MODULE-4	MAGNETIC FORCES
Force on a moving charge, differential current elements, Force between differential current elements, Numerical problems (Text: Chapter 9.1 to 9.3). Magnetic Materials: Magnetization and permeability, Magnetic boundary conditions, the magnetic circuit, problems (Text: Chapter 9.6 to 9.8) RBT Level: L1, L2, L3	

FORCE ON A MOVING CHARGE

In an electric field, the definition of the electric field intensity shows us that the force on a charged particle is

$$\mathbf{F}_e = Q\mathbf{E} \tag{8.1}$$

This shows that if Q is positive, $\mathbf{F}_e$ and $\mathbf{E}$ have the same direction.

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\mathbf{F}_m$ experienced by a charge Q moving with a velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

This clearly shows that $\mathbf{F}_m$ is perpendicular to both $\mathbf{u}$ and $\mathbf{B}$.

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by,

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

or

$$\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})} \tag{8.3}$$

This equation is known as the *Lorentz force equation,* and its solution is required in determining electron orbits in the magnetron, proton paths in the cyclotron, plasma characteristics in a magnetohydrodynamic (MHD) generator, or, in general, charged-particle motion in combined electric and magnetic fields.

TABLE 8.1 Force on a Charged Particle

State of Particle	E Field	B Field	Combined E and B Fields
Stationary	$Q\mathbf{E}$	—	$Q\mathbf{E}$
Moving	$Q\mathbf{E}$	$Q\mathbf{u} \times \mathbf{B}$	$Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

FORCE ON A DIFFERENTIAL CURRENT ELEMENT

we defined convection current density in terms of the velocity of the volume charge density,

$$\mathbf{J} = \rho_v \mathbf{u} \tag{8.5}$$

From eq. (7.5), we recall the relationship between current elements:

$$I\,d\mathbf{l} = \mathbf{K}\,dS = \mathbf{J}\,dv \tag{8.6}$$

Combining eqs. (8.5) and (8.6) yields

$$I\,d\mathbf{l} = \rho_v \mathbf{u}\,dv = dQ\,\mathbf{u}$$

Alternatively, $I\,d\mathbf{l} = \dfrac{dQ}{dt}d\mathbf{l} = dQ\dfrac{d\mathbf{l}}{dt} = dQ\,\mathbf{u}$

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Hence,

$$\boxed{I\,d\mathbf{l} = dQ\,\mathbf{u}} \tag{8.7}$$

This shows that an elemental charge dQ moving with velocity $\mathbf{u}$ (thereby producing convection current element $dQ\,\mathbf{u}$) is equivalent to a conduction current element $I\,d\mathbf{l}$. Thus the force on a current element $I\,d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is found from eq. (8.2) by merely replacing $Q\mathbf{u}$ by $I\,d\mathbf{l}$; that is,

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B} \tag{8.8}$$

If the current I is through a closed path L or circuit, the force on the circuit is given by

$$\boxed{\mathbf{F} = \oint_L I\, d\mathbf{l} \times \mathbf{B}} \tag{8.9}$$

$$d\mathbf{F} = \mathbf{K}\, dS \times \mathbf{B} \quad \text{or} \quad d\mathbf{F} = \mathbf{J}\, dv \times \mathbf{B} \tag{8.8$'$}$$

while eq. (8.9) becomes

$$\mathbf{F} = \int_S \mathbf{K}\, dS \times \mathbf{B} \quad \text{or} \quad \mathbf{F} = \int_v \mathbf{J}\, dv \times \mathbf{B} \tag{8.9$'$}$$

From eq. (8.8)

The **magnetic field B** is defined as the force per unit current element.

Alternatively, **B** may be defined from eq. (8.2) as the vector that satisfies $\mathbf{F}_m/q = \mathbf{u} \times \mathbf{B}$, just as we defined electric field **E** as the force per unit charge, $\mathbf{F}_e/q$. Both these definitions of **B** show that **B** describes the force properties of a magnetic field.

FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS

Let us now consider the force between two elements $I_1\, d\mathbf{l}_1$ and $I_2\, d\mathbf{l}_2$. According to Biot–Savart's law, both current elements produce magnetic fields. So we may find the force $d(d\mathbf{F}_1)$ on element $I_1\, d\mathbf{l}_1$ due to the field $d\mathbf{B}_2$ produced by element $I_2\, d\mathbf{l}_2$ as shown in Figure 8.1. From eq. (8.8),

$$d(d\mathbf{F}_1) = I_1\, d\mathbf{l}_1 \times d\mathbf{B}_2 \tag{8.10}$$

But from Biot–Savart's law,

$$d\mathbf{B}_2 = \frac{\mu_o I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2} \tag{8.11}$$

Hence,

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\, d\mathbf{l}_1 \times (I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{4\pi R_{21}^2} \tag{8.12}$$

This equation is essentially the law of force between two current elements and is analogous to Coulomb's law, which expresses the force between two stationary charges. From eq. (8.12), we obtain the total force $\mathbf{F}_1$ on current loop 1 due to current loop 2 shown in Figure 8.1 as

$$\mathbf{F}_1 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \tag{8.13}$$

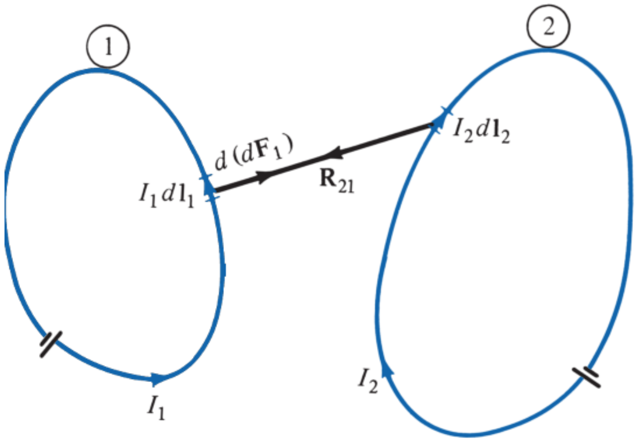

FIGURE 8.1 Force between two current loops.

The force $\mathbf{F}_2$ on loop 2 due to the magnetic field $\mathbf{B}_1$ from loop 1 is obtained from eq. (8.13) by interchanging subscripts 1 and 2. It can be shown that $\mathbf{F}_2 = -\mathbf{F}_1$; thus $\mathbf{F}_1$ and $\mathbf{F}_2$ obey Newton's third law that action and reaction are equal and opposite.

MAGNETIZATION AND PERMEABILITY

The **magnetization M**, in amperes per meter, is the magnetic dipole moment per unit volume.

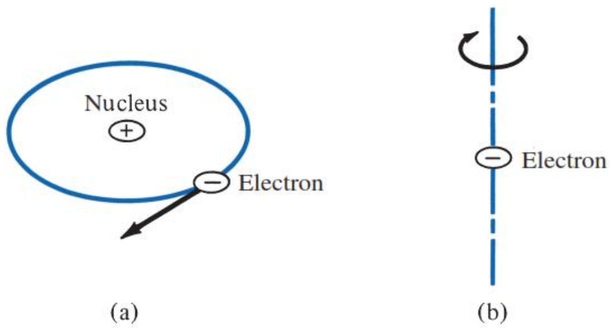

FIGURE 8.10 (**a**) Electron orbiting around the nucleus. (**b**) Electron spin.

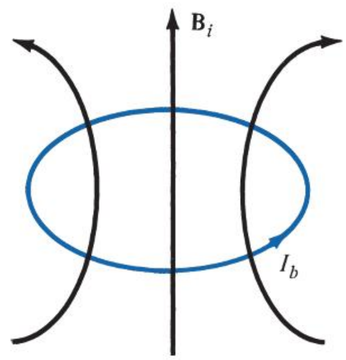

FIGURE 8.11 Circular current loop equivalent to electronic motion of Figure 8.10.

