MEK 1100 - Oblig 1

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1

Ball kastes ut fra origo.

Utgangsvinkel θ i forhold til den horisontale x-aksen.

Følger en bane gitt ved:

$x(t) = v_0 t cos(\theta)$

$y(t) = v_0 t sin(\theta) - \frac{1}{2} g t^2$

a)

Finn tiden t_m når ballen faller ned på bakken, og posisjonen $x(t_m) = x_m$ hvor dette skjer:

Når ballen treffer bakken er $y(t) = 0$.

Løser for å finne t ved dette punktet:

$$0 = v_0 t sin(\theta) - \frac{1}{2} g t^2$$

$$\frac{1}{2} g t^2 = v_0 t sin(\theta)$$

$$t^2 = 2 \frac{v_0 t sin(\theta)}{g}$$

$$t_m = 2 \frac{v_0 sin(\theta)}{g}$$

Deretter finner vi x_m ved å sette uttrykket for t_m inn i $x(t)$

$$x(t_m) = 2v_0 \Big(\frac{v_0 sin(\theta)}{g} \Big) cos(\theta) = \frac{2v_0^2 sin(\theta) cos(\theta)}{g} = \frac{v_0^2 sin(2\theta)}{g}$$

$t_m = 2 \frac{v_0 sin(\theta)}{g}$

$x_m = \frac{v_0^2 sin(2\theta)}{g}$

b)

Innfør dimensjonsløse variabler (x^*, y^*, z^*):

$$x^* = \frac{x}{x_m} = \frac{v_0 t cos(\theta) * g}{v_0^2 * 2sin(\theta)cos(\theta)} = \frac{gt}{2v_0 sin(\theta)}$$

$$y^* = \frac{y}{x_m} = \frac{(v_0 t sin(\theta) - \frac{1}{2}gt^2) * g}{v_0^2 * 2sin(\theta)cos(\theta)} = \frac{v_0 * gt sin(\theta)}{v_0^2 * 2sin(\theta)cos(\theta)} - \frac{\frac{1}{2}gt^2 * g}{v_0^2 * 2sin(\theta)cos(\theta)}$$

$$y^* = \frac{gt}{v_0 * 2cos(\theta)} - \frac{g^2t^2}{v_0^2 * 4sin(\theta)cos(\theta)}$$

$$y^* = \frac{gt}{2v_0 cos(\theta)} * \left(1 - \frac{gt}{2v_0 sin(\theta)}\right)$$

$\frac{gt}{2v_0 sin(\theta)} = x^*$

$$y^* = \frac{gt}{2v_0 cos(\theta)} * (1 - x^*)$$

$$y^* = \frac{gt}{2v_0 cos(\theta)} * \frac{sin(\theta)}{sin(\theta)} * (1 - x^*)$$

$$y^* = x^* tan(\theta)(1 - x^*)$$

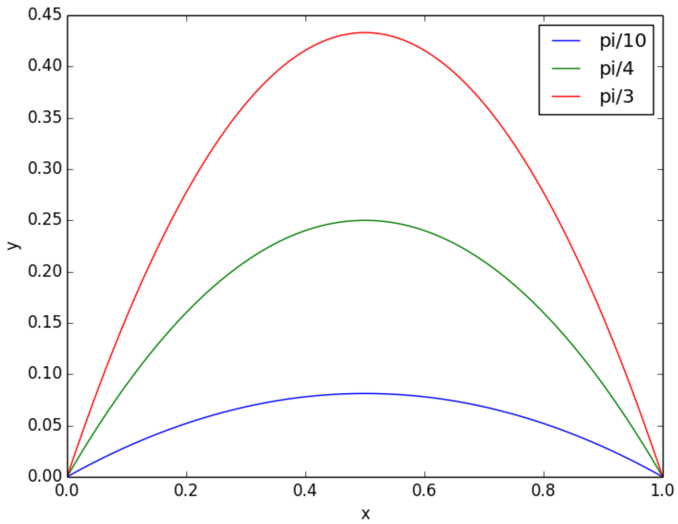
$$t^* = \frac{t}{t_m} = \frac{tg}{2v_0 sin(\theta)} = x^*$$

Ettersom θ beskrives ved å ta $\frac{buelengde}{radius}$, og dette ikke har noen benevning: $\left[\frac{avstand}{avstand}\right] = 1$, så trenger ikke θ å skaleres.

