▶ Preamble

$DE1 := KP(E1, K1) + KP(ID, E1) : DE2 := KP(E2, K2) + KP(ID, E2) :$

$DF1 := KP(F1, ID) + KP\left(\frac{1}{K1}, F1\right) : DF2 := KP(F2, ID) + KP\left(\frac{1}{K2}, F2\right) :$

$DK1 := KP(K1, K1) : DK2 := KP(K2, K2) :$

$DE12 := DE1 \cdot DE2 : DE21 := DE2 \cdot DE1 : DE121 := DE1 \cdot DE2 \cdot DE1 :$

$DE212 := DE2 \cdot DE1 \cdot DE2 : DE1212 := DE2 \cdot DE1 \cdot DE2 \cdot DE1 : DF12 := DF1 \cdot DF2 : DF21 := DF2 \cdot DF1 : DF121 := DF1 \cdot DF2 \cdot DF1 : DF212 := DF2 \cdot DF1 \cdot DF2 : DF1212 := DF2 \cdot DF1 \cdot DF2 \cdot DF1 :$

$basislist := [ID, F1, F2, F1 \cdot F2, F2 \cdot F1, F1 \cdot F2 \cdot F1, F2 \cdot F1 \cdot F2, F1 \cdot F2 \cdot F1 \cdot F2] :$

$DElist := [IDD, DE1, DE2, DE12, DE21, DE121, DE212, DE1212] : DFlist := [IDD, DF1, DF2, DF12, DF21, DF121, DF212, DF1212] :$

$\mathrm{phi} := Matrix(64) :$

for i **to** 8 **do**

  for j **to** 8 **do**

$\mathrm{phi}_{.., 8(i-1)+j} := \left(DFlist_i\right)_{.., 8(j-1)+1}$

  end do

end do: $phiinv := \frac{1}{\mathrm{phi}} :$

for i **to** 8 **do**

$conjDFlist_i := ssimplify\left(spmm\left(phiinv, spmm\left(DFlist_i, \mathrm{phi}\right)\right)\right) :$

end do:

$\mathrm{Omega} := ssimplify(spmm(phiinv, spmm(DE1212, spmm(DF1212, \mathrm{phi})))) :$

$\mathrm{zeta} := Matrix(64) :$

for i **to** 8 **do**

  for j **to** 8 **do**

$\mathrm{zeta}_{.., 8(i-1)+j} := conjDFlist_j.(\mathrm{Omega})_{.., i}$

  end do

end do: $\mathrm{zeta} := ssimplify(\mathrm{zeta}) : zetainv := \mathrm{zeta}^{-1} :$

$Gammainv := ssimplify(spmm(zetainv, phiinv)) : \mathrm{Gamma} := \frac{1}{Gammainv} :$

$R1 := simplify\left(DiagonalMatrix\left(\text{`~`}_q\left(Diagonal\left(\frac{2}{3} KP\left(H1, subs\left(\lambda = \Lambda, H1\right)\right) + \frac{2}{3} KP\left(H2, subs\left(\mu = \mathrm{M}, H2\right)\right) + \frac{1}{3} KP\left(H2, subs\left(\lambda = \Lambda, H1\right)\right) + \frac{1}{3} KP\left(H1, subs\left(\mu = \mathrm{M}, H2\right)\right)\right)\right)\right) \cdot (IdentityMatrix(64) + 2\,\mathrm{I}\,KP(E1, F1)) \cdot (IdentityMatrix(64) + 2\,\mathrm{I}\,KP(E3, F3)) \cdot (IdentityMatrix(64) + 2\,\mathrm{I}\,KP(E2, F2)), power\right) :$

$R := \text{`~`}_{simplify}\left(\text{`~`}_{expand}\left(subs\left(\mu = -\frac{2\,\mathrm{I}\ln(t2)}{\pi}, \lambda = -\frac{2\,\mathrm{I}\ln(t1)}{\pi}, \mathrm{M} = -\frac{2\,\mathrm{I}\ln(t2)}{\pi}, \Lambda = -\frac{2\,\mathrm{I}\ln(t1)}{\pi}, \frac{P \cdot R1}{R1_{8,8}}\right)\right)\right) :$

