MODULE-2	GAUSS'S LAW AND DIVERGENCE
Gauss 'law, Application of Gauss' law to Point Charge, line charge, Surface charge and Volume Charge, Point (differential) form of Gauss law, Divergence. Maxwell's First Equation (Electrostatics), Vector Operator ∇ and divergence theorem, Numerical Problems (Text: Chapter 3.2 to 3.7).Energy expended or work done in moving a point charge in an Electric field, The line integral ((Text: Chapter 4.1 and 4.2) Current and Current density, Continuity of current. (Text: Chapter 5.1, 5.2) RBT Level: L1, L2, L3	

GAUSS'S LAW

Gauss's[5] law constitutes one of the fundamental laws of electromagnetism.

Gauss's law states that the total electric flux ψ through any *closed* surface is equal to the total charge enclosed by that surface.

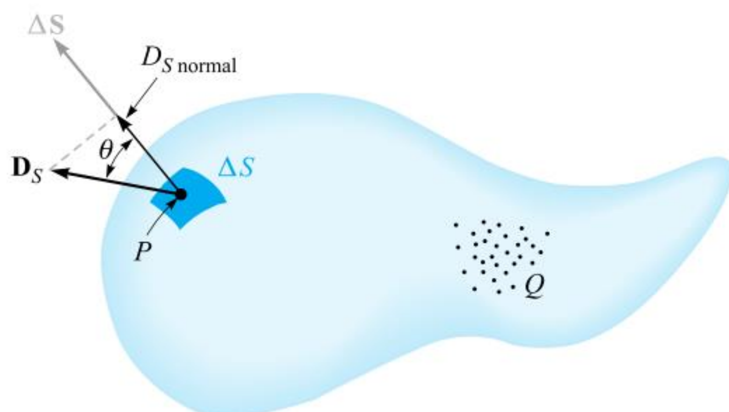

Figure 3.2 The electric flux density $\mathbf{D}_S$ at P arising from charge Q. The total flux passing through ΔS is $\mathbf{D}_S \cdot \Delta\mathbf{S}$.

At any point P, consider an incremental element of surface ΔS and let $\mathbf{D}_S$ make an angle θ with $\Delta\mathbf{S}$, as shown in Figure 3.2. The flux crossing ΔS is then the product of the normal component of $\mathbf{D}_S$ and $\Delta\mathbf{S}$,

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}}\Delta S = D_S \cos\theta\, \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

where we are able to apply the definition of the dot product developed in Chapter 1.

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

The resultant integral is a *closed surface integral*, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dx\,dy$, $\rho\,d\phi\,d\rho$, or $r^2 \sin\theta\,d\theta\,d\phi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we will always place an S below the integral sign to indicate a surface integral, although this is not actually necessary, as the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a *closed* surface. Such a surface is often called a *gaussian surface*. We then have the mathematical formulation of Gauss's law,

$$\boxed{\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q} \tag{5}$$

The charge enclosed might be several point charges, in which case

$$Q = \Sigma Q n$$

or a line charge,

$$Q = \int \rho_L \, dL$$

or a surface charge,

$$Q = \int_S \rho_S dS \qquad \text{(not necessarily a closed surface)}$$

or a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding, Gauss's law may be written in terms of the charge distribution as

$$\boxed{\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv} \tag{6}$$

a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

APPLICATIONS OF GAUSS'S LAW

We now consider how we may use Gauss's law,

$$\boxed{Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.

The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

Point Charge

Suppose a point charge Q is located at the origin. To determine $\mathbf{D}$ at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a spherical surface centered at the origin is the Gaussian surface in this case and is shown in Figure 4.13.

Since $\mathbf{D}$ is everywhere normal to the Gaussian surface, that is, $\mathbf{D} = D_r\mathbf{a}_r$, applying Gauss's law ($\Psi = Q_{\text{enc}}$) gives

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = D_r \oint_S dS = D_r\, 4\pi r^2 \tag{4.44}$$

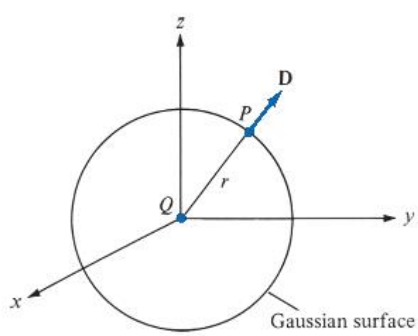

FIGURE 4.13 Gaussian surface about a point charge.

where $\oint dS = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2$ is the surface area of the Gaussian surface. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \tag{4.45}$$

as expected from eqs. (4.11) and (4.35).

