MATHEMATICAL BACKGROUND

AND

PROGRAMMING AIDS

FOR THE

PHYSICAL VULNERABILITY SYSTEM

FOR

NUCLEAR WEAPONS

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PREFACE

This reference document was written by Mr. Gilbert C. Binninger, Dr. Paul J. Castleberry, Jr. and Ms Patsy M. McGrady under the supervision of Mr. John W. Burfening of the Physical Vulnerability (Nuclear) Branch of the Targets Division, Defense Intelligence Agency. This document presents in detail the mathematics of selected portions of the Defense Intelligence Agency publication AP-550-1-2-69-INT, "Physical Vulnerability Handbook - Nuclear Weapons (U)," 1 June 1969. The appendices present the logic flow, necessary equations and constants needed for computerization of the PV Handbook. Illustrative programs written in BASIC language are included with each appendix.

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I. INTRODUCTION

The Defense Intelligence Agency publishes "Physical Vulnerability Handbook - Nuclear Weapons (U)," AP-550-1-2-INT, for the use of operational planners, target officers, and physical vulnerability analysts concerned with the employment of nuclear weapons and the prediction of their effects. The methodology presented in the PV Handbook allows for target hardness, weapon yield, weapon delivery system accuracy, height-of-burst, and doctrinal requirements to be considered in practical targeting problems.

The ancestors of the PV Handbook[1, 2] omitted mathematical formulations to decrease their bulk without detracting from their day-to-day effectiveness. The result is that today the "Physical Vulnerability Handbook - Nuclear Weapons (U)" presents only tabulated data, graphs, and other computational aids for solution of practical problems and almost completely omits the mathematical formulations needed to understand the system.

[1] "Target Analysis for Atomic Weapons", AF-628202, by the Physical Vulnerability Division, Directorate for Intelligence, Headquarters United States Air Force, PV TM-14, 30 June 1954.

[2] "Nuclear Weapons Employment Handbook (U)," AFM 200-8, by the Physical Vulnerability Division, Director for Targets, Assistant Chief of Staff, Intelligence, Headquarters United States Air Force, 1 May 1958, SECRET/FRD.

This publication is intended to be a reference work containing the detailed mathematical formulations which constitute the groundwork of the Physical Vulnerability system and in particular to assist those doing computer calculations. This presentation is especially needed to document Change 1 to the PV Handbook which was issued 1 September 1972. Change 1 dropped the circular coverage function and adopted the cumulative lognormal function to describe probability of damage as a function of distance from the detonation point.

II. ANALYTIC APPROXIMATION TO THE DISTANCE DAMAGE FUNCTION

A. Factors Leading to Choice

In 1951 the Hiroshima and Nagasaki data were analyzed[3] to determine the damaging overpressures and variation in damage with respect to overpressure for the various types of structures found at Hiroshima and Nagasaki.

These data were insufficient to determine the precise mathematical form of the probability of damage as a function of pressure. However, after the evaluation of several analytic approximations, the cumulative lognormal function (a cumulative normal function with logarithmic variable) was chosen as the best fit to the collected probability of damage versus pressure data.

Probability of damage versus range curves, referred to as distance damage functions, are generated for any desired height-of-burst (HOB) by combining the pressure-damage curves with pressure vs. range curves for the desired heights-of-burst (HOB). These distance damage functions are numerically integrated to obtain the weapon radii for Part I of AP-550-1-2-69-INT (See Appendix A).

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[3] "A Classification of Structures Based on Vulnerability to Blast from Atomic Bombs", Technical Memorandum No. 4, Physical Vulnerability Branch, Air Targets Division, Directorate for Intelligence, United States Air Force, 2 March 1951.

To simplify the probability of damage calculations in Part IV of AP-550-1-2-69-INT when weapon delivery error is included, the actual distance damage function described above is approximated by an analytic function. In order to give sufficiently accurate results for practical applications, this analytic function must comply with certain characteristics of the actual distance damage function. PV TM-14[4] showed that the chosen mathematical model must closely approximate the actual distance damage function in the maximum level of damage expected in the neighborhood of the ground zero and in the distance at which there is a 50% probability of damage. Also, the relative slope and general shape of the analytic function must agree within reasonable limits with the slope and shape of the actual distance damage curve.

Historically the Circular Coverage Function (CCF) has been used for this approximation. However, for targets having damage sigmas greater than 0.30 the CCF is unable to satisfy the requirement that probability of damage be near 100% at ground zero. Therefore, beginning with Change 1 to the 1969 PV Handbook, the complement of the cumulative lognormal function

[4] "Target Analysis for Atomic Weapons."

is used as the analytic approximation to the distance damage function. These two approximations for the distance damage function are virtually identical for $\sigma_d = .10$, differ at most 4 percentage points for $\sigma_d = .20$, and differ at most 6 percentage points for $\sigma_d = .30$. These differences do not cause significant differences in calculated probabilities of damage when weapon delivery error is included. However, significant differences in calculated probabilities of damage can occur for σ_d's of .40 and .50.

B. Analytic Form of the Lognormal Distance Damage Function

In Part IV of AP-550-1-2-69-INT the actual distance damage function is approximated by the complement of the cumulative lognormal function in order to simplify probability of damage calculations. The lognormal density function with variable r and parameters α and β is expressed mathematically as

$$p(r; \alpha, \beta) = \frac{1}{\sqrt{2\pi}\,\beta r} e^{-\frac{1}{2}\left[\frac{1}{\beta} \ln\left(\frac{\alpha}{r}\right)\right]^2}$$

for r > 0 where $\ln$ is the natural logarithm. The standard notation has r and α interchanged in the exponent, however, the forms are equivalent. This density function is plotted in Figure 1 below for several values of β for $\alpha = 1$.

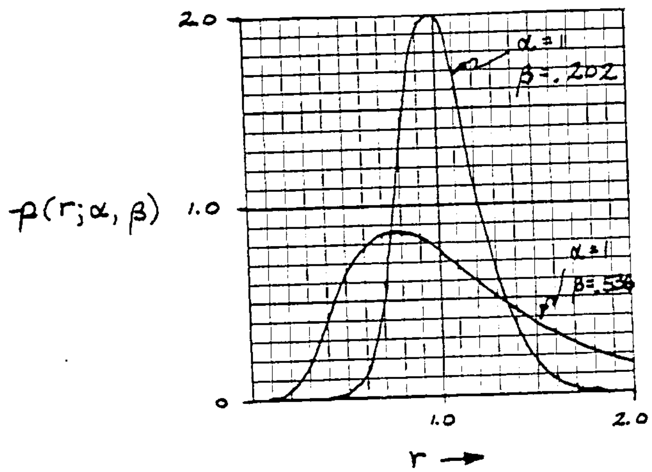


FIGURE 1.

It can be shown that $p(r;\alpha,\beta)$ is a density function over the domain $(0,\infty)$ by letting $y = \frac{1}{\beta}\ln\left(\frac{\alpha}{r}\right)$ and noting that

$$\lim_{r\to\infty}\left[\frac{1}{\beta}\ln\left(\frac{\alpha}{r}\right)\right] = -\infty \quad \text{and} \quad \lim_{r\to 0}\left[\frac{1}{\beta}\ln\left(\frac{\alpha}{r}\right)\right] = \infty$$

Therefore,

$$\int_0^\infty p(r;\alpha,\beta)\,dr = -\int_\infty^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = 1$$

The parameter α, which is the median of the lognormal density function, is r_{50}, the distance from ground zero at which there is a 50% chance of achieving a specified level of damage, i.e., $\int_0^\alpha p(r;\alpha,\beta)\,dr = 1/2$. The parameter β is the standard deviation of $\ln r$.

The cumulative lognormal function (i.e., the distribution function) is expressed as

$$P(r) = \int_0^r p(r;\alpha,\beta)\,dr = -\int_{\infty}^{z(r)} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = \int_{z(r)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$$

where $z(r) = \frac{1}{\beta} \ln\left(\frac{\alpha}{r}\right)$.

A graphical representation of P(r) is shown in figure 2.

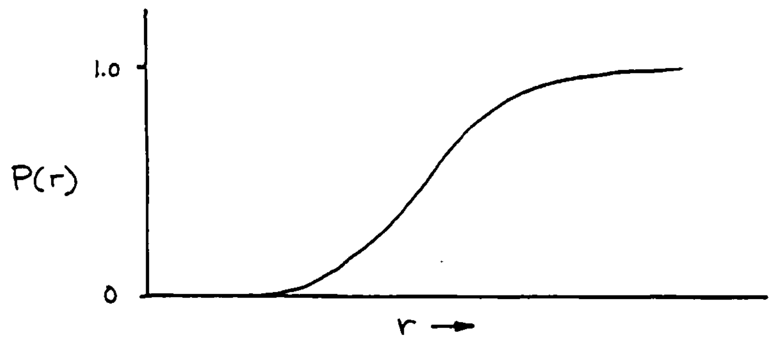


FIGURE 2

Since the distance damage function $P_d(r)$ is monotonically decreasing it is expressed as the complement of P(r).

$$P_d(r) = 1-P(r)$$

$$P_d(r) = 1 - \int_{z(r)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = \int_{-\infty}^{z(r)} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$$

where $z(r) = \frac{1}{\beta} \ln\left(\frac{d}{r}\right)$.

The distance damage function, $P_d(r)$, is depicted in figure 3.

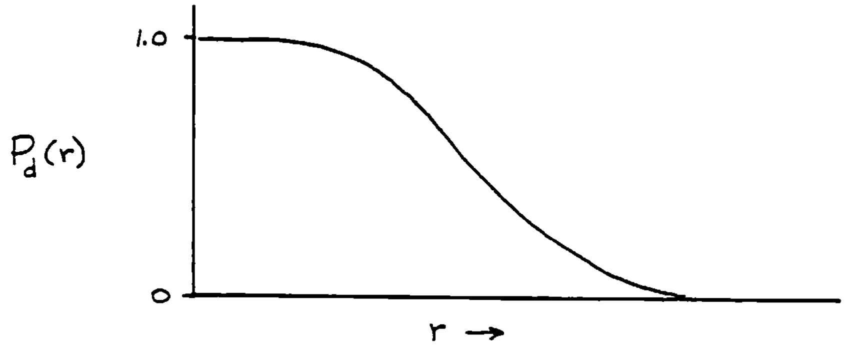


FIGURE 3

Note that the damage function $P_d(r)$ is the complement of a cumulative distribution function, not a density function, i.e.,

$$\int_0^\infty \left(-\frac{dP_d(r)}{dr}\right) dr = 1 \qquad \text{and} \qquad \int_0^\infty P_d(r)\, dr \neq 1 .$$

Since $P_d(r)$ is a cumulative function, it represents the probability that a target will receive <u>at least</u> a specified level of damage. For instance, if $P_d(r)$ is being used to describe severe damage, $1-P_d(r)$ is <u>not</u> the probability of survival, but the probability of achieving less than severe damage.

III. PARAMETERS USED TO DESCRIBE THE DISTANCE DAMAGE FUNCTION

A. Definitions of WR and σ_d, Parameters of the Distance Damage Function

Given the analytic form of $P_d(r)$ from Section II.B., the parameters α and β uniquely specify the damage function. Historically, however, the DIA PV methodology has used the two quantities weapon radius (WR) and distance damage sigma (σ_d) instead of α and β to describe $P_d(r)$. The quantities WR and σ_d are constructed from the first two moments about the origin of the lognormal density function. Specifically, $WR = \sqrt{\langle r^2 \rangle}$ and $\sigma_d^2 = \frac{\langle r^2\rangle - \langle r\rangle^2}{\langle r^2\rangle}$ where the first moment $\langle r\rangle$ and second moment $\langle r^2\rangle$ are defined below.

$$\langle r\rangle \equiv \int_0^\infty r\, p(r;\alpha,\beta)\, dr = \int_0^\infty r \frac{dP(r)}{dr}\, dr = \int_0^\infty r\left(-\frac{dP_d(r)}{dr}\right) dr = \int_0^\infty P_d(r)\, dr$$

$$\langle r^2\rangle \equiv \int_0^\infty r^2 p(r;\alpha,\beta)\, dr = \int_0^\infty r^2 \frac{dP(r)}{dr}\, dr = \int_0^\infty r^2\left(-\frac{dP_d(r)}{dr}\right) dr = \int_0^\infty 2r\, P_d(r)\, dr$$

The last transition on each line is achieved through integration by parts. The procedure for the numerical calculation of $\langle r\rangle$ and $\langle r^2\rangle$ for any distance damage function is presented in appendix A.

The functional relationship between α and β and WR and σ_d will be derived in part C below.

B. Conceptual Descriptions of WR and σ_d

The weapon radius is the square root of $\langle r^2 \rangle$, the second moment of the density function associated with the distance damage function. A more conceptual definition states: Given a uniform distribution of like targets, the WR is the radius of a circle centered at the GZ that contains as many targets undamaged to a specified level inside as there are targets damaged to a specified level outside. Thus if the undamaged targets inside the circle are replaced with the damaged targets outside the circle, the circle of radius WR would enclose an area entirely damaged to the specified level. Expressed mathematically,

$$\text{Number undamaged inside WR} = \int_0^{2\pi} \int_0^{WR} (1 - P_d(r))\, r\, dr\, d\theta$$

$$\text{Number damaged outside WR} = \int_0^{2\pi} \int_{WR}^{\infty} P_d(r)\, r\, dr\, d\theta$$

Equating these and simplifying,

$$WR^2 = \int_0^{\infty} 2r\, P_d(r)\, dr = \int_0^{\infty} r^2 p(r; \alpha, \beta)\, dr = \langle r^2 \rangle$$

i.e., WR^2 is equal to the second moment about the origin of the lognormal density function.

The mean area of effectiveness (M.A.E.) of a weapon is defined as the area over which a weapon on the average achieves at least a specified level of damage:

$$\text{M.A.E.} = \int_0^{2\pi}\int_0^{\infty} P_d(r)\, r\, dr\, d\theta = \pi\, WR^2$$

The mean area of effectiveness of a nuclear weapon is a circle of radius WR.

The second basic parameter of the PV system is the distance damage sigma, σ_d , a number between 0 and 1.

$$\sigma_d^2 = \frac{\langle r^2\rangle - \langle r\rangle^2}{\langle r^2\rangle} = 1 - \frac{\langle r\rangle^2}{\langle r^2\rangle}$$

σ_d^2 is a measure of the variance of the density function. It is the variance made dimensionless by division with the second moment. Small σ_d's indicate a relatively rapid fall off of the damage function. Large σ_d's indicate a more gradual fall off. The damage sigma should never be confused with the derivative (slope) of the damage function.

C. Evaluation of WR and σ_d for the Lognormal Distance Damage Function

In order to evaluate WR and σ_d in terms of the parameters α and β of the lognormal distance damage function, it is necessary to evaluate the first two moments of the lognormal density function.