c)

```python
import numpy as np
import matplotlib.pylab as plt

n = 0
N = 1000

t = np.linspace(0, 1, N)
x = np.zeros(N)
y = np.zeros(N)

##Init
v0 = 1.0
theta = [np.pi/10, np.pi/4, np.pi/3]

for n in range(3):
    for i in range(N):
        x[i] = t[i]
        y[i] = x[i]*np.tan(theta[n])*(1-t[i])
```

```
    plt.plot(x, y)

plt.xlabel(”x”)
plt.ylabel(”y”)
plt.legend([”pi/10”, ”pi/4”, ”pi/3”])
plt.savefig(”oppg1.png”)
plt.show()
```



Figur 1: PyPlot, plotter x* og y* for 3 forskjellige verdier for θ

Ved å se på denne grafen ser vi forholdet mellom høyden og lengden på banen ved forskjellige verdier av utfallsvinkelen θ. Verdiene kan brukes for alle verdier av g og v_0 fordi x^* er dimensjonsløs, og θ er den eneste variablen vi bruker for å beskrive grafen.

2

Hastighetsfeltet: $\vec{v} = v_x\vec{i} + v_y\vec{j} = xy\vec{i} + y\vec{j}$

a)

Finn strømlinjene:

Ser på kryssproduktet $\vec{v} \times d\vec{r}$, som gir 0 ettersom $\vec{v}$ og $d\vec{r}$ er parallelle.

Da får vi:

$$(v_x dy - vydx)\vec{k} = 0$$

$$v_x dy = v_y dx$$

$$xydy = ydx$$

Deler på xy:

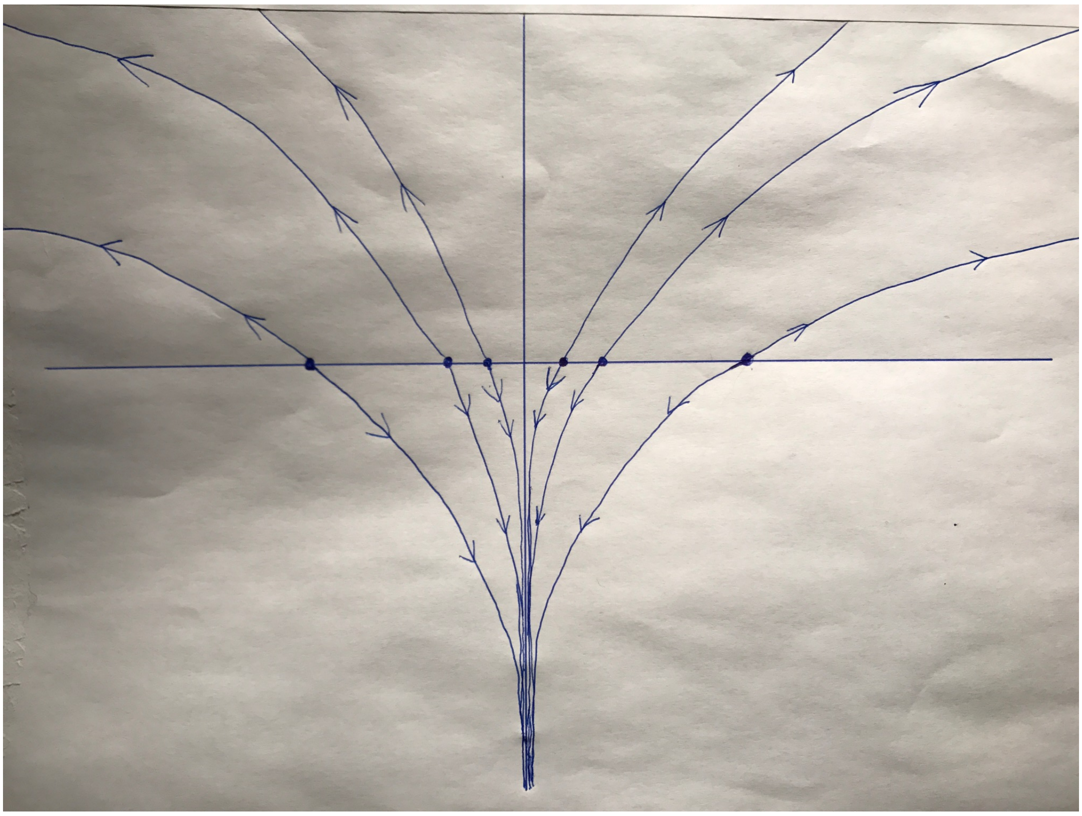
$$1dy = \frac{1}{x}dx$$

$$\int 1dy = \int frac1xdx$$

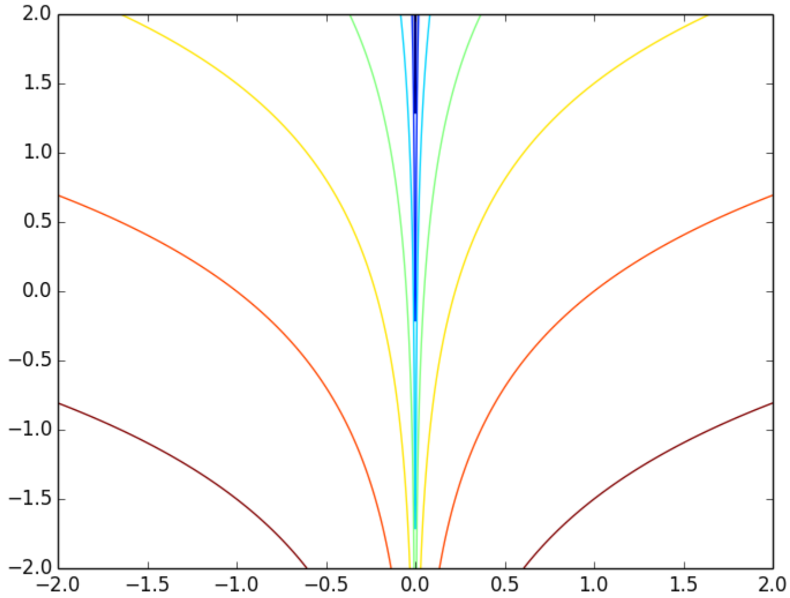
$$y = ln|x| + C$$

$$C = y - ln|x|$$

b)



```python
import numpy as np
import matplotlib.pylab as plt

N = 1000

x = np.linspace(-2, 2, N)
y = np.linspace(-2, 2, N)

x, y = np.meshgrid(x, y)
z = np.log(np.abs(x)) - y

plt.contour(x, y, z)
plt.savefig("oppg2.png")
plt.show()
```



c)

Vis at det **ikke** finnes en strømfunksjon ψ:

$$\frac{\delta(v_x)}{\delta x} + \frac{\delta(v_y)}{\delta y} = \frac{\delta(xy)}{\delta x} + \frac{\delta y}{\delta y} = y + 1$$
$$y + 1 \neq 0$$

Feltet er ikke divergens-fritt, så dermed har det ingen strømfunksjon.

3

$v_x = cos(x)sin(y)$

$v_y = -sin(x)cos(y)$

a)

Finn divergens og virvling.

Divergens:

$$\frac{\delta(v_x)}{\delta x} + \frac{\delta(v_y)}{\delta y}$$

$\frac{\delta(v_x)}{\delta x} = \frac{\delta(cos(x)sin(y))}{\delta x} = \frac{1}{2}cos(x+y) - \frac{1}{2}cos(x-y)$

$\frac{\delta(v_y)}{\delta y} = \frac{\delta(-sin(x)cos(y))}{\delta y} = \frac{1}{2}cos(x-y) - \frac{1}{2}cos(x+y)$

$$\frac{1}{2}cos(x+y) - \frac{1}{2}cos(x-y) + \frac{1}{2}cos(x-y) - \frac{1}{2}cos(x+y) = 0$$

Virvling:

$$\Big(\frac{\delta(v_y)}{\delta x} - \frac{\delta(v_x)}{\delta y}\Big)\vec{k}$$

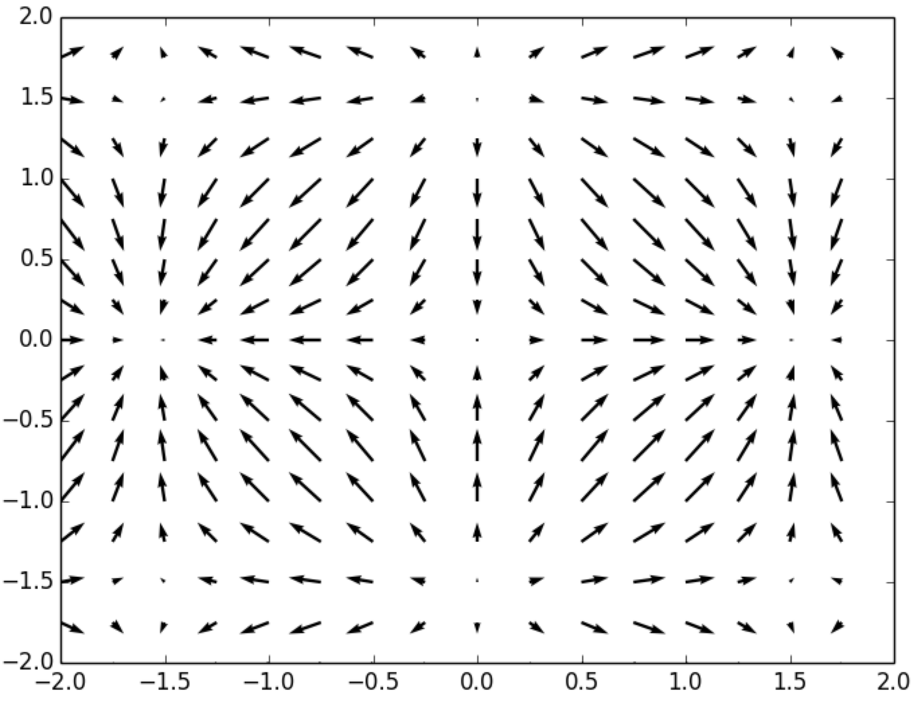
$$\frac{\delta(v_y)}{\delta x} = \frac{\delta(-sin(x)cos(y))}{\delta x} = -cos(x)cos(y)$$

$$\frac{\delta(v_x)}{\delta y} = \frac{\delta(cos(x)sin(x))}{\delta y} = cos(x)cos(y)$$

$$= \Big(-cos(x)cos(y) - cos(x)cos(y)\Big)\vec{k}$$

$$= \Big(-2cos(x)cos(y)\Big)\vec{k}$$

b)



```python
import numpy as np
import matplotlib.pyplot as plt


delta = 0.25
x, y = np.arange(-2, 2, delta), np.arange(-2, 2, delta)
vx, vy = np.meshgrid(np.cos(x) * np.sin(y), -np.sin(x) * np.cos(y))


plt.figure()
plt.quiver(x, y, vx, vy)
plt.savefig('oppg3.png')
```

c)

Integrerer hastighetsfeltet over et kurveintegral ved å dele opp kurven i fire sider med parametriseringen $\vec{r_i}$ hvor i er siden vi ser på.

Bruker formelen:

$$\int \vec{v} d\vec{r_i} = \int_a^b \vec{v}(\vec{r_i}(t)) * \vec{r_i}' dt$$

med $a = -\frac{\pi}{2}$ og $b = \frac{\pi}{2}$

Ser på parameterene:

$$\vec{r_1}(t) = (t, -\frac{\pi}{2}) \Rightarrow \vec{r_1}'(t) = (1, 0)$$

$$\vec{r_2}(t) = (\frac{\pi}{2}, t) \Rightarrow \vec{r_2}'(t) = (0, 1)$$

$$\vec{r_3}(t) = (-t, \frac{\pi}{2}) \Rightarrow \vec{r_3}'(t) = (-1, 0)$$

$$\vec{r_4}(t) = (-\frac{\pi}{2}, -t) \Rightarrow \vec{r_4}'(t) = (0, -1)$$

Bruker deretter formelen og regner ut $\int_a^b \vec{v}(\vec{r_i}(t)) * \vec{r_i}' dt$ for alle sider i:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vec{v}(\vec{r_1}(t)) * \vec{r_1}' dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(t) sin\Big(\frac{\pi}{2}\Big) dt = \Big[-sin(t)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -2$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vec{v}(\vec{r_2}(t)) * \vec{r_2}' dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -sin\Big(\frac{\pi}{2}\Big) cos(t) dt = \Big[-sin(t)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -2$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vec{v}(\vec{r_3}(t)) * \vec{r_3}' dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -cos(-t) sin\Big(\frac{\pi}{2}\Big) dt = \Big[sin(-t)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -2$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vec{v}(\vec{r_4}(t)) * \vec{r_4}' dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -sin\Big(-\frac{\pi}{2}\Big) cos(-t) dt = \Big[sin(-t)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -2$$