Infinite Line Charge

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine **D** at a point P, we choose a cylindrical surface containing P to satisfy the symmetry condition as shown in Figure 4.14. The electric flux density **D** is constant on and normal to the cylindrical Gaussian surface; that is, $\mathbf{D} = D_\rho \mathbf{a}_\rho$. If we apply Gauss's law to an arbitrary length ℓ of the line

$$\rho_L \ell = Q = \int_S \mathbf{D} \cdot d\mathbf{S} = D_\rho \int_S dS = D_\rho 2\pi\rho\ell \tag{4.46}$$

where $\int_S dS = 2\pi\rho\ell$ is the surface area of the Gaussian surface. Note that $\int \mathbf{D} \cdot d\mathbf{S}$ evaluated on the top and bottom surfaces of the cylinder is zero, since **D** has no z-component; that means that **D** is tangential to those surfaces. Thus

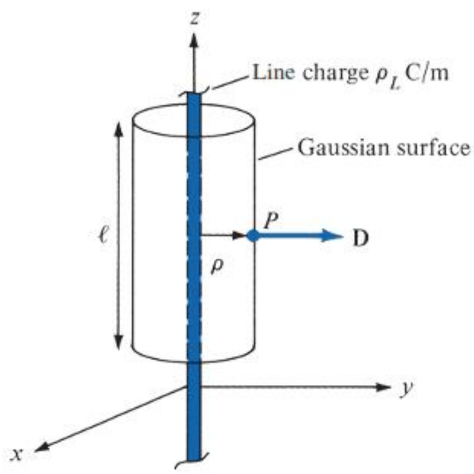

FIGURE 4.14 Gaussian surface about an infinite line charge.

COAXIAL CABLE

The problem of a coaxial cable is almost identical with that of the line charge and is an example that is extremely difficult to solve from the standpoint of Coulomb's law. Suppose that we have two coaxial cylindrical conductors, the inner of radius a and the outer of radius b, each infinite in extent (Figure 3.5). We will assume a charge distribution of ρ_S on the outer surface of the inner conductor.

Symmetry considerations show us that only the D_ρ component is present and that it can be a function only of ρ. A right circular cylinder of length L and radius ρ, where $a < \rho < b$, is necessarily chosen as the gaussian surface, and we quickly have

$$Q = D_S 2\pi\rho L$$

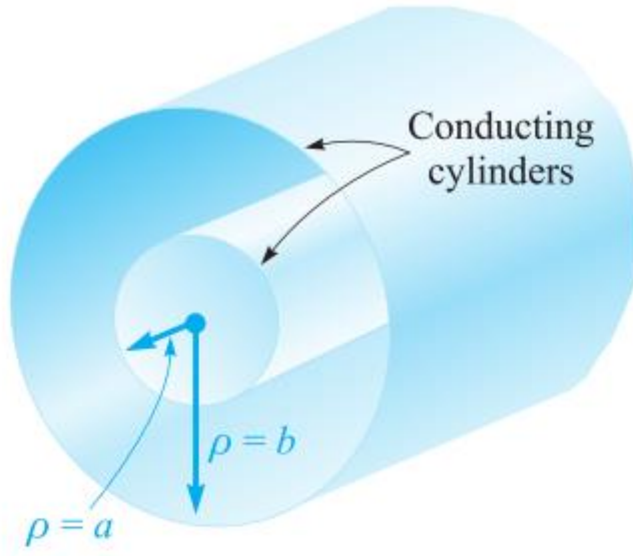

Figure 3.5 The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

The total charge on a length L of the inner conductor is

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_S a \, d\phi \, dz = 2\pi a L \rho_S$$

from which we have

$$D_S = \frac{a\rho_S}{\rho} \qquad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \qquad (a < \rho < b)$$

This result might be expressed in terms of charge per unit length because the inner conductor has $2\pi a\rho_S$ coulombs on a meter length, and hence, letting $\rho_L = 2\pi a\rho_S$,

$$\boxed{\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho}$$

and the solution has a form identical with that of the infinite line charge.

