On estimating the trace of quantum state powers

- Purification Estimation is BQP-complete problem?

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Chapter 1.

Introduction

Estimating purity and BQP- complete

Purity $\mathrm{tr}(\rho^2)$: SWAP test $\rightarrow O\left(\frac{1}{\varepsilon^2}\right)$ copies of ρ

$\mathrm{tr}(\rho^q) \rightarrow \exists$ efficient quantum algorithm when integer $q > 1$

Q.

1. $\exists$ efficient Q. A. for estimating $\mathrm{tr}(\rho^q)$ when non-integer $q > 1$?

2. Estimating $\mathrm{tr}(\rho^q)$ or purity
: Could fully capture the computational power of Q.C.? (BQP-complete ?)

Entropy Definitions

Quantum q-Tsallis entropy $S_q(\rho) = \frac{1-\mathrm{tr}(\rho^q)}{q-1}$

von Neumann entropy $\lim_{q\to 1} S_q(\rho) = S(\rho) = -\mathrm{tr}(\rho\log(\rho))$

Purity $\mathrm{tr}(\rho^2) = 1 - S_2(\rho)$

Definition 2.2 (q-Tsallis entropy and Shannon entropy)**.** *Let p be a probability distribution over $[N]$. The q-Tsallis entropy of p is defined by*

$$\mathrm{H}_q(p) := \frac{1-\sum_{x\in[N]} p(x)^q}{q-1} = -\sum_{x\in[N]} p(x)^q \ln_q(p(x)).$$

The Shannon entropy is the limiting case of the q-Tsallis entropy as $q \to 1$:

$$\mathrm{H}_1(p) := \lim_{q\to 1} \mathrm{H}_q(p) \quad \textit{and} \quad \lim_{q\to 1} \mathrm{H}_q(p) = \mathrm{H}(p) := -\sum_{x\in[N]} p(x)\ln(p(x)).$$

Estimating the trace of quantum state powers

In this section, we will establish reductions from the closeness testing of quantum states via the trace distance to testing via the quantum q-Tsallis entropy difference. Our proof crucially

Quantum q-Tsallis Entropy Difference Problem (TsallisQED$_q$)
Quantum q-Tsallis Entropy Approximation Problem (TsallisQEA$_q$)

Definition 5.1 (Quantum q-Tsallis Entropy Difference, TsallisQED$_q$). *Let Q_0 and Q_1 be quantum circuits acting on m qubits and having n specified output qubits, where $m(n)$ is a polynomial in n. Let ρ_i be the quantum state obtained by running Q_i on $|0\rangle^{\otimes m}$ and tracing out the non-output qubits. Let $g(n)$ be a positive efficiently computable function. Decide whether:*

- Yes: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{S}_q(\rho_0) - \mathrm{S}_q(\rho_1) \geq g(n)$;*
- No: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{S}_q(\rho_1) - \mathrm{S}_q(\rho_0) \geq g(n)$.*

Definition 5.2 (Quantum q-Tsallis Entropy Approximation, TsallisQEA$_q$). *Let Q be a quantum circuit acting on m qubits and having n specified output qubits, where $m(n)$ is a polynomial in n. Let ρ be the quantum state obtained by running Q on $|0\rangle^{\otimes m}$ and tracing out the non-output qubits. Let $g(n)$ and $t(n)$ be positive efficiently computable functions. Decide whether:*

- Yes: *A quantum circuit Q such that $\mathrm{S}_q(\rho) \geq t(n) + g(n)$;*
- No: *A quantum circuit Q such that $\mathrm{S}_q(\rho) \leq t(n) - g(n)$.*

(1) ConstRankTsallisQED$_q$: the ranks of ρ_0 and ρ_1 are at most $O(1)$.

(2) ConstRankTsallisQEA$_q$: the rank of ρ is at most $O(1)$.

Chapter 2.

Results

Computational hardness of TsallisQED$_q$ and TsallisQEA$_q$

	$q = 1$	$1 < q \leq 1+\frac{1}{n-1}$	$1+\Omega(1) \leq q \leq 2$	$q > 2$
TsallisQED$_q$	QSZK-complete [BASTS10]	QSZK-hard Theorem 1.2(2)	BQP-complete Theorems 1.1 and 1.2(1)	in BQP Theorem 1.1
TsallisQEA$_q$	NIQSZK-complete [BASTS10, CCKV08]	NIQSZK-hard* Theorem 1.2(2)	BQP-complete Theorems 1.1 and 1.2(1)	in BQP Theorem 1.1

Table 1: Computational hardness of TsallisQED$_q$ and TsallisQEA$_q$.

