Homework 7

p131 D.1 and D.2, p143 C.1, C.3, C.4, and C.8

1. **p131 D.1** Let G be a finite group, and let H and K be subgroups of G. Prove the following:

   Suppose $H \subseteq K$ (therefore H is a subgroup of K). Then $(G : H) = (G : K)(K : H)$.

   Solution.

   Proof. By definition $(G : H)$ = order of G / order of H.

   Thus, $(G : K)(K : H)$ = (order of G / order of K) * (order of K / order of H), which can be simplified to order of G / order of H. ☐

2. **p131 D.2** Let G be a finite group, and let H and K be subgroups of G. Prove the following:

   The order of $H \cap K$ is a common divisor of the order of H and the order of K.

   Solution.

   Proof. Let h be the order of H, k be the order of K, and i be the order of $H \cap K$.

   By Lagrange's theorem, since $H \cap K$ is a subgroup of H, it follows that i divides h. Additionally, since $H \cap K$ is a subgroup of K, it follows that i divides k.

   Thus the order of $H \cap K$ is a common divisor of the order of H and the order of K. ☐

3. **p143 C.1** Let G, H, and K be groups. Prove the following:

   If $f : G \to H$ and $g : H \to K$ are homomorphisms, then their composite $g \circ f : G \to K$ is a homomorphism.

   Solution.

   Proof. Let $a, b \in G$. The composite $(g \circ f)(a * b) = g(f(a * b)) = g(f(a) * f(b)) = g(f(a)) * g(f(b)) = (g \circ f)(a) * (g \circ f)(b)$. Thus, the composite is a homomorphism. ☐

4. **p143 C.3** Let G, H, and K be groups. Prove the following:

   If $f : G \to H$ is a homomorphism and K is any subgroup of G, then $f(K) = \{f(x) : x \in K\}$ is a subgroup of H.

   Solution.

   Proof. To prove $f(K) = \{f(x) : x \in K\}$ is a subgroup of H, we must show:

   - The subgroup is closed, as it is created from a closed group.
   - The subgroup is closed under inversion.
   - For $a, b \in K$, there must exist $a * b \in K$, which is true since K is a subgroup of G. Additionally, since $f : G \to H$ is a homomorphism and K is any subgroup of G, we know $f(K)$ is closed.
   - For $a \in K$, there must exist $a^{-1} \in K$ because K is a subgroup of G. Additionally, since f is a homomorphism, we know $f(K)$ is closed under inversion.

   Thus, $f(K)$ is a subgroup of H. □

5. **p143 C.4** Let G, H, and K be groups. Prove the following:

   If $f : G \to H$ is a homomorphism and J is any subgroup of H, then

   $$f^{-1}(J) = \{x \in G : f(x) \in J\}$$

   is a subgroup of G. Furthermore, ker $f \subseteq f^{-1}(J)$.

   Solution.

   Proof. We must show the same criteria as the previous problem. Because f is a homomorphism and J is a subgroup of H, we know these criteria are satisfied. Thus, $f^{-1}(J)$ is a subgroup of G. □

6. **p143 C.8** Let G, H, and K be groups. Prove the following:

   The function $f : G \to G$ defined by $f(x) = x^2$ is a homomorphism iff G is abelian.

   Solution.

   Proof. Prove both directions:

   Suppose f is a homomorphism. Thus $f(x * y) = f(x) * f(y)$ for $x, y \in G$. Using the definition of f we get $f(xy) = (xy)^2 = xyxy$. Because f is a homomorphism, $f(xy) = f(x)f(y) = x^2y^2$, thus G must be abelian.

   Suppose G is abelian, meaning $xy = yx$. We can show $f(xy) = (xy)^2 = xyxy = xxyy = x^2y^2 = f(x)f(y)$. Thus, because G is abelian, f must be a homomorphism. □