Because every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

and the surface charge on the outer cylinder is found as

$$2\pi bL\rho_{S,\text{outer cyl}} = -2\pi aL\rho_{S,\text{inner cyl}}$$

or

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b}\rho_{S,\text{inner cyl}}$$

What would happen if we should use a cylinder of radius ρ, $\rho > b$, for the gaussian surface? The total charge enclosed would then be zero, for there are equal and opposite charges on each conducting cylinder. Hence

$$0 = D_S 2\pi\rho L \qquad (\rho > b)$$

$$D_S = 0 \qquad (\rho > b)$$

An identical result would be obtained for $\rho < a$. Thus the coaxial cable or capacitor has no external field (we have proved that the outer conductor is a "shield"), and there is no field within the center conductor.

APPLICATION OF GAUSS'S LAW: DIFFERENTIAL VOLUME ELEMENT

Let us consider any point P, shown in Figure 3.6, located by a rectangular coordinate system. The value of $\mathbf{D}$ at the point P may be expressed in rectangular components, $\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at P, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law,

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q}$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the first of these in detail. Because the surface element is very small, **D** is essentially constant (over *this* portion of the entire closed surface) and

$$\begin{aligned}\int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y\, \Delta z\, \mathbf{a}_x \\ &\doteq D_{x,\text{front}} \Delta y\, \Delta z\end{aligned}$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$\begin{aligned}D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\end{aligned}$$

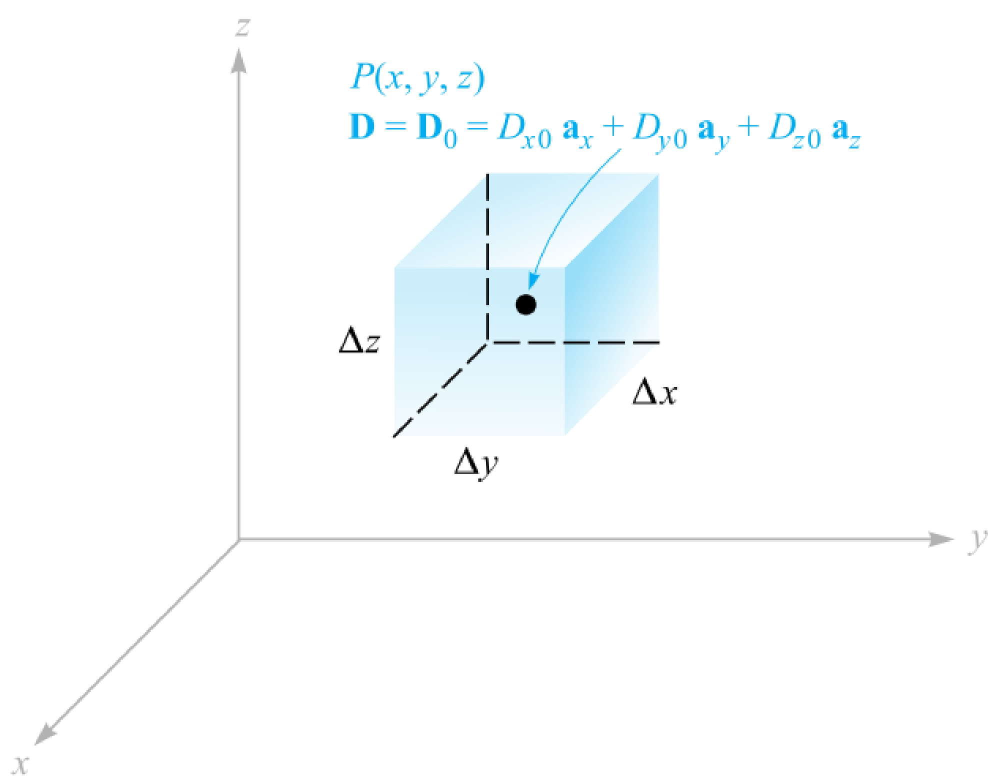

Figure 3.6 A differential-sized gaussian surface about the point P is used to investigate the space rate of change of **D** in the neighborhood of P.

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, as D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's-series expansion for D_x in the neighborhood of P.

