Homework 3

p63 D.1, p64 E.5 and E.6, p65 G.1 and G.2, p75 A.1, A.2, and A.5, p76 B.2

1. **p63 D.1.** Find the composite function, $f \circ g$ and $g \circ f$:

   $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sin(x)$
   $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = e^x$

   Solution. Let f, g be defined as above, and $x \in \mathbb{R}$ be any value.

   $f \circ g$

   $$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(e^x) \\ &= \sin(e^x) \end{aligned}$$

   $g \circ f$

   $$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(\sin(x)) \\ &= e^{\sin(x)} \end{aligned}$$

2. **p64 E.5.** f is a bijective function. Describe its inverse.

   $A = \{a, b, c, d\}, B = \{1, 2, 3, 4\}$ and $f : A \to B$ is given by:
   $\begin{pmatrix} a & b & c & d \\ 3 & 1 & 2 & 4 \end{pmatrix}$

   Solution. The inverse matrix $f^{-1} : B \to A$ is defined as:
   $\begin{pmatrix} 1 & 2 & 3 & 4 \\ b & c & a & d \end{pmatrix}$

   Since this matrix satisfies the property $\forall x \in B, f \circ f^{-1}(x) = x$, we can verify that it is the correct inverse of f.

3. **p64 E.6.** f is a bijective function. Describe its inverse.

   G is a group, $a \in G$, and $f : G \to G$ is defined by $f(x) = ax$.

   Solution. The inverse $f^{-1} : G \to G$ is defined as:

   $$f^{-1}(x) = a^{-1}x$$

   where $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$.

   This can be verified by the composition: $\forall x \in G, f \circ f^{-1}(x) = x$

   $$\begin{aligned} f \circ f^{-1}(x) &= f(f^{-1}(x)) \\ &= f(a^{-1}x) \\ &= aa^{-1}x \\ &= ex \\ &= x \end{aligned}$$

   $\square$

4. **p65 G.1.** Let A, B, and C by sets and let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is injective, then f is injective.

   Solution. Let f, g be defined as above.

   Proof. Assume, for the sake of contradiction, that f is not injective.
   Given that f is not injective, this implies there exists some $\alpha_1, \alpha_2 \in A$ and $\beta \in B$ such that $f(\alpha_1) = f(\alpha_2) = \beta$.
   However, since the composition $g \circ f$ is given to be injective, we know for any $\alpha_1, \alpha_2 \in A$ that $g \circ f(\alpha_1) \neq g \circ f(\alpha_2)$.
   Since $f(\alpha_1) = f(\alpha_2) = \beta$, this implies a contradiction that $g \circ f(\beta) \neq g \circ f(\beta)$, therefor f must be injective. $\square$

5. **p65 G.2.** Let A, B, and C by sets and let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is surjective, then g is surjective.

   Solution. Let f, g be defined as above.

   Proof. Assume, for the sake of contradiction, that g is not surjective.
   Given that g is not surjective, this implies there exists some $\zeta \in C$ such that $g(\beta) \neq \zeta$ for any $\beta \in B$.
   Because $g \circ f$ is surjective, we know for all $\zeta \in C, \exists \alpha \in A$ such that $g \circ f(\alpha) = \zeta$.
   Unfolding this statement, it implies $\exists \beta \in B, \alpha \in A$ such that $g(f(\alpha)) = g(\beta) = \zeta$, which is a contradiction to our original assumption. Therefor g must be surjective. $\square$

6. **p75 A.1.** Consider the following permutations f, g, and h in S_6

$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 5 & 4 & 2 \end{pmatrix}$ $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 5 & 4 \end{pmatrix}$

$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 5 & 2 \end{pmatrix}$

Solution. Compute the following:

$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 3 & 5 & 4 & 1 \end{pmatrix}$ $g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 5 & 4 \end{pmatrix}$

$h^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 4 & 5 & 3 \end{pmatrix}$

$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 2 & 4 & 5 \end{pmatrix}$ $g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 5 & 6 & 3 \end{pmatrix}$

7. **p75 A.2.** Given p75 A.1, compute the following:

$f \circ (g \circ h) =$

Solution.

$g \circ h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 6 & 5 & 3 \end{pmatrix}$

$f \circ (g \circ h) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 2 & 4 & 3 \end{pmatrix}$

8. **p75 A.5.** Given p75 A.1, compute the following:

$g \circ g \circ g =$

Solution.

$g \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}$

$g \circ (g \circ g) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}$

9. **p76 B.2.** List the elements of the cyclic subgroup of S_6 generated by:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$$

Solution. The following show the cycles of f:

$$f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 5 & 6 \end{pmatrix}$$

$$f^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 3 & 6 & 5 \end{pmatrix}$$

$$f^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$f^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$$