Homework 1

p24 B.5 and B.6, p29 A.1 and A.2, p39 A.1 and A.2, p40 C.3 and C.5

1. **p24 B.5.** $x * y = xy + 1$
   - **Commutative** Yes
   - **Associative** Yes
   - **Identity** Yes
   - **Inverses** No
2. **p24 B.6.** $x * y = \max\{x, y\}$ = the larger of the two numbers x and y
   - **Commutative** Yes
   - **Associative** Yes
   - **Identity** Yes
   - **Inverses** No

3. **p29 A.1.** Prove the following set is an abelian group:

   $x * y = x + y + k$ (k is a fixed constant), on the set $\mathbb{R}$ of the real numbers

   Solution.

   Let $x, y \in \mathbb{R}$ be real numbers. Forms in $*$ are **commutative** iff $x * y = y * x$.
   $x * y = x + y + k$
   $y * x = y + x + k$
   $x + y + k = y + x + k$
   Based upon the commutative nature of addition in the real numbers the above equations are equal $\square$

   Let $x, y, z \in \mathbb{R}$ be real numbers. Forms in $*$ are **associative** iff $(x * y) * z = y * (x * z)$.
   $(x * y) * z = (x + y + k) + z + k$
   $x * (y * z) = x + (y + z + k) + k$
   $x + y + k = y + x + k$
   Based upon the commutative nature of addition in the real numbers the above equations are equal $\square$

   Let $x, e \in \mathbb{R}$ be real numbers. Forms in $*$ have an **identify** iff $x * e = x$.
   $x * e = x + e + k = x$ is true for all x with $e = -k$ $\square$

   Let $x, x^{-1} \in \mathbb{R}$ be real numbers. Forms in $*$ have an **inverse** iff $x * x^{-1} = 1$.
   $x * x^{-1} = x + x^{-1} + k = 1$ is true for all x with $x^{-1} = -x - k + 1$ $\square$

   Thus, because the group is true for commutative, associative, identify, and inverse, it is an abelian group. $\square$

4. **p29 A.2.** Prove the following set is an abelian group:

   $x * y = \frac{xy}{2}$ on the set $\{x \in \mathbb{R} : x \neq 0\}$

   Solution.

   Let $x, y \in \{x \in \mathbb{R} : x \neq 0\}$ be real numbers. Forms in $*$ are **commutative** iff $x * y = y * x$.
   $x * y = \frac{xy}{2}$
   $y * x = \frac{yx}{2}$
   $\frac{xy}{2} = \frac{yx}{2}$
   Based upon the commutative nature of multiplication in the real numbers the above equations are equal $\square$

Let $x, y, z \in \{x \in \mathbb{R} : x \neq 0\}$ be real numbers. Forms in $*$ are **associative** iff $(x * y) * z = y * (x * z)$.
$(x * y) * z = \frac{(\frac{xy}{2})z}{2}$
$x * (y * z) = \frac{x(\frac{yz}{2})}{2}$
$\frac{(\frac{xy}{2})z}{2} = \frac{x(\frac{yz}{2})}{2}$
Based upon the commutative nature of multiplication in the real numbers the above equations are equal $\square$

Let $x, e \in \{x \in \mathbb{R} : x \neq 0\}$ be real numbers. Forms in $*$ have an **identify** iff $x * e = x$.
$x * e = \frac{xe}{2} = x$ is true for all x with $e = 2$

Let $x, x^{-1} \in \{x \in \mathbb{R} : x \neq 0\}$ be real numbers. Forms in $*$ have an **inverse** iff $x * x^{-1} = 1$.
$x * x^{-1} = \frac{xx^{-1}}{2} = 1$ is true for all x with $x^{-1} = 2/x$ $\square$

Thus, because the group is true for commutative, associative, identify, and inverse, it is an abelian group. $\square$

5. **p39 A.1.** Solve in terms of a, b, and c:

   $axb = c$

   Solution.

   $a = c/xb$
   $b = c/ax$
   $c = abx$

6. **p39 A.2.** Solve in terms of a, b, and c:

   $x^2b = xa^{-1}c$

   Solution.

   $a = \frac{xc}{x^2b}$
   $b = \frac{xa^{-1}c}{x^2}$
   $c = \frac{x^2b}{xa^{-1}}$

7. **p40 C.3.** Assuming that a and b commute, prove the following:

   a commutes with ab

   Solution.

   Let a, b be two numbers that commute. Thus, in $*$, $a * b = b * a$
   $a * ab = a * ba$
   $ab * a = a * ab = a * ba$
   Thus, based on the nature of commutativity, a commutes with ab $\square$

8. **p40 C.5.** Assuming that a and b commute, prove the following:

   xax^{-1} commutes with xbx^{-1}, for any $x \in G$

   Solution.

   Let a, b be two numbers that commute, $x \in G$. Thus, in $*$, $a * b = b * a$
   $xax^{-1} = \frac{xa}{x} = a$
   $xbx^{-1} = \frac{xb}{x} = b$
   Because a commutes with b, based on the nature of commutativity, xax^{-1} commutes with xbx^{-1} $\square$