Problem Set 2:
Quantum Learning and Complexity Theory (Summer 2025)

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Problem 1. Let ρ be a quantum state (i.e., a density matrix) and M be a Hermitian operator with operator norm $\|M\|_\infty \leq 1$. Define the function $f_{U,\rho} := \operatorname{tr}\left[M(U\rho U^\dagger)^{\otimes k}\right]$, where $U \in \mathcal{U}(d)$ is a d-dimensional unitary matrix. In this problem, you will prove that $f_{U,\rho}$ is Lipschitz with respect to U under the Hilbert–Schmidt norm.

(a) Show that for any unitaries $U, V \in \mathcal{U}(d)$, the difference of $f_{U,\rho}$ and $f_{V,\rho}$ can be bounded as

$$|f_{U,\rho} - f_{V,\rho}| \leq \left\|(U\rho U^\dagger)^{\otimes k} - (V\rho V^\dagger)^{\otimes k}\right\|_1 .$$

(b) Deduce that $\left\|(U\rho U^\dagger)^{\otimes k} - (V\rho V^\dagger)^{\otimes k}\right\|_1 \leq k \left\|U\rho U^\dagger - V\rho V^\dagger\right\|_1$.

(c) Show that $\left\|U\rho U^\dagger - V\rho V^\dagger\right\|_1 \leq 2\|\rho\|_2 \|U - V\|_2$.

(d) Conclude that the function $f_{U,\rho}$ has Lipschitz constant at most $L \leq 2k\|\rho\|_2 = 2k\sqrt{\operatorname{tr}[\rho^2]}$.

Problem 2. Let μ denote the uniform (Haar) measure over pure states on $\mathbb{C}^d$. For integers $a, b \geq 0$, consider the following operator:

$$\Phi_{a\to b}(\rho) := \mathbb{E}_{\psi\sim\mu}\left[\operatorname{tr}\left(\rho\psi^{\otimes a}\right) \cdot \psi^{\otimes b}\right],$$

which arises naturally in post-measurement analysis of symmetric measurements.

Show that, for any density operator ρ on $(\mathbb{C}^d)^{\otimes a}$,

$$\Phi_{a\to b}(\rho) \geq e^{-ab/d} \cdot \frac{\mathcal{S}_b}{\binom{d+b-1}{b}},$$

where $\mathcal{S}_b$ denotes the projector onto the symmetric subspace of $(\mathbb{C}^d)^{\otimes b}$.

Useful fact (Chiribella's Theorem): The operator $\Phi_{a\to b}(\rho)$ can be written as

$$\Phi_{a\to b}(\rho) = \frac{1}{\binom{d+a-1}{a}} \sum_{s=0}^{\min(a,b)} \frac{\binom{a}{s}\binom{d+b-1}{b-s}}{\binom{d+a+b-1}{b}} \cdot \operatorname{Clone}_{s\to b}\left(\operatorname{tr}_{a-s}[\rho]\right),$$

where the optimal cloning map $\operatorname{Clone}_{s\to b}$ satisfies

$$\operatorname{Clone}_{s\to b}(\sigma) = \frac{\binom{d+s-1}{s}}{\binom{d+b-1}{b}} \cdot \mathcal{S}_b\left(\sigma \otimes \mathbb{I}^{\otimes(b-s)}\right)\mathcal{S}_b.$$

(Hint: Try restricting the above sum to the $s = 0$ term to obtain a lower bound.)

Problem 3. Let $|\psi\rangle \in \mathbb{C}^d$ be a Haar-random pure state. In this problem, you will compute moments of expectation values $\langle\psi|A|\psi\rangle = \langle\psi|\, A\, |\psi\rangle$ for Hermitian matrices $A, B, C, D \in \mathbb{C}^{d\times d}$.

(a) Show that

$$\mathop{\mathbb{E}}_{\psi\sim\mathbb{C}^d} \langle\psi|A|\psi\rangle\ \langle\psi|B|\psi\rangle = \frac{1}{d(d+1)}\left(\operatorname{Tr}(A)\operatorname{Tr}(B) + \operatorname{Tr}(AB)\right).$$

(b) Show that

$$\mathop{\mathbb{E}}_{\psi\sim\mathbb{C}^d} \langle\psi|A|\psi\rangle\ \langle\psi|B|\psi\rangle\ \langle\psi|C|\psi\rangle = \frac{1}{d(d+1)(d+2)}\Big(\operatorname{Tr}(A)\operatorname{Tr}(B)\operatorname{Tr}(C) + \operatorname{Tr}(AB)\operatorname{Tr}(C)$$
$$+ \operatorname{Tr}(A)\operatorname{Tr}(BC) + \operatorname{Tr}(C)\operatorname{Tr}(AB) + \operatorname{Tr}(ABC) + \operatorname{Tr}(ACB)\Big).$$

(c) Show that

$$\mathop{\mathbb{E}}_{\psi\sim\mathbb{C}^d} \langle\psi|A|\psi\rangle\ \langle\psi|B|\psi\rangle\ \langle\psi|C|\psi\rangle\ \langle\psi|D|\psi\rangle = \frac{1}{d(d+1)(d+2)(d+3)}\sum_{\pi\in S_4}\operatorname{Tr}\left(A\otimes B\otimes C\otimes D\cdot P_d(\pi)\right),$$

where $P_d(\pi)$ denotes the natural permutation representation of $\pi \in S_4$ acting on $(\mathbb{C}^d)^{\otimes 4}$.

(d) Use the general formula

$$\mathop{\mathbb{E}}_{\psi\sim\mathbb{C}^d} |\psi\rangle\langle\psi|^{\otimes k} = \frac{1}{(d+k-1)\cdots d}\sum_{\pi\in S_k} P_d(\pi)$$

to justify your answers to parts (a)–(c), by tracing with appropriate tensor product operators.

