THE EXPERIMENTAL LAW OF COULOMB

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is:

1. Along the line joining them
2. Directly proportional to the product Q_1Q_2 of the charges
3. Inversely proportional to the square of the distance R between them.[3]

Expressed mathematically,

$$F = \frac{k\,Q_1Q_2}{R^2} \tag{4.1}$$

where k is the proportionality constant whose value depends on the choice of system of units. In SI units, charges Q_1 and Q_2 are in coulombs (C), the distance R is in meters (m), and the force F is in newtons (N) so that $k = 1/4\pi\varepsilon_o$. The constant ε_o is known as the *permittivity of free space* (in farads per meter) and has the value

$$\boxed{\begin{aligned} \varepsilon_o &= 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}\ \text{F/m} \\ \text{or} \quad k &= \frac{1}{4\pi\varepsilon_o} \simeq 9 \times 10^9\ \text{m/F} \end{aligned}} \tag{4.2}$$

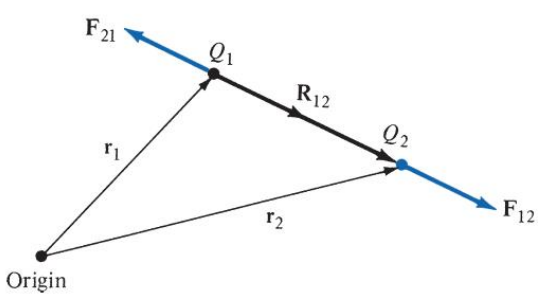

FIGURE 4.1 Coulomb vector force on point charges Q_1 and Q_2.

Thus eq. (4.1) becomes

$$F = \frac{Q_1Q_2}{4\pi\varepsilon_o R^2} \tag{4.3}$$

If point charges Q_1 and Q_2 are located at points having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then the force $\mathbf{F}_{12}$ on Q_2 due to Q_1, shown in Figure 4.1, is given by

$$\boxed{\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\,\mathbf{a}_{R_{12}}} \tag{4.4}$$

where

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \tag{4.5a}$$

$$R = |\mathbf{R}_{12}| \tag{4.5b}$$

$$\mathbf{a}_{R_{12}} = \frac{\mathbf{R}_{12}}{R} \tag{4.5c}$$

By substituting eq. (4.5) into eq. (4.4), we may write eq. (4.4) as

$$\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^3}\,\mathbf{R}_{12} \tag{4.6a}$$

or

$$\mathbf{F}_{12} = \frac{Q_1 Q_2\,(\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o |\mathbf{r}_2 - \mathbf{r}_1|^3} \tag{4.6b}$$

It is worthwhile to note that

1. As shown in Figure 4.1, the force $\mathbf{F}_{21}$ on Q_1 due to Q_2 is given by

$$\mathbf{F}_{21} = |\mathbf{F}_{12}|\mathbf{a}_{R_{21}} = |\mathbf{F}_{12}|(-\mathbf{a}_{R_{12}})$$

or

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \tag{4.7}$$

since

$$\mathbf{a}_{R_{21}} = -\mathbf{a}_{R_{12}}$$

2. Like charges (charges of the same sign) repel each other, while unlike charges attract. This is illustrated in Figure 4.2.

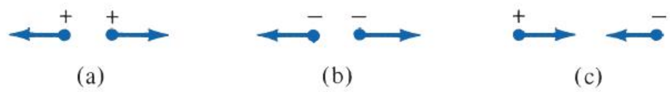

FIGURE 4.2 (**a**), (**b**) Like charges repel. (**c**) Unlike charges attract.

3. The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that is, Q_1 and Q_2 must be point charges.
4. Q_1 and Q_2 must be static (at rest).
5. The signs of Q_1 and Q_2 must be taken into account in eq. (4.4). For like charges, $Q_1 Q_2 > 0$. For unlike charges, $Q_1 Q_2 < 0$.
6. Charges cannot be created or destroyed; the quantity of total charge remains constant.

