcal{B}$. So $\mathcal{B}$ is an upper bound for X.

Claim 2: $\mathcal{B}$ is in $\mathcal{L}(V)$.

Zorn's Lemma: Since every totally ordered subset of $\mathcal{L}(V)$ has an upper bound in $\mathcal{L}(V)$, $\mathcal{L}(V)$ has a maximal element.

Claim 1: A maximal element of $\mathcal{L}(V)$ is a basis for V.

Every vector space has a basis

Let $\mathcal{L}(V)$ be the set of all linearly independent subsets of V partially ordered by set containment.

Let $X = \{\mathcal{B}_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{L}(V)$. That is for any sets $\mathcal{B}_i, \mathcal{B}_j \in X$, either $\mathcal{B}_i \subseteq \mathcal{B}_j$ or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$. For all $i \in I$, $\mathcal{B}_i \subseteq \mathcal{B}$. So $\mathcal{B}$ is an upper bound for X.

Claim 2: $\mathcal{B}$ is in $\mathcal{L}(V)$.

Zorn's Lemma: Since every totally ordered subset of $\mathcal{L}(V)$ has an upper bound in $\mathcal{L}(V)$, $\mathcal{L}(V)$ has a maximal element.

Claim 1: A maximal element of $\mathcal{L}(V)$ is a basis for V.

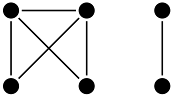
Therefore, V has a basis.

Connection: Graph Theory

A graph is a set of vertices and a set of edges joining some pairs of the vertices.

Connection: Graph Theory

A graph is a set of vertices and a set of edges joining some pairs of the vertices.

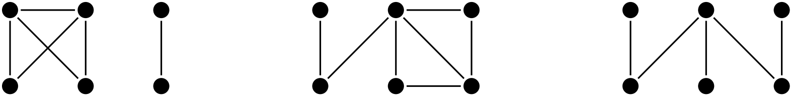


Connection: Graph Theory

A graph is a set of vertices and a set of edges joining some pairs of the vertices.

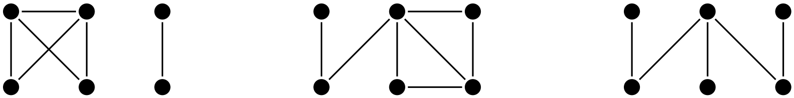
Connection: Graph Theory

A graph is a set of vertices and a set of edges joining some pairs of the vertices.



Connection: Graph Theory

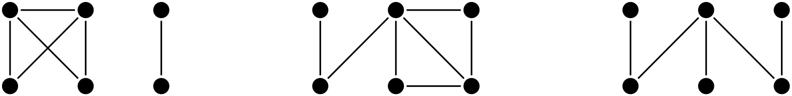
A graph is a set of vertices and a set of edges joining some pairs of the vertices.



A graph is connected if there is a path from any vertex to any other vertex.

Connection: Graph Theory

A graph is a set of vertices and a set of edges joining some pairs of the vertices.

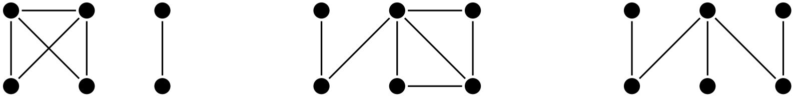


A graph is connected if there is a path from any vertex to any other vertex.

A graph is a tree if there is a unique path from any vertex to any other vertex.

Connection: Graph Theory

A graph is a set of vertices and a set of edges joining some pairs of the vertices.



A graph is connected if there is a path from any vertex to any other vertex.

A graph is a tree if there is a unique path from any vertex to any other vertex.

Note: A tree is a connected graph which does not contain any cycles.

Connection: Graph Theory

A contradiction

Connection: Graph Theory

A contradiction

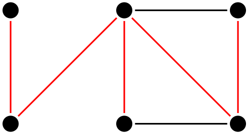
Fife Council are good graph theorists.

Connection: Graph Theory

A spanning tree of a graph is a subgraph which is a tree and which includes every vertex.

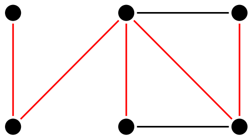
Connection: Graph Theory

A spanning tree of a graph is a subgraph which is a tree and which includes every vertex.



Connection: Graph Theory

A spanning tree of a graph is a subgraph which is a tree and which includes every vertex.

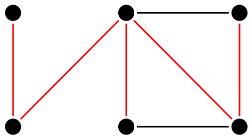


Theorem

Every connected graph has a spanning tree.

Connection: Graph Theory

A spanning tree of a graph is a subgraph which is a tree and which includes every vertex.



Theorem

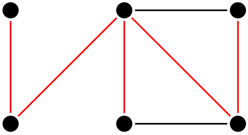
Every connected graph has a spanning tree.

Sketch of Proof

Show that every maximal subtree is a spanning tree.

Connection: Graph Theory

A spanning tree of a graph is a subgraph which is a tree and which includes every vertex.



Theorem

Every connected graph has a spanning tree.

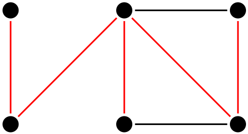
Sketch of Proof

Show that every maximal subtree is a spanning tree.

Show that the union of any nest of trees is a tree.

Connection: Graph Theory

A spanning tree of a graph is a subgraph which is a tree and which includes every vertex.



Theorem

Every connected graph has a spanning tree.

Sketch of Proof

Show that every maximal subtree is a spanning tree.

Show that the union of any nest of trees is a tree.

Conclude, by Zorn's Lemma, that every graph has a spanning tree. ∎

Closing Remarks

Closing Remarks

Any questions?