Problem 4. Let U be an n-qubit unitary, and define the equality projector

$$\Pi^{\mathsf{eq}} := \sum_{x\in[N]} |x\rangle\langle x| \otimes |x\rangle\langle x|, \quad \text{where } N = 2^n.$$

Let $|\mathsf{EPR}_N\rangle := \frac{1}{\sqrt{N}}\sum_{x\in[N]} |x\rangle\otimes|x\rangle$ be the maximally entangled state.

(a) Show that for any unitary 2-design $\mathfrak{D}$,

$$\mathop{\mathbb{E}}_{U\sim\mathfrak{D}}\left[(U\otimes U)^\dagger\, \Pi^{\mathsf{eq}}\,(U\otimes U)\right] = \frac{2}{N+1}\cdot \Pi_{\mathsf{sym}}^{N,2},$$

where $\Pi_{\mathsf{sym}}^{N,2}$ is the projector onto the symmetric subspace of $(\mathbb{C}^N)^{\otimes 2}$.

(b) Show that

$$\mathop{\mathbb{E}}_{U\sim\mathfrak{D}}\left[(U\otimes \overline{U})^\dagger\, \Pi^{\mathsf{eq}}\,(U\otimes \overline{U})\right] = \left(\frac{2}{N+1}\cdot \Pi_{\mathsf{sym}}^{N,2}\right)^{T_B}.$$

where X^{T_B} denotes the partial transpose of X with respect to the second register.

(c) Show that

$$\mathop{\mathbb{E}}_{U\sim\mathfrak{D}}\left[(U\otimes \overline{U})^\dagger\, \Pi^{\mathsf{eq}}\,(U\otimes \overline{U})\right] = |\mathsf{EPR}_N\rangle\langle\mathsf{EPR}_N| + \frac{1}{N+1}\left(\mathsf{Id} - |\mathsf{EPR}_N\rangle\langle\mathsf{EPR}_N|\right).$$

Problem 5. In this problem, we explore the moments of random pure states sampled from subspaces orthogonal to a fixed vector, and study their interactions with projected permutation operators.

Let $|\psi\rangle \in \mathbb{C}^d$ be an arbitrary unit vector, and let $\mathbb{C}^{d-1}_{\psi_\perp}$ denote the $(d-1)$-dimensional subspace orthogonal to $|\psi\rangle$. Consider pure states $|\psi'\rangle$ sampled uniformly (according to Haar measure) from this subspace.

(a) Show that the first and second moments of $|\psi'\rangle$ sampled from $\mathbb{C}^{d-1}_{\psi_\perp}$ are given by:

$$\mathop{\mathbb{E}}_{\psi' \sim \mathbb{C}^{d-1}_{\psi_\perp}} |\psi'\rangle\langle\psi'| = \frac{I - |\psi\rangle\langle\psi|}{d-1}, \qquad \mathop{\mathbb{E}}_{\psi' \sim \mathbb{C}^{d-1}_{\psi_\perp}} |\psi'\rangle\langle\psi'|^{\otimes 2} = \frac{1}{d(d-1)} \left[(I - |\psi\rangle\langle\psi|)^{\otimes 2} + \mathrm{SWAP}_\psi \right],$$

where

$$\mathrm{SWAP}_\psi := (I - |\psi\rangle\langle\psi|)^{\otimes 2} \mathrm{SWAP} (I - |\psi\rangle\langle\psi|)^{\otimes 2}$$

is the SWAP operator projected onto the orthogonal subspace.

(Hint: Use the fact that the LHS is proportional to the orthogonal projector onto the symmetric subspace of $(\mathbb{C}^{d-1}_{\psi_\perp})^{\otimes 2}$.)

(b) Let $f := |\langle\phi|\psi\rangle|^2$ for some unit vector $|\phi\rangle \in \mathbb{C}^d$. Show that

$$\mathop{\mathbb{E}}_{\phi' \sim \mathbb{C}^{d-1}_{\phi_\perp}} |\langle\phi'|\psi\rangle|^4 = \frac{2(1-f)^2}{d(d-1)}.$$

(Hint: Express the fourth power as a trace of a tensor product and apply the second moment from part (a).)

(c) Consider $\phi' \sim \mathbb{C}^{d-1}_{\phi_\perp}$ and $\psi' \sim \mathbb{C}^{d-1}_{\psi_\perp}$ sampled independently. Show that

$$\mathbb{E}\left[|\langle\phi'|\psi'\rangle|^2 \cdot |\langle\phi'|\psi\rangle|^2 \right] = \frac{(1-f)(d+2f-2)}{d(d-1)^2}.$$

(Hint: Use the tensor product of the first and second moment operators, and compute the trace of a product of three operators.)

(d) Prove that

$$\mathop{\mathbb{E}}_{\phi', \psi'} |\langle\phi'|\psi'\rangle|^4 = \frac{2(d-2+f)^2 + 2(d-2+f^2)}{d^2(d-1)^2}.$$

(Hint: Expand the fourth moment as a trace over a product of projected swap operators. Compute each of the four terms separately:)

- $(I - |\phi\rangle\langle\phi|)^{\otimes 2} \cdot (I - |\psi\rangle\langle\psi|)^{\otimes 2}$
- $\mathrm{SWAP}_\phi \cdot (I - |\psi\rangle\langle\psi|)^{\otimes 2}$
- $(I - |\phi\rangle\langle\phi|)^{\otimes 2} \cdot \mathrm{SWAP}_\psi$
- $\mathrm{SWAP}_\phi \cdot \mathrm{SWAP}_\psi$