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the resultant force $\mathbf{F}$ on a charge Q located at point $\mathbf{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence,

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + F_3 + \cdots + \mathbf{F}_N \\ &= \frac{QQ_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{QQ_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{QQ_N(\mathbf{r} - \mathbf{r}_N)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}\end{aligned}$$

or

$$\boxed{\mathbf{F} = \frac{Q}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}} \tag{4.8}$$

ELECTRIC FIELD INTENSITY

> The **electric field intensity** (or **electric field strength**) **E** is the force that a unit positive charge experiences when placed in an electric field.

Thus

$$\mathbf{E} = \lim_{Q \to 0} \frac{\mathbf{F}}{Q} \tag{4.9}$$

or simply

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{Q}} \tag{4.10}$$

For $Q > 0$, the electric field intensity $\mathbf{E}$ is obviously in the direction of the force $\mathbf{F}$ and is measured in newtons per coulomb or volts per meter. The electric field intensity at point $\mathbf{r}$ due to a point charge located at $\mathbf{r}'$ is readily obtained from eqs. (4.6) and (4.10) as

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3} \tag{4.11a}$$

or simply

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r} \tag{4.11b}$$

For N point charges $Q_1, Q_2, \ldots, Q_N$ located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the electric field intensity at point $\mathbf{r}$ is obtained from eqs. (4.8) and (4.10) as

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \mathrm{E}_3 + \cdots + \mathbf{E}_N \\ &= \frac{Q_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{Q_N(\mathbf{r} - \mathbf{r}_N)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}\end{aligned}$$

or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3} \tag{4.12}$$

FIELD ARISING FROM A CONTINUOUS VOLUME CHARGE DISTRIBUTION

So far we have considered only forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume, as illustrated in Figure 4.5.

It is customary to denote the line charge density, surface charge density, and volume charge density by ρ_L (in C/m), ρ_S (in C/m^2), and ρ_v (in C/m^3), respectively. These must not be confused with ρ (without subscript), used for radial distance in cylindrical coordinates.

The charge element dQ and the total charge Q due to these charge distributions are obtained from Figure 4.5 as

$$dQ = \rho_L\, dl \rightarrow Q = \int_L \rho_L\, dl \quad \text{(line charge)} \tag{4.13a}$$

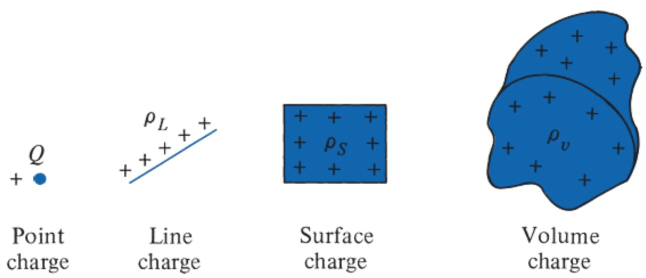

FIGURE 4.5 Various charge distributions and charge elements.

$$dQ = \rho_S\, dS \rightarrow Q = \int_S \rho_S\, dS \quad \text{(surface charge)} \tag{4.13b}$$

$$dQ = \rho_v\, dv \rightarrow Q = \int_v \rho_v\, dv \quad \text{(volume charge)} \tag{4.13c}$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_S, and ρ_v may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution. We treat dQ as a point charge. Thus by replacing Q in eq. (4.11) with charge element $dQ = \rho_L\, dl$, $\rho_S\, dS$, or $\rho_v\, dv$ and integrating, we get

$$\mathbf{E} = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon_o R^2}\mathbf{a}_R \quad \text{(line charge)} \tag{4.14}$$

$$\mathbf{E} = \int_S \frac{\rho_S\, dS}{4\pi\varepsilon_o R^2}\mathbf{a}_R \quad \text{(surface charge)} \tag{4.15}$$

$$\mathbf{E} = \int_v \frac{\rho_v\, dv}{4\pi\varepsilon_o R^2}\mathbf{a}_R \quad \text{(volume charge)} \tag{4.16}$$

It should be noted that R^2 and $\mathbf{a}_R$ vary as the integrals in eqs. (4.14) to (4.16) are evaluated. We shall now apply these formulas to some specific charge distributions.