$blockR := ssimplify(spmm(Gammainv, spmm(R, \mathrm{Gamma}))) :$

$r := Matrix(8) :$

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for i to 8 do
    for j to 8 do
        r[i,j] := blockR[8*i, 8*j]
    end do
end do: r; with(PolynomialTools) : CoefficientVector( CharacteristicPolynomial(r, x), x);
    simplify( CharacteristicPolynomial(r, x) );
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$$\begin{bmatrix} t1^2\,t2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -t2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{t1^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{t2^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{t1^2\,t2^2} \end{bmatrix}$$

$$\left[\left[1\right],\right.$$

$$\left[-\frac{t1^4\,t2^4 - t1^4\,t2^2 - t1^2\,t2^4 - t1^2 - t2^2 + 1}{t1^2\,t2^2}\right],$$

$$\left[-\frac{1}{t1^4\,t2^4}\left(t1^8\,t2^6 + t1^6\,t2^8 - t1^6\,t2^6 + t1^6\,t2^4 + t2^6\,t1^4 - t1^6\,t2^2 - 4\,t1^4\,t2^4 - t1^2\,t2^6\right.\right.$$
$$\left.\left. + t1^4\,t2^2 + t1^2\,t2^4 - t1^2\,t2^2 + t1^2 + t2^2\right)\right],$$

$$\left[-\frac{1}{t1^4\,t2^4}\left(t1^8\,t2^8 + t1^8\,t2^4 + 3\,t1^6\,t2^6 + t1^4\,t2^8 - 3\,t1^6\,t2^4 - 3\,t2^6\,t1^4 + 2\,t1^4\,t2^4\right.\right.$$
$$\left.\left. - 3\,t1^4\,t2^2 - 3\,t1^2\,t2^4 + t1^4 + 3\,t1^2\,t2^2 + t2^4 + 1\right)\right],$$

$$\left[-\frac{1}{t1^4\,t2^4}\left(2\,\left(t1^8\,t2^6 + t1^6\,t2^8 - t1^6\,t2^6 + t1^6\,t2^4 + t2^6\,t1^4 - t1^6\,t2^2 - 3\,t1^4\,t2^4 - t1^2\,t2^6\right.\right.\right.$$
$$\left.\left.\left. + t1^4\,t2^2 + t1^2\,t2^4 - t1^2\,t2^2 + t1^2 + t2^2\right)\right)\right],$$

$$\left[-\frac{1}{t1^4\,t2^4}\left(t1^8\,t2^8 + t1^8\,t2^4 + 3\,t1^6\,t2^6 + t1^4\,t2^8 - 3\,t1^6\,t2^4 - 3\,t2^6\,t1^4 + 2\,t1^4\,t2^4\right.\right.$$
$$\left.\left. - 3\,t1^4\,t2^2 - 3\,t1^2\,t2^4 + t1^4 + 3\,t1^2\,t2^2 + t2^4 + 1\right)\right],$$

$$\left[-\frac{1}{t1^4\,t2^4}\left(t1^8\,t2^6+t1^6\,t2^8-t1^6\,t2^6+t1^6\,t2^4+t2^6\,t1^4-t1^6\,t2^2-4\,t1^4\,t2^4-t1^2\,t2^6 \right.\right.$$
$$\left.\left.+t1^4\,t2^2+t1^2\,t2^4-t1^2\,t2^2+t1^2+t2^2\right)\right], \qquad ,$$
$$\left[-\frac{t1^4\,t2^4-t1^4\,t2^2-t1^2\,t2^4-t1^2-t2^2+1}{t1^2\,t2^2} \right],$$
$$\left[1 \right] \qquad , \qquad \Bigg]$$
$$\frac{\left(x^2+1\right)\left(t2^2+x\right)\left(t2^2\,x+1\right)\left(t1^2+x\right)\left(t1^2\,x+1\right)\left(-t1^2\,t2^2+x\right)\left(x\,t1^2\,t2^2-1\right)}{t1^4\,t2^4} \tag{1}$$