By definition, the first moment, $\langle r \rangle$, of the lognormal density function is

$$\langle r \rangle = \int_0^\infty r\, p(r;\alpha,\beta)\, dr = \int_0^\infty r \frac{1}{\sqrt{2\pi}\,\beta r}\, e^{-\frac{1}{2}\left[\frac{1}{\beta}\ln\left(\frac{\alpha}{r}\right)\right]^2} dr$$

As was stated in Section II.B., the parameter α is r_{50}, the range where there is a 50% probability of damage. Letting $y = \frac{1}{\beta}\ln\frac{r_{50}}{r}$ which is equivalent to letting $r = r_{50}\, e^{-\beta y}$, the above equation becomes

$$\langle r \rangle = \int_\infty^{-\infty} \frac{r\, e^{-y^2/2}}{\sqrt{2\pi}\,\beta r} \left(-\beta r_{50}\, e^{-\beta y}\right) dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, r_{50}\, e^{-\beta y}\, dy$$

Completing the square,

$$\langle r \rangle = r_{50} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y^2 + 2\beta y + \beta^2)}\, e^{\beta^2/2}\, dy$$

$$= r_{50}\, e^{\beta^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y+\beta)^2}\, dy$$

Since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y+\beta)^2}\, dy = 1$, the first moment is

$$\boxed{\langle r \rangle = r_{50}\, e^{\beta^2/2}}$$

The second moment, $\langle r^2 \rangle$, of the lognormal density function is defined as

$$\langle r^2 \rangle = \int_0^\infty r^2 p(r; \alpha, \beta)\, dr$$

Again using the substitution $r = r_{50}\, e^{-\beta y}$, this becomes

$$\langle r^2 \rangle = - \int_\infty^{-\infty} \frac{r_{50}^2}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y^2 + 4\beta y)}\, dy$$

By completing the square,

$$\langle r^2 \rangle = r_{50}^2\, e^{2\beta^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y + 2\beta)^2}\, dy$$

Since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y+2\beta)^2}\, dy = 1$, the second moment is

$$\boxed{\langle r^2 \rangle = r_{50}^2\, e^{2\beta^2}}$$

Using the above results for $\langle r^2 \rangle$ and the definition $WR = \sqrt{\langle r^2 \rangle}$, the WR is given by

$$\boxed{WR = r_{50}\, e^{\beta^2} = \alpha\, e^{\beta^2}}$$

The above results for $\langle r \rangle$ and $\langle r^2 \rangle$ can be substituted into the equation for σ_d^2 to obtain

$$\boxed{\sigma_d^2 = \frac{\langle r^2 \rangle - \langle r \rangle^2}{\langle r^2 \rangle} = 1 - e^{-\beta^2}}$$

This can be inverted to obtain β for the lognormal distance damage function if the distance damage sigma is known. The inversion gives

$$\boxed{\beta = \sqrt{-\ln(1 - \sigma_d^2)}}$$

The expressions for WR and σ_d^2 can be combined to give $r_{50} \equiv \alpha$ if WR and σ_d are known. Specifically,

$$\boxed{r_{50} \equiv \alpha = WR(1 - \sigma_d^2)}$$

IV. COMBINED WEAPON EFFECTS

When a target is primarily susceptible to one weapon effect, then the WR and σ_d of that effect alone are used in predicting damage to the target. For example, a single story wood frame dwelling is primarily susceptible to overpressure and therefore it is designated a P (for overpressure) target. In this case the probability of damage to the target is calculated using only the appropriate WR and σ_d for overpressure.

In some cases, however, two or more weapon effects may significantly contribute toward damaging the target. For example, personnel casualties may result from blast effects and/or radiation effects. In these situations the weapon radius has a larger value than it would have for either effect alone. The DIA methodology assumes the independence of effects. Therefore, the combined effects weapon radius is estimated without allowance for the possible greater susceptibility of a target to one effect because of exposure to another.

If three independent effects (i = 1, 2, 3) are considered, and the damage function for each is given by

$$P_{d_i}(r) = \int_{-\infty}^{z_i(r)} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

where $$z_i(r) = \frac{1}{\beta_i} \ln \left(\frac{WR_i \, e^{-\beta_i^2}}{r} \right)$$

and $\sigma_{d_i}^2 = 1 - e^{-\beta_i^2}$,

then the combined distance damage function is

$$
\begin{aligned}
P_d(r) &= 1-(1-P_{d_1}(r))(1-P_{d_2}(r))(1-P_{d_3}(r)) \\
&= 1-\{1-[1-(1-P_{d_1}(r))(1-P_{d_2}(r))]\}\{1-P_{d_3}(r)\} \\
&= 1-(1-P_{d_{12}}(r))(1-P_{d_3}(r))
\end{aligned}
$$

where $P_{d_{12}}(r) = 1-(1-P_{d_1}(r))(1-P_{d_2}(r))$.

The above formulation represents the combining of two of the effects and then combining that resultant effect with the third.

The weapon radius of two combined effects, WR_{12}, is[5]

$$WR_{12}^2 = \int_0^\infty 2r\,P_{d_{12}}(r)\,dr = WR_1^2 + WR_2^2 - 2\int_0^\infty r\,P_{d_1}(r)\,P_{d_2}(r)\,dr$$

$$WR_{12}^2 = WR_1^2\,B\left(z_2'(WR_1 e^{\beta_1^2})\right) + WR_2^2\,B\left(z_1'(WR_2 e^{\beta_2^2})\right) \qquad \text{(EQU. 1)}$$

where

$$B(z_i'(r)) = \int_{z_i'(r)}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy$$

[5]The derivations of equations 1 and 2 are presented in detail in Appendix B.

and

$$Z_i'(r) = \frac{1}{\sqrt{\beta_1^2+\beta_2^2}} \ln\left(\frac{WR_i\, e^{-\beta_i^2}}{r}\right).$$

The first moment when two independent events are combined, $\langle r_{12}\rangle$, is derived in the same manner.

$$\langle r_{12}\rangle = \langle r_1\rangle + \langle r_2\rangle - \int_0^\infty P_{d_1}(r)\, P_{d_2}(r)\, dr$$

$$\boxed{\langle r_{12}\rangle = \langle r_1\rangle B\left(Z_2'(WR_1)\right) + \langle r_2\rangle B\left(Z_1'(WR_2)\right)} \qquad \text{(EQU. 2)}$$

The distance damage sigma for two combined effects, $\sigma_{d_{12}}$, is found using

$$\boxed{\sigma_{d_{12}}^2 = 1 - \frac{\langle r_{12}\rangle^2}{\langle r_{12}^2\rangle}}$$

A computer program for calculating the weapon radius and damage sigma for two combined effects is presented in Appendix C.

V. WEAPON DELIVERY ERROR

The circular error probable (CEP) is a measure of weapon system accuracy. It is the radius of a circle centered at the desired ground zero (DGZ) within which 50% of the impact points will fall if the distribution of impact points is assumed to be normally distributed about the DGZ. Expressed mathematically:

$$\frac{1}{2} = \int_0^{2\pi} \int_0^{CEP} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \, r \, dr \, d\theta$$

"The parameter of the impact point distribution ~~bution~~ is often expressed in terms of the CEP as follows:

$$\frac{1}{2} = -e^{-r^2/2\sigma^2} \Big|_0^{CEP}$$

$$= 1 - e^{-CEP^2/2\sigma^2}$$

$$\ln\left(\frac{1}{2}\right) = -\frac{CEP^2}{2\sigma^2}$$

$$\boxed{\sigma = 0.84932 \; CEP}$$

or

$$\boxed{CEP = 1.1774 \; \sigma}$$

VI. PROBABILITY OF DAMAGE CALCULATIONS

A. Point Targets

By using the damage function presented in Section II.B, it is possible to predict the probability of damage to a point target if the offset distance from the detonation point to the target is known. In general, however, all that is known is the distance from the aim point to the target. The actual distance from the weapon detonation point to the target is uncertain due to inaccuracies in the weapon delivery system. Thus the results of an attack can not be predicted with certainty; all that can be predicted in most cases is what is most likely to happen, or what will happen on the average. The PV system calculates an average probability of damage by weighting the probability of damage for each possible detonation point by the probability that the weapon lands at that detonation point. The probability curves in Part IV of AP-550-1-2-69-INT were derived assuming that the distribution of actual detonation points about the aim point is described by the circular normal distribution, i.e., the probability that a weapon lands in a small area, ΔA, a distance ρ from the DGZ is given by

$$\frac{1}{2\pi\sigma^2} e^{-\rho^2/2\sigma^2} \Delta A$$

"where 0.6551*σ is the standard deviation of the circular normal distribution. The distance of the impact point from the DGZ, ρ, is related to x, the distance the DGZ is offset from the target, and to r, the distance of the target from the impact point, by

$$\rho^2 = r^2 + x^2 - 2rx\cos\theta$$

This relationship is geometrically illustrated in figure 4 below.

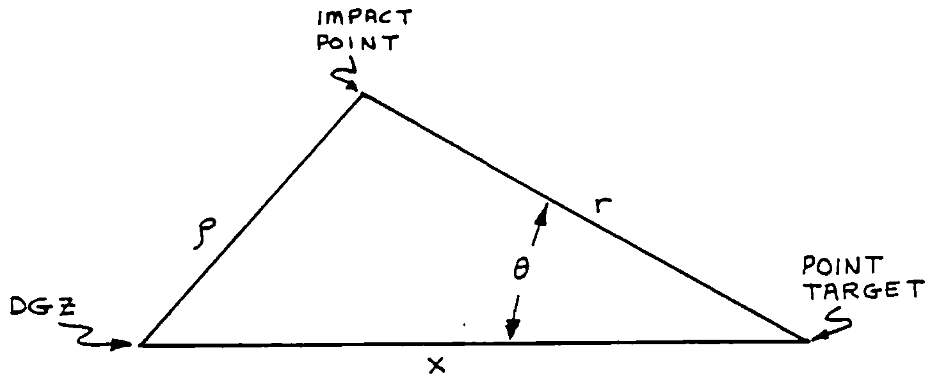


Figure 4

If $P_d(r)$ is the probability of damage to a point target located a distance r from the actual impact point, then

$$P_d(r) \cdot \frac{1}{2\pi\sigma^2} e^{-\rho^2/2\sigma^2} \Delta A$$

is the probability that the weapon falls in the small area ΔA and damages the target to the specified level. Therefore, the sum of all possible terms like that above is the probability of achieving the desired level of damage, P, to a point target located a distance x from the DGZ.

$$P = \sum_{\substack{\text{all} \\ \text{area}}} P_d(r) \cdot \frac{1}{2\pi\sigma^2} e^{-\rho^2/2\sigma^2} \Delta A$$

$$\boxed{P = \int_0^{2\pi} \int_0^{\infty} P_d(r) \cdot \frac{1}{2\pi\sigma^2} e^{-\frac{(r^2 + x^2 - 2rx\cos\theta)}{2\sigma^2}} r\,dr\,d\theta}$$

To calculate P in Part IV of AP-550-1-2-69-INT, $P_d(r)$ is approximated by the complement of the cumulative lognormal function presented in Section II.B.

$$P_d(r) = \int_{-\infty}^{z(r)} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

where $z(r) = \frac{1}{\beta} \ln\left(\frac{r_{50}}{r}\right) = \frac{1}{\beta} \ln\left(\frac{WR\, e^{-\beta^2}}{r}\right)$.

The value of $P_d(r)$ is calculated at increments of r by use of the error function (erf) definition

$$P_d(r) = .5 + .5 \frac{|z|}{z} \operatorname{erf}\left(\frac{|z|}{\sqrt{2}}\right)$$

where $z = \frac{1}{\beta} \ln\left(\frac{WR\, e^{-\beta^2}}{r}\right)$.

The erf can be approximated by $\operatorname{erf}(u) = 1 - \left(\frac{1}{D}\right)^{16}$

where $u > 0$, $D = 1 + \sum_{i=1}^{6} e_i u^i$

and $e_1 = 0.0705230784$

$e_2 = 0.0422820123$

$e_3 = 0.0092705272$

$e_4 = 0.0001520143$

$e_5 = 0.0002765672$

$e_6 = 0.0000430638$

The term $\frac{1}{2\pi\sigma^2}\int_0^{2\pi} e^{-\frac{(r^2+x^2-2rx\cos\theta)}{2\sigma^2}}\, r\,dr\,d\theta$ is found by using series expansions[6] of the zeroth order modified Bessel function of the first kind, $\frac{1}{2\pi}\int_0^{2\pi}\exp(\frac{rx}{\sigma^2}\cos\theta)\,d\theta$, then multiplying by $r\exp(-\frac{1}{2\sigma^2}(r^2+x^2))$. If $\frac{rx}{\sigma^2} \le 3.75$ then

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{rx}{\sigma^2}\cos\theta\right)d\theta = 1 + 3.5156229*j_1 + 3.0899424*j_1^2 + 1.2067492*j_1^3 + 0.2659732*j_1^4 + 0.0360768*j_1^5 + 0.0045813*j_1^6 + \varepsilon$$

where $\varepsilon < 1.6 \times 10^{-7}$ and $j_1 = \left(\frac{rx}{3.75\sigma^2}\right)^2$.

If $\frac{rx}{\sigma^2} > 3.75$ then

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{rx}{\sigma^2}\cos\theta\right)d\theta = \left(\frac{rx}{\sigma^2}\right)^{1/2}\exp\left(\frac{rx}{\sigma^2}\right)\Big[0.39894228 + 0.01328592*j_2 + 0.00225319*j_2^2 - 0.00157565*j_2^3 + 0.00916281*j_2^4 - 0.02057706*j_2^5 + 0.02635537*j_2^6 - 0.01647633*j_2^7 + 0.00392377*j_2^8 + \varepsilon\Big]$$

$\left(\frac{rx}{\sigma^2}\right)^{-1/2}$

[6]E. E. Allen, Analytical Approximation, Math, Tables Aids Comp. 8, 240-241 (1954).

where $\varepsilon < 1.9 \times 10^{-7}$ and $j_2 = \dfrac{3.75\sigma^2}{r^*x}$.

The original integral for the average probability of damage, P, may be thought of in the following manner.

$$P = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_0^{\infty} P_d(r)\, e^{-\frac{r^2 + x^2 - 2rx\cos\theta}{2\sigma^2}}\, r\,dr\,d\theta = \int_a^b f(r)\,dr$$

where a and b are selected such that when $r < a$, $f(r) \approx 0$ and when $r > b$, $f(r) \approx 0$. The integral $\int_a^b f(r)\,dr$ can be evaluated using Gauss-Legendre quadrature. Using this method:

$$\int_a^b f(r)\,dr = \frac{(b-a)}{2} \sum_{i=1}^{n} w_i\, f\left(\frac{(b-a)z_i + b + a}{2}\right) + \varepsilon$$

where w_i and z_i have been derived for various n and ε is the accumulated error. Use of n=10 will result in $\varepsilon \leq .0001$.

Appendix E contains a computer program based on the above formulation for calculating the probability of damage to a point target at any distance from the DGZ.

Appendix F presents a computer program for calculating the required offset distance from the DGZ to the target to achieve a specified probability of damage.

B. Area Targets

Average probabilities of damage to area targets are obtained by dividing the area target into small cells which can be treated as point targets, calculating the probability of damage to each cell, weighting these probabilities by the area of the cell or the portion of the target in the cell, and averaging the results. This approach is discussed in the following paragraphs for normally distributed and uniformly distributed area targets.

1. Circular Normal Distribution

Some area targets, such as population centers, exhibit a concentration of target elements in the center which tends to become less as the distance from the target center increases. The distribution of target elements in this case can be well described by a circular normal distribution. A P-95 circle is used in Part IV of AP-550-1-2-69-INT to describe a normally distributed population. The P-95 is the radius of the smallest circle which encompasses at least 95 percent of the population being considered. If the target is not a population target, the 95% radius may be referred to as an R-95 or other equivalent term. Expressed mathematically

$$.95 = \int_0^{2\pi} \int_0^{P\text{-}95} \frac{1}{2\pi\sigma_t^2} e^{-\frac{t^2}{2\sigma_t^2}} \, t \, dt \, d\theta$$

which gives the target sigma, σ_t , as $\frac{P-95}{2.44}$.

The average probability of damaging the target is obtained by dividing the target into J small cells having a proportion a_j of the target elements, calculating P_j at the center of each cell and then finding the average probability of damage P to the target by

$$P = \frac{\sum_{j=1}^{J} P_j a_j}{\sum_{j=1}^{J} a_j}$$

This method of calculating the probability is quite slow, however. A faster, preferred method involves combining the target and weapon delivery distributions into a joint distribution and then calculating the probability as if for a point target using the CEP of the joint distribution. This approach is possible because both the delivery error, f(r), and the target density, f(t), are normally distributed and thus their joint density function f(r,t) is also normally distributed. The variance of a joint distribution of independent random variables is the sum of the variances of those random variables. Since the delivery error and target density are independent, f(r,t) = f(r) * f(t), then $\sigma_a^2 = \sigma^2 + \sigma_t^2$ where σ_a^2 describes the variance of the joint distribution, σ^2 describes the variance of the delivery error, and σ_t^2 describes the variance of the target density. Therefore, the CEP of the joint distribution, the adjusted CEP (CEP_a), is given by

$$CEP_a = 1.1774\,\sigma_a = \sqrt{CEP^2 + .231\,(P\text{-}95)^2}$$

where $.231(P\text{-}95)^2 = \left(\dfrac{1.1774\ (P\text{-}95)}{2.44}\right)^2 = \left(1.1774\ \sigma_t\right)^2$ is the target CEP^2.

