Team RAH Final Project Report

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The following report shows our numerical solutions to the 1D and 2D heat equations. For the material that we chose, which is a common type of brick, it has a thermal diffusivity of 0.52. The generic form of these two equations can be seen below:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{2}$$

$$\frac{\partial u}{\partial t} = \alpha^2 [\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}] \tag{1}$$

Where u(x, t) is the temperature in a material at (position x, and time t), and k is the constant representing the thermal diffusivity of the material. Our main motivation was to be able to understand how heat dissipates across a surface in a single or two dimensions, and ultimately, the movement of heat in an interface of two semiconductor materials with different diffusivity constants.

In order to solve the 1D heat equation numerically, we rearranged the first-order partial derivative present in it (u with respect to t) to take on the form of our forward difference equation that we have used for numerical differentiation. The generic forward difference equation can be seen below.

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

It was important for us to be able to refactor the second-order partial derivatives similarly to when we used the forward difference method. For these second-order partial derivatives (u with respect to x and u with respect to y) we used the second derivative numerical method that takes on the form of:

$$f''(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

By utilizing the aforementioned finite difference general equations we were able to obtain solutions for the 1 and 2D heat equations with the successful incorporation of initial conditions through these formulas:

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) \approx \frac{u(x_i+h,t_j)-2u(x_i,t_j)+u(x_i-h,t_j)}{h^2} \tag{4}$$

In our Matlab code, We used matrix M that takes in six variables to be formed (IC, N, h, k, l, alpha, t) which are the initial condition, number of points, step size, step size for x, length, and then alpha and t are defining parameters for the diffusivity constant. Our update formula for M ended up as follows:

$$M(i,j+1) = (1-\frac{2\cdot\alpha^2\cdot k}{h^2})\cdot M(i,j)+\frac{\alpha^2\cdot k}{h^2}\cdot(M(i+1,j)+M(i-1,j)) \tag{5}$$

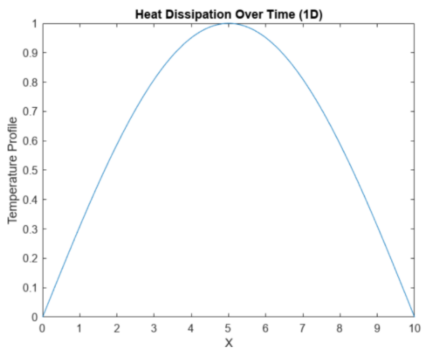

Figure 1: At $t = 0$

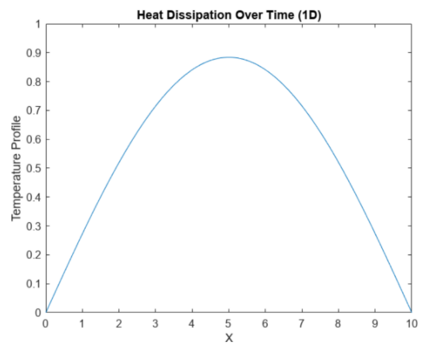

Figure 2: At $t = t_{max}$

Figure 3: These are the results of our MATLAB code

This plot shows the thermal dissipation of bricks with thermal diffusivity of about 0.52 $\frac{mm^2}{s}$ over time, Where the initial condition was $\sin(\frac{x\cdot\pi}{l})$ which represents a sinusoidal homogeneous initial condition. This represents a real-life scenario in which a brick is held under a lighter which then heats it up at a specific position denominated as its peak (in the middle), and the heat source dies off over time.

The main derivation goes from equation (4) to equation (5). The goal is to solve for the only unknown in the equation in terms of the known. The unknown outlined in equation (5) is the j+1 time step which is the time step in the future, which depends on all previous time steps.

This can happen through the rearrangement of the terms

We start by showing the centered second difference equation for the partial derivatives with respect to the spatial variables. We then recognize that in order to implement a partial derivative with respect to x, we must proceed with only x, and the same can be said for y and t. Finally, we used forward difference to approximate the partial derivative with respect to t.

