CSCI 2820: Linear Algebra & CS applications
Fall 2024

due date: Friday, September 20 at 10:00 am

HW 4F: Linear Systems, Linear Combinations, & Matrix Equations

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The goal of this assignment is to convince you that the following 3 problems are actually all just different ways of framing the same problem:

1. Solving a system of linear number equations
2. Solving a linear vector equation – i.e. determining whether one vector can be written as a linear combination of other vectors
3. Solving a matrix equation – i.e. given a matrix A and a vector $\vec{\mathbf{b}}$, determine if there is a vector $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$

1 Understanding Linear Systems

Lay & McDonald, Chapter 1.3 exercises 5-10.

In exercises 5, 6, 9, and 10, write the system as a matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ as well.

5. The vector equation:

$$x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

   The matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

$$\begin{bmatrix} 6 & -3 \\ -1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

6. The vector equation:

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   The matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

$$\begin{bmatrix} -2 & 8 & 1 \\ 3 & 5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

9. The vector equation:

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   The matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

$$\begin{bmatrix} 0 & 1 & 5 \\ 4 & 6 & -1 \\ -1 & 3 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

10. The vector equation:

$$x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

   The matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

$$\begin{bmatrix} 4 & 1 & 3 \\ 1 & -7 & -2 \\ 8 & 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

2 Solving Linear Systems

a. Lay & McDonald, Chapter 1.1 exercises 3-4, 11-12

b. Lay & McDonald, Chapter 1.3 exercise 26

3. Find the point (x_1, x_2) that lies on the lines:

$$x_1 + 5x_2 = 7$$
$$x_1 - 2x_2 = -2$$

Solution: Solving this system, we find $x_1 = 3$ and $x_2 = 4/5$. So the point is $(3, 4/5)$.

4. Find the point of intersection of the lines:

$$x_1 - 5x_2 = 1$$
$$3x_1 - 7x_2 = 5$$

Solution: Solving this system, we find $x_1 = 6$ and $x_2 = 1$. So the point of intersection is $(6, 1)$.

11. Solve the system:

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

Solution: Using Gaussian elimination, we find $x_1 = 2$, $x_2 = 1$, and $x_3 = -2$.

12. Solve the system:

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}$$

Solution: After row reduction, the solution is $x_1 = -1$, $x_2 = 2$, and $x_3 = -3$.

26. Let

$$A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$$

and let W be the set of all linear combinations of the columns of A.

   a. Is $\mathbf{b} \in W$? **Solution:** No, $\mathbf{b}$ is not in W because the system $A\mathbf{x} = \mathbf{b}$ does not have a solution.

   b. Show that the third column of A is in W. **Solution:** The third column is in W because it is part of the span of the columns of A.

3 Matrix Equations & Column Space

Lay & McDonald, Chapter 1.4 exercises 15-20, 25, 35-36

15. Let

$$A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solution: The determinant of A is zero, so the system has solutions only when b_1 and b_2 are dependent. If $b_1 = -2b_2$, then there is a solution.

16. Repeat Exercise 15 with

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution: Using Gaussian elimination, we find that a solution exists only if $b_1 + b_2 - 2b_3 = 0$.

17. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b} \in \mathbb{R}^4$? **Solution:** Perform Gaussian elimination to determine the number of pivots. If the number of pivots is equal to the number of rows, then there is a solution for each $\mathbf{b} \in \mathbb{R}^4$.

18. Do the columns of B span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y} \in \mathbb{R}^4$? **Solution:** Perform Gaussian elimination on B. If there are 4 pivot positions, then the columns span $\mathbb{R}^4$, and the system has a solution for each $\mathbf{y}$.

19. Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$? **Solution:** If A has full rank, its columns span $\mathbb{R}^4$. Perform Gaussian elimination to verify.

20. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$? **Solution:** If the rank of B is 3, then the columns span $\mathbb{R}^3$ but not $\mathbb{R}^4$.

25. Given

$$\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$$

Find c_1, c_2, c_3 such that

$$\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$$

Solution: Solving this system, we find $c_1 = 1$, $c_2 = 2$, and $c_3 = -1$.

35. Let A be a 3×4 matrix, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be vectors in $\mathbb{R}^3$, and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$, for some vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^4$. What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent? **Solution:** Since $\mathbf{w} = A(\mathbf{x}_1 + \mathbf{x}_2)$, the system is consistent due to the linearity of matrix multiplication.

36. Let A be a 5×3 matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 4\mathbf{z}$ is consistent? **Solution:** Since $A\mathbf{y} = \mathbf{z}$, multiplying both sides by 4 gives $A(4\mathbf{y}) = 4\mathbf{z}$, ensuring the system is consistent with $\mathbf{x} = 4\mathbf{y}$.