Bounds on query and sample complexities for estimating $S_q(\rho)$

Rank r

Regime of q	Query Complexity		Sample Complexity	
	Upper Bound	Lower Bound	Upper Bound	Lower Bound
$q \geq 1 + \Omega(1)$	$O(1/\epsilon^{1+\frac{1}{q-1}})$ Theorem 3.2	$\Omega(1/\sqrt{\epsilon})$ Theorem 5.12	$\widetilde{O}(1/\epsilon^{3+\frac{2}{q-1}})$ Theorem 3.3	$\Omega(1/\epsilon)$ Theorem 5.15
$1 < q \leq 1 + \frac{1}{n-1}$	$\widetilde{O}(r/\epsilon^2)$ [WZL24]	$\Omega(r^{0.17-c})$[7] Theorem 5.13	$\widetilde{O}(r^2/\epsilon^5)$[8] [WZ24c]	$\Omega(r^{0.51-c'})$[7] Theorem 5.16
$q = 1$	$\widetilde{O}(r/\epsilon^2)$[9] [WGL+24]	$\widetilde{\Omega}(\sqrt{r})$ [BKT20]	$\widetilde{O}(r^2/\epsilon^5)$[8] [WZ24c]	$\Omega(r/\epsilon)$ [WZ24c]

Reductions for TsallisQED$_q$ and TsallisQEA$_q$

Problem	Regime of q	Reduction from	New inequalities
ConstRank TsallisQED$_q$ Theorem 1.2(1)	$1 \le q \le 2$	PureQSD is BQP-hard adapted from [RASW23]	$\mathrm{H}_q\left(\frac{1}{2}\right) - \mathrm{H}_q\left(\frac{1-\mathrm{T}}{2}\right) \le \mathrm{QJT}_q \le \mathrm{H}_q\left(\frac{1}{2}\right)\mathrm{T}^q$ Theorem 4.1
TsallisQED$_q$ Theorem 1.2(2)	$1 \le q \le 1 + \frac{1}{n-1}$	QSD is QSZK-hard [Wat02, Wat09]	$\mathrm{H}_q\left(\frac{1}{2}\right) - \mathrm{H}_q\left(\frac{1-\mathrm{T}}{2}\right) \le \mathrm{QJT}_q$ Theorem 4.1
TsallisQEA$_q$ Theorem 1.2(2)	$q = 1 + \frac{1}{n-1}$	QSCMM is NIQSZK-hard [Kob03, BASTS10, CCKV08]	$\left(1 - \mathrm{T} - \frac{1}{2^n}\right) \ln_q(2^n) \le \mathrm{S}_q \le \ln_q(2^n(1-\mathrm{T}))$ Lemma 4.10

Chapter 3.

Purification Estimation is in BQP

Quantum estimator for $S_q(\rho)$

Theorem 1.1 (Quantum estimator for q-Tsallis entropy). *Given quantum query access to the state-preparation circuit of an n-qubit quantum state ρ, for any $q \geq 1+\Omega(1)$, there is a quantum algorithm for estimating $\mathrm{S}_q(\rho)$ to additive error 0.001 with query complexity $O(1)$. Moreover, if the description of the state-preparation circuit is of size $\mathrm{poly}(n)$, then the time complexity of the quantum algorithm is $\mathrm{poly}(n)$. Consequently, for any $q \geq 1+\Omega(1)$, $\textsc{TsallisQED}_q$ and $\textsc{TsallisQEA}_q$ are in BQP.*

Previous quantum algorithms
: time complexity $\exp(n)$

Desired additive error ε

Query complexity $O(1/\varepsilon^{1+\frac{1}{q-1}})$ or $\mathrm{poly}(1/\varepsilon)$

Time complexity $\tilde{O}(L/\varepsilon^{1+\frac{1}{q-1}}))$ or $\mathrm{poly}(n, 1/\varepsilon)$
when the state-preparation circuit of ρ is the size $L(n) = \mathrm{poly}(1/\varepsilon)$

Sample complexity upper bound $\tilde{O}(L/\varepsilon^{3+\frac{2}{q-1}}))$ or $\mathrm{poly}(1/\varepsilon)$

Efficient quantum algorithms for estimating $S_q(\rho)$

Algorithm 1 A framework for estimating q-Tsallis entropy for $q \geq 1 + \Omega(1)$ (query access).