A computer program based on this formulation for calculating probabilities of damage to normally distributed area targets is presented in Appendix E.

2. Uniform Distribution

For uniformly distributed area targets, the method used to calculate precise probabilities of damage again begins with the division of the area target into small cells. All of the cells that are equidistant from the DGZ will have equal probabilities of damage. Referring to figure 5, if concentric circles are drawn about the DGZ with radius r_i where $r_i = r_{i-1} + \Delta r$ and where Δr is small, then the target cells located in the annulus between r_i and r_{i-1} will have equal probabilities of damage.

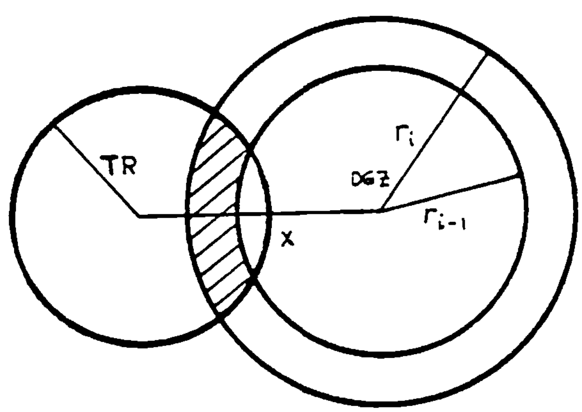


Figure 5

If a_i is the area of intersection of the $i^{\underline{th}}$ annulus and the target, then the average probability of damage to the uniform target is

$$P = \frac{\sum_{i=1}^{I} P_i a_i}{\sum_{i=1}^{I} a_i}$$

where the sum of the a_i is equal to $\pi(TR)^2$ and TR is the target radius.

A computer program for calculating probabilities of damage to uniformly distributed area targets based on the above method is presented in Appendix G.

Good approximations to probabilities of damage to uniformly distributed area targets can be obtained by treating these targets as point targets and replacing the weapon delivery CEP with an adjusted CEP, CEP_a, where

$$CEP_a = \sqrt{CEP^2 + .4\ TR^2} \quad \text{for} \quad TR \leq (WR + X + CEP)$$

or

$$CEP_a = \sqrt{CEP^2 + .5\ TR^2} \quad \text{for} \quad TR > (WR + X + CEP).$$

These CEP adjustments were empirically derived. They have no basis in mathematical theory as does the adjustment for normally distributed area targets. Probabilities of damage calculated using these adjustments will usually be within 2 or 3 percentage points of the rigorously calculated results.

These CEP adjustments can be easily incorporated into the computer program of Appendix E for rapid calculations of probabilities of damage.

C. Single Shot Probability

Single shot probability, SSP, measures the relative frequency with which a weapon can be placed within a designated area assuming that with a large number of independent trials there would be a circular normal distribution of impact points. For a circular normal distribution, the probability that an impact point falls in an infinitesimal area dA a distance ρ from the DGZ is $\frac{1}{2\pi\sigma^2} e^{-\rho^2/2\sigma^2}$ dA. As discussed in section V, the standard deviation of this distribution, σ is equal to .84932 CEP. The SSP is found by integrating this density function over the designated target area, A.

$$SSP = \frac{1}{2\pi\sigma^2} \int e^{-\rho^2/2\sigma^2} dA$$

If the target area is circular, it is easiest to perform this integration using circular cylindrical coordinates (r,θ) centered on the target. The general case where the aim point is offset a distance x from the center of a target of radius TR is shown in figure 6.

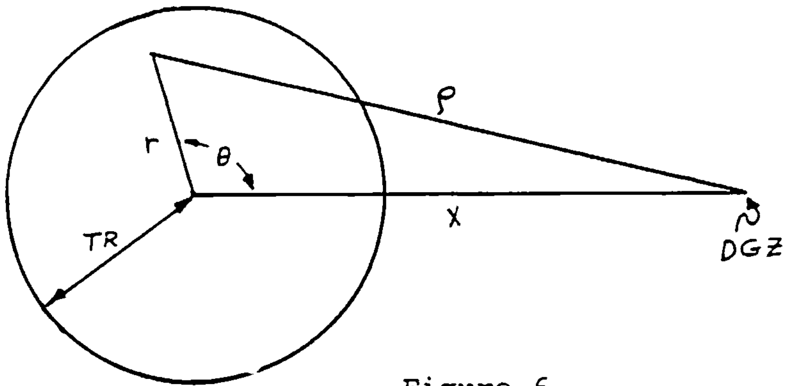


Figure 6

To calculate the SSP, ρ must be expressed in terms of the variable of integration, r. Using the geometry of figure 6, the relationship between ρ and r is

$$\rho^2 = r^2 + x^2 - 2rx\cos\theta$$

The SSP is then

$$SSP = \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_0^{TR} e^{-\frac{r^2+x^2-2rx\cos\theta}{2\sigma^2}}\, r\,dr\,d\theta$$

$$= e^{-x^2/2\sigma^2}\left\{\frac{1}{\sigma^2}\int_0^{TR} I_0\left(\frac{rx}{\sigma^2}\right) r\, e^{-r^2/2\sigma^2}\,dr\right\}$$

where $I_0\left(\frac{rx}{\sigma^2}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{rx}{\sigma^2}\cos\theta}\,d\theta$ is the zeroth order modified Bessel function of the first kind discussed earlier in this section.

If the aim point is located at the center of the circular target (i.e., x=o) then the above equation reduces to

$$SSP = \frac{1}{\sigma^2}\int_0^{TR} r\, e^{-r^2/2\sigma^2}\,dr$$

$$= 1 - e^{-.5(TR/\sigma)^2}$$

In section V the CEP is defined such that $\frac{1}{2} = 1 - e^{-CEP^2/2\sigma^2}$. This definition can be used to eliminate σ from the above expression for SSP to obtain

$$\boxed{SSP = 1 - .5^{(TR/CEP)^2}}$$

D. Equivalent Target Area

For certain special classes of targets such as bridges, dams, locks, runways, etc., a specified degree of damage to some part of the target satisfies the damage objective. For example, for a bridge the collapse of one span is usually the damage objective. The exact determination of the probability of damaging such targets is possible but quite laborious.[7] Approximate answers can be obtained with much less effort by employment of the equivalent target area (ETA) approximation. The ETA is defined as an area such that the probability of placing the GZ in the area is equal to the probability of doing the desired level of damage to the target.

For a rectangular target the ETA is approximated by adding marginal strips around each edge of the target of width equal to the weapon radius for that aspect of the target. Then the probability of damage to a rectangle having a length ℓ and width w is approximately the probability of placing a weapon in

[7] "Target Analysis for Atomic Weapons", p. 5-1.

a rectangular area of length $(\ell+2WR_\ell)$ and width of $(w+2WR_w)$ where WR_ℓ is the weapon radius associated with the length VN and WR_w is the weapon radius associated with the width VN.

This approximation is improved if the actual CEP of weapon delivery is replaced by an adjusted CEP for each aspect of the target which compensates for the variation of damage probability with distance. The width adjusted CEP, CEP_w, is given by $CEP_w = 1.1774\ \sigma_w = \sqrt{CEP^2 + (1.1774\ \sigma_d)^2\ WR_w{}^2}$, and the length adjusted CEP, CEP_ℓ, is given by $CEP_\ell = 1.1774\ \sigma_\ell = \sqrt{CEP^2 + (1.1774\ \sigma_d)^2\ WR_\ell^2}$ where σ_d is the damage sigma having a value between 0.10, and 0.50, and σ_w and σ_ℓ are the adjusted standard deviations of the adjusted delivery function.

If the DGZ is located at the origin of the x; y coordinate system of figure 7, the probability that a weapon falls in the semi-infinite plane from x = a to x = ∞ is

$$P_a = \frac{1}{2\pi\sigma_w\sigma_\ell}\int_{-\infty}^{\infty}\int_{a}^{\infty} e^{-x^2/2\sigma_\ell^2}\ e^{-y^2/2\sigma_w^2}\ dx\,dy$$

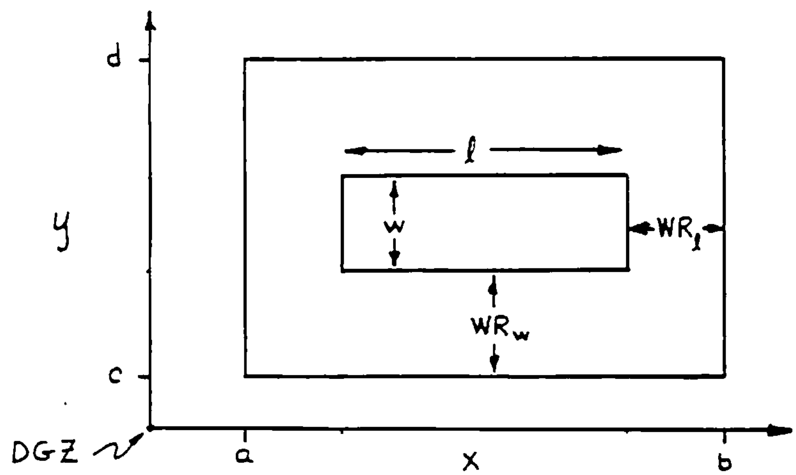


Figure 7

Holding "x" constant and integrating with respect "y" yields

$$P_a = \frac{1}{\sqrt{2\pi}\,\sigma_\ell} \int_a^\infty e^{-x^2/2\sigma_\ell^2}\, dx = .5 - \frac{1}{\sqrt{2\pi}\,\sigma_\ell} \int_0^a e^{-x^2/2\sigma_\ell^2}\, dx$$

Let $t = x/\sigma_\ell$, $dx = \sigma_\ell dt$. Therefore,

$$P_a = .5 - \frac{1}{\sqrt{2\pi}} \int_0^{a/\sigma_\ell} e^{-t^2/2}\, dt$$

$$= .5 - .5\,\frac{|a|}{a}\,\mathrm{erf}\left(\frac{|a|}{\sqrt{2}\,\sigma_\ell}\right)$$

where $|a|$ is the absolute value of a, i.e. $|-5| = 5$.

The probability that the weapon falls in the semi-infinite strip of width (b-a) is (P_a-P_b), where $b-a = \ell + 2WR_\ell$ (see figure 7). The probability that the weapon falls in the semi-infinite strip of width (d-c) is (P_c-P_d), where $d-c = w + 2WR_w$. Therefore, the probability that the weapon falls in the rectangle defined by x = a to b, y = c to d is P, where $P = (P_a-P_b)\,(P_c-P_d)$.

$$\boxed{P = .25\left[\frac{|b|}{b}\,\mathrm{erf}\left(\frac{|b|}{\sqrt{2}\,\sigma_\ell}\right) - \frac{|a|}{a}\,\mathrm{erf}\left(\frac{|a|}{\sqrt{2}\,\sigma_\ell}\right)\right]\left[\frac{|d|}{d}\,\mathrm{erf}\left(\frac{|d|}{\sqrt{2}\,\sigma_w}\right) - \frac{|c|}{c}\,\mathrm{erf}\left(\frac{|c|}{\sqrt{2}\,\sigma_w}\right)\right]}$$

This result is easily programmed using the series expansion for the error function from Section IV. A.

The ETA for a circular target is constructed by adding a ring of width equal to WR around the target. The probability of damage is then calculated using the method of part C, Single Shot Probability, where TR is taken to be the target radius, and the CEP is replaced by an adjusted CEP (CEP_a) where

$$CEP_a = \sqrt{CEP^2 + (1.1774\ \sigma_d)^2\ WR^2} \quad .$$

VII. VULNERABILITY NUMBER (VN) CODING SYSTEM

A. Vulnerability Numbers

In the VN system, a target's susceptibility to blast damage is indicated by a combination of numbers and letters. The vulnerability number (VN) consists of a two-digit number reflecting the target hardness relative to a specified damage level, a letter indicating predominant sensitivity to overpressure (P) or dynamic pressure (Q), and a K factor. The two digit numerical value scale of the VN is an arbitrary classification describing a target's hardness. It is a linear function of the logarithm of the peak pressure from a 20 KT weapon that would have a 50% probability of damaging a randomly oriented target to the desired level. The base yield was chosen to be 20 KT instead of the more convenient 1 KT because the original system was developed from the Hiroshima-Nagasaki data assuming that the yields of the Hiroshima and Nagasaki weapons were 20 KT. The appropriate damage sigma for P targets unless otherwise specified is $\sigma_d = .20$. The appropriate damage sigma for Q targets unless otherwise specified is $\sigma_d = .30$. The K factor allows for hardness adjustments to be made to account for the effects of variations in blast wave duration due to different weapon yields. Each VN must also have a specified damage-level criterion, such as "collapse," "24-hour recovery time," "severe damage to contents," "moderate structural damage," etc.

The completely arbitrary coding relationship for P type targets is

$$p_{50} = 1.1216 \ (1.2)^{PVN}$$

where p_{50} is the peak overpressure from a 20 KT weapon that has a 50% probability of achieving the desired level of damage to the target. This relationship was established so that a p_{50} of 10 psi corresponds to a PVN of 12. With each integer increase in the PVN the associated p_{50} increases 20%.

This coding relationship may be inverted to obtain

$$PVN = \frac{\log_{10} p_{50} - \log_{10}(1.1216)}{\log_{10}(1.2)}$$

or

$$\boxed{PVN = 12.63 \log_{10} p_{50} - .63}$$

Since the peak overpressure at a given range is uncertain to roughly ± 20%, this coding relationship insures that P type target hardnesses are not specified more precisely than justified by the pressure-range data. This scale conveniently allows for the complete pressure range of interest to be coded by a two-digit number.

The dynamic pressure coding scale was chosen using the approximate form of the Rankine-Hugoniot equation,

$q = .023\ p^2$. The scale was defined so that the dynamic pressure required for a 50% probability of damage, q_{50}, for the VN of interest is equal to $.023\ p_{50}{}^2$ where p_{50} is from the numerically equal P VN. Therefore, the VN's are "tied" at the 50% probability but not at other probabilities. Thus,

$$q_{50} = .023\ (p_{50}{}^2)$$

$$= .023\ (1.1216*(1.2)^{QVN})^2$$

$$q_{50} = .02893*(1.44)^{QVN}$$

The above relationship establishes a 44% difference in peak dynamic pressures between adjacent VN's. This can be inverted to give

$$Q\,VN = \frac{\log_{10} q_{50} - \log_{10}(.02893)}{\log_{10}(1.44)}$$

or

$$\boxed{Q\,VN = 6.31 \log_{10} q_{50} + 9.72}$$

This relationship insures that the complete dynamic pressure range of interest is covered. Since the dynamic pressure at a given range is only known to within about ± 40%, Q type target hardnesses are also not specified more accurately than justified by the pressure range data.

Care should be taken not to use the equation $q = .023\ p^2$ to calculate the peak dynamic pressure corresponding to a given peak overpressure. For overpressures below about 10 psi the equation $q = .023\ p^2$ is a good approximation to the Rankine-Hugoniot relation $q = \frac{5}{2} \frac{p^2}{7p_0 + p}$ where p_0 is the ambient atmospheric pressure. This Rankine-Hugoniot relation was derived[8] assuming an ideal shock front. It only fits the available experiment data fairly well for zero heights-of-burst (HOB). The correct determination of the q given p or vice versa for a given HOB must be through the horizontal ground range using pressure-range-HOB curves such as figures I-4 and I-9 of AP 550-1-2-69-INT.

B. K Factor

As previously mentioned, the blast wave duration varies with weapon yield. The increased blast duration associated with larger yields may cause targets to fail at lower pressure levels, while at small yields the reduced blast duration may necessitate higher pressures for target failure. To account for this yield dependence, the PV system uses K-factors for both P and Q targets. The K-factor is an integer from 0 to 9 which adjusts the base VN to reflect the sensitivity of the

[8]Samuel Glasstone, "The Effects of Nuclear Weapons", Air Force Pamphlet No. 136-1-3, Department of the Air Force, April 1962.

target to the different pressure-time pulse shapes for yields other than 20 KT. A K factor of 0 indicates a target that is not sensitive to blast wave duration and can be expected to fail at the same pressure regardless of weapon yield. A K factor of 9 indicates a target that is very sensitive to blast wave duration and can be expected to fail at quite different pressures at various yields.