We now have

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

Consider now the integral over the back surface,

$$\begin{aligned} \int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y \, \Delta z \end{aligned}$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

giving

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

and these results may be collected to yield

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v \tag{7}$$

The expression is an approximation which becomes better as Δv becomes smaller, and in the following section we shall let the volume Δv approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv and have as a result the approximation (7) stating that

$$\boxed{\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v} \tag{8}$$

DIVERGENCE AND MAXWELL'S FIRST EQUATION

We will now obtain an exact relationship from (7), by allowing the volume element $\Delta\nu$ to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta\nu\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta\nu}=\lim_{\Delta\nu\to 0}\frac{Q}{\Delta\nu}=\rho_\nu \tag{9}$$

in which the charge density, ρ_ν, is identified in the second equality.

The methods of the previous section could have been used on any vector $\mathbf{A}$ to find $\oint_S \mathbf{A}\cdot d\mathbf{S}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)=\lim_{\Delta\nu\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta\nu} \tag{10}$$

where $\mathbf{A}$ could represent velocity, temperature gradient, force, or any other vector field.

This operation appeared so many times in physical investigations in the last century that it received a descriptive name, *divergence*. The divergence of $\mathbf{A}$ is defined as

$$\boxed{\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta\nu\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta\nu}} \tag{11}$$

and is usually abbreviated div $\mathbf{A}$. The physical interpretation of the divergence of a vector is obtained by describing carefully the operations implied by the right-hand side of (11), where we shall consider $\mathbf{A}$ to be a member of the flux-density family of vectors in order to aid the physical interpretation.

The divergence of the vector flux density $\mathbf{A}$ *is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.*

Writing (9) with our new term, we have

$$\boxed{\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right) \qquad \text{(rectangular)}} \tag{12}$$

$$\boxed{\text{div } \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}} \tag{13}$$

$$\boxed{\text{div } \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \, D_\theta) + \frac{1}{r \sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}} \tag{14}$$

Finally, we can combine Eqs. (9) and (12) and form the relation between electric flux density and charge density:

$$\boxed{\text{div } \mathbf{D} = \rho_\nu} \tag{15}$$

THE VECTOR OPERATOR ∇ AND THE DIVERGENCE THEOREM

If we remind ourselves again that divergence is an operation on a vector yielding a scalar result, just as the dot product of two vectors gives a scalar result, it seems possible that we can find something that may be dotted formally with **D** to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Obviously, this cannot be accomplished by using a dot *product*; the process must be a dot *operation*.

With this in mind, we define the *del operator* ∇ as a *vector operator*,

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z} \tag{16}$$

Similar *scalar operators* appear in several methods of solving differential equations where we often let D replace d/dx, D^2 replace d^2/dx^2, and so forth.[4] We agree on defining ∇ that it shall be treated in every way as an ordinary vector with the one important exception that partial derivatives result instead of products of scalars.

Consider $\nabla \cdot \mathbf{D}$, signifying

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

We first consider the dot products of the unit vectors, discarding the six zero terms, and obtain the result that we recognize as the divergence of **D**:

$$\boxed{\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \mathrm{div}(\mathbf{D})}$$

By the definition of Gauss's Law, we have,

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$$

By applying divergence theorem to the middle term in eq.

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D} \, dv} \tag{17}$$

which may be stated as follows:

> *The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.*

ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD

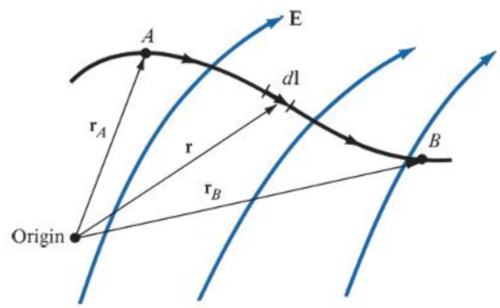

FIGURE 4.18 Displacement of point charge Q in an electrostatic field **E**.

Suppose we wish to move a charge Q a distance $d\mathbf{L}$ in an electric field $\mathbf{E}$. The force on Q arising from the electric field is

$$\boxed{\mathbf{F}_E = Q\mathbf{E}} \tag{1}$$

where the subscript reminds us that this force arises from the field. The component of this force in the direction $d\mathbf{L}$ which we must overcome is

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

where $\mathbf{a}_L$ = a unit vector in the direction of $d\mathbf{L}$.

The force that we must apply is equal and opposite to the force associated with the field,

$$F_{\text{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

and the expenditure of energy is the product of the force and distance. That is, the differential work done by an external source moving charge Q is $dW = -Q\mathbf{E} \cdot \mathbf{a}_L dL$, or

$$\boxed{dW = -Q\mathbf{E} \cdot d\mathbf{L}} \tag{2}$$

where we have replaced $\mathbf{a}_L dL$ by the simpler expression $d\mathbf{L}$.

