Australian National University

# Fuzzy Logic

## Introduction and Basic

COMP4660/8420 - Neural Networks, Deep Learning and Bio-inspired Computing

iHcc Human Centred Computing

# Schedule

- Introduction and Basics
- Fuzzy rules and rule base systems
- Fuzzy clustering
- Sparse and hierarchical fuzzy systems
- Fuzzy signatures

# Contents

- What is Fuzzy Logic?
- Introduction to Fuzzy Sets
  - Fuzzy Representation
  - Linguistic Variables and Values
  - Characteristics of Fuzzy Sets
  - Properties
  - Operations
- Examples

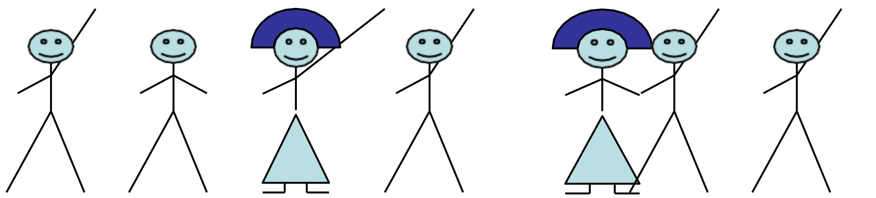
# Background

- In Classical mathematics an element can (exclusively) either belong to a set or not belong to a set
- Impossible to represent much of human discourse. How is one to represent notions like:
  - large profit
  - high pressure
  - tall man
  - moderate temperature
- Ordinary set-theoretic representations will require the maintenance of a crisp differentiation in a very artificial manner:
  - high
  - not quite high
  - very high … etc.

# Background

- Prof. Lotfi A. Zadeh introduced “Fuzzy Sets” in 1965.
- Experts rely on **common sense** when they solve problems.
- **How can we represent expert knowledge that uses vague and ambiguous terms in a computer?**
- Fuzzy logic is not logic that is fuzzy, but logic that is used to describe fuzziness.
- Fuzzy logic is the theory of fuzzy sets.
- Fuzzy logic is based on the idea that all things admit of **degrees**.

# An Example



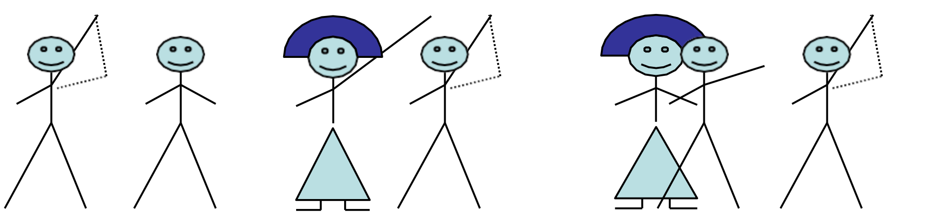
1 0 1 1 0 1 1

"Who has a driver's license?"

A subset of X = A (Crisp) Set

(X) = CHARACTERISTIC FUNCTION

## CRISP SET



0.7 0 1.0 0.8 0 0.4 0.2

"Who can drive very well?"

(X) = MEMBERSHIP FUNCTION

## FUZZY SET

# Why Fuzzy Logic?

- Boolean logic isn't always most suitable
  - Some knowledge or information are imprecise
- Boolean logic uses sharp distinctions
  - e.g. We may say, Tony is tall because his height is 181 cm. If we drew a line at 180 cm, we would find that David, who is 179 cm, is short.
- Fuzzy set theory is more natural

# Fuzzy Applications

- Theory of fuzzy sets and fuzzy logic has been applied to problems in a variety of fields:
  - taxonomy; topology; linguistics; logic; automata theory; game theory; pattern recognition; medicine; law; decision support; information retrieval; etc.
- And more recently fuzzy machines have been developed including:
  - automatic train control; tunnel digging machinery; washing machines; rice cookers; vacuum cleaners; air conditioners, etc.

# Fuzzy Applications

**Advertisement: …**

- Extraklasse Washing Machine - 1200 rpm. The Extraklasse machine has a number of features which will make life easier for you.
- Fuzzy Logic detects the type and amount of laundry in the drum and allows only as much water to enter the machine as is really needed for the loaded amount. And less water will heat up quicker - which means less energy consumption.
- **Foam detection**
  Too much foam is compensated by an additional rinse cycle: If Fuzzy Logic detects the formation of too much foam in the rinsing spin cycle, it simply activates an additional rinse cycle. Fantastic!
- **Imbalance compensation**
  In the event of imbalance, Fuzzy Logic immediately calculates the maximum possible speed, sets this speed and starts spinning. This provides optimum utilization of the spinning time at full speed.
- **Washing without wasting - with automatic water level adjustment**
- Fuzzy automatic water level adjustment adapts water and energy consumption to the individual requirements of each wash programme, depending on the amount of laundry and type of fabric.

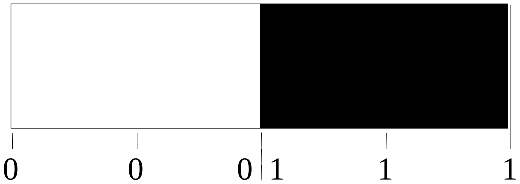
EXTRA KLASSE from Siemens

# Why Fuzzy Logic?

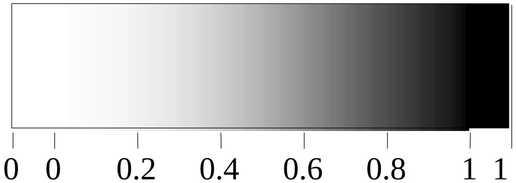
- resembles human reasoning
  - approximate information and uncertainty to generate decisions
- mathematical representation of uncertainty and vagueness
- formalized tools for dealing with the imprecision intrinsic to many problems

# Fuzzy Sets

- Fuzzy logic is a set of mathematical principles for knowledge representation based on **degrees of membership**.
- Unlike two-valued Boolean logic, fuzzy logic is **multi-valued.**
  - Uses **degrees of membership** .
  - continuum of logical values between 0 (completely false) and 1 (completely true).




