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University of Illinois Spring 2020

CS 446/ECE 449 Machine Learning
Homework 8: Variational Auto-Encoders

Due on Tuesday April 21 2020, noon Central Time

1. **[17 points]** Variational Auto-Encoders (VAEs)

(a) (3 points) We want to maximize the log-likelihood $\log p_\theta(x)$ of a model $p_\theta(x)$ which is parameterized by θ. To this end we introduce a joint distribution $p_\theta(x, z)$ and an approximate posterior $q(z|x)$ and reformulate the log-likelihood via

$$\log p_\theta(x) = \log \sum_z q(z|x) \frac{p_\theta(x, z)}{q(z|x)}.$$

Use Jensen's inequality to obtain a bound on the log likelihood and divide the bound into two parts, one of which is the Kullback-Leibler (KL) divergence

$$\mathrm{KL}(q(z|x), p(z)).$$

Your answer:

$$① \log \sum_z q(z|x) \frac{P_\theta(x,z)}{q(z|x)} \geq ② \sum_z q(z|x) \log \frac{P_\theta(x|z) P_\theta(z)}{q(z|x)}$$

b/c Jensen's inequality for concave function, so ② is lower bound of ①

$$= \sum_z q(z|x) \log \frac{P_\theta(z)}{q(z|x)} + \sum_z q(z|x) \log P_\theta(x|z)$$

$$= -\mathrm{KL}(q(z|x), p(z)) + \mathbb{E}_q[\log P_\theta(x|z)]$$

(b) (2 points) State at least two properties of the KL divergence.

Your answer:

I) KL divergence is always greater than or equal to 0
II) If the two distributions, q and p, are identical, KL divergence is exactly 0

(c) (2 points) Let

$$q(z|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(z - \mu_q)^2\right).$$

What is the value for the KL-divergence $\mathrm{KL}(q(z|x), q(z|x))$ and why?

Your answer:

The value would be 0 because the KL divergence of two identical distributions is 0

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(d) (3 points) Further, let

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(z-\mu_p)^2\right).$$

Note the difference of the means for $p(z)$ and $q(z|x)$ while their standard deviation is identical. What is the value for the KL-divergence $\mathrm{KL}(q(z|x), p(z))$ in terms of μ_p, μ_q and σ?

Your answer:

$$KL = \sum_z q(z|x) \log \frac{q(z|x)}{p(z)}$$

$$= \sum_z q(z|x) \log \left[\frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(z-\mu_q)^2\right)}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(z-\mu_p)^2\right)} \right]$$

$$= \sum_z q(z|x) \left[\frac{-(z-\mu_q)^2}{2\sigma^2} + \frac{(z-\mu_p)^2}{2\sigma^2} \right]$$

$$= \sum_z q(z|x) \cdot \left[\frac{2z\mu_q - \mu_q^2 - 2z\mu_p + \mu_p^2}{2\sigma^2} \right]$$

$$= \sum_z q(z|x) \cdot \left[\frac{\mu_p^2 - \mu_q^2 + 2z\mu_q - 2z\mu_p}{2\sigma^2} \right]$$

$$= q(z|x) \left[\frac{\mu_p^2 - \mu_q^2}{2\sigma^2} \right] dz + \sum_z q(z|x) \left[\frac{z\mu_q - z\mu_p}{\sigma^2} \right]$$

$$= \frac{\mu_p^2 - \mu_q^2}{2\sigma^2} + \sum_z q(z|x)\, z \left[\frac{\mu_q - \mu_p}{\sigma^2} \right]$$

$$= \frac{\mu_p^2 - \mu_q^2}{2\sigma^2} + \mu_q \left(\frac{\mu_q - \mu_p}{\sigma^2} \right)$$

$$= \boxed{\frac{\mu_p^2 + \mu_q^2 - 2\mu_q\mu_p}{2\sigma^2}}$$

(e) (4 points) Now, let $q(z|x)$ and $p(z)$ be arbitrary probability distributions. We want to find that $q(z|x)$ which maximizes

$$\sum_z q(z|x) \log p_\theta(x|z) - \mathrm{KL}(q(z|x), p(z))$$

subject to $\sum_z q(z|x) = 1$. Ignore the non-negativity constraints. State the Lagrangian and compute its stationary point, *i.e.*, solve for $q(z|x)$ which depends on $p_\theta(x|z)$ and $p(z)$. Make sure to get rid of the Lagrange multiplier.

Your answer:

$$L = \sum_z q(z|x) \log p_\theta(x|z) - \sum_z q(z|x) \log \frac{q(z|x)}{p(z)} + \lambda\left[\sum_z (q(z|x)) - 1\right]$$

$$\frac{\partial L}{\partial q(z|x)} := \sum_z \log p_\theta(x|z) - \sum_z \left(1 + \log \frac{q(z|x)}{p(z)}\right) + \lambda N = 0$$

$$\sum_z \log p_\theta(x|z) - \left(\sum_z 1 + \sum_z \log q(z|x) - \sum_z \log p(z)\right) + \lambda N = 0$$

$$\sum_z \log p_\theta(x|z) - N - \sum_z \log q(z|x) - \sum_z \log p(z) + \lambda N \Rightarrow$$

$$\sum_z \log q(z|x) = \sum_z (\log p_\theta(x|z) - 1 - \log p(z) + \lambda)$$

$$\log q(z|x) = \log \frac{p_\theta(x|z)}{p(z)} - 1 + \lambda$$

$$q(z|x) = \exp\left(\log \frac{p(x|z)}{p(z)} - 1 + \lambda\right)$$

$$q(z|x) = \frac{p(x|z)}{p(z)} e^{\lambda - 1}$$

$$\sum_z q(z|x) = \sum_z \frac{p(x|z)}{p(z)} e^{\lambda-1} = 1$$

$$e^{\lambda-1} \sum_z \frac{p(x|z)}{p(z)} = 1$$

$$e^{\lambda-1} = \frac{1}{\sum_z \frac{p(x|z)}{p(z)}}$$

$$\lambda = 1 - \log \sum_z \frac{p(x|z)}{p(z)}$$

$$q(z|x) = \frac{p(x|z)}{p(z)} \exp\left(-\log \sum_z \frac{p(x|z)}{p(z)}\right)$$

$$\boxed{q^*(z|x) = \frac{p(x|z)}{p(z) \sum_z \frac{p(x|z)}{p(z)}}}$$

(f) (1 point) Which of the following terms should $q(z|x)$ be equal to: (1) $p(z)$; (2) $p_\theta(x|z)$; (3) $p_\theta(z|x)$; (4) $p_\theta(x,z)$.

Your answer:

$p_\theta(z|x)$

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(g) (2 points) Provide the code for implementing the 'reparameterize' function in A8_VAE.py.

Your answer:

```
def reparameterize(self, mu, logvar):
    ##############################
    ## implement the reparameterize function
    ##############################
    std = torch.sqrt(torch.exp(logvar))
    eps = torch.randn(mu.shape)
    return mu + std * eps
```