Input: A quantum circuit Q that prepares a purification of an n-qubit mixed quantum state ρ, and a precision parameter $\epsilon \in (0, 1)$.

Output: A single bit $b \in \{0, 1\}$ such that $\Pr[b = 0] \approx \frac{1}{2} + \frac{1}{8}\operatorname{tr}(\rho^q)$.

1: Implement a unitary operator U_ρ that is a block-encoding of ρ by Lemma 2.28, using $O(1)$ queries to Q.

2: Let $P(x)$ be a polynomial that approximates $\frac{1}{4}x^{q-1}$ in the range $[0, 1]$, where $P(x)$ is determined according to ϵ, n, and q. More precisely, for constant $q > 1$, $P(x)$ is chosen by Lemma 3.1.

3: Implement a unitary operator $U_{P(\rho)}$ that is a block-encoding of $P(\rho)$ by quantum singular value transformation (Lemma 2.26), using $O(\deg(P))$ queries to U_ρ.

4: Perform the Hadamard test on ρ and $U_{P(\rho)}$ by Lemma 2.29, and return the measurement outcome.

Theorem 3.2 (Trace estimation of quantum state constant powers via queries)**.** *Suppose that Q is a unitary operator that prepares a purification of mixed quantum state ρ. For every $q \geq 1 + \Omega(1)$, there is a quantum query algorithm that estimates $\operatorname{tr}(\rho^q)$ to within additive error ϵ by using $O(1/\epsilon^{1+\frac{1}{q-1}})$ queries to Q.*

Efficient quantum algorithms for estimating $S_q(\rho)$

Algorithm 2 A framework for estimating q-Tsallis entropy for $q > 1 + \Omega(1)$ (sample access).

Input: Independent and identical samples of an n-qubit mixed quantum state ρ, and parameters $q > 1$ and $\delta, \epsilon_p, \delta_p \in (0, 1)$.

Output: A single bit $b \in \{0, 1\}$ such that $\Pr[b = 0] \approx \frac{1}{2} + \frac{1}{2^{q+3}} \operatorname{tr}(\rho^q)$.

1: **function** $\texttt{ApproxPower}(q, \epsilon_p, \delta_p)^U$
   Input: A unitary $(1, a, 0)$-block-encoding U of A, and parameters $q > 1, \epsilon_p, \delta_p \in (0, 1)$.
   Output: A unitary operator $\widetilde{U}$.
2: Let $P(x)$ be a polynomial of degree $d = O(1/\epsilon_p^{\frac{1}{q-1}})$ such that $\max_{x\in[0,1]}|P(x) - \frac{1}{2}x^{q-1}| \le \epsilon_p$ and $\max_{x\in[-1,1]}|P(x)| \le 1$ (by Lemma 3.1).
3: Construct a unitary $(1, a + 2, \delta_p)$-block-encoding $\widetilde{U}$ of $\frac{1}{2}P(A)$ (by Lemma 2.26).
4: **return** $\widetilde{U}$.
5: **end function**

6: Let b' be the outcome of the Hadamard test (by Lemma 2.29) performing on the quantum state ρ and $\mathsf{Samplize}_\delta\langle \texttt{ApproxPower}(q, \epsilon_p, \delta_p)^U \rangle[\rho]$ (as if it were unitary).
7: **return** b'.

Theorem 3.3 (Trace estimation of quantum state constant powers via samples). *For every $q \ge 1 + \Omega(1)$, there is a quantum sample algorithm that estimates $\operatorname{tr}(\rho^q)$ to within additive error ϵ by using $\widetilde{O}(1/\epsilon^{3+\frac{2}{q-1}})$ samples of ρ.*

Chapter 4.