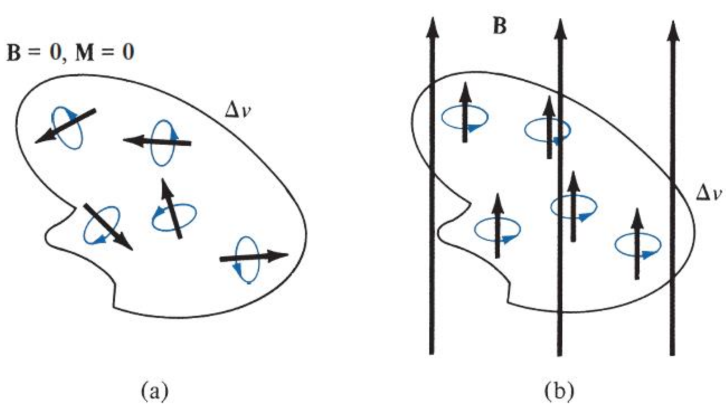

FIGURE 8.12 Magnetic dipole moment in a volume Δv: (**a**) before **B** is applied, (**b**) after **B** is applied.

Let us begin by defining the magnetization **M** in terms of the magnetic dipole moment **m**. The bound current I_b circulates about a path enclosing a differential area $d\mathbf{S}$, establishing a dipole moment ($\mathrm{A}\cdot\mathrm{m}^2$),

$$\mathbf{m} = I_b d\mathbf{S}$$

If there are n magnetic dipoles per unit volume and we consider a volume Δv, then the total magnetic dipole moment is found by the vector sum

$$\mathbf{m}_{\text{total}} = \sum_{i=1}^{n\Delta v} \mathbf{m}_i \tag{19}$$

Each of the $\mathbf{m}_i$ may be different. Next, we define the *magnetization* **M** as the *magnetic dipole moment per unit volume,*

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \mathbf{m}_i$$

and see that its units must be the same as for **H**, amperes per meter.

Now let us consider the effect of some alignment of the magnetic dipoles as the result of the application of a magnetic field. We shall investigate this alignment along a closed path, a short portion of which is shown in Figure 8.9. The figure shows several magnetic moments **m** that make an angle θ with the element of path $d\mathbf{L}$; each moment consists of a bound current I_b circulating about an area $d\mathbf{S}$. We are therefore

Thus the differential change in the net bound current I_B over the segment $d\mathbf{L}$ will be

$$dI_B = nI_b d\mathbf{S}\cdot d\mathbf{L} = \mathbf{M}\cdot d\mathbf{L} \tag{20}$$

and within an entire closed contour,

$$I_B = \oint \mathbf{M}\cdot d\mathbf{L} \tag{21}$$

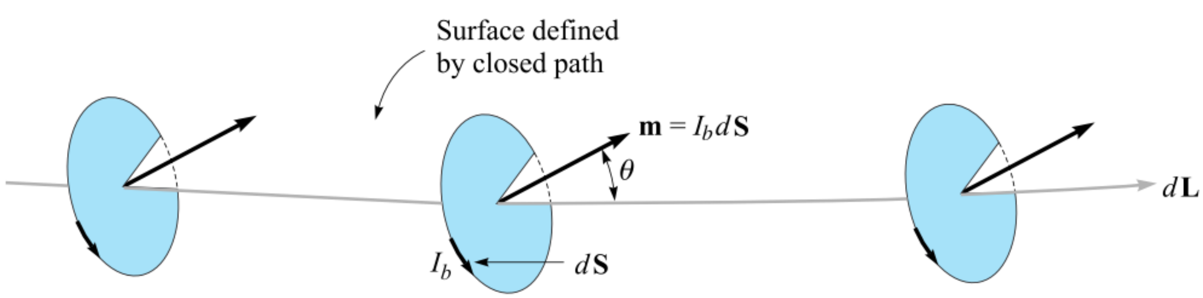

Figure 8.9 A section $d\mathbf{L}$ of a closed path along which magnetic dipoles have been partially aligned by some external magnetic field. The alignment has caused the bound current crossing the surface defined by the closed path to increase by $nI_b d\mathbf{S}\cdot d\mathbf{L}$ A.

We thus write Ampère's circuital law in terms of the *total* current, bound plus free,

$$\oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} = I_T \tag{22}$$

where

$$I_T = I_{\mathbf{B}} + I$$

and I is the total *free* current enclosed by the closed path.