The adjustment factor R is the ratio of the pressure (either overpressure, p(Y), or dynamic pressure, q(Y), required for a 50% probability of damage at yield Y to the pressure required at 20 KT (p(20) or q(20)). The K factor is related to the adjustment factor, R, in the following manner[9]

$$R = 1 - \frac{K}{10}\left(1 - \frac{t_{do}}{t_d}\right) = \frac{p(Y)}{p(20)} \quad \text{or} \quad \frac{q(Y)}{q(20)}$$

where

t_{do} = positive phase blast wave duration for 20 KT

t_d = positive phase blast wave duration for yield Y

$t_d \approx .45 * \dfrac{Y^{1/3}}{p^{1/2}}$ for overpressure

$t_d \approx .105 * \dfrac{Y^{1/3}}{q^{1/3}}$ for dynamic pressure.

[9] "Development of the K-factor in the VN System," PV-105-61, Air Force Intelligence Center, Assistant Chief of Staff, Intelligence, 25 January 1961.

Therefore, for P type targets,

$$R = 1 - \frac{K}{10} + \frac{K}{10}\left(\frac{p(Y)}{p(20)}\right)^{1/2}\left(\frac{20}{Y}\right)^{1/3} = 1 - \frac{K}{10} + \frac{K}{10} R^{1/2}\left(\frac{20}{Y}\right)^{1/3}$$

For Q type targets,

$$R = 1 - \frac{K}{10} + \frac{K}{10}\left(\frac{q(Y)}{q(20)}\right)^{1/3}\left(\frac{20}{Y}\right)^{1/3} = 1 - \frac{K}{10} + \frac{K}{10} R^{1/3}\left(\frac{20}{Y}\right)^{1/3}.$$

These equations are solved for R for a given K factor and yield by iteration to the desired accuracy using a first guess of 2 for PVN's or a first guess of 3 for QVN's.

The adjustment factor R is used to determine the adjusted VN (VN_a) using the PVN and QVN coding relationship as shown below. For P VN's, $VN_a = VN+A$, where

$$A = \frac{\log(R)}{\log(1.2)}$$

$$= 12.63 \log_{10} R$$

$$= 5.485 \log_e R$$

For Q VN's,

$$A = \frac{\log(R)}{\log(1.44)}$$

$$= 6.315 \log_{10} R$$

$$= 2.742 \log_e R$$

C. VN-Probability Relations

It is often necessary to know not only p_{50} for a given VN and yield, but also the pressures for other probabilities of damage. The method used to obtain the pressure p_a for a probability of damage a% for the adjusted VN of v2 is discussed below.

For P type targets, the VN coding relationship gives

$$p_{50} = 1.1216\ (1.2)^{v2}$$

The analysis of the Hiroshima-Nagasaki and Nevada test data[10] resulted in the adoption of the following relationship between the overpressure required to damage a structure to a given level and the pressure which gives a 50% probability of damage, p_{50}.

$$p_a = p_{50}\, e^{.297\ b}$$

The probability a is given by

$$a = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-x^2/2}\, dx$$

where b is the probability a expressed as probits-5. (A probit has the magnitude of the standard deviation. Minus 5 probits is defined as a = 0%, 5 probits is 50%, and 10 probits

[10] "A Classification of Structures Based on Vulnerability to Blast from Atomic Bombs".

corresponds to a = 100%.)

Similarily, for dynamic pressure,

$$q_{50} = .02893\ (1.44)^{v2}$$

$$q_a = q_{50}\ e^{1.042\ b}$$

In addition to coding p_{50}'s and q_{50}'s, adjusted VN's code the scaled weapon radii, SWR[11], to be used in calculating $P_d(r)$, the distance damage function. SWR for each VN are calculated using the pressure - probability functions given above by combining them with pressure-range-HOB data. SWR's calculated in this manner are listed in AP-550-1-2-69-INT for PVN's and QVN's at various scaled heights of burst, SHOB. To facilitate computer applications, these calculated SWR's have been approximated with polynominals of the form SWR = f(v2) for various scaled heights-of-burst. The polynomials have the form

$$SWR = \exp\left(\sum_{j=0}^{n} k_j\ (v2)^j \right)$$

where n varies from 2 to 7 depending on the SHOB. The constants k_j and a further description of the curve fits are

[11]Scaled weapon radii and scaled height-of-burst have dimensions $ft/KT^{1/3}$.

included in Appendix D.

Numerical Calculation of Moments

The moments presented in Section III.A. can be calculated numerically if the damage function $P_d(r)$ is known at the ranges r_i either from experimental data or from the analytic approximation to $P_d(r)$ from Section II.B.

Consider the contribution to the first moment from the small section of $P_d(r)$ from r_{i-1} to r_i shown in figure A.1.

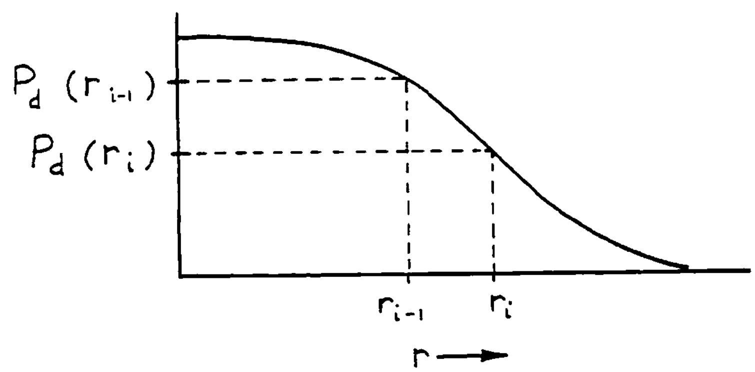


Figure A.1.

Over the small range from r_{i-1} to r_i, accurate estimates for $P_d(r)$ can be obtained by linear interpolation between the values of P_d at r_{i-1} and at r_i. Therefore,

$$P_d(r) \approx P_d(r_i) - \left(P_d(r_i) - P_d(r_{i-1})\right)\left(\frac{r_i - r}{r_i - r_{i-1}}\right)$$

The contribution to $\langle r \rangle$ from this section of the damage function is

$$\langle r \rangle_i \doteq \int_{r_{i-1}}^{r_i} \left[P_d(r_i) - \left(P_d(r_i) - P_d(r_{i-1})\right)\left(\frac{r_i - r}{r_i - r_{i-1}}\right)\right] dr$$

$$\langle r \rangle_i \doteq \frac{1}{2}\left(P_d(r_i) + P_d(r_{i-1})\right)\left(r_i - r_{i-1}\right)$$

Since the first moment is the summation of all possible terms like that above,

$$\boxed{\langle r \rangle = \sum_{i=1}^{n} \langle r \rangle_i \doteq \sum_{i=1}^{n} \frac{1}{2}\left(P_d(r_i) + P_d(r_{i-1})\right)\left(r_i - r_{i-1}\right)}$$

where $r_0 = 0$.

Similarily, the second moment $\langle r^2 \rangle$ is approximated by

$$\langle r^2 \rangle \doteq \sum_{i=1}^{n} 2\Big\{ P_d(r_i)(r_i - r_{i-1})\left(r_{i-1} + \frac{1}{2}(r_i - r_{i-1})\right) + $$

$$+\left(P_d(r_{i-1}) - P_d(r_i)\right)\left(\frac{r_i - r_{i-1}}{2}\right)\left(r_{i-1} + \frac{1}{3}(r_i - r_{i-1})\right)\Big\}$$

This form may be rewritten as

$$\boxed{\langle r^2 \rangle \doteq \sum_{i=1}^{n} \frac{1}{2}\left(P_d(r_i) + P_d(r_{i-1})\right)\left(r_i^2 - r_{i-1}^2\right) + \frac{1}{6}\left(P_d(r_i) - P_d(r_{i-1})\right)\left(r_i - r_{i-1}\right)^2}$$

where again $r_0 = 0$.

As explained in Section II.A., the weapon radii $(WR = \sqrt{\langle r^2 \rangle})$ presented in Part I of AP-550-1-2-69-INT are calculated numerically using this formulation. The damage sigmas $\left(\sigma_d = \sqrt{1 - \frac{\langle r \rangle^2}{\langle r^2 \rangle}}\right)$ calculated numerically for P type targets range from .10 to .20

while for Q type targets the damage sigmas range from .20 to .30. For probability calculations, the distance damage function is approximated by the analytic function of Section II.B. The parameter WR of this function is taken to be the numerically calculated WR, while the parameter σ_d is assumed to be .20 for P targets or .30 for Q targets. This analytic approximation to $P_d(r)$ will always have $P_d(o) = 1$. However, experimental data may not have $P_d(o) = 1$ due to the hardness of the target and the particular yield and HOB chosen. In such cases, accurate probabilities of damage can not be obtained using the analytic approximation of $P_d(r)$.

Derivation of Combined Effects Formulas; Section IV

The derivation of Equation 1 from Section IV is as follows: From Section IV,

$$WR_{12}^2 = WR_1^2 + WR_2^2 - 2\int_0^\infty r\, P_{d_1}(r)\, P_{d_2}(r)\, dr$$

Let $I = 2\int_0^\infty r\, P_{d_1}(r)\, P_{d_2}(r)\, dr$

Using integration by parts, let

$$u = P_{d_1}(r)\, P_{d_2}(r) \qquad \text{and} \qquad dv = 2r\,dr$$

then $\quad du = P_{d_1}(r)\, d\, P_{d_2}(r) + P_{d_2}(r)\, d\, P_{d_1}(r)$

and $\quad v = r^2$.

I then becomes

$$I = r^2 P_{d_1}(r) P_{d_2}(r)\Big|_0^\infty - \int_0^\infty r^2 P_{d_1}(r)\, d\, P_{d_2}(r) - \int_0^\infty r^2 P_{d_2}(r)\, d\, P_{d_1}(r)$$

$$= -\int_0^\infty r^2 P_{d_1}(r)\, d\, P_{d_2}(r) - \int_0^\infty r^2 P_{d_2}(r)\, d\, P_{d_1}(r) \qquad \text{(EQU. B-1)}$$

Since the terms are symmetrical in Equation B-1, it is necessary only to deal with one and apply the findings to both integrals.

Let $\mathrm{II} = -\int_0^\infty r^2 P_{d_1}(r)\, d\, P_{d_2}(r)$

From Section II.B. we know that $P_{d_1}(r) = \int_{-\infty}^{z_1(r)} \phi(y)\, dy$

where $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$,

$$z_1(r) = \frac{1}{\beta_1} \ln\left(\frac{WR_1 e^{-\beta_1^2}}{r}\right)$$

and $$\left(\frac{d P_{d_2}(r)}{d r}\right) dr = -\left(\frac{1}{\sqrt{2\pi}\, \beta_2 r} \exp\left[-\frac{1}{2}\left(z_2(r)\right)^2\right]\right) dr$$

Then, $$\mathrm{II} = -\int_0^\infty r^2 \left(\int_{-\infty}^{z_1(r)} \phi(y)\, dy\right)\left(-\frac{1}{\sqrt{2\pi}\, \beta_2 r} \exp\left[-\frac{1}{2}(z_2(r))^2\right]\right) dr \qquad \text{(EQU. B-1)}$$

Since $d(\ln(r)) = \frac{dr}{r}$ and $r^2 = e^{2\ln(r)}$, the integration of Equation B-2 may be made with respect to ln(r) rather than r.

$$\lim_{r\to\infty} \ln(r) = \infty \qquad \lim_{r\to 0} \ln(r) = -\infty$$

$$\mathrm{II} = -\int_{-\infty}^{\infty} e^{2\ln(r)} \left(\int_{-\infty}^{z_1(\ln r)} \phi(y)\, dy\right)\left(-\frac{1}{\sqrt{2\pi}\, \beta_2} \exp\left[-\frac{1}{2}\left(z_2(\ln r)\right)^2\right]\right) d\ln(r)$$

Combine the exponential terms and complete the square to obtain:

$$\mathrm{II} = e^{\ln(WR_2^2)} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z_1(\ln r)} \phi(y)\, dy\right) \exp\left(-\frac{1}{2\beta_2^2}\left(\ln r - \ln(WR_2 e^{\beta_2^2})\right)\right)^2 \frac{d \ln r}{\sqrt{2\pi}\, \beta_2}$$

If we let $y = \frac{x - \ln r}{\beta_1}$ and $dy = \frac{dx}{\beta_1}$, then

$$\text{II} = e^{\ln(wR_2^2)} \int_{-\infty}^{\infty} \left(\left(\int_{-\infty}^{\ln(wR_1 e^{-\beta_1^2})} \phi\left(\frac{x - \ln r}{\beta_1}\right) \frac{dx}{\beta_1} \right) \exp\left(-\frac{1}{2\beta_2^2}\left(\ln r - \ln(wR_2 e^{\beta_2^2})\right)^2\right) \frac{d \ln r}{\sqrt{2\pi}\,\beta_2}$$

Interchange the integral signs, combine exponential terms and complete the square to obtain

$$\text{II} = e^{\ln(wR_2^2)} \int_{-\infty}^{\ln(wR_1 e^{-\beta_1^2})} \left(\left(\int_{-\infty}^{\infty} \exp\left[-\frac{\beta^2}{2\beta_1^2\beta_2^2}(\ln r - B)^2\right] \frac{d \ln r}{\sqrt{2\pi}\,\beta_2} \right) \exp\left[-\frac{\beta^2}{2\beta_1^2\beta_2^2}(C - B^2)\right] \frac{dx}{\sqrt{2\pi}\,\beta_1}$$

where $B = \dfrac{x\beta_2^2 + \beta_1^2 \ln(wR_2 e^{\beta_2^2})}{\beta^2}$

$$C = \frac{x^2\beta_2^2 + \beta_1^2 \left(\ln(wR_2 e^{\beta_2^2})\right)^2}{\beta^2}$$

and $\quad \beta^2 = \beta_1^2 + \beta_2^2$.

Holding the variable "x" constant and integrating with respect to ln(r) yields a value of $\frac{\beta_1}{\beta}$ for the inner integral. Therefore,

$$\text{II} = \frac{e^{\ln(wR_2^2)}}{\beta} \int_{-\infty}^{\ln(wR_1 e^{-\beta_1^2})} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\beta^2}{2\beta_1^2\beta_2^2}(C - B^2)\right] dx$$

Simplifying the exponential under the integral sign this becomes

$$\text{II} = \frac{e^{\ln(wR_2^2)}}{\beta} \int_{-\infty}^{\ln(wR_1 e^{-\beta_1^2})} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2\beta^2}\left(x - \ln(wR_2 e^{\beta_2^2})\right)^2\right] dx$$

Let $\rho = \dfrac{x - \ln(WR_2 e^{\beta_2^2})}{\beta}$, then

$$II = e^{\ln(WR_2^2)} \int_{-\infty}^{\alpha_1} \phi(\rho)\, d\rho$$

where $\alpha_1 = \frac{1}{\beta} \ln\left(\frac{WR_1}{WR_2} e^{-\beta^2}\right) = z_1'\left(r = WR_2 e^{\beta_2^2}\right)$. We then obtain

$$II = WR_2^2 \int_{-\infty}^{\alpha_1} \phi(\rho)\, d\rho$$

Employing the same procedure, the other integral in Equation B-1 is:

$$WR_1^2 \int_{-\infty}^{\alpha_2} \phi(\xi)\, d\xi$$

where $\alpha_2 = \frac{1}{\beta} \ln\left(\frac{WR_2}{WR_1} e^{-\beta^2}\right) = z_2'\left(r = WR\, e^{\beta_1^2}\right)$.