$dim := 8$:
$hred := ssimplify(\,(tr2(spmm(Gammainv, spmm(KP(ID, K1^{-2}.K2^{-2}), \mathrm{Gamma}))))\,)$:
$char := CharacteristicPolynomial(r^2, x)$:
$sequencevector := Vector(degree(char, x)+1)$:
for i **from** 0 **to** $degree(char, x)$ **do**
  $sequencevector_{i+1} := x(n+i)$:
end do:
$recursion := DotProduct(sequencevector, CoefficientVector(char, x), conjugate = false)$:
$invariants := [\,]$: **for** i **from** 0 **to** $\frac{1}{2}\,degree(char, x)$ **do**
  $invariants := [op(invariants), simplify(expand(simplify(Trace(hred \cdot r^{2i+1}))))]$:
end do:
$IC := \{recursion\}$: **for** i **to** $\frac{1}{2}\,degree(char, x)$ **do**
  $IC := IC \cup \{x(-i) = expand(invariants_i), x(i-1) = expand(invariants_i)\}$:
end do:
$T2n := simplify(rsolve(IC, x))$ assuming $n :: integer$

$$T2n := \frac{t2^2\left(t1^2-1\right)t1^{-4n}}{\left(t2^2+1\right)\left(t1^4+1\right)\left(t1^2\,t2^2-1\right)}+\frac{t1^2\left(t2^2-1\right)t2^{-4n}}{\left(t1^2+1\right)\left(t1^2\,t2^2-1\right)\left(t2^4+1\right)} \tag{2}$$
$$+\frac{t1^{4+4n}\,t2^2\left(t1^2-1\right)}{\left(t2^2+1\right)\left(t1^4+1\right)\left(t1^2\,t2^2-1\right)}+\frac{\left(t1^2\,t2^2+1\right)(t1\,t2)^{-4n}}{\left(t1^4\,t2^4+1\right)\left(t2^2+1\right)\left(t1^2+1\right)}$$
$$+\frac{t1^2\,t2^{4+4n}\left(t2^2-1\right)}{\left(t1^2+1\right)\left(t1^2\,t2^2-1\right)\left(t2^4+1\right)}+\frac{t1^4\,t2^4\left(t1^2\,t2^2+1\right)(t1\,t2)^{4n}}{\left(t1^4\,t2^4+1\right)\left(t2^2+1\right)\left(t1^2+1\right)}$$

$$termt1 := simplify\left(\frac{t1^{4+4n}\,t2^2\left(t1^2-1\right)}{\left(t2^2+1\right)\left(t1^4+1\right)\left(t1^2\,t2^2-1\right)}+\frac{t2^2\left(t1^2-1\right)t1^{-4n}}{\left(t2^2+1\right)\left(t1^4+1\right)\left(t1^2\,t2^2-1\right)}\right)$$
assuming $n :: integer$

$$termt1 := \frac{\left(t1^2-1\right)t2^2\left(t1^{4+4n}+t1^{-4n}\right)}{\left(t2^2+1\right)\left(t1^4+1\right)\left(t1^2\,t2^2-1\right)} \tag{3}$$

$$termt2 := simplify\left(\frac{t1^2\,t2^{4+4n}\left(t2^2-1\right)}{\left(t1^2+1\right)\left(t1^2\,t2^2-1\right)\left(t2^4+1\right)}+\frac{t1^2\left(t2^2-1\right)t2^{-4n}}{\left(t1^2+1\right)\left(t1^2\,t2^2-1\right)\left(t2^4+1\right)}\right)$$
assuming $n :: integer$

$$termt2 := \frac{(t2^2-1)\,t1^2\,(t2^{4+4n}+t2^{-4n})}{(t1^2+1)\,(t1^2\,t2^2-1)\,(t2^4+1)} \tag{4}$$

$$termt12 := simplify\left(\frac{(t1^2\,t2^2+1)\,(t1\,t2)^{-4n}}{(t1^4\,t2^4+1)\,(t2^2+1)\,(t1^2+1)}+\frac{t1^4\,t2^4\,(t1^2\,t2^2+1)\,(t1\,t2)^{4n}}{(t1^4\,t2^4+1)\,(t2^2+1)\,(t1^2+1)},\ power, symbolic\right) \text{assuming } n :: integer$$

$$termt12 := \frac{(t1^2\,t2^2+1)\,(t1^{-4n}\,t2^{-4n}+t1^{4+4n}\,t2^{4+4n})}{(t1^4\,t2^4+1)\,(t2^2+1)\,(t1^2+1)} \tag{5}$$