Homework 2

p39 B.5 and B.6, p40 C.7, p48 A.1, p49 B.4, p50 D.1 and D.3

1. **p39 B.5.** Prove the following is true in every group G, or give a counterexample. For every $x \in G$, there is some $y \in G$ such that $x = y^2$. (This is the same as saying that every element of G has a "square root.")

   Solution.

   Answer

2. **p39 B.6.** Prove the following is true in every group G, or give a counterexample. For any two elements x and y in G, there is an element z in G such that $y = xz$

   Solution.

   Let $x, y \in G$. We know $y = xz$
   $x^{-1}y = x^{-1}xz$
   $x^{-1}y = ez$
   $x^{-1}y = z$

   Because $x \in G$, we know $x^{-1} \in G$. Since the group G is closed under the group operation, we know $x^{-1}y \in G$.

   Thus $z \in G$, since $x^{-1}y \in G$ and $x^{-1}y = z$. $\square$

3. **p40 C.7.** Assuming that a and b commute, prove the following:
   $ab = ba$ iff $aba^{-1}b^{-1} = e$

   Solution.

   Given that a and b commute:
   $aba^{-1}b^{-1} = baa^{-1}b^{-1}$
   $baa^{-1}b^{-1} = beb^{-1}$
   $beb^{-1} = bb^{-1}$
   $bb^{-1} = e$
   Thus, $aba^{-1}b^{-1} = e$

If this property did not hold true, then a and b must not commute.

4. **p48 A.1.** Determine whether or not H is a subgroup of G:

   $G = \langle \mathbb{R}, + \rangle, H = \{\log a : a \in \mathbb{Q}, a > 0\}$

   Solution.

   H **is/is not** a subgroup of G.

5. **p49 B.4.** Solve in terms of a, b, and c:

   $G = \langle \mathscr{C}(\mathbb{R}), + \rangle, H = \{f \in \mathscr{C}(\mathbb{R}) : \int_0^1 f(x)dx = 0\}$

   Solution.

   Answer

6. **p50 D.1.** Let G be a group

   If H and K are subgroups of a group G, prove that $H \cap K$ is a subgroup of G. (Remember that $x \in H \cap K$ iff $x \in H$ and $x \in K$.)

   Solution. To prove $H \cap K$ is a subgroup of G, we mush show:
   if $f \in S$ then $f^{-1} \in S$
   if $f, g \in S$ then $fg \in S$

   Let H, K be subgroups of G.
   We know by definition that for every $x \in H \cap K$, $x \in H$ and $x \in K$
   Since H and K are subgroups, we also know $x^{-1} \in H$ and $x^{-1} \in K$, thus $x^{-1} \in H \cap K$ □

   For $x, y \in H \cap K$, we know $x, y \in H$ and $x, y \in K$
   Since H and K are subgroups, they are closed under their operation. Thus $xy \in H$ and $xy \in K$
   Thus $xy \in H \cap K$. □

   Since we have shown $x^{-1} \in H \cap K$ and $xy \in H \cap K$, we can conclude that $H \cap K$ is a subgroup of G. □

7. **p50 D.3.** Let G be a group

   By the *center* of a group G we mean the set of all elements of G which commute with every element of G, that is,

   $C = \{a \in G : ax = xa \text{ for every } x \in G\}$

   Prove that C is a subgroup of G.

   Solution.

   - Given $f \in C$, show $f^{-1} \in C$:
     Given $f \in C, a \in G$, by definition of *center*, $af = fa \forall f \in G$
     $f^{-1}af = f^{-1}fa$
     $f^{-1}af = ea = a$
     $f^{-1}aff^{-1} = af^{-1}$
     $f^{-1}ae = f^{-1}a = af^{-1}$
     Thus, by definition of C, $f^{-1} \in C$ □

   - Given $f, g \in C, a \in G$ show $fg \in C$:
     We know $f(ga) = (ga)f$ since $ga \in G$ and $f \in C$

$f(ga) = g(af)$ by associativity
$f(ga) = (af)g$ since $g \in C$ and $af \in G$
$f(ga) = afg$, thus by definition of C, $fg \in C$ $\square$

- Since we have shown $f^{-1} \in C$ and $fg \in C$, we can conclude that C is a subgroup of G. $\square$