We get the following:

$$\alpha^2[\frac{u(i+1,j,n)-2u(i,j,n)+u(i-1,j,n)}{\Delta x^2}+\frac{u(i,j+1,n)-2u(i,j,n)+u(i,j-1,n)}{\Delta y^2}]$$

$$=\frac{u(i,j,n+1)-u(i,j,n)}{\Delta t} \qquad (6)$$

from then, we rearrange equation (6) with the goal of solving for the only unknown, which is the point in the future governed by u(i,j,n+1), so the equation is rearranged as follows:

$$u(i,j,n+1)=$$

$$\alpha^2\Delta t[\frac{u(i+1,j,n)-2u(i,j,n)+u(i-1,j,n)}{\Delta x^2}+\frac{u(i,j+1,n)-2u(i,j,n)+u(i,j-1,n)}{\Delta y^2}]+u(i,j,n)$$

from then we need to recognize that there are common terms between the expression on the L.H.S, resulting in the following:

$$u(i,j,n+1)=c_1\cdot[u(i+1,j,n)-2\cdot u(i,j,n)+u(i-1,j,n)]$$
$$+c_2\cdot[u(i,j+1,n)-2\cdot u(i,j,n)+u(i,j-1,n)]+u(i,j,n)$$

,where

$$c1=\frac{\alpha^2\cdot\Delta t}{\Delta x^2}, \qquad c2=\frac{\alpha^2\cdot\Delta t}{\Delta y^2};$$

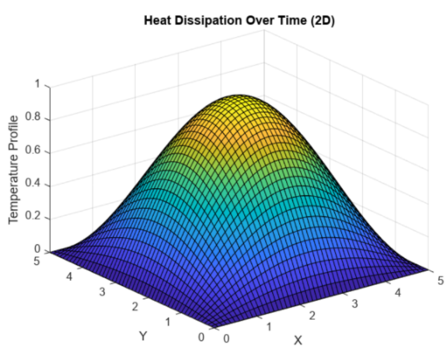

Figure 4: At $t=0$

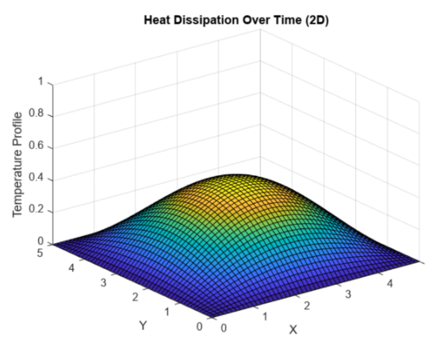

Figure 5: At $t=t_{max}$

Figure 6: These are the results of our MATLAB code

To get these figures, we used similar logic to the 1D heat equation initial condition and simply extended it to the 2D case.

Crank-Nicholson method

Different from the finite difference above, the Crank-Nicholson method is an implicit method that helps increase the solution's stability.

For 1D heat equations:

$$\frac{\partial u}{\partial t} = \alpha^2 \cdot \frac{\partial^2 x}{\partial x^2}$$

We transform the partial derivative into finite difference:

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

$$\frac{\partial^2 x}{\partial x^2} = \frac{1}{2} \cdot \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{\Delta x^2} \right)$$

As can be seen, there are multiple terms in the future step which means we cannot solve for it explicitly.

After grouping the terms with the known part on the right and the unknown (future) term on the left, we get:

$$\begin{aligned} &-\alpha^2 \Delta t \cdot u_{i+1,j+1} + (2\Delta x^2 + 2\alpha^2 \Delta t) \cdot u_{i,j+1} - \alpha^2 \Delta t \cdot u_{i-1,j+1} \\ &= \alpha^2 \Delta t \cdot u_{i+1,j} + (2\Delta x^2 - 2\alpha^2 \Delta t) \cdot u_{i,j} + \alpha^2 \Delta t \cdot u_{i-1,j} \end{aligned}$$

Let:

$$a = -\alpha^2 \cdot \Delta t, \qquad b = 2\Delta x^2 + 2 * \alpha^2 \cdot \Delta t, \qquad C_n = R.H.S$$

Then:

$$\begin{bmatrix} a & b & a & 0 & 0 & \dots & 0 \\ 0 & a & b & a & 0 & \dots & 0 \\ 0 & 0 & a & b & a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a & b & a \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix}$$