Substituting into the equation for WR_{12}^2 from Section IV,

$$WR_{12}^2 = WR_1^2 + WR_2^2 - WR_1^2 \int_{-\infty}^{\alpha_2} \phi(\xi)\, d\xi - WR_2^2 \int_{-\infty}^{\alpha_1} \phi(\rho)\, d\rho$$

$$WR_{12}^2 = WR_1^2 \left(1 - \int_{-\infty}^{\alpha_2} \phi(\xi)\, d\xi\right) + WR_2^2 \left(1 - \int_{-\infty}^{\alpha_1} \phi(\rho)\, d\rho\right)$$

$$\boxed{WR_{12}^2 = WR_1^2 \int_{\alpha_2}^{\infty} \phi(\xi)\, d\xi + WR_2^2 \int_{\alpha_1}^{\infty} \phi(\rho)\, d\rho}$$

where

$$\int_{\alpha_1}^{\infty} \phi(\rho)\, d\rho = B\left(z_1'(WR_2 e^{\beta_2^2})\right)$$

and

$$\int_{\alpha_2}^{\infty} \phi(\xi)\, d\xi = B\left(z_2'(WR_1 e^{\beta_1^2})\right).$$

This result is equation 1 of Section IV.

The mean for combined weapon effects as presented in Section IV is

$$\langle r_{12} \rangle = \int_0^\infty \left(P_{d_1}(r) + P_{d_2}(r) - P_{d_1}(r)\, P_{d_2}(r) \right) dr$$

$$= \langle r_1 \rangle + \langle r_2 \rangle - \int_0^\infty P_{d_1}(r)\, P_{d_2}(r)\, dr$$

Let $I = \int_0^\infty P_{d_1}(r)\, P_{d_2}(r)\, dr$.

Using integration by parts, let

$$u = P_{d_1}(r)\, P_{d_2}(r) \quad \text{and} \quad dv = dr .$$

Then $du = P_{d_1}(r)\, d\, P_{d_2}(r) + P_{d_2}(r)\, d\, P_{d_1}(r)$

and $v = r$.

Our integral is then

$$I = -\int_0^\infty r\, P_{d_1}(r)\, d\, P_{d_2}(r) - \int_0^\infty r\, P_{d_2}(r)\, d\, P_{d_1}(r) .$$

Since the integrals are symmetric, it is sufficient to examine only one integral and apply the results to both.

$$\text{Let } II = -\int_0^\infty r\, P_{d_1}(r)\, d\, P_{d_2}(r)$$

where

$$P_{d1}(r) = \int_{-\infty}^{z_1(r)} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy \quad \text{and} \quad z_1(r) = \frac{1}{\beta_1} \ln\left(\frac{W R_1 e^{-\beta_1^2}}{r} \right) .$$

We also know that

$$\frac{dP_{d_2}(r)}{dr}\,dr = \frac{-\exp\left[-\frac{1}{2\beta_2^2}\left(\ln(WR_2e^{-\beta_2^2})-\ln r\right)^2\right]}{\sqrt{2\pi}\,\beta_2 r}\,dr$$

Therefore, II becomes

$$II = \int_0^\infty r\left(\int_{-\infty}^{z_1(r)}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right)\left(\frac{\exp\left[-\frac{1}{2\beta_2^2}\left(\ln(WR_2e^{-\beta_2^2})-\ln r\right)^2\right]}{\sqrt{2\pi}\,\beta_2 r}\right)dr$$

Changing the variable "r" to ln(r) such that $r = e^{\ln r}$ and $d\ln r = \frac{dr}{r}$, this becomes

$$II = \int_{-\infty}^\infty e^{\ln r}\left(\int_{-\infty}^{z_1(r)}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right)\left(\frac{\exp\left[-\frac{1}{2\beta_2^2}\left(\ln(WR_2e^{-\beta_2^2})-\ln r\right)^2\right]}{\sqrt{2\pi}\,\beta_2}\right)d\ln r$$

Combining exponential terms and completing the square,

$$II = \langle r_2\rangle\int_{-\infty}^\infty \left(\int_{-\infty}^{z_1(r)}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right)\left(\frac{\exp\left[-\frac{1}{2\beta_2^2}\left(\ln r-\ln WR_2\right)^2\right]}{\sqrt{2\pi}\,\beta_2}\right)d\ln r$$

Letting $y = \frac{x-\ln r}{\beta_1}$ then $dy = \frac{dx}{\beta_1}$ and when $y=z_1(r)$ then $x = \ln(WR_1e^{-\beta_1^2})$. Therefore,

$$II = \langle r_2\rangle\int_{-\infty}^\infty \left(\int_{-\infty}^{\ln(WR_1e^{-\beta_1^2})}\frac{1}{\sqrt{2\pi}\,\beta_1}\exp\left[-\frac{1}{2\beta_1^2}(x-\ln r)^2\right]dx\right)\left(\frac{\exp\left[-\frac{1}{2\beta_2^2}\left(\ln r-\ln WR_2\right)^2\right]}{\sqrt{2\pi}\,\beta_2}\right)d\ln r$$

Combining exponential terms and completing the square

$$II = \langle r_2\rangle\int_{-\infty}^\infty \left(\int_{-\infty}^{\ln(WR_1e^{-\beta_1^2})}\frac{\exp\left[-\frac{1}{2}\left(\frac{\beta}{\beta_1\beta_2}\right)^2(C-B^2)\right]}{\sqrt{2\pi}\,\beta_1}dx\right)\left(\frac{\exp\left[-\frac{1}{2}\left(\frac{\beta}{\beta_1\beta_2}\right)^2(\ln r-B)^2\right]}{\sqrt{2\pi}\,\beta_2}\right)d\ln r$$

where $$B = \frac{x\beta_2^2 + \beta_1^2 \ln WR_2}{\beta^2}$$

$$C = \frac{x^2\beta_2^2 + \beta_1^2(\ln WR_2)^2}{\beta^2}$$

and $\beta^2 = \beta_1^2 + \beta_2^2$.

Holding "x" constant and integrating with respect to ln(r)

$$II = \langle r_2 \rangle \int_{-\infty}^{\ln(WR_1 e^{-\beta_1^2})} \frac{\exp\left[-\frac{1}{2\beta^2}(x - \ln WR_2)^2\right]}{\sqrt{2\pi}\,\beta}\, dx$$

Letting $\rho = \frac{x - \ln WR_2}{\beta}$ and $d\rho = \frac{dx}{\beta}$ then

$$II = \langle r_2 \rangle \int_{-\infty}^{\alpha_1} \frac{1}{\sqrt{2\pi}} e^{-\rho^2/2}\, d\rho \qquad \text{where } \alpha_1 = \frac{1}{\beta} \ln\left(\frac{WR_1 e^{-\beta_1^2}}{WR_2}\right).$$

Therefore, the first moment for combined effects is

$$\langle r_{12} \rangle = \langle r_1 \rangle + \langle r_2 \rangle - \langle r_2 \rangle \int_{-\infty}^{\alpha_1} \frac{1}{\sqrt{2\pi}} e^{-\rho^2/2}\, d\rho - \langle r_1 \rangle \int_{-\infty}^{\alpha_2} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}\, d\xi$$

$$= \langle r_1 \rangle \int_{\alpha_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}\, d\xi + \langle r_2 \rangle \int_{\alpha_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\rho^2/2}\, d\rho$$

$$\boxed{\langle r_{12} \rangle = \langle r_1 \rangle \left\{ B\left(z_2'(WR_1)\right) \right\} + \langle r_2 \rangle \left\{ B\left(z_1'(WR_2)\right) \right\}}$$

where

$$B(z_i'(r)) = \int_{\alpha_i}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$\alpha_i = z_i'(r) = \frac{1}{\beta} \ln\left(\frac{WR_i e^{-\beta_i^2}}{r}\right)$$

This result is equation 2 of Section IV.

Calculation of Weapon Radii for Combined Effects

As explained in Section IV when two or more weapon effects significantly contribute toward damaging a target, a combined WR and σ_d should be calculated and the P.D. computation then based on this combined WR and σ_d. This procedure is most often used in estimating personnel casualties/fatalities when radiation and blast have comparable ranges of effect.

The example program takes two WR's and their associated damage sigmas and calculates the WR and damage sigma of the combined effects.

WR_{12}, the combined WR, $\langle r_{12}\rangle$, the combined mean radius, and $\sigma_{d_{12}}$, the resultant damage sigma of two damaging effects, may be calculated using the equations of Section IV.

$$WR_{12}^2 = WR_1^2 \int_{z_2(WR_1 e^{\beta_1^2})}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + WR_2^2 \int_{z_1(WR_2 e^{\beta_2^2})}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$\langle r_{12}\rangle = \langle r_1\rangle \int_{z_2(WR_1)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \langle r_2\rangle \int_{z_1(WR_2)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$\sigma_{d_{12}}^2 = 1 - \frac{\langle r_{12}\rangle^2}{WR_{12}^2}$$

where $$z_i(d) = \frac{1}{\beta_3} \ln\left(\frac{WR_i e^{-\beta_i^2}}{d}\right)$$

$$\beta_3 = \sqrt{\beta_1^2 + \beta_2^2}$$

$$\int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = .5 - .5 \frac{|z|}{z} \operatorname{erf}\left(\frac{|z|}{\sqrt{2}}\right).$$

In the example program:

$$f1 = \int_{z1}^{\infty} e^{-y^2/2} dy$$

$$f2 = \int_{z2}^{\infty} e^{-y^2/2} dy$$

and fna(v) = erf(v)

t is an indexing variable. If t=0 then

$z1 = z_1(w2)$

$z2 = z_2(w1)$

$f1 = f1(z_1(w2))$

$f2 = f2(z_2(w1))$

and $r = \langle r_{12} \rangle = w1*\exp(-\beta_1^2/2)*f2+w2*\exp(-\beta_2^2/2)*f1$

since $r_i = WR_i*\exp(-\beta_i^2/2)$

After r is calculated then:

$z1 = z_1(w2*\exp(\beta_2^2))$

$z2 = z_2(w1*\exp(\beta_1^2))$

t is then set to 1 and:

$f1 = f1(z_1(w2\exp(\beta_2^2)))$

$f2 = f2(z_2(w1\exp(\beta_1^2)))$

and $w = WR_{12} = \sqrt{w1^2*f2 + w2^2*f1}$

After r and w have been calculated, the combined damage sigma may be derived:

$$s = \sqrt{1 - r^2/w^2}$$

```
05 rem COMPUTES THE COMBINED WR AND SIGMA : Given 2 wr and s
10 data .07052 30784, .04228 20123, .00927 05272
20 data .00015 20143,  .00027 65672,  .00004 30638
30 read a,b,c,d,e,f
40 def fna(x) = 1 - 1/(1+a*x+b*x^2+c*x^3+d*x^4+e*x^5+f*x^6)^16
50 print "input wr(1),s(1),wr(2),s(2)",
60 input w1,s1,w2,s2
65 let t = 0
70 let b1 = sqr(- log(1-s1^2))
80 let b2 = sqr(- log(1-s2^2))
90 let b3 = sqr(b1^2+b2^2)
100 let z2 = (1/b3)*log(w2*exp(-b2^2)/w1)
110 let z1 = (1/b3)*log(w1*exp(-b1^2)/w2)
120 let f1 = .5-.5*z1/abs(z1)*fna(abs(z1)/1.41421)
130 let f2 = .5 - .5*z2/abs(z2)*fna(abs(z2)/1.41421)
135 if t > 0 then 180
140 let r = w1*exp(-b1^2/2)*f2 + w2*exp(-b2^2/2)*f1
150 print "<r> = " int(r+.5)
160 let z2 = (1/b3)*log(w2*exp(-b2^2)/w1/exp(b1^2))
170 let z1 = (1/b3)*log(w1*exp(-b1^2)/w2/exp(b2^2))
173 let t = t + 1
175 go to 120
180 let w = w1^2*f2 + w2^2*f1
190 let w = sqr(w)
200 print "WR = "int(w+.5)
210 let s = sqr(1-r^2/w^2)
220 print "sigma = "int(100*s+.5)/100
230 print
240 print
250 go to 50
400 end
```

```
READY

*run


input wr(1),s(1),wr(2),s(2)        ?
3500,.3,3000,.4
<r> =  3709
WR =  3873
sigma =   .29


input wr(1),s(1),wr(2),s(2)        ?
10500,.4,8000,.4
<r> =     10644
WR =    11375
sigma =   .35


input wr(1),s(1),wr(2),s(2)        ?
5000,.4,3500,.4
<r> =  4953
WR =  5306
sigma =   .36


input wr(1),s(1),wr(2),s(2)        ?
1200,.3,1000,.3
<r> =  1245
WR =  1289
sigma =   .26
```

Weapon Radii Determination for P and Q Targets Including K-Factor Adjustments

Scaled weapon radii (SWR) for PVN's and QVN's numerically calculated at various heights of burst as explained in Appendix A may be approximated by polynominal curve fits of the form

$$\log(\mathrm{SWR}) = \sum_{j=0}^{n} k_j (v2)^j$$

where v2 = adjusted VN

and n varies from 2 to 7 depending on the scaled height of burst

The constants k_j are calculated for SWR's associated with PVN's and QVN's at each 100 ft/kT$^{1/3}$ increment in SHOB from surface to 900 ft/kT$^{1/3}$. Fits at other HOB's are not provided because SWR's were calculated only at 100 ft/kT$^{1/3}$ increments in scaled HOB. This increment supplies sufficient accuracy when interpolating for a non-tabulated HOB. Fits are not provided for higher HOB's because higher HOB's would be above the "knee" in the HOB curves for pressures of interest.

For PVN's at scaled heights of zero ft., 100 ft, 800 ft. and 900 ft., two equations per SHOB are required. These are third and fourth degree polynominals with one fitting VN's <7.5 and the other fitting VN's >7.5. However, at all other overpressure SHOB's and at each dynamic pressure SHOB, one equation gives an accurate fit. These polynominals are of the sixth or seventh degree.

One advantage to this polynominal fit is that it gives a better interpolation for fractional adjusted VN's than can be achieved by interpolating between SWR's as given in a list.

A disadvantage is that optimum SHOB's must be chosen even to 100 ft. If the true optimum SHOB were 150 ft., the program would provide for calculating the WR at 100 ft. and at 200 ft. and then interpolate between these two to arrive at a WR for 150 ft. Thus if 100 ft. and 200 ft. were less than optimum, the WR computed at 150 ft. will also be less than optimum. Therefore, to obtain the largest WR possible using the given curve fits, select the SHOB even to 100 ft. which gives the largest WR. This calculated WR will generally not differ from the actual optimum WR by more than 5 ft/kT$^{1/3}$.

The following program is designed so that VN, type target, K factor, appropriate damage sigma, yield, and height of burst are input for each case. The appropriate damage sigma, s, is input to enable WR's to be computed for targets which do not have SIGMA-20 associated with their PVN or SIGMA-30 associated with their QVN.

The VN adjustment is performed using the following variable definitions.

v1 = base VN

k = k factor

Y = yield, in kilotons

v = adjustment to the base VN = A*log(R)

v2 = v1+v = the adjusted VN

The adjustment to the VN is derived by solving the equation

$$R = 1 - \left(\frac{k}{10}\right) + \left(\frac{k}{10}\right)\left(\frac{20}{Y}\right)^{1/3} R^{e} \qquad \text{(EQU. D-1)}$$

where e = 1/2 for PVN

e = 1/3 for QVN

The solution for R may be achieved by an iterative process. Modify Equation B-1 to read

$$r2 = 1 - \left(\frac{k}{10}\right) + \left(\frac{k}{10}\right)\left(\frac{20}{Y}\right)^{1/3} r1^{e} \qquad \text{(EQU. B-2)}$$

Initially set r1 = 2 if PVN

r1 = 3 if QVN

Solve for r2. If $|r2-r1| > .001$ then let r1 = r2 and again solve Equation B-2 for r2. Continue this process until $|r2-r1| < .001$

The adjustment, v is equal to $A*\log_e(r2)$

where A = 5.485 for PVN

A = 2.742 for QVN

The adjusted VN is then

v2 = v1+v

The WR is calculated using the following variable definitions.

h = actual height of burst

$h1$ = scaled height of burst = $h/Y^{1/3}$

$h(1)$ = int(h1/100)*100, i.e. the nearest even 100 less than h1

$h(2)$ = h(1)+100, i.e. the nearest even 100 greater than h1

$w(i)$ = scaled weapon radius at h(i)

w = SWR at h1

$w1$ = WR scaled to correct yield and damage sigma

s = appropriate damage sigma, .2, .3, .4, .5

k0,...k7 = appropriate constants for the polynominal curve fit; stored for each 100 SHOB up to 900 ft.