Purification Estimation is BQP-hard problem

Quantum state Distinguishability

Definition 2.6 (Trace distance). *The trace distance between two quantum states ρ_0 and ρ_1 is*

$$\mathrm{T}(\rho_0, \rho_1) := \frac{1}{2}\mathrm{tr}(|\rho_0 - \rho_1|) = \frac{1}{2}\mathrm{tr}\left(\left((\rho_0 - \rho_1)^\dagger(\rho_0 - \rho_1)\right)^{1/2}\right).$$

We begin by defining the closeness testing of quantum states with respect to the trace distance, denoted as $\mathrm{QSD}[\alpha, \beta]$,[26] along with a variant of this promise problem, as described in Definition 2.14. In particularly, we say that $\mathcal{P} = (\mathcal{P}_{\text{yes}}, \mathcal{P}_{\text{no}})$ is a *promise* problem, if it satisfies the conditions $\mathcal{P}_{\text{yes}} \cap \mathcal{P}_{\text{no}} = \emptyset$ and $\mathcal{P}_{\text{yes}} \cup \mathcal{P}_{\text{no}} \subseteq \{0,1\}^*$.

Definition 2.14 (Quantum State Distinguishability, QSD, adapted from [Wat02, Section 3.3]). *Let Q_0 and Q_1 be quantum circuits acting on m qubits ("input length") and having n specified output qubits ("output length"), where $m(n)$ is a polynomial function of n. Let ρ_i denote the quantum state obtained by running Q_i on state $|0\rangle^{\otimes m}$ and tracing out the non-output qubits. Let $\alpha(n)$ and $\beta(n)$ be efficiently computable functions. Decide whether:*

- Yes: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{T}(\rho_0, \rho_1) \geq \alpha(n)$;*
- No: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{T}(\rho_0, \rho_1) \leq \beta(n)$.*

Furthermore, we denote the restricted version, where ρ_0 and ρ_1 are pure states, as PureQSD.

Definition 2.15 (Quantum State Closeness to Maximally Mixed State, QSCMM, adapt from [Kob03, Section 3]). *Let Q be a quantum circuit acting on m qubits and having n specified output qubits, where $m(n)$ is a polynomial function of n. Let ρ denote the quantum state obtained by running Q on state $|0\rangle^{\otimes m}$ and training out the non-output qubits. Let $\alpha(n)$ and $\beta(n)$ be efficiently computable functions. Decide whether:*

- Yes: *A quantum circuit Q such that $\mathrm{T}(\rho, (I/2)^{\otimes n}) \leq \beta(n)$;*
- No: *A quantum circuit Q such that $\mathrm{T}(\rho, (I/2)^{\otimes n}) \geq \alpha(n)$.*

Input models and reductions

- **White-box input model**: The input of the problem QSD consists of descriptions of polynomial-size quantum circuits Q_0 and Q_1. Specifically, for $b \in \{0,1\}$, the description of Q_b includes a sequence of polynomially many 1- and 2-qubit gates.
- **Black-box input model**: In this model, instead of providing the descriptions of the quantum circuits Q_0 and Q_1, only query access to Q_b is allowed, denoted as O_b for $b \in \{0,1\}$. For convenience, we also allow query access to $Q_b^\dagger$ and controlled-Q_b, denoted by $O_b^\dagger$ and controlled-O_b, respectively.

- **Karp reduction**. A deterministic polynomial-time computable function f is called a *Karp reduction* from a promise problem $\mathcal{P}$ to another promise problem $\mathcal{P}'$ if, for every x, the following holds: $x \in \mathcal{P}_{\text{yes}}$ if and only if $f(x) \in \mathcal{P}'_{\text{yes}}$, and $x \in \mathcal{P}_{\text{no}}$ if and only if $f(x) \in \mathcal{P}'_{\text{no}}$.
- **Turing reduction**. A promise problem $\mathcal{P}$ is *Turing-reducible* to a promise problem $\mathcal{P}'$ if there exists a deterministic polynomial-time oracle machine $\mathcal{A}$ such that, for every function f that solves $\mathcal{P}'$ it holds that $\mathcal{A}^f$ solves $\mathcal{P}$. Here, $\mathcal{A}^f(x)$ denotes the output of machine $\mathcal{A}$ on input x when given oracle access to f.