Returning to the charge in the electric field, the work required to move the charge a finite distance must be determined by integrating,

$$\boxed{W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}} \tag{3}$$

THE LINE INTEGRAL

The integral expression for the work done in moving a point charge Q from one position to another, Eq. (3), is an example of a line integral, which in vector-analysis notation always takes the form of the integral along some prescribed path of the dot product of a vector field and a differential vector path length $d\mathbf{L}$. Without using vector analysis we should have to write

$$W = -Q \int_{\text{init}}^{\text{final}} E_L \, dL$$

where E_L = component of $\mathbf{E}$ along $d\mathbf{L}$.

This procedure is indicated in Figure 4.1, where a path has been chosen from an initial position B to a final position[1] A and a *uniform electric field* is selected for simplicity. The path is divided into six segments, $\Delta\mathbf{L}_1, \Delta\mathbf{L}_2, \ldots, \Delta\mathbf{L}_6$, and the components of $\mathbf{E}$ along each segment are denoted by $E_{L1}, E_{L2}, \ldots, E_{L6}$. The work involved in moving a charge Q from B to A is then approximately

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

or, using vector notation,

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

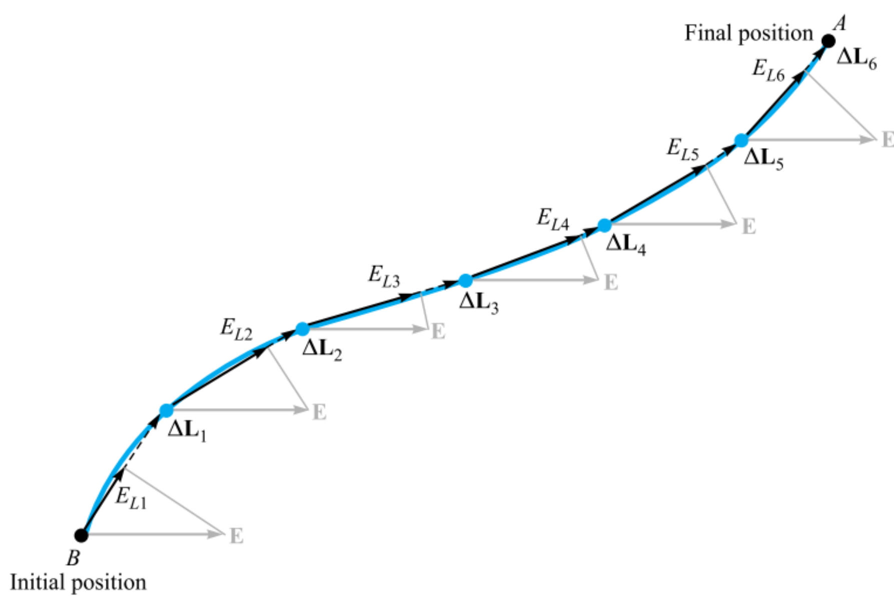

Figure 4.1 A graphical interpretation of a line integral in a uniform field. The line integral of **E** between points B and A is independent of the path selected, even in a nonuniform field; this result is not, in general, true for time-varying fields.

and because we have assumed a uniform field,

$$\mathbf{E}_1 = \mathbf{E}_2 = \cdots = \mathbf{E}_6$$
$$W = -Q\mathbf{E} \cdot (\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \cdots + \Delta\mathbf{L}_6)$$

What is this sum of vector segments in the preceding parentheses? Vectors add by the parallelogram law, and the sum is just the vector directed from the initial point B to the final point A, $\mathbf{L}_{BA}$. Therefore

$$W = -Q\mathbf{E} \cdot \mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)} \tag{4}$$

Remembering the summation interpretation of the line integral, this result for the uniform field can be obtained rapidly now from the integral expression

$$W = -Q \int_B^A \mathbf{E} \cdot d\mathbf{L} \tag{5}$$

as applied to a uniform field

$$W = -Q\mathbf{E} \cdot \int_B^A d\mathbf{L}$$

where the last integral becomes $\mathbf{L}_{BA}$ and

$$W = -Q\mathbf{E} \cdot \mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$

CURRENT AND CURRENT DENSITY

The **current** (in amperes) through a given area is the electric charge passing through the area per unit time.