Combining these last three equations, we obtain an expression for the free current enclosed,

$$I = I_T - I_{\mathbf{B}} = \oint \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \cdot d\mathbf{L} \tag{23}$$

We may now define $\mathbf{H}$ in terms of $\mathbf{B}$ and $\mathbf{M}$,

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{24}$$

and we see that $\mathbf{B} = \mu_0 \mathbf{H}$ in free space where the magnetization is zero. This relationship is usually written in a form that avoids fractions and minus signs:

$$\boxed{\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})} \tag{25}$$

We may now use our newly defined $\mathbf{H}$ field in (23),

$$I = \oint \mathbf{H} \cdot d\mathbf{L} \tag{26}$$

obtaining Ampère's circuital law in terms of the free currents.

Using the several current densities, we have

$$I_{\mathbf{B}} = \int_S \mathbf{J}_B \cdot d\mathbf{S}$$

$$I_T = \int_S \mathbf{J}_T \cdot d\mathbf{S}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

With the help of Stokes' theorem, we may therefore transform (21), (26), and (22) into the equivalent curl relationships:

$$\nabla \times \mathbf{M} = \mathbf{J}_B$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J}_T$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{27}$$

The relationship between **B**, **H**, and **M** expressed by (25) may be simplified for linear isotropic media where a magnetic susceptibility χ_m can be defined:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}} \tag{28}$$

Thus we have

$$\begin{aligned} \mathbf{B} &= \mu_0(\mathbf{H} + \chi_m \mathbf{H}) \\ &= \mu_0 \mu_r \mathbf{H} \end{aligned}$$

where

$$\mu_r = 1 + \chi_m \tag{29}$$

is defined as the *relative permeability* μ_r. We next define the *permeability* μ:

$$\mu = \mu_0 \mu_r \tag{30}$$

and this enables us to write the simple relationship between **B** and **H**,

$$\mathbf{B} = \mu \mathbf{H} \tag{31}$$

MAGNETIC BOUNDARY CONDITIONS

Figure 8.10 shows a boundary between two isotropic homogeneous linear materials with permeabilities μ_1 and μ_2. The boundary condition on the normal components

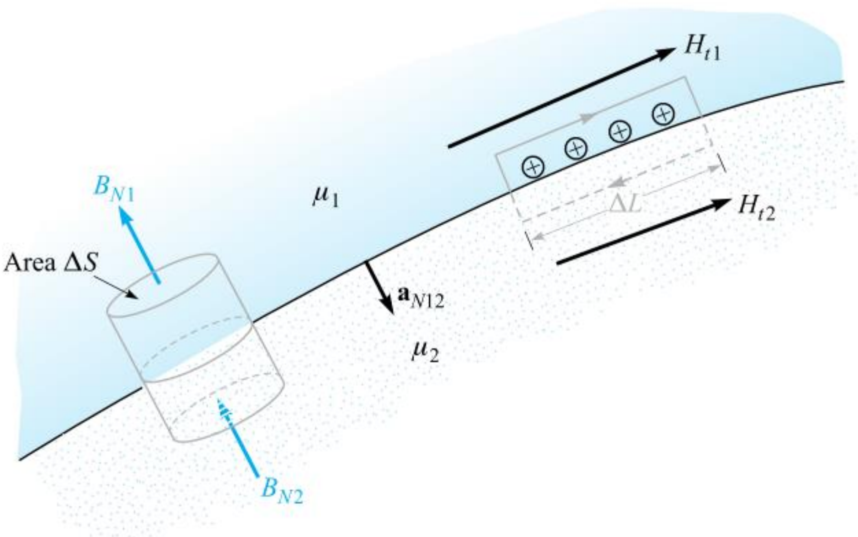

Figure 8.10 A gaussian surface and a closed path are constructed at the boundary between media 1 and 2, having permeabilities of μ_1 and μ_2, respectively. From this we determine the boundary conditions $B_{N1} = B_{N2}$ and $H_{t1} - H_{t2} = K$, the component of the surface current density directed into the page.

is determined by allowing the surface to cut a small cylindrical gaussian surface. Applying Gauss's law for the magnetic field from Section 7.5,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

we find that

$$B_{N1}\Delta S - B_{N2}\Delta S = 0$$

or

$$\boxed{B_{N2} = B_{N1}} \tag{32}$$

Thus

$$H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \tag{33}$$

The normal component of **B** is continuous, but the normal component of **H** is discontinuous by the ratio μ_1/μ_2.