QSD and Computational hardness

Lemma 2.16 (QSD is QSZK-hard). *Let $\alpha(n)$ and $\beta(n)$ be efficiently computable functions satisfying $\alpha^2(n) - \beta(n) \geq 1/O(\log n)$. For any constant $\tau \in (0, 1/2)$, $\mathrm{QSD}[\alpha, \beta]$ is QSZK-hard under Karp reduction when $\alpha(n) \leq 1 - 2^{-n^\tau}$ and $\beta(n) \geq 2^{-n^\tau}$ for every $n \in \mathbb{N}$.*

Lemma 2.17 (PureQSD is BQP-hard). *Let $\alpha(n)$ and $\beta(n)$ be efficiently computable functions such that $\alpha(n) - \beta(n) \geq 1/\mathrm{poly}(n)$. For any polynomial $l(n)$, let $n' := n+1$, $\mathrm{PureQSD}[\alpha(n'), \beta(n')]$ is BQP-hard when $\alpha(n') \leq 1 - 2^{-l(n'-1)}$ and $\beta(n') \geq 2^{-l(n'-1)}$ for every integer $n' \geq 2$. Specifically, by choosing $l(n'-1) = n'$, it holds that: For every integer $n' \geq 2$,*

$$\mathrm{PureQSD}\left[1 - 2^{-n'}, 2^{-n'}\right] \text{ is BQP-hard under Karp reduction.}$$

Lemma 2.18 (QSCMM is NIQSZK-hard, adapted from [CCKV08, Section 8.1]).

For any $n \geq 3$, $\mathrm{QSCMM}[1/n, 1 - 1/n]$ is NIQSZK-hard under Karp reduction.

PureQSD and ConstRankTsallisQED$_q$ for $1 \leq q \leq 2$

The reduction in Lemma 5.4 is from the trace distance between two n-qubit pure states (PureQSD) to the quantum q-Tsallis entropy difference between two new constant-rank $(n+1)$-qubit states (ConstRankTsallisQED$_q$), for $1 \leq q \leq 2$.

Lemma 5.4 (PureQSD $\leq$ ConstRankTsallisQED$_q$). *Let Q_0 and Q_1 be quantum circuits acting on n qubits and having the same number of output qubits. Let $|\psi_i\rangle$ be the quantum state obtained by running Q_i on $|0\rangle^{\otimes n}$. For any $b \in \{0,1\}$, there is a new quantum circuit Q'_b acting on $n+3$ qubits, using $O(1)$ queries to controlled-Q_0 and controlled-Q_1, as well as $O(1)$ one- and two-qubit gates. The circuit Q'_b prepares a new quantum state ρ'_b, which has constant rank and acts on $n' := n+1$ qubits, such that for any efficiently computable functions $\alpha(n)$ and $\beta(n)$, where $\beta(n) + \sqrt{1-\alpha(n)^2} < 1$, and any $q \in [1,2]$, the following holds:*

$$\mathrm{T}(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|) \geq \alpha(n) \;\Rightarrow\; \mathrm{S}_q(\rho'_0) - \mathrm{S}_q(\rho'_1) \geq g_q(n') = g_q(n+1),$$
$$\mathrm{T}(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|) \leq \beta(n) \;\Rightarrow\; \mathrm{S}_q(\rho'_1) - \mathrm{S}_q(\rho'_0) \geq g_q(n') = g_q(n+1),$$

where $g_q(n+1) := 2^{-q} \cdot \mathrm{H}_q(1/2) \cdot \left(1 - \beta(n)^q - \sqrt{1-\alpha(n)^2}\right)$.

Theorem 5.7 (ConstRankTsallisQED$_q$ is BQP-hard for $1 \leq q \leq 2$). *For any $q \in [1,2]$ and any $n \geq 3$, the following holds:*

$$\forall g_q(n) \in \left[\frac{1}{\mathrm{poly}(n)}, 2^{-q}\mathrm{H}_q\left(\frac{1}{2}\right)\left(1 - 2^{-\frac{n}{2}+\frac{7}{5}}\right)\right],\ \text{ConstRankTsallisQED}_q[g_q(n)] \textit{ is BQP-hard.}$$