Ettersom summen av disse blir -8, er sirkulasjonen om randen til kvadratet -8

d)

Dette feltet er divergensfritt(i motsetning til forrige oppgave), og dermed så har feltet en strømfunksjon.

$$cos(x)sin(y) = -\frac{\delta\psi}{\delta y}$$
$$\int -cos(x)sin(y)dy = \int \frac{\delta\psi}{\delta y}$$
$$-cos(x)\int sin(y)dy = \psi$$
$$\psi = cos(x)cos(y) + C_1$$

$$-sin(x)cos(y) = \frac{\delta\psi}{\delta x}$$
$$\int -sin(x)cos(y)dx = \int \frac{\delta\psi}{\delta x}$$
$$-cos(y)\int sin(x)dx = \psi$$
$$\psi = cos(x)cos(y) + C_2$$

Dermed får vi:

$$\psi = cos(x)cos(y)$$

e)

$$\psi(x,y) \approx \psi(x_0,y_0)+\Big(\frac{\delta\psi}{\delta x}\Big)(x-x_0)+\Big(\frac{\delta\psi}{\delta y}\Big)(y-y_0)+\Big(\frac{\delta^2\psi}{2\delta x^2}\Big)(x-x_0)^2+\Big(\frac{\delta^2\psi}{2\delta y^2}\Big)(y-y_0)^2+\Big(\frac{\delta^2\psi}{\delta x\delta y}\Big)(x-x_0)(y-y_0)$$

$$\frac{\delta\psi}{\delta x} = -sin(x)cos(y),\ \frac{\delta\psi}{\delta x}(0,0) = 0$$
$$\frac{\delta\psi}{\delta y} = -cos(x)sin(y),\ \frac{\delta\psi}{\delta x}(0,0) = 0$$
$$\frac{\delta^2\psi}{\delta x^2} = -cos(x)cos(y),\ \frac{\delta\psi}{\delta x}(0,0) = -1$$
$$\frac{\delta^2\psi}{\delta y^2} = -cos(x)cos(y),\ \frac{\delta\psi}{\delta x}(0,0) = -1$$
$$\frac{\delta^2\psi}{\delta x\delta y} = -sin(x)sin(y),\ \frac{\delta\psi}{\delta x}(0,0) = 0$$

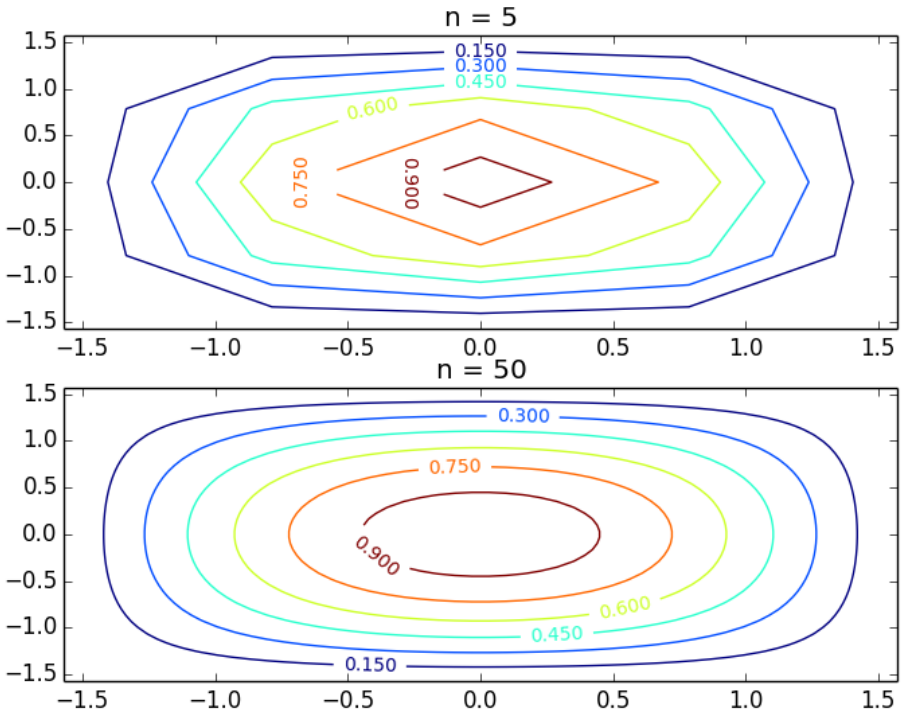
Setter dette inn i formelen over for å få taylorekspansjonen.