A scaled weapon radius is computed at h(1) and at h(2). If h1 = h(1) then a computation is made only at h(1).

$$g(i) = k0 + \sum_{j=1}^{n} k_j (v2)^j$$

$$w(i) = \exp(g(i))$$

The SWR for h1 is derived by linearly interpolating between w(1) and w(2).

$$w = w(1) + e * (w(2) - w(1))$$

$$\text{where } e = (h1 - h(1))/100$$

W is then scaled to the correct yield and damage sigma to give the actual WR. SWR's for PVN's are based on a damage sigma of .20 and SWR's for QVN's are based on a damage sigma of .30. If the appropriate damage sigma, s, differs from this generality, the suitable adjustment is based on

$$WR = r_{50}/(1-s^2)$$

$$s1 = 1.04 = 1/(1-.2^2) \text{ for PVN's}$$

$$s1 = 1.10 = 1/(1-.3^2) \text{ for QVN's}$$

Thus, $w * Y^{1/3}/s1 = r_{50}$ = range such that $P_d(r_{50}) = .5$

$$a1 = 1/(1-s^2)$$

$$w1 = r_{50} * a1 = w * Y^{1/3} * a1/s1$$

```
?
?

*list

05 rem          NWRCAL- Calculates Weapon Radii
10 print"INPUT VN,T,K,S,Yield,Hob"
20 input v1,t$,k,s,y,h
30 if t$ = "p" then 90
40 let e = (1/3)
50 let s1 = 1/(1-.3^2)
60 let a = 2.742
70 let r1 = 3
80 go to 130
90 let e = .5
100 let s1 = 1/(1-.2^2)
110 let a = 5.405
120 let r1 = 2
130 let r2 = 1-(k/10)+(k/10)*(20/y)^(1/3)*(r1)^e
140 if abs(r2-r1)< .001 then 170
150 let r1 = r2
160 go to 130
170 let v = a*log(r2)
180 let v2 = v1+v
190 let h1 = h/y^(1/3)
200 let h(1) =int(h1/100)*100
210 for i = 1 to 2
220 if t$ = "p" then 380
230 go to 1430
232 let g(i) = k0 +k1*v2+k2*v2^2 +k3*v2^3 + k4*v2^4
234 go to 250
240 let g(i) = k0+k1*v2+k2*v2^2+k3*v2^3+k4*v2^4+k5*v2^5+k6*v2^6+k7*v2^7
250 let w(i) = exp(g(i))
260 if abs(h1-h(1))< .01 then 280
270 go to 300
280 let i =2
290 let e = 0
300 next i
310 let a1 = 1/(1-s^2)
320 let w = w(1)+e*(w(2)-w(1))
325 let r5 = w*y^(1/3)/s1
330 let w1 = r5*a1
340 print "                    WR = "int(w1+.5)
350 print
360 print
365 go to 20
370 print"NO VALID WR=== VN TOO LARGE FOR HOB"
372 print
375 go to 20
380 rem this gives parameters for the pwrs
390 if h1 <=900 then 420
400 print"HOB too large for program"
410 go to 10
```

```
420 let h(2) = h(1) + 100.
430 let e = (h1-h(1))/100
440 if h(i) >= 100 then 550
442 if v2 > 7.5 then 470
450 let k0 = 8.206936
452 let k1 = -9.8662222e-02
454 let k2 = -4.2705319e-03
456 let k3 = 44.67361000e-05
458 let k4 = 0
460 go to 232
470 let k0 = 8.263243
480 let k1 = -1.2109524e-01
490 let k2 = 12.74266e-04
500 let k3 = -9.2065496e-06
502 let k4 = 0
540 go to 232
550 if h(i) >=200 then 660
555 rem CONSTANTS FOR 100'
557 if v2 > 51 then 370
560if v2 > 7.5 then 600
565 let k0 = 8.29123
570 let k1 = -1.132939e-01
575 let k2 = 31.19908e-05
580 let k3 = 0
590 let k4 = 0
595 go to 232
600 let k0 = 8.29959
610 let k1 = -1.1043358e-01
620 let k2 = -4.8494085e-04
625 let k3 = 65.8301e-06
630 let k4 = -9.1680378e-07
640 go to 232
660 if h(i) >= 300 then 770
665 rem CONSTANTS FOR 200'_______ONE EQ. for ALL VNS
670 if v2 > 41 then 370
680 let k0 = 8.395223
690 let k1 = -1.4717856e-01
700 let k2 = 12.74489e-03
710 let k3 = -2.0632771e-03
720 let k4 = 16.67591e-05
730 let k5 = -6.89342e-06
740 let k6 = 14.23714e-08
750 let k7 = -1.1675015e-09
760 go to 240
770 if h(i) >=400 then 880
775 if v2 > 34 then 370
780 let k0 = 8.419584
790 let k1 = -9.9827816e-02
800 let k2 = -4.1872797e-03
810 let k3 = 54.49084e-05
820 let k4 = -3.758352e-05
830 let k5 = 14.00969e-07
840 let k6 = -2.0170989e-08
850 let k7 = 0
870 go to 240
```

```
880 if h(i) >=500 then 990
885 remCONSTANTS FOR 400'__________ONE EQ. FOR ALL VNS
890 if v2 > 30 then 370
900 let k0 = 8.499489
905 let k1 = -1.0965211e-01
910 let k2 = -3.4445747e-03
920 let k3 = 72.61706e-05
930 let k4 = -7.10905e-05
940 let k5 = 33.19013e-07
950 let k6 = -5.6685057e-08
960 let k7 = 0
980 go to 240
990 if h(i) >= 600 then 1100
992 rem CONSTANTS FOR 500'______ONE EQ.for ALL VNS
995 if v2 > 27 then 370
1000 let k0 = 8.525985
1010 let k1 = -6.3120552e-02
1020 let k2 = -2.5622191e-02
1030 let k3 = 54.26447e-04
1040 let k4 = -5.926339e-04
1050 let k5 = 34.85504e-06
1060 let k6 = -1.0228646e-06
1070 let k7 = 11.4432e-09
1090 go to 240
1100 if h(i) >=700 then 1210
1105 rem CONSTANTS FOR 600'__________ ONE EQ. for ALLVNS
 1107 if v2 > 25 then 370
1110 let k0 = 8.586222
1120 let k1 = -1.002711e-01
1130 let k2 = -9.9171759e-03
1135 let k3 = 26.0232e-04
1140 let k4 = -3.6028224e-04
1150 let k5 = 28.02515e-06
1160 let k6 = -1.0826364e-06
1170 let k7 = 15.41557e-09
1200 go to 240
1210 if h(i) >=800 then 1320
1212 if v2 > 22 then 370
1215 rem CONSTANTS FOR 700'______ONE EQ. for ALL VN's
1220 let k0 = 8.655962
1225 let k1 = -1.3679886e-01
1230 let k2 = 14.26281e-03
1235 let k3 = -4.0929993e-03
1240 let k4 = 50.28125e-05
1245 let k5 = -2.5712239e-05
1250 let k6 = 43.79003e-08
1253 let k7 = 0
1260 go to 240
1320 if h(i) >=900 then 1380
1321 if v2 > 21 then 370
1322 rem CONSTANTS FOR 800
1324 if v2 > 7.5 then 1350
1326 let k0 = 8.681285
1330 let k1 = -1.1432864e-01
1335 let k2 = -1.7888686e-03
```

```
1340 let k3 = 15.95909e-05
1342 let k4 = 0
1345 go to 232
1350 let k0 = 12.51342
1355 let k1 = -1.516344
1360 let k2 = 17.69944e-02
1365 let k3 = -8.900835e-03
1370 let k4 = 14.00736e-05
1375 go to 232
1380 rem CONSTANTS FOR 900`
1381 if v2 > 20 then 370
1382 if v2 > 7.5 then 1400
1385 let k0 = 8.719654
1387 let k1 = -1.2158526e-01
1390 let k2 = 12.03604e-04
1392 let k3 = -1.3863281e-04
1394 let k4 = 0
1395 go to 232
1400 let k0 = 13.47289
1405 let k1 = -1.971983
1407 let k2 = 25.47267e-02
1409 let k3 = -1.4325115e-02
1410 let k4 = 26.40371e-05
1415 go to 232
1430 rem THIS SECTION PRESENTS PARAMETERS FOR QWRS
1440 if h1<= 900 then 1470
1450 print"HOB is too large for program"
1460 go to 10
1470 let h(2) = h(1) +100
1480 let e = (h1-h(1))/100
1490 if h(i) >= 100 then 1600
1494 rem CONSTANTS FOR 0 HOB
1496 if v2 > 35 then 370
1500 let k0 = 8.315159
1510 let k1 = -.1060868
1520 let k2= .0005224
1530 let k3 = -.000313
1540 let k4 = 3.22649e-05
1550 let k5 = -1.23227e-06
1560 let k6 = 1.96707e-08
1570 let k7 = -1.05880e-10
1590 goto240
1600 if h(i) >=200 then 1700
1604 remCONSTANTS FOR 100` SHOB
1606 if v2 > 35 then 370
1610 let k0 = 8.376082
1620 let k1 = -.1042945
1630 let k2 = -.0012014
1640 let k3 = -3.91136e-05
1650 let k4 = 1.28757e-05
1660 let k5 = -4.97579e-07
1670 let k6 = 5.77257e-09
1680 let k7 = 0
1695 go to 240
1700 if h(i) >= 300 then 1810
```

```
1704 rem CONSTANTS FOR 200` SHOB
1705 if v2> 35 then 370
1710 let k0 = 8.42024
1720 let k1 = -1.09473e-01
1730 let k2 = 14.62288e-04
1740 let k3 = -5.969792e-04
1750 let k4 = 66.97002e-06
1760 let k5 = -3.0149459e-06
1770 let k6 = 61.88228e-09
1780 let k7 = -4.866633e-10
1800 go to 240
1810 if h(i) >=400 then 1920
1812 rem CONSTANTS FOR 300` SHOB
1815 if v2 > 35 then 370
1820 let k0 = 8.485315
1830 let k1 = -.1031393
1840 let k2 = -.0034114
1850 let k3 = .0003087
1860 let k4 = -1.07267e-05
1870 let k5 = 3.15662e-07
1880 let k6 = -5.56646e-09
1890 let k7 = 0
1910 go to 240
1920 if h(i) >=500 then 2030
1922 remCONSTANTS FOR 400` SHOB
1925 if v2 > 31 then 370
1930 let k0 = 8.576003
1940 let k1 = -.1039885
1950 let k2 = -.0065788
1960 let k3 = .0012382
1970 let k4 = -.0001333
1980 let k5 = 8.01387e-06
1990 let k6 = -2.34684e-07
2000 let k7 = 2.51295e-09
2020 go to 240
2030 if h(i) >=600 then 2140
2032 rem CONSTANTS FOR 500`SHOB
2035 if v2 > 28 then 370
2040 let k0 = 8.643504
2050 let k1 = -.1110564
2060 let k2 = -.0041904
2070 let k3 = .0006644
2080 let k4 = -7.76848e-05
2090 let k5 = 5.98695e-06
2100 let k6 = -2.27079e-07
2110 let k7 = 3.00626e-09
2130 go to 240
2140 if h(i) >=700 then 2250
2142 rem CONSTANTS FOR 600` SHOB
2145 IF V2 > 26 THEN 370
2150 let k0 = 8.686697
2160 let k1 = -.1164822
2170 let k2 = .0003634
2180 let k3 = -.0006169
2190 let k4 = 8.57541e-05
```

```
2200 let k5 = -4.07263e-06
2210 let k6 = 5.66402e-08
2220 let k7 = 0
2240 go to 240
2250 if h(i) >=800 then 2360
2252 rem CONSTANTS FOR 700` SHOB
2255 if v2 > 25 then 370
2260 let k0 = 8.707449
2270 let k1 = -.1175502
2280 let k2 = .0023483
2290 let k3 = -.0013054
2300 let k4 = .0001909
2310 let k5 = -1.15200e-05
2320 let k6 = 2.83079e-07
2330 let k7 = -2.44704e-09
2350 go to 240
2360 if h(i) = 900 then 2470
2362 rem CONSTANTS FOR 800` SHOP
2365 if v2 > 23 then 370
2370 let k0 = 8.736328
2380 let k1 = -.1151805
2390 let k2 = .0021175
2400 let k3 = -.0015218
2410 let k4 = .0002654
2420 let k5 = -1.96750e-05
2430 let k6 = 6.18015e-07
2440 let k7 = -7.20562e-09
2460 go to 240
2470 rem Constants for 900 SHOB
2475 if v2 > 22 then 370
2480 let k0 = 8.793042
2490 let k1 = -.1154885
2500 let k2 = .0001871
2510 let k3 = -.0011008
2520 let k4 = .0002357
2530 let k5 = -2.01562e-05
2540 let k6 = 6.97520e-07
2550 let k7 = -8.74866e-09
2570 go to 240
2620 end

READY

*
```

```
13,p,0,.2,1,500
                    WR =  1204

?
13,p,0,.2,1,400
                    WR =  1114

?
13,p,0,.2,1,900
                    WR =  1048

?
?

*


run


INPUT VN,T,K,S,Yield,Hob
?
13,p,0,.2,1,100
                    WR =   993

?
15,p,3,.2,100,3500
                    WR =  4929

?
8,q,0,.3,1,100
                    WR =  1777

?
11,q,3,.3,1000,5000
                    WR =    15822

?
18,p,3,.3,150,700
                    WR =  3832

?
16,q,5,.4,500,5000
                    WR =  8098
```

Computation of Probability of Damage to Point Targets or Normally Distributed Area Targets

This appendix explains the basic probability of damage calculation program presented in Section VI for point targets or normally distributed area targets located either at the DGZ or offset some distance "x." The example program also provides for calculating probability of damage to a target when CEP = 0, i.e., when the impact point is known. The methodology outlined here may be adapted to the calculations of probabilities of damage to uniform area targets as explained in Section VI. B.2.

The probability of damaging a point target located a distance "x" from the DGZ is

$$P = \int_0^{2\pi}\int_0^{\infty} P_d(r)\,\frac{1}{2\pi\sigma^2}\,e^{-\frac{r^2+x^2-2rx\cos\theta}{2\sigma^2}}\,r\,dr\,d\theta$$

$$\text{where } P_d(r) = \int_{-\infty}^{z(r)} \frac{1}{\sqrt{2\pi}}\,e^{-y^2/2}\,dy$$

$$z(r) = \frac{1}{\beta}\ln\left(\frac{WRe^{-\beta^2}}{r}\right)$$

$$\sigma = \text{CEP}/1.1774$$

The probability of damaging a normally distributed area target can also be described by the above equation if σ is replaced by σ_A where

$$\sigma_A = \frac{CEP_A}{1.1774}$$

$$CEP_A = \sqrt{CEP^2 + .231*(P-95)^2}$$

As expressed above P = f(β,WR,x,σ,P-95). In computing P.D. slightly different parameters are input. They are

$s = \sigma_d$ = damage sigma (.1, .2, .3, .4, .5)

where β = sqr $(-\ln(1-s^2))$

wl = WR

xl = x = offset distance, which may be zero

c = CEP = measure of delivery error. (When P.D. from GZ is desired, input c = 0)

σ = CEP/1.1774

$c_a = CEP_A$ = adjusted CEP,

and $\sigma_A = \dfrac{CEP_A}{1.1774}$ where σ_A^2 describes the variance of the joint distribution of delivery error and target density.

tl = P-95 = radius of circle which encompasses 95% of the target being considered.

In the example program $CEP_A = c_a = \sqrt{c^2+.231*(tl)^2}$. Except for the case of calculating $P_d(r)$ from a GZ to a point target, all variables are standarized by division by $\sigma_A = \dfrac{c_a}{1.1774}$.

If the target is a point target then tl = 0 and c_a = c. If the probability of damage to a normally distributed area target from a GZ is desired then c = 0 and $c_a = \sqrt{.231*(tl)^2}$.

