Homework 2

p39 B.5 and B.6, p40 C.7, p48 A.1, p49 B.4, p50 D.1 and D.3

1. **p39 B.5.** Prove the following is true in every group G, or give a counterexample. For every $x \in G$, there is some $y \in G$ such that $x = y^2$. (This is the same as saying that every element of G has a "square root.")

   Solution.

   False. There are many groups where this is not true.

   Consider the Klein four-group:

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

   In the Klein four-group, for any $y \in G$, $y^2 = e$. Thus, this is not true. □

2. **p39 B.6.** Prove the following is true in every group G, or give a counterexample. For any two elements x and y in G, there is an element z in G such that $y = xz$

   Solution.

   Let $x, y \in G$. Assume $y = xz$
   $x^{-1}y = x^{-1}xz$
   $x^{-1}y = ez$
   $x^{-1}y = z$

   Because $x \in G$, we know $x^{-1} \in G$. Since the group G is closed under the group operation, we know $x^{-1}y \in G$.

   Thus we know there exists $z \in G$, where $z = x^{-1}y$, such that $y = xz$. □

3. **p40 C.7.** Assuming that a and b commute, prove the following:
   $ab = ba$ iff $aba^{-1}b^{-1} = e$

Solution.

- Assume $ab = ba$, show $aba^{-1}b^{-1} = e$:
  $(ab)a^{-1} = (ba)^{-1}$
  $aba^{-1} = be = b$
  $(aba^{-1})b^{-1} = bb^{-1}$
  $aba^{-1}b^{-1} = e$ $\square$

- Assume $aba^{-1}b^{-1} = e$, show $ab = ba$
  $(aba^{-1}b^{-1})b = eb$
  $aba^{-1}e = eb$
  $aba^{-1} = b$
  $aba^{-1}a = ba$
  $abe = ab = ba$ $\square$

- Thus, $ab = ba$ iff $aba^{-1}b^{-1} = e$ $\square$

4. **p48 A.1.** Determine whether or not H is a subgroup of G:
   $G = \langle \mathbb{R}, + \rangle, H = \{\log a : a \in \mathbb{Q}, a > 0\}$
   Solution.
   H **is** a subgroup of G, because given $x \in H \to x^{-1} \in H$ and given $x, y \in H \to xy \in H$.

5. **p49 B.4.** Show that H is a subgroup of G:
   $G = \langle \mathscr{C}(\mathbb{R}), + \rangle, H = \{f \in \mathscr{C}(\mathbb{R}) : \int_0^1 f(x)dx = 0\}$
   Solution.
   Answer

6. **p50 D.1.** Let G be a group

   If H and K are subgroups of a group G, prove that $H \cap K$ is a subgroup of G. (Remember that $x \in H \cap K$ iff $x \in H$ and $x \in K$.)

   Solution. To prove $H \cap K$ is a subgroup of G, we mush show:
   if $f \in S$ then $f^{-1} \in S$
   if $f, g \in S$ then $fg \in S$

   Let H, K be subgroups of G.
   We know by definition that for every $x \in H \cap K$, $x \in H$ and $x \in K$
   Since H and K are subgroups, we also know $x^{-1} \in H$ and $x^{-1} \in K$, thus $x^{-1} \in H \cap K$ $\square$

   For $x, y \in H \cap K$, we know $x, y \in H$ and $x, y \in K$
   Since H and K are subgroups, they are closed under their operation. Thus $xy \in H$ and $xy \in K$
   Thus $xy \in H \cap K$. $\square$

   Since we have shown $x^{-1} \in H \cap K$ and $xy \in H \cap K$, we can conclude that $H \cap K$ is a subgroup of G. $\square$

7. **p50 D.3.** Let G be a group

   By the *center* of a group G we mean the set of all elements of G which commute with every element of G, that is,

   $C = \{a \in G : ax = xa \text{ for every } x \in G\}$

   Prove that C is a subgroup of G.

   Solution.

   - Given $f \in C$, show $f^{-1} \in C$:
     Given $f \in C, a \in G$, by definition of *center*, $af = fa \forall f \in G$
     $f^{-1}af = f^{-1}fa$
     $f^{-1}af = ea = a$
     $f^{-1}aff^{-1} = af^{-1}$
     $f^{-1}ae = f^{-1}a = af^{-1}$
     Thus, by definition of C, $f^{-1} \in C$ $\square$

   - Given $f, g \in C, a \in G$ show $fg \in C$:
     We know $f(ga) = (ga)f$ since $ga \in G$ and $f \in C$

$f(ga) = g(af)$, by associativity
$f(ga) = (af)g$, since $g \in C$ and $af \in G$
$f(ga) = afg$, thus by definition of C, $fg \in C$ $\square$

- Since we have shown $f^{-1} \in C$ and $fg \in C$, we can conclude that C is a subgroup of G. $\square$