MODULE-5	FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION
Faraday's law of Electromagnetic Induction- Integral form and Point form, Numerical problems. Inconsistency of Ampere's law with continuity equation, displacement current, Conduction current, Derivation of Maxwell's equations in point form, and integral form, Maxwell's equations for different media, Numerical problems (Text: Chapter10.1 to10.4) Uniform Plane Wave: Wave propagation in free space, Uniform plane wave, Derivation of plane wave equations from Maxwell's equations,Poynting's Theorem and wave power, Skin effect or Depth of penetration, Numerical problems. (Text: Chapter 12.1, 12.3, 12.4) RBT Level: L1, L2, L3	

FARADAY'S LAW

Faraday discovered that the **induced emf,** V_{emf} (in volts) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

Faraday's law is customarily stated as

$$\boxed{\text{emf} = -\frac{d\Phi}{dt}\ \text{V}} \tag{1}$$

Equation (1) implies a closed path, although not necessarily a closed conducting path; the closed path, for example, might include a capacitor, or it might be a purely imaginary line in space. The magnetic flux is that flux which passes through any and every surface whose perimeter is the closed path, and $d\Phi/dt$ is the time rate of change of this flux.

A nonzero value of $d\Phi/dt$ may result from any of the following situations:

1. A time-changing flux linking a stationary closed path
2. Relative motion between a steady flux and a closed path
3. A combination of the two

The minus sign is an indication that the emf is in such a direction as to produce a current whose flux, if added to the original flux, would reduce the magnitude of the emf. This statement that the induced voltage acts to produce an opposing flux is known as *Lenz's law*.[3]

If the closed path is that taken by an N-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and let

$$\boxed{\text{emf} = -N\frac{d\Phi}{dt}} \tag{2}$$

where Φ is now interpreted as the flux passing through any one of N coincident paths.

We need to define emf as used in (1) or (2). The emf is obviously a scalar, and (perhaps not so obviously) a dimensional check shows that it is measured in volts. We define the emf as

$$\boxed{\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L}} \tag{3}$$

Replacing Φ in (1) with the surface integral of **B**, we have

$$\boxed{\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}} \tag{4}$$

We first consider a stationary path. The magnetic flux is the only time-varying quantity on the right side of (4), and a partial derivative may be taken under the integral sign,

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{5}$$

Before we apply this simple result to an example, let us obtain the point form of this integral equation. Applying Stokes' theorem to the closed line integral, we have

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

where the surface integrals may be taken over identical surfaces. The surfaces are perfectly general and may be chosen as differentials,

$$(\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

and

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{6}$$

TRANSFORMER AND MOTIONAL ELECTROMOTIVE FORCES

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single turn ($N = 1$), eq. (9.1) becomes

$$V_{\text{emf}} = -N\frac{d\Phi}{dt} \tag{9.4}$$

In terms of $\mathbf{E}$ and $\mathbf{B}$, eq. (9.4) can be written as

$$V_{\text{emf}} = \oint_L \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{9.5}$$

The variation of flux with time as in eq. (9.1) or eq. (9.5) may be caused in three ways:

1. By having a stationary loop in a time-varying **B** field
2. By having a time-varying loop area in a static **B** field
3. By having a time-varying loop area in a time-varying **B** field

Stationary Loop in Time-Varying B Field (Transformer emf)

In Figure 9.3 a stationary conducting loop is in a time-varying magnetic **B** field. Equation (9.5) becomes

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}} \tag{9.6}$$

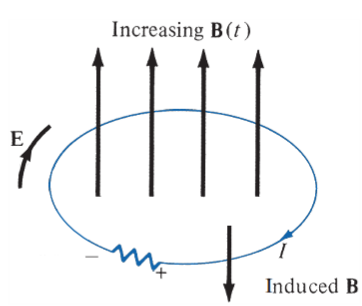

FIGURE 9.3 Induced emf due to a stationary loop in a time-varying **B** field.

This emf induced by the time-varying current (producing the time-varying **B** field) in a stationary loop is often referred to as *transformer emf* in power analysis, since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. (9.6), we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{9.7}$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{9.8}$$

This is one of the Maxwell's equations for time-varying fields.

Moving Loop in Static B Field (Motional emf)

When a conducting loop is moving in a static **B** field, an emf is induced in the loop. We recall from eq. (8.2) that the force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

We define the *motional electric field* $\mathbf{E}_m$ as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B} \tag{9.9}$$

If we consider a conducting loop, moving with uniform velocity **u** as consisting of a large number of free electrons, the emf induced in the loop is

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}} \tag{9.10}$$

This type of emf is called *motional emf* or *flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators.