For the Dirichlet boundary condition, the value for the bound is known, (i.e. u_1 and u_n is known). Consequently, the matrix equation can be simplified to:

$$\begin{bmatrix} b & a & 0 & 0 & \dots & 0 \\ a & b & a & 0 & \dots & 0 \\ 0 & a & b & a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a & b \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} C_1 - a \cdot u_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n - a \cdot u_n \end{bmatrix}$$

By solving the u vector in each iteration, we achieve the desired result for the solution. Also, since the matrix A is the same for all iterations, instead of solving $Au = C$, we can first get A^{-1}, then calculate $u = b \cdot A^{-1}$

Comparison of stability:

For the explicit method, it becomes unstable after the time difference passes through the Courant–Friedrichs–Lewy condition(CFL condition). However, with the Crank Nicholson method applied here, stability remains.

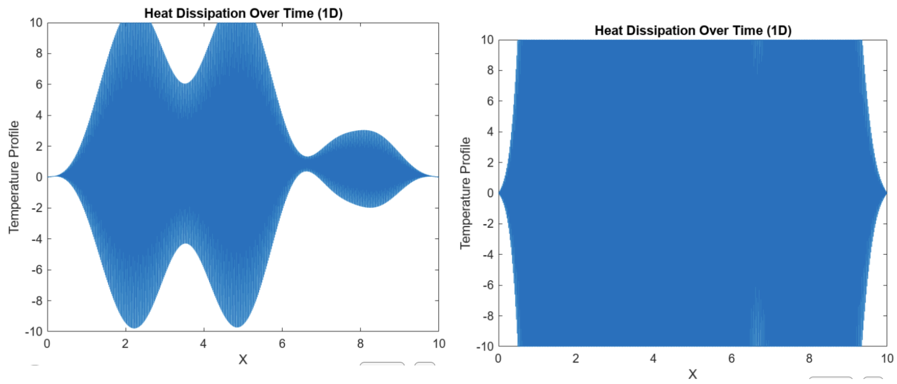

Figure 7: These are the results when forward difference becomes unstable

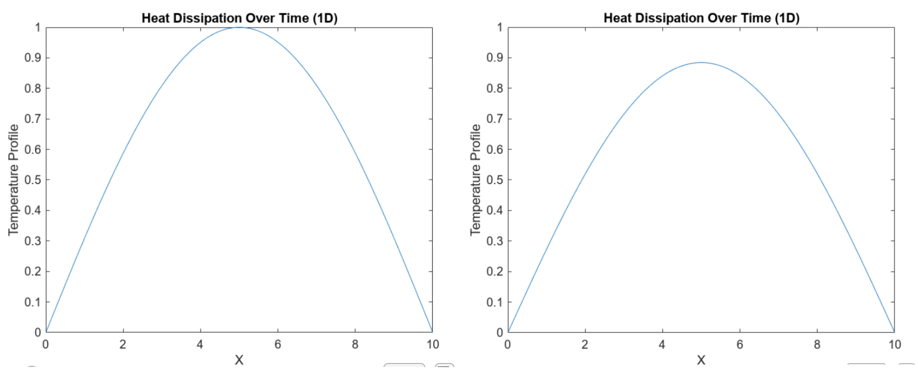

Figure 8: These are the results showing Crank-Nicholson method is still stable with identical conditions to forward difference method

Conclusion

From what we intended to do, we achieved the following:

1. Solving the simple 1D heat equation with finite difference method.
2. Solving the 2D heat equation using the forward difference/finite difference method.
3. Solving the simple 1D heat equation with Crank-Nicholson

For future modifications, we read about how to incorporate external heat sources through the Robin initial condition. Throughout this report, we show an analytical solution to incorporate external heat sources using the 1D and 2D heat equations, solving them with two different methods: Forward and center finite differences, and the Crank Nicholson method.

The heat equation has many applications in many fields. In our case, we successfully showed the power of solving the heat equation for some brick-like materials, but the solution very well could be generalized to any material using any heat source. This can be achieved with the incorporation of different thermal diffusivity and initial conditions but still the same numerical methods. We began by solving the 1D heat equation, then added a second dimension, and we were then able to properly incorporate boundary conditions numerically.