

## Homework 1

p24 B.5 and B.6, p29 A.1 and A.2, p39 A.1 and A.2, p40 C.3 and C.5

1. **p24 B.5.**  $x * y = xy + 1$

- **Commutative** Yes
- **Associative** Yes
- **Identity** Yes
- **Inverses** No

2. **p24 B.6.**  $x * y = \max\{x, y\}$  = the larger of the two numbers  $x$  and  $y$

- **Commutative** Yes
- **Associative** Yes
- **Identity** Yes
- **Inverses** No

3. **p29 A.1.** Prove the following set is an abelian group:

$x * y = x + y + k$  ( $k$  is a fixed constant), on the set  $\mathbb{R}$  of the real numbers

**Solution.**

Let  $x, y \in \mathbb{R}$  be real numbers. Forms in  $*$  are **commutative** iff  $x * y = y * x$ .

$$x * y = x + y + k$$

$$y * x = y + x + k$$

$$x + y + k = y + x + k$$

Based upon the commutative nature of addition in the real numbers the above equations are equal  $\square$

Let  $x, y, z \in \mathbb{R}$  be real numbers. Forms in  $*$  are **associative** iff  $(x * y) * z = y * (x * z)$ .

$$(x * y) * z = (x + y + k) + z + k$$

$$x * (y * z) = x + (y + z + k) + k$$

$$x + y + k = y + x + k$$

Based upon the commutative nature of addition in the real numbers the above equations are equal  $\square$

Let  $x, e \in \mathbb{R}$  be real numbers. Forms in  $*$  have an **identity** iff  $x * e = x$ .

$$x * e = x + e + k = x \text{ is true for all } x \text{ with } e = -k \quad \square$$

Let  $x, x^{-1} \in \mathbb{R}$  be real numbers. Forms in  $*$  have an **inverse** iff  $x * x^{-1} = 1$ .

$$x * x^{-1} = x + x^{-1} + k = 1 \text{ is true for all } x \text{ with } x^{-1} = -x - k + 1 \quad \square$$

Thus, because the group is true for commutative, associative, identity, and inverse, it is an abelian group.  $\square$

4. **p29 A.2.** Prove the following set is an abelian group:

$x * y = \frac{xy}{2}$  on the set  $\{x \in \mathbb{R} : x \neq 0\}$

**Solution.**

Let  $x, y \in \{x \in \mathbb{R} : x \neq 0\}$  be real numbers. Forms in  $*$  are **commutative** iff

$$x * y = y * x.$$

$$x * y = \frac{xy}{2}$$

$$y * x = \frac{yx}{2}$$

$$\frac{xy}{2} = \frac{yx}{2}$$

Based upon the commutative nature of multiplication in the real numbers the above equations are equal  $\square$

Let  $x, y, z \in \{x \in \mathbb{R} : x \neq 0\}$  be real numbers. Forms in  $*$  are **associative** iff  $(x * y) * z = y * (x * z)$ .

$$(x * y) * z = \frac{(\frac{xy}{2})z}{2}$$

$$x * (y * z) = \frac{x(\frac{yz}{2})}{2}$$

$$\frac{(\frac{xy}{2})z}{2} = \frac{x(\frac{yz}{2})}{2}$$

Based upon the commutative nature of multiplication in the real numbers the above equations are equal  $\square$

Let  $x, e \in \{x \in \mathbb{R} : x \neq 0\}$  be real numbers. Forms in  $*$  have an **identify** iff  $x * e = x$ .  $x * e = \frac{xe}{2} = x$  is true for all  $x$  with  $e = 2$

Let  $x, x^{-1} \in \{x \in \mathbb{R} : x \neq 0\}$  be real numbers. Forms in  $*$  have an **inverse** iff  $x * x^{-1} = 1$ .

$$x * x^{-1} = \frac{xx^{-1}}{2} = 1 \text{ is true for all } x \text{ with } x^{-1} = 2/x \quad \square$$

Thus, because the group is true for commutative, associative, identify, and inverse, it is an abelian group.  $\square$

5. **p39 A.1.** Solve in terms of  $a$ ,  $b$ , and  $c$ :

$$axb = c$$

**Solution.**

$$a = c/bx$$

$$b = c/ax$$

$$c = abx$$

6. **p39 A.2.** Solve in terms of  $a$ ,  $b$ , and  $c$ :

$$x^2b = xa^{-1}c$$

**Solution.**

$$a = \frac{xc}{x^2b}$$

$$b = \frac{xa^{-1}c}{x^2}$$

$$c = \frac{x^2b}{xa^{-1}}$$

7. **p40 C.3.** Assuming that  $a$  and  $b$  commute, prove the following:

$a$  commutes with  $ab$

**Solution.**

Let  $a, b$  be two numbers that commute. Thus, in  $*$ ,  $a * b = b * a$

$$a * ab = a * ba$$

$$ab * a = a * ab = a * ba$$

Thus, based on the nature of commutativity,  $a$  commutes with  $ab$   $\square$

8. **p40 C.5.** Assuming that  $a$  and  $b$  commute, prove the following:

$axa^{-1}$  commutes with  $xbx^{-1}$ , for any  $x \in G$

**Solution.**

Let  $a, b$  be two numbers that commute. Thus, in  $*$ ,  $a * b = b * a$

$$axa^{-1} = \frac{xa}{x} = a$$

$$xbx^{-1} = \frac{xb}{x} = b$$

Because  $a$  commutes with  $b$ , based on the nature of commutativity,  $axa^{-1}$  commutes with  $xbx^{-1}$   $\square$