Moving Loop in Time-Varying Field

In the general case, a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining eqs. (9.6) and (9.10) gives the total emf as

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}} \tag{9.15}$$

or from eqs. (9.8) and (9.14),

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{9.16}$$

DISPLACEMENT CURRENT

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{9.17}$$

But the divergence of the curl of any vector field is identically zero (see Example 3.10). Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \tag{9.18}$$

The continuity of current in eq. (5.43), however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \tag{9.19}$$

Thus eqs. (9.18) and (9.19) are obviously incompatible for time-varying conditions. We must modify eq. (9.17) to agree with eq. (9.19). To do this, we add a term to eq. (9.17) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \tag{9.20}$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \tag{9.21}$$

In order for eq. (9.21) to agree with eq. (9.19),

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{9.22a}$$

or

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}} \tag{9.22b}$$

Substituting eq. (9.22b) into eq. (9.20) results in

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{9.23}$$

This is Maxwell's equation (based on Ampère's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ is known as *displacement current density* and $\mathbf{J}$ is the conduction current

Based on the displacement current density, we define the *displacement current* as

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \tag{9.24}$$

This example, shown in Figure 9.10, serves to illustrate

the need for the displacement current. Applying an unmodified form of Ampère's circuit law to a closed path L shown in Figure 9.10(a) gives

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = I \tag{9.25}$$

where I is the current through the conductor and S_1 is the flat surface bounded by L. If we use the balloon-shaped surface S_2 that passes between the capacitor plates, as in Figure 9.10(b),

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = 0 \tag{9.26}$$

because no conduction current $(\mathbf{J} = 0)$ flows through S_2. This is contradictory in view of the fact that the same closed path L is used. To resolve the conflict, we need to include

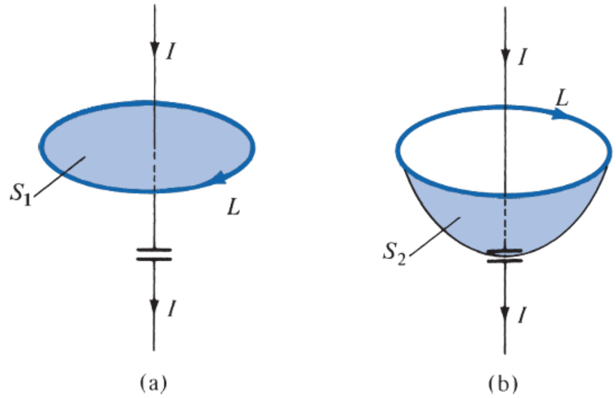

FIGURE 9.10 Two surfaces of integration showing the need for $\mathbf{J}_d$ in Ampère's circuit law.

the displacement current in Ampère's circuit law. The total current density is $\mathbf{J} + \mathbf{J}_d$. In eq. (9.25), $\mathbf{J}_d = \mathbf{0}$, so that the equation remains valid. In eq. (9.26), $\mathbf{J} = \mathbf{0}$, so that

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J}_d \cdot d\mathbf{S} = \frac{d}{dt}\int_{S_2} \mathbf{D} \cdot d\mathbf{S} = \frac{dQ}{dt} = I \tag{9.27}$$

So we obtain the same current for either surface, although it is conduction current in S_1 and displacement current in S_2.

MAXWELL'S EQUATIONS IN POINT FORM

We have already obtained two of Maxwell's equations for time-varying fields,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{20}$$

and

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{21}$$

The remaining two equations are unchanged from their non-time-varying form:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_\nu} \tag{22}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{23}$$

Equation (22) essentially states that charge density is a source (or sink) of electric flux lines. Note that we can no longer say that *all* electric flux begins and terminates on charge, because the point form of Faraday's law (20) shows that **E**, and hence **D**, may have circulation if a changing magnetic field is present. Thus the lines of electric flux may form closed loops. However, the converse is still true, and every coulomb of charge must have one coulomb of electric flux diverging from it.

Equation (23) again acknowledges the fact that "magnetic charges," or poles, are not known to exist. Magnetic flux is always found in closed loops and never diverges from a point source.

These four equations form the basis of all electromagnetic theory. They are partial differential equations and relate the electric and magnetic fields to each other and to their sources, charge and current density. The auxiliary equations relating **D** and **E**,

$$\boxed{\mathbf{D} = \epsilon \mathbf{E}} \tag{24}$$

relating **B** and **H**,

$$\boxed{\mathbf{B} = \mu \mathbf{H}} \tag{25}$$

defining conduction current density,

$$\boxed{\mathbf{J} = \sigma \mathbf{E}} \tag{26}$$

and defining convection current density in terms of the volume charge density ρ_ν,

$$\boxed{\mathbf{J} = \rho_\nu \mathbf{v}} \tag{27}$$

are also required to define and relate the quantities appearing in Maxwell's equations.