FIELD OF A LINE CHARGE

Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in Figure 4.6. The charge element dQ associated with element $dl = dz$ of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz$$

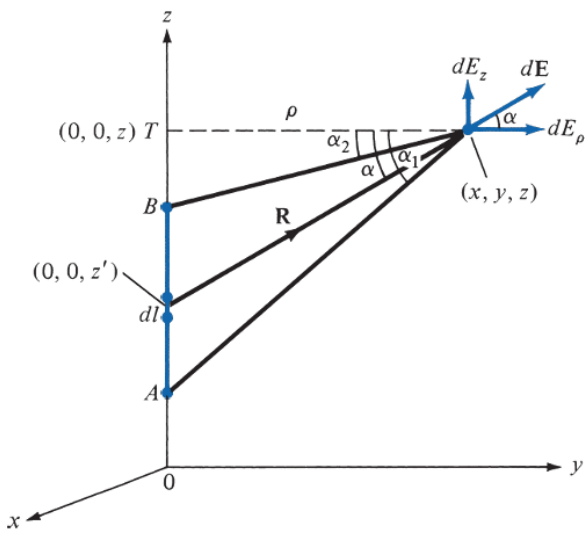

FIGURE 4.6 Evaluation of the **E** field due to a line charge.

and hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L \, dz \tag{4.17}$$

The electric field intensity **E** at an arbitrary point $P\,(x, y, z)$ can be found by using eq. (4.14). It is important that we learn to derive and substitute each term in eqs. (4.14) to (4.16) for a given charge distribution. It is customary to denote the field point[4] by (x, y, z) and the source point by (x', y', z'). Thus from Figure 4.6,

$$dl = dz'$$
$$\mathbf{R} = (x, y, z) - (0, 0, z') = x\mathbf{a}_x + y\mathbf{a}_y + (z - z')\mathbf{a}_z$$

or

$$\mathbf{R} = \rho\mathbf{a}_\rho + (z - z')\mathbf{a}_z$$
$$R^2 = |\mathbf{R}|^2 = x^2 + y^2 + (z - z')^2 = \rho^2 + (z - z')^2$$
$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{\rho\mathbf{a}_\rho + (z - z')\mathbf{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}$$

Substituting all this into eq. (4.14), we get

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int \frac{\rho \mathbf{a}_\rho + (z - z')\mathbf{a}_z}{[\rho^2 + (z - z')^2]^{3/2}} \, dz' \tag{4.18}$$

To evaluate this, it is convenient that we define α, α_1, and α_2 as in Figure 4.6.

$$R = [\rho^2 + (z - z')^2]^{1/2} = \rho \sec \alpha$$

$$z' = OT - \rho \tan \alpha, \quad dz' = -\rho \sec^2 \alpha \, d\alpha$$

Hence, eq. (4.18) becomes

$$\begin{aligned} \mathbf{E} &= \frac{-\rho_L}{4\pi\varepsilon_o} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^2 \alpha \, [\cos \alpha \, \mathbf{a}_\rho + \sin \alpha \, \mathbf{a}_z] \, d\alpha}{\rho^2 \sec^2 \alpha} \\ &= -\frac{\rho_L}{4\pi\varepsilon_o \rho} \int_{\alpha_1}^{\alpha_2} [\cos \alpha \, \mathbf{a}_\rho + \sin \alpha \, \mathbf{a}_z] \, d\alpha \end{aligned} \tag{4.19}$$

Thus for a *finite line charge,*

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o \rho} [-(\sin \alpha_2 - \sin \alpha_1)\mathbf{a}_\rho + (\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_z] \tag{4.20}$$

As a special case, for an *infinite line charge,* point B is at $(0, 0, \infty)$ and A at $(0, 0, -\infty)$ so that $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$; the z-component vanishes and eq. (4.20) becomes

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o \rho} \mathbf{a}_\rho} \tag{4.21}$$

Bear in mind that eq. (4.21) is obtained for an infinite line charge along the z-axis so that ρ and $\mathbf{a}_\rho$ have their usual meaning. If the line is not along the z-axis, ρ is the perpendicular distance from the line to the point of interest, and $\mathbf{a}_\rho$ is a unit vector along that distance directed from the line charge to the field point.