A CEP of zero denotes a situation in which the impact point is known and it is necessary to compute the probability of damage to a target located "xl" distance from the impact point. This probability is calculated by letting

$$z(xl) = 1/\beta * \ln (wl*\exp(-\beta^2)/xl)$$

then

$$P_d(xl) = \int_{-\infty}^{z(xl)} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= .5 + .5 \frac{|z|}{z} \operatorname{erf}\left(\frac{|z|}{\sqrt{2}}\right) .$$

The erf may be approximated as shown in Section VI.A.3. The approximation is fna(v) in the example program. If z>3.87 then P_d=1 and if z < -3.87 then P_d = 0 since:

$$\int_{-\infty}^{3.87} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy > .9999$$

and

$$\int_{-\infty}^{-3.87} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy < .0001$$

If $|z| < 5 \times 10^{-6}$ then $P_d = .5$. This is to avoid an exponential underflow on the computer, since

$$\mathrm{erf}\left(\frac{|z|}{\sqrt{2}}\right) = fna(v) = 1 - \frac{1}{D^{16}}$$

where $v = \frac{|z|}{\sqrt{2}}$

and $D = 1 + e_1 v + e_2 v^2 + e_3 v^3 + e_4 v^4 + e_5 v^5 + e_6 v^6$

When $|z| < 5 \times 10^{-6}$, D^{16} will contain negative exponents too large for the computer on which these example programs were written to handle and an exponent underflow message would be printed. To alleviate this problem, P_d is set equal to .5 when $|z| < 5 \times 10^{-6}$.

When delivery error and/or target density must be considered, all parameters are standardized by division with

$$\sigma_A = CEP_A/1.1774$$

$$w3 = \frac{WR}{\sigma_A} = 1.1774 * w1/c_a$$

$$x = \frac{XA}{\sigma_A} = 1.1774 * x1/c_a.$$

$$r = (\text{distance from impact point to target})/\sigma_A$$

The probability of damage P is

$$P = \int_0^{\infty} f(r)\,dr$$

where $f(r) = P_d(r)\; r\; e^{-\frac{1}{2}(r^2+x^2)} \frac{1}{2\pi}\int_0^{2\pi} e^{rx\cos\theta}\,d\theta$

Since it is necessary to evaluate f(r) only as long as f(r) is significantly greater than 0,

$$\int_0^\infty f(r)\,dr = \int_a^b f(r)\,dr$$

where $f(r) \simeq 0$ whenever $r<a$ or $r>b$.

Wherever r = 9.25 * WR * damage sigma (σ_d = .2, .3, .4, or .5), $P_d(r) \le .0005$, and it is unnecessary to evaluate the integral beyond the point b = 9.25 * WR * σ_d.

Impact points are assumed to be normally distributed about the DGZ. The probability that a weapon would impact farther than 4σ from the DGZ is given by

$$\frac{1}{2\pi}\int_4^\infty r e^{-r^2/2}\,dr = .00005$$

Since the DGZ is located "x" distance from the target, there is no significant contribution to P.D. when r < (x-4) or r > (x+4). Therefore, a = max (0,x-4) and ~~b = min (10*WR*σ_d, x+4)~~.
min [illegible]

The integral $\int_a^b f(r)dr$ may be evaluated using a 10 point Gauss-Legendre quadrature formula i.e.,

$$\int_a^b f(r)\,dr \doteq \frac{(b-a)}{2}\sum_{i=1}^{10} w_i f(r_i)$$

where

$$r_i = .5\left[(b-a)z_i + b + a\right].$$

The base points, z_i, are symmetrically placed with respect to the midpoint of the interval of integration. The weights, w_i, are the same for each symmetric pair of z_i. It is customary to designate $z_{\pm 1}$ for the pair of symmetric points nearest the midpoint, $z_{\pm 2}$ for the next symmetric pair, etc. The notation $z_{\pm n} = J$, $w_{\pm n} = K$ means $z_n = J$, $z_{-n} = -J$, $w_n = K$, $w_{-n} = K$.

10-Point Formula

$z_{\pm 1}$ = 0.14887 4339	$w_{\pm 1}$ = .29552 42247
$z_{\pm 2}$ = 0.43339 53941	$w_{\pm 2}$ = .26926 67193
$z_{\pm 3}$ = 0.67940 95683	$w_{\pm 3}$ = .21908 63625
$z_{\pm 4}$ = 0.86506 33667	$w_{\pm 4}$ = .14945 13492
$z_{\pm 5}$ = 0.97390 65285	$w_{\pm 5}$ = .06667 13433

At each r, z = 1/β ln(wl*exp(-β²)/r) and pl = P_d(r) is calculated as previously explained using fna(v). If h=0, i.e., x=0, then f(r) = pl*r*exp(-r²/2)

If h = (x*r) < 3.75 then j = (h/3.75)²

and f(r) = pl*r*exp(-(x²+r²)/2)*fnb(j)

$$\text{where fnb}(j) = \frac{1}{2\pi}\int_0^{2\pi} \exp(xr\cos\theta)\, d\theta$$

If h = x*r > 3.75 then j = 3.75/h

$$\text{and } f(r) = pl*r*\exp(-(x-r)^2/2) * \frac{fnc(j)}{\sqrt{h}}$$

$$\text{where } \frac{\text{fnc}(j)*\exp(xr)}{\sqrt{h}} = \frac{1}{2\pi}\exp(xr\cos\theta)\,d\theta$$

f(r) is then weighted by w_i and summed into f according to the Gauss-Legendre quadrature formula

$$P = .5(b-a)*f$$

```
SYSTEM ?
basic
old or new-old newprob
FILE newprob CLASSIFIED UNT

ready
*list

10 rem CALCULATES P.D. TO POINT TARGETS ----10-PT. Gaussian-Legendre
20 dim z(5),w(5),f(5,2)
30 read e1,e2,e3,e4,e5,e6
40 data .0705 2308, .0422 8201, .00927053, .00015201
45 data .0002 7657 , .00004306
50 def fna(v) = 1-1/(1+v*(e1+v*(e2+v*(e3+v*(e4+v*(e5+v*e6))))))^16
 60 read b1,b2,b3,b4,b5,b6
70 data 3.5156229, 3.0899424, 1.2067492
80 data .2659732, .0360768, .0045813
90 def fnb(v) =1+v*(b1+v*(b2+v*(b3+v*(b4+v*(b5+v*b6)))))
100 read c0,c1,c2,c3,c4,c5,c6,c7,c8
110 data .39894228,.01328592, .00225319, .00157565
120 data .00916281, .02057706, .02635537
130 data .01647633, .00392377
140 def fnc(v) = c0+v*(c1+v*(c2-v*(c3-v*(c4-v*(c5-v*(c6-v*(c7-v*c8)))))))
150 for I = 1 to 5
160 read z(i),w(i)
170 next i
180 data .148874339,.295524225   ,.433395394, .269266719
185 data .679409568, .219086363,   .865063367, .149451349
190 data .973906529, .066671344
200 print"INPUT SIGMA-  (decimal form)"
210 input s
220 print"INPUT WR, X, CEP, P-95"
230 input w1,x1,c,t1
240 let b = sqr(-log(1-s^2))
245 let c = sqr(c^2+.231*t1^2)
250 if c > 0 then 430
260 if x1 > 0 then 290
270 let p = 1
280 go to 860
290 let z = (1/b)*log((w1*exp(-b^2))/x1)
300 if z > 3.87 then 370
310 if abs(z) < 5e-06 then 390
320 if z < (-3.87 ) then 410
330 let u = abs(z)/sqr(2)
340 let e = fna(u)
350 let p = .5 + .5*abs(z)/z*e
360 go to 860
370 let p = 1
380 go to 860
390 let p = .5
400 go to 860
410 let p = 0
420 go  to 860
```


S27050

```
430 let w3 = 1.1774*w1/c
440 let x = 1.1774*x1/c
450 let f = 0
460 let b9 = 1.06*w3*exp (2.[illegible])
470 if b9 < 4 then 530
480 let a9 = x-4
490 if a9 > 0 then 510
500 let a9 = 0
510 let b9 = x+4
520 go to 540
530 let a9 = 0
540 for i = 1 to 5
550 for n = 1 to 2
560 let r = .5*(z(i)*(-1)^n*(b9-a9) + b9+a9)
570 let z = (1/b)*log((w3*exp(-b^2))/r)
580 if z >3.87 then 650
590 if abs(z) < 5e-06 then 670
600 if z < (-3.87) then 690
610 let u = abs(z)/sqr(2)
620 let e = fna(u)
630 let p1 = .5+.5*abs(z)/z*e
640 go to 710
650 let p1 = 1
660 go to 710
670 let p1 = .5
680 go to 710
690 let p1 = 0
700 go to 710
710 let h = x*r
720 if h = 0 then 800
730 if h > 3.75 then 770
740 let j = (h*h)/(3.75*3.75)
750 let f(i,n) = p1*r*exp(-(x^2+r^2)/2)*fnb(j)
760 go to 920
770 let j = 3.75/h
780 let f(i,n) = p1*r*exp(-(x-r)^2/2)*fnc(j)/sqr(h)
790 go to 920
800 let f(i,n) = p1*r*exp(-r^2/2)
910 go to 920
920 let f = f+w(i)*f(i,n)
930 next n
940 next i
950 let p = .5*(b9-a9)*f
960 print int(1000*p+.5)/1000
970 print
980 go to 230
990 end

READY

*
```

```
?

*run


INPUT SIGMA-  (decimal form)
?
.3
INPUT VR, X, CEP, P-95
?
3500,2500,1000,0
        .686

?
2250,0,1500,500
        .699

?
3000,1000,500,0
        .989

?
1500,0,1500,0
        .456

?
1500,0,500,0
        .969

?
1500  ,0,500,1000
        .987

?
```

Computation of the Offset Distance for a Specified Probability of Damage

Situations arise when it is desirable to know the maximum offset distance, x_M at which an aim point may be placed and still maintain a specified probability of damaging a target. It is possible to find this maximum offset distance by an iterative process of increasing the offset distance and calculating P.D. until the lowest acceptable P.D., P_m, is found. However, this method is slow and would be much too time consuming if many cases had to be considered.

Equations have been developed which yield a rapid calculation of x_M when $45\% < P_m < 95\%$. Two equations are given for each damage sigma, an equation of a curve is used for smaller values of WR/CEP and a linear equation for larger values. Constants in the equations are functions of P_m.

Let x_M' denote the offset distance such that $P(x_M') = P_m$ and let x_M be the offset distance as determined by the equations. The following statements of accuracy generally hold.

$$|x_M' - x_M| \le 0.10 \text{ CEP}$$

In the range of the equations of a curve, i.e. for smaller WR/CEP:

$$|P_m - P(x_M)| \le .02$$

In the range of the linear equations, i.e., for larger WR/CEP:

$$\left| P_m - P(x_M) \right| \leq .01$$

There are two exceptions to this general accuracy, both occur at $\sigma_d = .50$. When P_m is in the range of 60% then max $\left|P_m - P(x_M)\right| < 2.5\%$. When P is the range of 50% then max $\left|P_m - P(x_M)\right| < 5\%$. If greater accuracy is required, it is suggested that the equations be used to derive x_M, then x_M used as an initial starting value in the P.D. computer program, increasing or decreasing x_m as necessary to find x_M'.

The following equations are used for determining x_M as $f(\sigma_d, WR, CEP, P_m)$. For SIGMA-20 targets, when WR/CEP ≤ 3, then

$$x_M = CEP * A*(w1-w0)^B$$

where

$$w1 = WR/CEP$$

$$w0* = .58 + 1.7*P_m^2$$

$$A = 5.102 - 14.9435*P_m + 22.515*P_m^2 - 11.674*P_m^3$$

$$B = -5.218 + 27.37*P_m + -41.766*P_m^2 + 20.664*P_m^3$$

*w0 indicates the smallest WR/CEP which can achieve P_m at the DGZ. Therefore in the sample program if w1 < w0 a statement is printed saying "P cannot be achieved with this weapon."

When WR/CEP > 3,

$$x_M = CEP*(A + B*w1)$$

where

$$w1 = WR/CEP$$

$$A = -(.0153 + 1.184*P_m^4)$$

$$B = 1.195 - .456*P_m$$

For Sigma-30 targets, when WR/CEP ≤ 3.5,

$$x_M = CEP*\left(A+B*(w1-w0)+C*(w-w0)^2\right)$$

where

$$w1 = WR/CEP$$

$$w0 = (1.522 - 1.2058*P_m)^{-1}$$

$$A = 1.146 - 1.85*P_m+.975*P_m^2$$

$$B = 1.4$$

$$C = -.414 + 1.126*P_m - 1.14*P_m^2$$

When WR/CEP > 3.5,

$$x_M = CEP*(A + B + w1),$$

where

$$w1 = WR/CEP$$

$$A = -(.00662 + .93 * P_m^4)$$

$$B = 1.256 - .678*P_m$$

For SIGMA-40 targets, when WR/CEP $\leq$ 3.5,

$$x_M = CEP*\left(A+B(w1-w0) + C(w1-w0)^2\right)$$

where

$$w1 = WR/CEP$$

$$w0 = (1.467 - 1.217*P_m)^{-1}$$

$$A = Pm/(-2.73 + 8.033 *P_m)$$

$$B = 1.7 - .4115 *P_m$$

$$C = -(.508 - 1.184*P_m + 1.118*P_m^2)$$

When WR/CEP > 3.5,

$$x_M = CEP *(A + B * w1)$$

where

$$w1 = WR/CEP$$

$$A = .026 - 1.01*P_m^4$$

$$B = 1.234 - .785*P_m$$

For Sigma-50 targets, when WR/CEP $\leq$ 5,

$$x_M = CEP*\left(A+B(w1-w0)+C*(w1-w0)^2\right)$$

where

$$w1 = WR/CEP$$

$$w0 = (1.34 - 1.15*P_m)^{-1}$$

$$A = .28$$

$$B = 2.152 - 1.524*P_m$$

$$C = -(.0252 + .35*P_m - .33*P_m^2)$$

When WR/CEP > 5,

$$x_M = CEP*(A + B * wl)$$

where

$$wl = WR/CEP$$

$$A = .113 - .864*P_m^{4}$$

$$B = 1.17 - .858 * P_m$$

```
3
*list

03 rem CALCULATES MAX. OFFSET FOR MIN. REQUIRED P.D.
05 print"INPUT S"
10 input s
15 print"INPUT P, WR, CEP"
20 input p,w,c0
25 let w1 = w/c0
30 if s = .2 then 200
35 if s = .3 then 300
40 if s = .4 then 400
45 if s = .5 then 500
50 print"nonappropriate sigma"
55 go to 10
200 if w1> 3 then 250
210 let w0 = .58+1.7*p^2
212  if w1< w0 then 580
215 let a = 5.102-14.9435*p+22.515*p^2-11.674*p^3
220 let b = -5.218 + 27.37*p -41.766*p^2 +20.664*p^3
230 let x1 = a*(w1-w0)^b
240 go to 600
250 let a = -(.0153 + 1.184*p^4)
255 let b = 1.195 - .456*p
260 let x1 = a+b*w1
 265 go to 600
300 if w1 > 3.5 then 350
305 let w0 = (1.522-1.2058*p)^(-1)
307 if w1< w0 then 580
310 let a = 1.146-1.85*p+.975*p^2
315 let b = 1.4
320 let c = -.414 + 1.126*p -1.14*p^2
325 let x1 = a+b*(w1-w0) + c*(w1-w0)^2
330 go to 600
350 let a = -(.00662 +.93*p^4)
355 let b = 1.256 -.678*p
360 let x1 = a+b*w1
 365 go to 600
400 if w1 > 3.5 then 450
405 let w0 = (1.467-1.217*p)^(-1)
407 if w1 < w0 then 580
410 let a = p/(-2.73 + 8.033*p)
415 let b = 1.7-.4115*p
420 let c = -(.508 -1.194*p + 1.118*p^2)
```

S27050

```
425 let x1 = a+b*(w1-w0) + c*(w1-w0)^2
430 go to 600
450 let a = .026 -1.01*p^4
455 let b = 1.234-.785*p
460 let x1 = a+b*w1
 465 go to 600
500 if w1 > 5 then 550
505 let w0 = (1.34-1.15*p)^(-1)
507 if w1 < w0 then 580
510 let a = .28
515 let b = 2.152-1.524*p
520 let c = -(.0252 +.35*p -.33*p^2)
525 let x1 = a+b*(w1-w0) +c*(w1-w0)^2
530 go to 600
550 let a = .113-.864*p^4
 555 let b = 1.17-.958*p
560 let x1 = a+b*w1
 565 go to 600
580 printp,w," p cannot be achieved with this weapon"
585 go to 20
600 if x1> 0 then 605
 603 let x1 = 0
605 let x = x1*c0
610 print"              "x
620 print
630 go to 20
999 end

ready

*run



INPUT S
?.2
INPUT P, WR, CEP
?.8,4500,1000
                    3235.634

?.9,5500,1500
                    3127.116

?.75,10000,2500
                    7555.187

?.6,10000,2000
                    8876.507

?.5,2000,1000
                    1794.269
```

```
?.95,2000,1000
        .95              2000
 p cannot be achieved with this weapon
?.35,2000,1000
                    593.1782

?
*run



INPUT S
?.3
INPUT P, WP, CEP
?.75,4500,3000
        .75              4500
 p cannot be achieved with this weapon
?.75,9000,3000
                    5515.112

?.35,5000,1500
                    2457.137

?
```

Computation of Probability of Damage to Circular Area Targets with Uniform Density

To compute probability of damage to circular area targets with uniformly distributed elements, concentric circles are drawn about the DGZ (or GZ if CEP is 0) with radii $r_i = r_{i-1} + \Delta r$ where Δr is a small increment. The i^{th} annulus is defined as $2\pi r_i^2 - 2\pi r_{i-1}^2$. The probability of damage to any target element located in the i^{th} annulus is P_i. If a_i is the area of intersection of the i^{th} annulus with the target then the average probability of damage to the uniform circular target is

$$P = \frac{\sum_{i=1}^{I} P_i a_i}{\sum_{i=1}^{I} a_i}$$

where TR = target radius,

$$\sum_{i=1}^{I} a_i = \pi TR^2$$

and I is such that a circle of radius r_I about the DGZ encompasses the target.