If we do not have "nice" materials to work with, then we should replace (24) and (25) with the relationships involving the polarization and magnetization fields,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}} \tag{28}$$

$$\boxed{\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})} \tag{29}$$

For linear materials we may relate **P** to **E**

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \tag{30}$$

and **M** to **H**

$$\mathbf{M} = \chi_m \mathbf{H} \tag{31}$$

Finally, because of its fundamental importance we should include the Lorentz force equation, written in point form as the force per unit volume,

$$\boxed{\mathbf{f} = \rho_\nu (\mathbf{E} + \mathbf{v} \times \mathbf{B})} \tag{32}$$

The following chapters are devoted to the application of Maxwell's equations to several simple problems.

MAXWELL'S EQUATIONS IN INTEGRAL FORM

Integrating (20) over a surface and applying Stokes' theorem, we obtain Faraday's law,

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}} \tag{33}$$

and the same process applied to (21) yields Ampère's circuital law,

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}} \tag{34}$$

Gauss's laws for the electric and magnetic fields are obtained by integrating (22) and (23) throughout a volume and using the divergence theorem:

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv} \tag{35}$$

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0} \tag{36}$$

These four integral equations enable us to find the boundary conditions on **B**, **D**, **H**, and **E**, which are necessary to evaluate the constants obtained in solving Maxwell's equations in partial differential form. These boundary conditions are in general unchanged from their forms for static or steady fields, and the same methods may be used to obtain them. Between any two real physical media (where **K** must be zero on the boundary surface), (33) enables us to relate the tangential **E**-field components,

$$E_{t1} = E_{t2} \tag{37}$$

and from (34),

$$H_{t1} = H_{t2} \tag{38}$$

The surface integrals produce the boundary conditions on the normal components,

$$D_{N1} - D_{N2} = \rho_S \tag{39}$$

and

$$B_{N1} = B_{N2} \tag{40}$$

It is often desirable to idealize a physical problem by assuming a perfect conductor for which σ is infinite but $\mathbf{J}$ is finite. From Ohm's law, then, in a perfect conductor,

$$\mathbf{E} = 0$$

and it follows from the point form of Faraday's law that

$$\mathbf{H} = 0$$

for time-varying fields. The point form of Ampère's circuital law then shows that the finite value of $\mathbf{J}$ is

$$\mathbf{J} = 0$$

and current must be carried on the conductor surface as a surface current $\mathbf{K}$. Thus, if region 2 is a perfect conductor, (37) to (40) become, respectively,

$$E_{t1} = 0 \tag{41}$$

$$H_{t1} = K \quad (\mathbf{H}_{t1} = \mathbf{K} \times \mathbf{a}_N) \tag{42}$$

$$D_{N1} = \rho_s \tag{43}$$

$$B_{N1} = 0 \tag{44}$$

where $\mathbf{a}_N$ is an outward normal at the conductor surface.

THE UNIFORM PLANE WAVE

A uniform plane wave is a type of electromagnetic wave that propagates through space with constant amplitude and phase across any plane perpendicular to the direction of propagation.

WAVE PROPAGATION IN FREE SPACE

When considering electromagnetic waves in free space, we note that the medium is *sourceless* ($\rho_v = \mathbf{J} = 0$). Under these conditions, Maxwell's equations may be written in terms of **E** and **H** only as

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{4}$$

In this case,

$$\boxed{\sigma = 0, \quad \varepsilon = \varepsilon_o, \quad \mu = \mu_o} \tag{10.45}$$

we obtain

$$\alpha = 0, \beta = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} \tag{10.46a}$$

$$u = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c, \lambda = \frac{2\pi}{\beta} \tag{10.46b}$$

where $c \simeq 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM waves travel in free space at the speed of light is significant. It provides some evidence that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

By substituting the constitutive parameters in eq. (10.45) into eq. (10.33), $\theta_\eta = 0$ and $\eta = \eta_o$, where η_o is called the *intrinsic impedance of free space* and is given by

$$\boxed{\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\ \Omega} \tag{10.47}$$

$$\mathbf{E} = E_o \cos(\omega t - \beta z)\ \mathbf{a}_x \tag{10.48a}$$

then

$$\mathbf{H} = H_o \cos(\omega t - \beta z)\mathbf{a}_y = \frac{E_o}{\eta_o}\cos(\omega t - \beta z)\mathbf{a}_y \tag{10.48b}$$

The plots of **E** and **H** are shown in Figure 10.7(a). In general, if $\mathbf{a}_E$, $\mathbf{a}_H$, and $\mathbf{a}_k$ are unit vectors along the **E** field, the **H** field, and the direction of wave propagation; it can be shown that (see Problem 10.69).