FIELD OF A SHEET OF CHARGE

Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_S. The charge associated with an elemental area dS is

$$dQ = \rho_S \, dS \tag{4.22}$$

From eq. (4.15), the contribution to the **E** field at point $P(0, 0, h)$ by the charge dQ on the elemental surface 1 shown in Figure 4.7 is

$$d\mathbf{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\,\mathbf{a}_R \tag{4.23}$$

From Figure 4.7,

$$\mathbf{R} = \rho(-\mathbf{a}_\rho) + h\mathbf{a}_z, \quad R = |\mathbf{R}| = [\rho^2 + h^2]^{1/2}$$

$$\mathbf{a}_R = \frac{\mathbf{R}}{R}, \quad dQ = \rho_S \, dS = \rho_S \, \rho \, d\phi \, d\rho$$

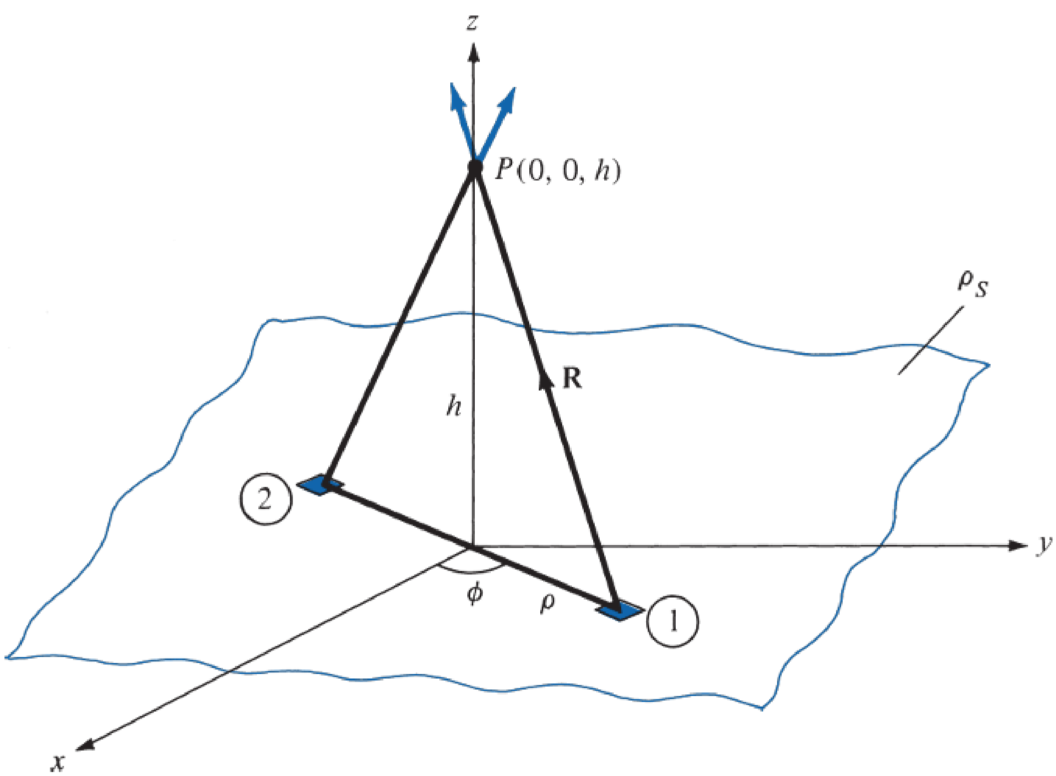

FIGURE 4.7 Evaluation of the **E** field due to an infinite sheet of charge.

Substitution of these terms into eq. (4.23) gives

$$d\mathbf{E} = \frac{\rho_S\,\rho\, d\phi\, d\rho\,[-\rho\mathbf{a}_\rho + h\mathbf{a}_z]}{4\pi\varepsilon_o[\rho^2 + h^2]^{3/2}} \tag{4.24}$$

Owing to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2 whose contribution along $\mathbf{a}_\rho$ cancels that of element 1, as illustrated in Figure 4.7. Thus the contributions to E_ρ add up to zero so that $\mathbf{E}$ has only z-component. This can also be shown mathematically by replacing $\mathbf{a}_\rho$ with $\cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$. Integration of $\cos\phi$ or $\sin\phi$ over $0 < \phi < 2\pi$ gives zero. Therefore,

$$\begin{aligned}
\mathbf{E} &= \int_S dE_z = \frac{\rho_S}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\rho\, d\phi}{[\rho^2 + h^2]^{3/2}}\mathbf{a}_z \\
&= \frac{\rho_S h}{4\pi\varepsilon_o}2\pi\int_0^\infty [\rho^2 + h^2]^{-3/2}\frac{1}{2}d(\rho^2)\mathbf{a}_z \\
&= \frac{\rho_S h}{2\varepsilon_o}\left\{-[\rho^2 + h^2]^{-1/2}\right\}_0^\infty \mathbf{a}_z
\end{aligned}$$