The relationship between the normal components of **M**, of course, is fixed once the relationship between the normal components of **H** is known. For linear magnetic materials, the result is written simply as

$$M_{N2} = \chi_{m2} \frac{\mu_1}{\mu_2} H_{N1} = \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1} \tag{34}$$

Next, Ampère's circuital law

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

is applied about a small closed path in a plane normal to the boundary surface, as shown to the right in Figure 8.10. Taking a clockwise trip around the path, we find that

$$H_{t1} \Delta L - H_{t2} \Delta L = K \Delta L$$

where we assume that the boundary may carry a surface current **K** whose component normal to the plane of the closed path is K. Thus

$$\boxed{H_{t1} - H_{t2} = K} \tag{35}$$

The directions are specified more exactly by using the cross product to identify the tangential components,

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{N12} = \mathbf{K}$$

where $\mathbf{a}_{N12}$ is the unit normal at the boundary directed from region 1 to region 2. An equivalent formulation in terms of the vector tangential components may be more convenient for **H**:

$$\mathbf{H}_{t1} - \mathbf{H}_{t2} = \mathbf{a}_{N12} \times \mathbf{K}$$

For tangential **B**, we have

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \tag{36}$$

The boundary condition on the tangential component of the magnetization for linear materials is therefore

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K \tag{37}$$

The last three boundary conditions on the tangential components are much simpler, of course, if the surface current density is zero. This is a free current density, and it must be zero if neither material is a conductor.

THE MAGNETIC CIRCUIT

We begin with the electrostatic potential and its relationship to electric field intensity,

$$\mathbf{E} = -\nabla V \tag{38a}$$

The scalar magnetic potential has already been defined, and its analogous relation to the magnetic field intensity is

$$\boxed{\mathbf{H} = -\nabla V_m} \tag{38b}$$

The electric potential difference between points A and B may be written as

$$V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{L} \tag{39a}$$

and the corresponding relationship between the mmf and the magnetic field intensity,

$$\boxed{V_{mAB} = \int_A^B \mathbf{H} \cdot d\mathbf{L}} \tag{39b}$$

Ohm's law for the electric circuit has the point form

$$\mathbf{J} = \sigma \mathbf{E} \tag{40a}$$

and we see that the magnetic flux density will be the analog of the current density,

$$\boxed{\mathbf{B} = \mu \mathbf{H}} \tag{40b}$$

To find the total current, we must integrate:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{41a}$$

A corresponding operation is necessary to determine the total magnetic flux flowing through the cross section of a magnetic circuit:

$$\boxed{\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}} \tag{41b}$$

We then defined resistance as the ratio of potential difference and current, or

$$V = IR \tag{42a}$$

and we shall now define *reluctance* as the ratio of the magnetomotive force to the total flux; thus

$$\boxed{V_m = \Phi \mathfrak{R}} \tag{42b}$$

where reluctance is measured in ampere-turns per weber ($\mathrm{A \cdot t/Wb}$). In resistors that are made of a linear isotropic homogeneous material of conductivity σ and have a uniform cross section of area S and length d, the total resistance is

$$R = \frac{d}{\sigma S} \tag{43a}$$

If we are fortunate enough to have such a linear isotropic homogeneous magnetic material of length d and uniform cross section S, then the total reluctance is

$$\boxed{\mathfrak{R} = \frac{d}{\mu S}} \tag{43b}$$

The only such material to which we shall commonly apply this relationship is air.

Finally, let us consider the analog of the source voltage in an electric circuit. We know that the closed line integral of $\mathbf{E}$ is zero,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

In other words, Kirchhoff's voltage law states that the rise in potential through the source is exactly equal to the fall in potential through the load. The expression for magnetic phenomena takes on a slightly different form,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I_{\text{total}}$$

for the closed line integral is not zero. Because the total current linked by the path is usually obtained by allowing a current I to flow through an N-turn coil, we may express this result as

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = NI} \tag{44}$$