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$$

or

$$\mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

or

$$\boxed{\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k} \tag{10.49}$$

Both **E** and **H** fields (or EM waves) are everywhere normal to the direction of wave propagation, $\mathbf{a}_k$. That means that the fields lie in a plane that is transverse or orthogonal to the

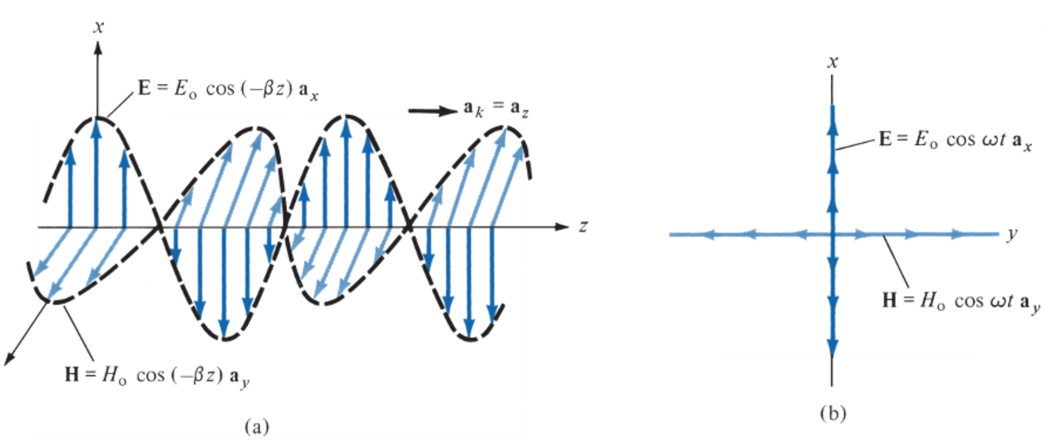

FIGURE 10.7 Plots of **E** and **H** (**a**) as functions of z at $t = 0$; and (**b**) at $z = 0$. The arrows indicate instantaneous values.

direction of wave propagation. They form an EM wave that has no electric or magnetic field components along the direction of propagation; such a wave is called a *transverse electromagnetic* (TEM) wave. A combination of **E** and **H** is called a *uniform plane wave* because **E** (or **H**) has the same magnitude throughout any transverse plane, defined by $z =$ constant. The direction in which the electric field points is the *polarization* of a TEM wave.

POYNTING'S THEOREM AND WAVE POWER

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the ohmic losses.

In order to find the power flow associated with an electromagnetic wave, it is necessary to develop a power theorem for the electromagnetic field known as the Poynting theorem. It was originally postulated in 1884 by an English physicist, John H. Poynting.

The development begins with one of Maxwell's curl equations, in which we assume that the medium may be conductive:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{63}$$

Next, we take the scalar product of both sides of (63) with **E**,

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{64}$$

We then introduce the following vector identity, which may be proved by expansion in rectangular coordinates:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \nabla \times \mathbf{H} + \mathbf{H} \cdot \nabla \times \mathbf{E} \tag{65}$$

Using (65) in the left side of (64) results in

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{66}$$

where the curl of the electric field is given by the other Maxwell curl equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Therefore

$$-\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

or

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} \tag{67}$$

The two time derivatives in (67) can be rearranged as follows:

$$\epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) \tag{68a}$$

and

$$\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) \tag{68b}$$

With these, Eq. (67) becomes

$$\boxed{-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right)} \tag{69}$$

Finally, we integrate (69) throughout a volume:

$$-\int_{\text{vol}} \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv = \int_{\text{vol}} \mathbf{J} \cdot \mathbf{E}\, dv + \int_{\text{vol}} \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) dv + \int_{\text{vol}} \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) dv$$

The divergence theorem is then applied to the left-hand side, thus converting the volume integral there into an integral over the surface that encloses the volume. On the right-hand side, the operations of spatial integration and time differentiation are interchanged. The final result is

$$\boxed{-\oint_{\text{area}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_{\text{vol}} \mathbf{J} \cdot \mathbf{E}\, dv + \frac{d}{dt}\int_{\text{vol}} \frac{1}{2}\mathbf{D} \cdot \mathbf{E}\, dv + \frac{d}{dt}\int_{\text{vol}} \frac{1}{2}\mathbf{B} \cdot \mathbf{H}\, dv} \tag{70}$$