WR, target radius, offset distance of the center of the target to the DGZ and CEP are input into the example program as wl, tl, xl, and c. Computation, however, is based on the parameters w3, t3, and x3. If CEP = 0 then

$$w3 = WR = w1$$

$$t3 = t1 = \text{target radius}$$

$$x3 = x1 = \text{offset distance}$$

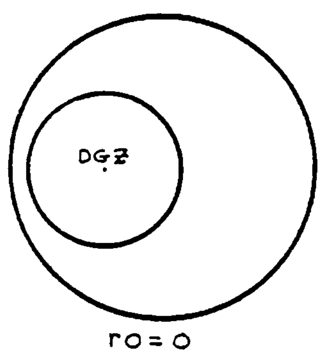
If CEP > 0, then

$$w3 = \frac{1.1774*w1}{c} = \frac{WR}{\sigma}$$

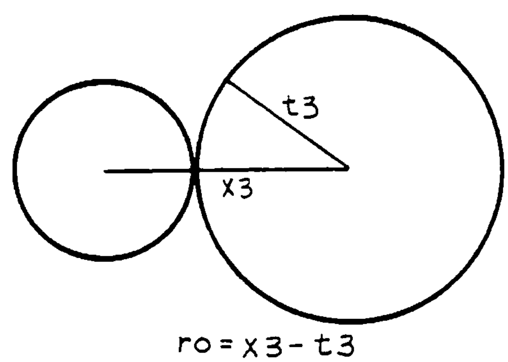
$$t3 = \frac{1.1774*t1}{c} = \frac{TR}{\sigma}$$

$$x3 = \frac{1.1774*x1}{c} = \frac{\text{offset distance}}{\sigma}$$

The radius of the i^{th} concentric circle about the DGZ is r(i) = r0 + i*k1. r0 is the initial value of r(i) and is 0 if the DGZ is located inside the target area. Referring to figure G-1, if the DGZ is located outside the target area then r0 = x3-t3. In the example program k1 = Δr = t3/10.



DGZ Inside Target



DGZ Outside Target

Figure G-1

If the intersection of the i^{th} concentric circle about the DGZ with the target is s(i), then the intersection of the i^{th} annulus with the target is a(i) = s(i)-s(i-1). (See figure G-2)

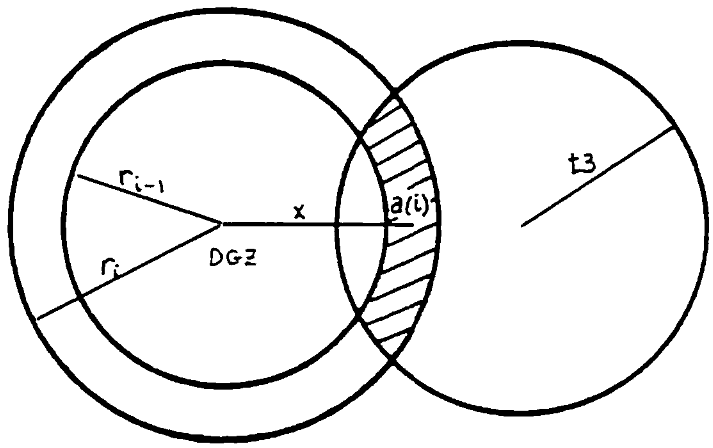


Figure G-2

The program uses the notation x = r(i) - .5*kl, where x is the average offset distance of target elements in the i^{th} annulus.

The average probability of damage p(i) to target elements in the i^{th} annulus is computed as follows.

If the CEP = 0, then $p(i) = P_d(x) = \int_{-\infty}^{z(x)} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$

where $z(x) = \frac{1}{\beta} \ln\left(\frac{W3\, e^{-\beta^2}}{x}\right)$

If CEP > 0, then $p(i) = \int_0^{2\pi}\int_0^{\infty} \frac{P_d(r)}{2\pi}\, r \exp(-.5(r^2+x^2-2rx\cos\theta))\,dr\,d\theta$

The probability p(i) is multiplied by a(i) and this product is summed for all i.

This process is continued by incrementing r(i), computing s(i), a(i), and p(i) for each r(i), and summing p(i)*a(i) until r(i) = t3 + x3, i.e., until the concentric circles about the DGZ cover the total target area.

The average P.D. to the target, P, is then

$$P = \frac{\sum_{i=1} p(i)\, a(i)}{\pi\,(t3)^2}$$

The area of intersection of two circles, s(i), is calculated as follows. If the DGZ is inside the target, then r(i) < t3 = x3.

In this case $s(i) = \pi\, r(i)^2$ (See figure G-3)

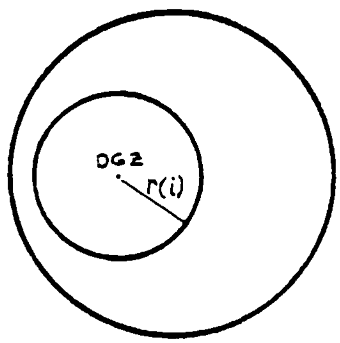
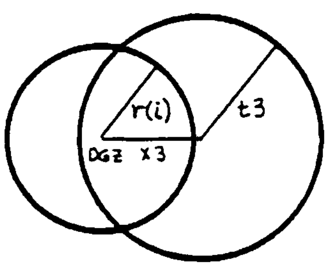
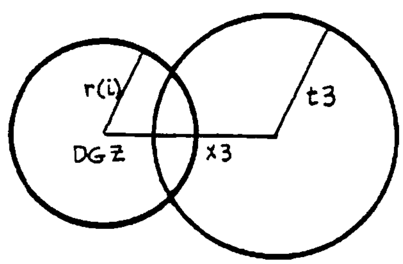


Figure G-3

For a DGZ inside or outside the target where r(i) is greater than (t3 - x3) (See figure G-4),



DGZ Inside Target



DGZ Outside Target

Figure G-4

the following notation is used

d1 = the cartesian coordinate in the horizontal direction of the intersection of the i^{th} circle about the DGZ with the target, based on a coordinate system centered at the DGZ.

$$= \frac{r(i)^2 - t3^2 + x3^2}{2 * x3}$$

d2 = the cartesian coordinate in the horizontal direction of the intersection of the i^{th} circle about the DGZ with the target, based on a coordinate system centered at the target center.

$$= d1 - x3$$

y1 = y2 = the cartesian coordinate in the vertical direction of the intersection of the i^{th} circle about the DGZ with the target. It is the same if the coordinate system is centered at the DGZ or at the target center.

$$y1 = \sqrt{r(i)^2 - d1^2}$$

$$y2 = \sqrt{t3^2 - d2^2}$$

$$\left.\begin{array}{l} a1 = \text{atn } (y1/d1) \\ a2 = \text{atn } (y2/d2) \end{array}\right\} \text{ where atn is the arctangent function}$$

In the example program, if y1/d1 or y2/d2 is negative the computer places the arctangent in the fourth quadrant. Since an angle in the second quadrant is desired, π radians are added. Therefore,

if d1 < 0 then a1 = atn (y1/d1) + π

if d2 < 0 then a2 = atn (y2/d2) + π

if d1 or d2 = 0 then a1 or a2 is $\pi/2$ radians

s(i) = s1 + s2

where s1 = $r(i)^2$*(a1-.5sin(2(a1))

s2 = $t3^2 * \pi$ - $t3^2$*(a2-.5sin(2*a2))

```
READY

*list

05 dim x3(25),r(25),a(25),s(25),f(5,2)
06 dim z(5),w(5)
10 read e1,e2,e3,e4,e5,e6
15 data .0705230784, .0422820123 ,.009270572
20 data .0001520143, .0002765672, .0000430638
25 def fna(v) = 1-1/(1+v*(e1+v*(e2+v*(e3+v*(e4+v*(e5+v*e6))))))^16
30 read b1,b2,b3,b4,b5,b6
35 data 3.5156229, 3.0899424, 1.2067492
40 data .2659732, .0360768, .0045813
45 def fnb(v) = 1+v*(b1+v*(b2+v*(b3+v*(b4+v*(b5+v*b6)))))
50 read c0,c1,c2,c3,c4,c5,c6,c7,c8
55 data .39894228 , .01328592 , .00225319, .00157565
60 data .00916281, .02057706, .02635537
65 data .01647633, .00392377
70 def fnc(v) =c0+v*(c1+v*(c2-v*(c3-v*(c4-v*(c5-v*(c6-v*(c7-v*c8)))))))
71 for i2 = 1 to 5
72 read z(i2),w(i2)
73 next i2
74 data .148874339, .295524225 , .433395394, .269266719
75 data .679409568, .219086363, .865063367, .149451349
76 data .983906529, .066671344
78 print" input SIGMA - (decimal form)"
80 input s
85 print" input WR,TR,X,CEP"
90 input w1,t1,x1,c
95 let b = sqr(-log(1-s^2))
100 if c = 0 then 125
105 let x3 = 1.1774*x1/c
110 let w3 = 1.1774*w1/c
115 let t3 = 1.1774*t1/c
120 go to 140
125 let x3 = x1
130 let w3 = w1
135 let t3 = t1
140 let p2 = 0
145 let i = 0
150 let k1 = t3/10
155 if x3 > 0 then 170
160 go sub 320
165 go to 90
170 if x3 > t3 then 190
175 rem DGZ INSIDE TARGET
180 let r0 = 0
185 go to 205
190 rem DGZ OUTSIDE TARGET
195 let r0 = x3-t3
200 rem r(i) IS THE RADIUS OF THE ITH CONCENTRIC CIRCLE ABOUT THE DGZ
205 let i = i+1
210 let r(i) = r0 +i*k1
215 if x3 +r(i) >t3 then 255
```

```
220 rem ANNULUS CONTAINED INSIDE TARGET
225 let s(i) =3.14159*r(i)^2
230 go to 255
235 rem ANNULUS NOT CONTAINED INSIDE TARGET
240 if r(i) < t3+x3 then 250
245 let r(i) = t3+x3
250 go sub 685
 255 let a(i) = s(i) - s(i-1)
260 rem     x IS THEN AVERAGE OFFSET DISTANCE OF TARGET ELEMENTS ---
261 REM I
265 let x = r(i) -.5*k1
270 go sub 385
 275 let p2 = p2+p*a(i)
280 if abs(d1-r(i))< .01 then 290
285 go to 205
290 let p2 = p2/(3.14159*t3^2)
300 print using 305,p2
305:                  #.###
310 print
315 go to 90
320 rem DCZ AT CENTER OF TARGET
325 for i = 1 to 10
330 let r(i) = i*k1
335 let x = r(i) -.5*k1
340 let s(i) = 3.14159*r(i)^2
345 let a(i) = s(i)-s(i-1)
350  go sub 385
355 let p2 = p2 +p*a(i)
360 next i
365 let p2 = p2/(3.14159*t3^2)
370 print using 305,p2
380 return
385rem CALCULATES PD
390 if c = 0 then 610
405 let f = 0
410 [illegible]
415 if b9< 4 then 433
420 let a9 = x-4
423 if a9 > 0 then 428
425 let a9 = 0
428 let b9 = x+4
430 go to 435
433 let a9 = 0
435 for i2 = 1 to 5
438 for n = 1 to 2
440 let r = .5*(z(i2)*(-1)^n*(b9-a9)+b9+a9)
445 let z = (1/b)*log((w3*exp(-b^2))/r)
450 if z >3.87 then 485
455 if abs(z) < 5e-06 then 495
460 if z < (-3.87) then 505
465 let u = abs(z)/sqr(2)
470 let e = fna(u)
475 let p1 = .5+.5*abs(z)/z*e
480 go to 515
485 let p1 = 1
```

```
490 go to 515
495 let p1 = .5
500 go to 515
505 let p1 = 0
510 go to 515
515 let h = x*r
520 if h = 0 then 570
522 if h > 3.75 then 535
525 let j = (h*h)/(3.75*3.75)
527 let f(i2,n) = p1*r*exp(-(x^2+r^2)/2)*fnb(j)
530 go to 590
535 let j = 3.75/h
540 let f(i2,n) = p1*r*exp(-(x-r)^2/2)*fnc(j)/sqr(h)
545 go to 575 delete
550 go to 590
570 let f(i2,n) = p1*r*exp(-r^2/2)


590 let f = f+w(i2)*f(i2,n)
592 next n
595 next i2
600 let p = .5*(b9-a9)*f
605 go to 680
610 rem CALCULATES Pd FROM GZ
615 let z = 1/b*log((w1*exp(-b^2))/x)
620  if z > 3.87 then 655
625 if abs(z) < 5e-06 then 665
630 if z < (-3.87 ) then 675
635 let u = abs(z)/sqr(2)
640 let e = fna(u)
645 let p = .5+.5*abs(z)/z*e
650 go to 680
655 let p = 1
660 go to 680
665 let p = .5
670 go to 680
675 let p = 0
680 return
685 rem CALCULATES INTERSECTION WHEN ONE CIRCLE IS NOT CONTAINED
690 rem INSIDE THE OTHER
695 let d1 = (r(i)^2 -t3^2 +x3^2)/(2*x3)
700 let y1 = sqr(r(i)^2 -d1^2)
705 let d2 = d1-x3
710 let y2 = sqr(t3^2-d2^2)
715 if d2 => 0 then 730
720 let a2 = atn(y2/d2) + (355/113)
725 go to 750
730 if d2 > 0 then 745
735 let a2 = (355/113)*.5
740 go to 750
745 let a2 = atn(y2/d2)
750 if d1=> 0 then 765
755 let a1 = atn(y1/d1) + (355/113)
760 go to 785
```

```
765 if d1> 0 then 780
770  let a1 = (355/113)*.5
775 go to 785
780 let a1 = atn(y1/d1)
785 let s1 = r(i)^2*(a1-.5*sin(2*a1))
790 let h = a1*180/3.14159
795 let h1 = a2*180/3.14159
800 let s2 = t3^2*(355/113)-t3^2*(a2-.5*sin(2*a2))
805 let s(i) = s1+s2
810 let a(i) = s(i)-s(i-1)
815 return

825 end

READY

*
```

```
 input SIGMA - (decimal form)
?
.2
 input WR,TR,X,CEP
?
3500,1500,4000,1800
                         .298

?
5000,2000,4000,1500
                         .615

?
1500,450,500,2000
                         .299

?
1500,4500,0,1000
                         .112
```