QSD and TsallisQED$_q$ for $1 \leq q \leq 2$

5.2.1 QSD $\leq$ TsallisQED$_q$ for $1 \leq q \leq 2$

Lemma 5.5 (QSD $\leq$ TsallisQED$_q$). *Let Q_0 and Q_1 be quantum circuits acting on m qubit, defined in Definition 5.1, that prepares the purification of n-qubit mixed states ρ_0 and ρ_1, respectively. For any $b \in \{0,1\}$, there is a new quantum circuits Q'_b acting on $m+3$ qubits, requiring $O(1)$ queries to controlled-Q_0 and controlled-Q_1, as well as $O(1)$ one- and two- qubit gates, that prepares a new n'-qubit mixed state ρ'_b, where $n' := n+1$, such that: For any ρ_0 and ρ_0 satisfying $\max\{\mathrm{S}_q(\rho_0), \mathrm{S}_q(\rho_1)\} \leq \gamma(n)$ with $\mathrm{S}_q(I/2) \leq \gamma(n) \leq \mathrm{S}_q((I/2)^{\otimes n})$, any $\varepsilon(n) \in (0, 1/2)$, and any $q \in [1,2]$, there is a $g(n) > 0$ with appropriate ranges of γ, ε, and n such that*

$$\begin{aligned} \mathrm{T}(\rho_0,\rho_1) \geq 1-\varepsilon(n) \quad &\Rightarrow\ \mathrm{S}_q(\rho'_0) - \mathrm{S}_q(\rho'_1) \geq g_q(n') = g_q(n+1), \\ \mathrm{T}(\rho_0,\rho_1) \leq \varepsilon(n) \quad &\Rightarrow\ \mathrm{S}_q(\rho'_1) - \mathrm{S}_q(\rho'_0) \geq g_q(n') = g_q(n+1), \end{aligned}$$

where $g_q(n) := \frac{1}{2}\mathrm{H}_q\left(\frac{1}{2}\right) - \gamma(n)\left(\frac{1}{2}-\frac{1}{2^q}\right) - \left(\frac{1}{2}+\frac{1}{2^q}\right)\left(\frac{\varepsilon(n)^q}{2^q}\ln_q\left(2^n\right)+\mathrm{H}_q\left(\frac{1}{2}\right)\sqrt{\varepsilon(n)(2-\varepsilon(n))}\right)$.

Theorem 5.9 (TsallisQED$_q$ is QSZK-hard for $1 < q \leq 1+\frac{1}{n-1}$). *For any $q \in \left(1, 1+\frac{1}{n-1}\right]$ and any $n \geq 90$, it holds that*

$$\forall g(n) \in [1/\operatorname{poly}(n), 1/400],\ \mathrm{TsallisQED}_q[g(n)] \text{ is QSZK-hard.}$$

QSCMM and TsallisQEA$_q$ for $q = 1 + 1/(n-1)$

5.2.2 QSCMM $\leq$ TsallisQEA$_q$ for $q(n) = 1 + \frac{1}{n-1}$

Lemma 5.6 (QSCMM $\leq$ TsallisQEA$_q$). *Let Q be a quantum circuit acting on m qubit, defined in Definition 5.2, that prepares the purification of n-qubit mixed states ρ, respectively. For any ρ, any $n \geq 5$, and any $q(n) = 1 + 1/(n-1)$, let $t(n) := \frac{1}{4}\left(3n - n^{1+\frac{1}{n}} - 1\right)$, we have:*

$$\mathrm{T}\left(\rho, (I/2)^{\otimes n}\right) \leq 1/n \quad \Rightarrow \quad \mathrm{S}_q(\rho) > t(n) + 1/150,$$
$$\mathrm{T}\left(\rho, (I/2)^{\otimes n}\right) \geq 1 - 1/n \quad \Rightarrow \quad \mathrm{S}_q(\rho) < t(n) - 1/150.$$

Theorem 5.11 (TsallisQEA$_q$ is NIQSZK-hard for $q = 1 + \frac{1}{n-1}$). *For any $n \geq 5$, it holds that:*

$$\forall g(n) \in [1/\operatorname{poly}(n), 1/150], \ \mathrm{TsallisQEA}_{1+\frac{1}{n-1}} \text{ with } g(n) \text{ is NIQSZK-hard.}$$