Equation (70) is known as Poynting's theorem. The sum

of the expressions on the right must therefore be the total power flowing *into* this volume, and so the total power flowing *out* of the volume is

$$\boxed{\oint_{\text{area}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \quad \text{W}} \tag{71}$$

where the integral is over the closed surface surrounding the volume. The cross product $\mathbf{E} \times \mathbf{H}$ is known as the Poynting vector, $\mathbf{S}$,

$$\boxed{\mathbf{S} = \mathbf{E} \times \mathbf{H} \quad \text{W/m}^2} \tag{72}$$

which is interpreted as an instantaneous power density, measured in watts per square meter (W/m^2). The direction of the vector $\mathbf{S}$ indicates the direction of the instantaneous

PROPAGATION IN GOOD CONDUCTORS: SKIN EFFECT

Plane waves in good conductors comprise another special case of that considered in Section 10.3. A perfect, or good conductor, is one in which $\sigma \gg \omega\varepsilon$, so that $\frac{\sigma}{\omega\varepsilon} \gg 1$; that is,

$$\boxed{\sigma \simeq \infty, \quad \varepsilon = \varepsilon_o, \quad \mu = \mu_o\mu_r} \tag{10.50}$$

Hence, eqs. (10.23) and (10.24) become

Hence, eqs. (10.23) and (10.24) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \tag{10.51a}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \quad \lambda = \frac{2\pi}{\beta} \tag{10.51b}$$

Also, from eq. (10.32),

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \tag{10.52}$$

and thus **E** leads **H** by 45°. If

$$\mathbf{E} = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x \tag{10.53a}$$

then

$$\mathbf{H} = \frac{E_o}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z}\cos(\omega t - \beta z - 45^\circ)\mathbf{a}_y \tag{10.53b}$$

Therefore, as the **E** (or **H**) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ, shown in Figure 10.8, through which the wave amplitude decreases to a factor e^{-1} (about 37% of the original value) is called *skin depth* or *penetration depth* of the medium; that is,

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

or

$$\delta = \frac{1}{\alpha} \tag{10.54a}$$

The **skin depth** is a measure of the depth to which an **EM** wave can penetrate the medium.

Equation (10.54a) is generally valid for any material medium. For good conductors, eqs. (10.51a) and (10.54a) give

$$\boxed{\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha}} \tag{10.54b}$$

The illustration in Figure 10.8 for a good conductor is exaggerated. However, for a partially conducting medium, the skin depth can be quite large. Note from eqs. (10.51a), (10.52), and (10.54b) that for a good conductor,

$$\eta = \frac{1}{\sigma\delta}\sqrt{2}\, e^{j\pi/4} = \frac{1 + j}{\sigma\delta} \tag{10.55}$$

Noting that for good conductors we have $\alpha = \beta = \frac{1}{\delta}$, eq. (10.53a) can be written as

$$\mathbf{E} = E_o e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)\mathbf{a}_x$$

showing that δ measures the exponential damping of the wave as it travels through the conductor. The skin depth in copper at various frequencies is shown in Table 10.2. From Table 10.2, we notice that the skin depth decreases with increasing frequency. Thus, **E** and **H** can hardly propagate through good conductors.

The phenomenon whereby field intensity in a conductor rapidly decreases is known as the *skin effect*. It is a tendency of charges to migrate from the bulk of the conducting material to the surface, resulting in higher resistance. The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface.

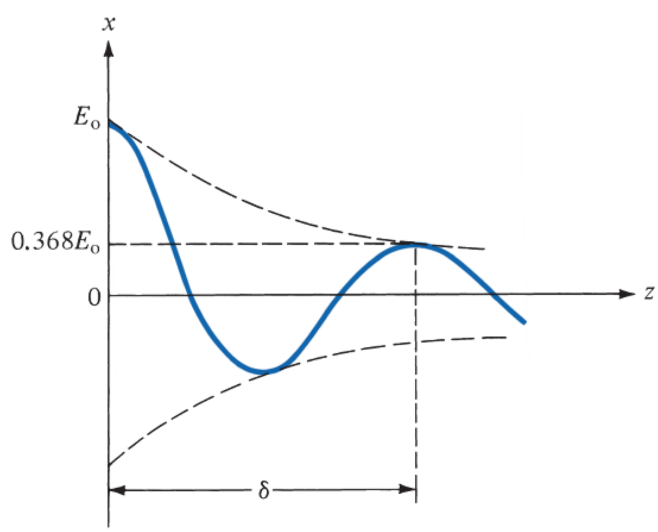

FIGURE 10.8 Illustration of skin depth.