That is,

$$I = \frac{dQ}{dt} \tag{1}$$

Thus in a current of one ampere, charge is being transferred at a rate of one coulomb per second.

We now introduce the concept of *current density* **J**. If current ΔI flows through a planar surface ΔS, the current density is

$$\mathbf{J} = \frac{\Delta I}{\Delta S}$$

or

$$\Delta I = J \Delta S$$

assuming that the current density is perpendicular to the surface. If the current density is not normal to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

Total current is obtained by integrating,

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{S}} \tag{2}$$

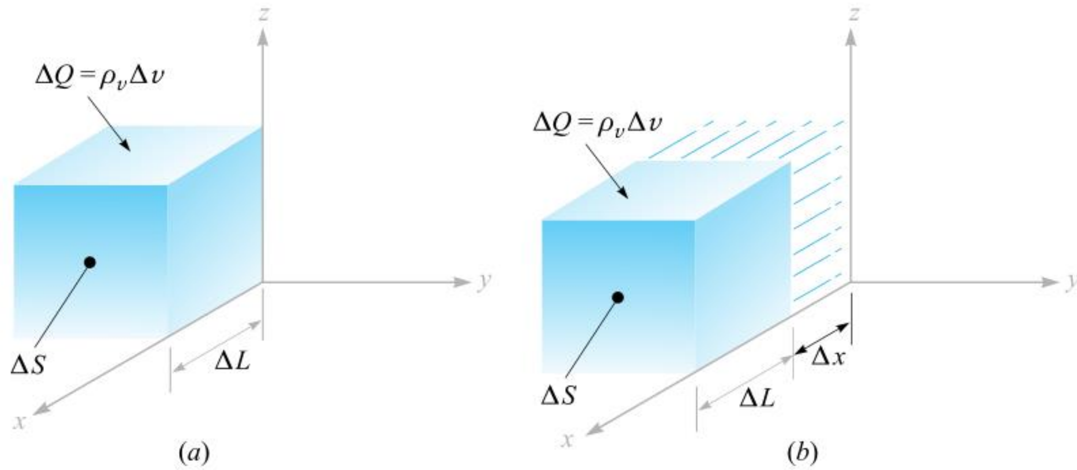

Figure 5.1 An increment of charge, $\Delta Q = \rho_v \Delta S \Delta L$, which moves a distance Δx in a time Δt, produces a component of current density in the limit of $J_x = \rho_v v_x$.

Current density may be related to the velocity of volume charge density at a point. Consider the element of charge $\Delta Q = \rho_v \Delta v = \rho_v \, \Delta S \, \Delta L$, as shown in Figure 5.1*a*. To simplify the explanation, assume that the charge element is oriented with its edges parallel to the coordinate axes and that it has only an x component of velocity. In the time interval Δt, the element of charge has moved a distance Δx, as indicated in Figure 5.1*b*. We have therefore moved a charge $\Delta Q = \rho_v \, \Delta S \, \Delta x$ through a reference plane perpendicular to the direction of motion in a time increment Δt, and the resulting current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta x}{\Delta t}$$

As we take the limit with respect to time, we have

$$\Delta I = \rho_v \, \Delta S \, v_x$$

where v_x represents the x component of the velocity $\mathbf{v}$.[2] In terms of current density, we find

$$J_x = \rho_v \, v_x$$

and in general

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}} \tag{3}$$

This last result shows clearly that charge in motion constitutes a current. We call this type of current a *convection current,* and $\mathbf{J}$ or $\rho_v \mathbf{v}$ is the *convection current density*.

CONTINUITY OF CURRENT

The continuity equation follows from this principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

and this *outward flow* of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i, then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} \tag{4}$$

It might be well to answer here an often-asked question. "Isn't there a sign error? I thought $I = dQ/dt$." The presence or absence of a negative sign depends on what current and charge we consider. In circuit theory we usually associate the current flow *into* one terminal of a capacitor with the time rate of increase of charge on that plate. The current of (4), however, is an *outward-flowing* current.

Equation (4) is the integral form of the continuity equation; the differential, or point, form is obtained by using the divergence theorem to change the surface integral into a volume integral:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v \, dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

from which we have our point form of the continuity equation,

$$\boxed{(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t}} \tag{5}$$

which is called the *continuity of current equation* or just *continuity equation.*