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\mathbf{a}_z \tag{4.25}$$

that is, $\mathbf{E}$ has only z-component if the charge is in the xy-plane. Equation (4.25) is valid for $h > 0$; for $h < 0$, we would need to replace $\mathbf{a}_z$ with $-\mathbf{a}_z$. In general, for an *infinite sheet* of charge

$$\boxed{\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\mathbf{a}_n} \tag{4.26}$$

where $\mathbf{a}_n$ is a unit vector normal to the sheet. From eq. (4.25) or (4.26), we notice that the electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P. In a parallel-plate capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\mathbf{a}_n + \frac{-\rho_S}{2\varepsilon_o}(-\mathbf{a}_n) = \frac{\rho_S}{\varepsilon_o}\mathbf{a}_n \tag{4.27}$$

ELECTRIC FLUX DENSITY

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement flux,* or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\boxed{\Psi = Q}$$

and the electric flux Ψ is measured in coulombs.

We can obtain more quantitative information by considering an inner sphere of radius a and an outer sphere of radius b, with charges of Q and $-Q$, respectively (Figure 3.1). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

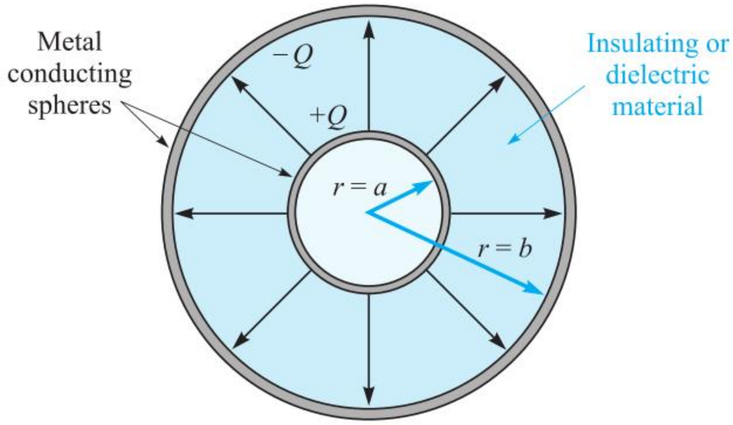

Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of D are not functions of the dielectric between the spheres.

field intensity **E**. The direction of **D** at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Referring again to Figure 3.1, the electric flux density is in the radial direction and has a value of

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

and at a radial distance r, where $a \leq r \leq b$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

Electric flux density, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we will use the term consistently.

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric

Suppose a new vector field **D** is defined by

$$\boxed{\mathbf{D} = \varepsilon_o \mathbf{E}}$$

electric flux Ψ in terms of **D**, namely,

$$\Psi = \int_S \mathbf{D} \cdot d\mathbf{S} \tag{4.36}$$

In SI units, one line of electric flux emanates from $+1$ C and terminates on -1 C. Therefore, the electric flux is measured in coulombs. Hence, the vector field **D** is called the *electric flux density* and is measured in coulombs per square meter. For historical reasons, the electric flux density is also called *electric displacement.*

From eq. (4.35), it is apparent that all the formulas derived for **E** from Coulomb's law in Sections 4.2 and 4.3 can be used in calculating **D**, except that we have to multiply those formulas by ε_o. For example, for an infinite sheet of charge, eqs. (4.26) and (4.35) give

$$\mathbf{D} = \frac{\rho_S}{2} \mathbf{a}_n \tag{4.37}$$

and for a volume charge distribution, eqs. (4.16) and (4.35) give

$$\mathbf{D} = \int_v \frac{\rho_v \, dv}{4\pi R^2} \mathbf{a}_R \tag{4.38}$$

Note from eqs. (4.37) and (4.38) that **D** is a function of charge and position only; it is independent of the medium.