

## Homework 2

p39 B.5 and B.6, p40 C.7, p48 A.1, p49 B.4, p50 D.1 and D.3

1. **p39 B.5.** Prove the following is true in every group  $G$ , or give a counterexample.  
For every  $x \in G$ , there is some  $y \in G$  such that  $x = y^2$ . (This is the same as saying that every element of  $G$  has a "square root.")

**Solution.**

False. There are many groups where this is not true.

Consider the Klein four-group:

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

In the Klein four-group, for any  $y \in G$ ,  $y^2 = e$ . Thus, this is not true.  $\square$

2. **p39 B.6.** Prove the following is true in every group  $G$ , or give a counterexample.  
For any two elements  $x$  and  $y$  in  $G$ , there is an element  $z$  in  $G$  such that  $y = xz$

**Solution.**

Let  $x, y \in G$ . Assume  $y = xz$

$$x^{-1}y = x^{-1}xz$$

$$x^{-1}y = ez$$

$$x^{-1}y = z$$

Because  $x \in G$ , we know  $x^{-1} \in G$ . Since the group  $G$  is closed under the group operation, we know  $x^{-1}y \in G$ .

Thus we know there exists  $z \in G$ , where  $z = x^{-1}y$ , such that  $y = xz$ .  $\square$

3. **p40 C.7.** Assuming that  $a$  and  $b$  commute, prove the following:  
 $ab = ba$  iff  $aba^{-1}b^{-1} = e$

**Solution.**

- Assume  $ab = ba$ , show  $aba^{-1}b^{-1} = e$ :

$$(ab)a^{-1} = (ba)a^{-1}$$

$$aba^{-1} = be = b$$

$$(aba^{-1})b^{-1} = bb^{-1}$$

$$aba^{-1}b^{-1} = e$$

□

- Assume  $aba^{-1}b^{-1} = e$ , show  $ab = ba$

$$(aba^{-1}b^{-1})b = eb$$

$$aba^{-1}e = eb$$

$$aba^{-1} = b$$

$$aba^{-1}a = ba$$

$$abe = ab = ba$$

□

- Thus,  $ab = ba$  iff  $aba^{-1}b^{-1} = e$

□

4. **p48 A.1.** Determine whether or not  $H$  is a subgroup of  $G$ :

$$G = \langle \mathbb{R}, + \rangle, H = \{\log a : a \in \mathbb{Q}, a > 0\}$$

**Solution.**

$H$  is a subgroup of  $G$ , because given  $x \in H \rightarrow x^{-1} \in H$  and given  $x, y \in H \rightarrow xy \in H$ .

5. **p49 B.4.** Show that  $H$  is a subgroup of  $G$ :

$$G = \langle \mathcal{C}(\mathbb{R}), + \rangle, H = \{f \in \mathcal{C}(\mathbb{R}) : \int_0^1 f(x)dx = 0\}$$

**Solution.**

- Given  $x \in H$ , show  $x^{-1} \in H$ :

$$x \in \mathcal{C}(\mathbb{R}) \quad \int_0^1 f(x)dx = 0$$

Since the  $x^{-1}$  still satisfies an integral of 0, we know  $x^{-1} \in H$ . □

- Given  $x, y \in H$  show  $xy \in H$ :

$$x, y \in \mathcal{C}(\mathbb{R})$$

$$\int_0^1 f(x)dx = 0$$

$$\int_0^1 f(y)dy = 0$$

$$\int_0^1 f(x)dx + \int_0^1 f(y)dy = 0$$

Thus, since adding the two integrals still produces 0, the result is also in  $H$ .

$$xy \in H$$
□

- Since we have shown  $x^{-1} \in H$  and  $xy \in H$ , we can conclude that  $H$  is a subgroup of  $G$ . □

6. **p50 D.1.** Let  $G$  be a group

If  $H$  and  $K$  are subgroups of a group  $G$ , prove that  $H \cap K$  is a subgroup of  $G$ .  
(Remember that  $x \in H \cap K$  iff  $x \in H$  and  $x \in K$ .)

**Solution.** To prove  $H \cap K$  is a subgroup of  $G$ , we must show:

if  $f \in S$  then  $f^{-1} \in S$

if  $f, g \in S$  then  $fg \in S$

Let  $H, K$  be subgroups of  $G$ .

We know by definition that for every  $x \in H \cap K$ ,  $x \in H$  and  $x \in K$

Since  $H$  and  $K$  are subgroups, we also know  $x^{-1} \in H$  and  $x^{-1} \in K$ , thus  $x^{-1} \in H \cap K$  □

For  $x, y \in H \cap K$ , we know  $x, y \in H$  and  $x, y \in K$

Since  $H$  and  $K$  are subgroups, they are closed under their operation. Thus  $xy \in H$  and  $xy \in K$

Thus  $xy \in H \cap K$ . □

Since we have shown  $x^{-1} \in H \cap K$  and  $xy \in H \cap K$ , we can conclude that  $H \cap K$  is a subgroup of  $G$ . □

7. **p50 D.3.** Let  $G$  be a group

By the *center* of a group  $G$  we mean the set of all elements of  $G$  which commute with every element of  $G$ , that is,

$$C = \{a \in G : ax = xa \text{ for every } x \in G\}$$

Prove that  $C$  is a subgroup of  $G$ .

**Solution.**

- Given  $f \in C$ , show  $f^{-1} \in C$ :

Given  $f \in C, a \in G$ , by definition of *center*,  $af = fa \forall f \in G$

$$f^{-1}af = f^{-1}fa$$

$$f^{-1}af = ea = a$$

$$f^{-1}aff^{-1} = af^{-1}$$

$$f^{-1}ae = f^{-1}a = af^{-1}$$

Thus, by definition of  $C$ ,  $f^{-1} \in C$  □

- Given  $f, g \in C, a \in G$  show  $fg \in C$ :

We know  $f(ga) = (ga)f$  since  $ga \in G$  and  $f \in C$

$f(ga) = g(af)$ , by associativity

$f(ga) = (af)g$ , since  $g \in C$  and  $af \in G$

$f(ga) = afg$ , thus by definition of  $C$ ,  $fg \in C$

□

- Since we have shown  $f^{-1} \in C$  and  $fg \in C$ , we can conclude that  $C$  is a subgroup of  $G$ . □