1. Determine whether the following proofs are correct. Why or why not?

(a) Theorem: If n is an even integer, then n^2 is an even integer.

Proof. Since this theorem discusses an even integer n, we pick 2 to be our even

integer for this proof. Clearly $2 = 2 \cdot 1$, so 2 is even. Next, we see that $2^2 = 4$ is

also clearly even. Thus, for any even integer n, n^2 is an even integer.

This proof is wrong, it does not show that n^2 is an even integer for all integers and only uses one example.

(b) Theorem: For any integer n, if n^3 is an odd integer, than n is an odd integer.

Proof. We suppose that n^3 is an odd integer. By definition, $n^3 = 2k + 1$ for some

integer k. Next, we let $n = 2j + 1$, for some integer j. By definition, this shows

that n is an odd integer.

This proof is also wrong, showing $n^3 = 2k + 1$ does not show how they make the solution of $n = 2j + 1$

2. Consider the statement "If $3n + 2$ is even for some integer n, then n is even". Prove this

statement using the given proof technique.

(a) Proof by contrapositive

Assuming n is odd, n can be shown as $n = 2k + 1$ for some integer k. Substituting n into $3n + 2$ gives us $6k + 5$. This is not divisible by two, so its not even. Therefore, if n is odd, $3n + 2$ is odd.

(b) Proof by contradiction

Assuming that $3n + 2$ is even for some integer n, but n is odd. If n is odd then $n = 2k + 1$ for some integer k. Substituting n into $3n + 2$ we get $6k + 5$ which is odd. However for some integer n $3n + 2$ is even leading to a contradiction, so n must be even.

3. Prove the following statements.

(a) Use a proof by contradiction to show that if r is a non-zero rational number and

Assuming that r is a non-zero rational number and s is an irrational numbers, then rs is rational. rs can be expressed as $rs = \frac{c}{d}$ where c and d are integers with no common factors and $d \neq 0$. Then, $rs = \frac{a}{b} \cdot s$, and $s = c \cdot \frac{b}{d} \cdot a$. Since a, b, c, and d are all integers this contradicts the assumption that s is irrational. So, if r is a non-zero rational number and s is an irrational number, then rs must be irrational.$

s is an irrational number, then rs is irrational. Recall that a real number r is

rational if it can be written as $r = \frac{a}{b}$, where a and b are integers with no common

factors and $b \neq 0$.

(b) Show that if a and b are distinct rational numbers, then there is an irrational number between a and b.

If $m = \frac{1}{2}$, then the value $a + \frac{1}{2} \cdot (b - a)$ represents a number exactly halfway between a and b. $c = \frac{1}{2}a + \frac{1}{2}b$. Assuming the contradiction, that c is a rational number and can be expressed as

$\frac{p}{q}$ where p and q are integers with no common factors and $q \neq 0$. But if $\frac{1}{2}a + \frac{1}{2}b = \frac{p}{q}$, showing that $a + b = 2\frac{p}{q}$ or $a + b = 2c$, which must be rational. However, this contradicts that a and b are distinct ration numbers as they must be the same number. Therefore c must be irrational and there exists an irrational number c between an distinct rational numbers a and b.

4. Use backward reasoning to show that $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$ for all positive real numbers x and y

1. $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$
2. $(\sqrt{\frac{x^2+y^2}{2}})^2 \geq (\frac{x+y}{2})^2$
3. $\frac{x^2+y^2}{2} \geq \frac{(x+y)^2}{4}$
4. $2(x^2 + y^2) \geq (x + y)^2$
5. $2x^2 + 2y^2 \geq x^2 + 2xy + y^2$
6. $x^2 - 2xy + y^2 \geq 0$
7. $(x - y)^2 \geq 0$
8. Since the left side must be positive this inequality is always true. Therefore, working backwards we can see that $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$ must be true for all positive real numbers x and y.

5. We say that a number, e, is the additive identity if $e + x = x + e = x$ for all numbers x. Prove that the additive identity exists and is unique.

1. $0 + x = x + 0 = x$
2. $e_1 + x = x + e1 = x$

   $e_2 + x = x + e2 = x$
3. $e_1 + e_2 = e_2 + e1 = e_1$

   $e_1 + e_2 = e_1 + e1$

   $e_1 + e_2 + e_1$

   $e_2 = 0$

4. $e_1 + e_2 = e_1 + e1 = e_2$
$e_1 + e_2 = e_2$
$e_1 = 0$