$$\psi(0,0) \approx \psi(0,0) + \Big(\frac{\delta\psi}{\delta x}\Big)x + \Big(\frac{\delta\psi}{\delta y}\Big)y + \Big(\frac{\delta^2\psi}{2\delta x^2}\Big)x^2 + \Big(\frac{\delta^2\psi}{2\delta y^2}\Big)y^2 + \Big(\frac{\delta^2\psi}{\delta x\delta y}\Big)xy$$

$$
\begin{aligned}
&= 1 + 0 + 0 - \frac{x^2}{2} - \frac{y^2}{2} \\
&= 1 - \frac{x^2 - y^2}{2}
\end{aligned}
$$

4

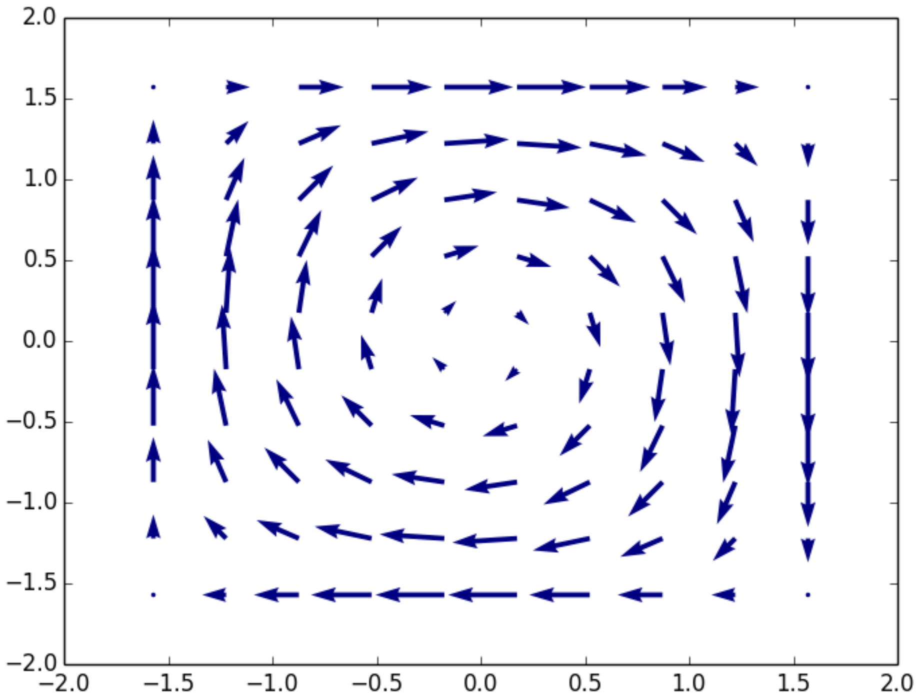
a)




```
import matplotlib.pyplot as plt
from streamfun import streamfun

#n = 5
x_5, y_5, psi_5 = streamfun(5)

#n = 50
x_50, y_50, psi_50 = streamfun(50)


plt.figure()
plt.subplot(211)
contour_plot = plt.contour(x_5, y_5, psi_5)
plt.clabel(contour_plot, inline=1, fontsize=10)
plt.title('n = 5')


plt.subplot(212)
contour_plot_2 = plt.contour(x_50, y_50, psi_50)
plt.clabel(contour_plot_2, inline=1, fontsize=10)
plt.title('n = 50')
plt.savefig('strlin.png')

plt.show()
```

b)

Valgte n=10 for å få passende lesbarhet av plottet.




```python
import numpy as np

def velfield(n):

    x = np.linspace(-0.5 * np.pi, 0.5 * np.pi, n)
    [x, y] = np.meshgrid(x, x)
    vx = np.cos(x) * np.sin(y)
    vy = -np.sin(x) * np.cos(y)

    return x, y, vx, vy
```

```python
from velfield import velfield
import matplotlib.pyplot as plt

x, y, vx, vy = velfield(10)

plt.figure()
plt.quiver(x, y, vx, vy, 1.5)
plt.savefig('velfield.png')
```