(*a*) Boolean Logic.




(*b*) Multi-valued Logic.

# Fuzzy Sets

- The concept of a **set** is fundamental to mathematics
- Our own language is also the supreme expression of sets. For example, *car* indicates the *set of cars*. When we say a car, we mean one out of the set of cars.
- An example is height

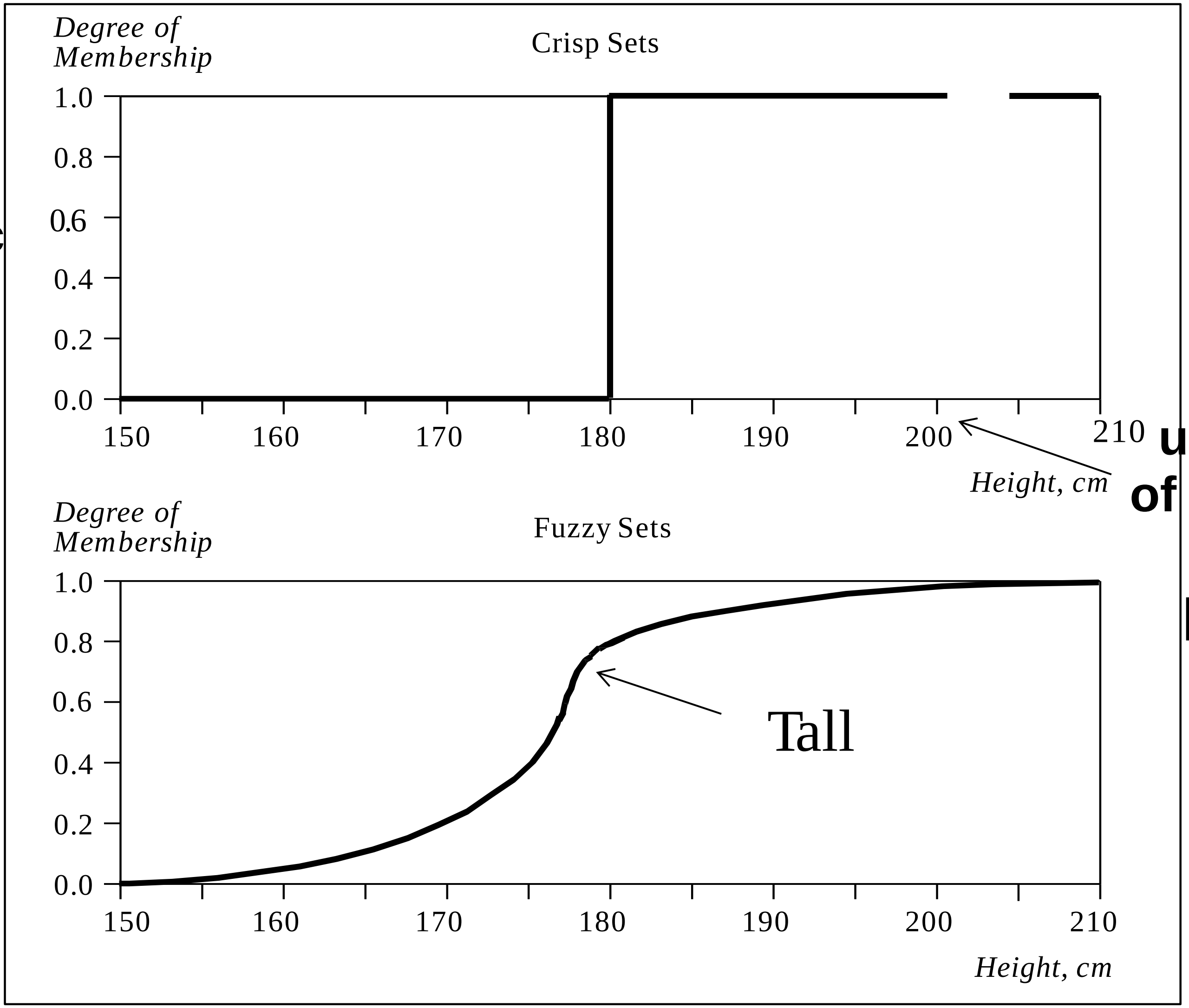
# Fuzzy Set *TALL*

| Name | Height, cm | Degree of Membership | |
|---|---|---|---|
| | | *Crisp* | *Fuzzy* |
| Chris | 208 | 1 | 1.00 |
| Mark | 205 | 1 | 1.00 |
| John | 198 | 1 | 0.98 |
| Tom | 181 | 1 | 0.51 |
| David | 179 | 0 | 0.49 |
| Mike | 172 | 0 | 0.24 |
| Bob | 167 | 0 | 0.15 |
| Steven | 158 | 0 | 0.06 |
| Bill | 155 | 0 | 0.01 |
| Peter | 152 | 0 | 0.00 |

# Crisp Vs Fuzzy Sets

Characteristic value

Membership value




universe of discourse

height”

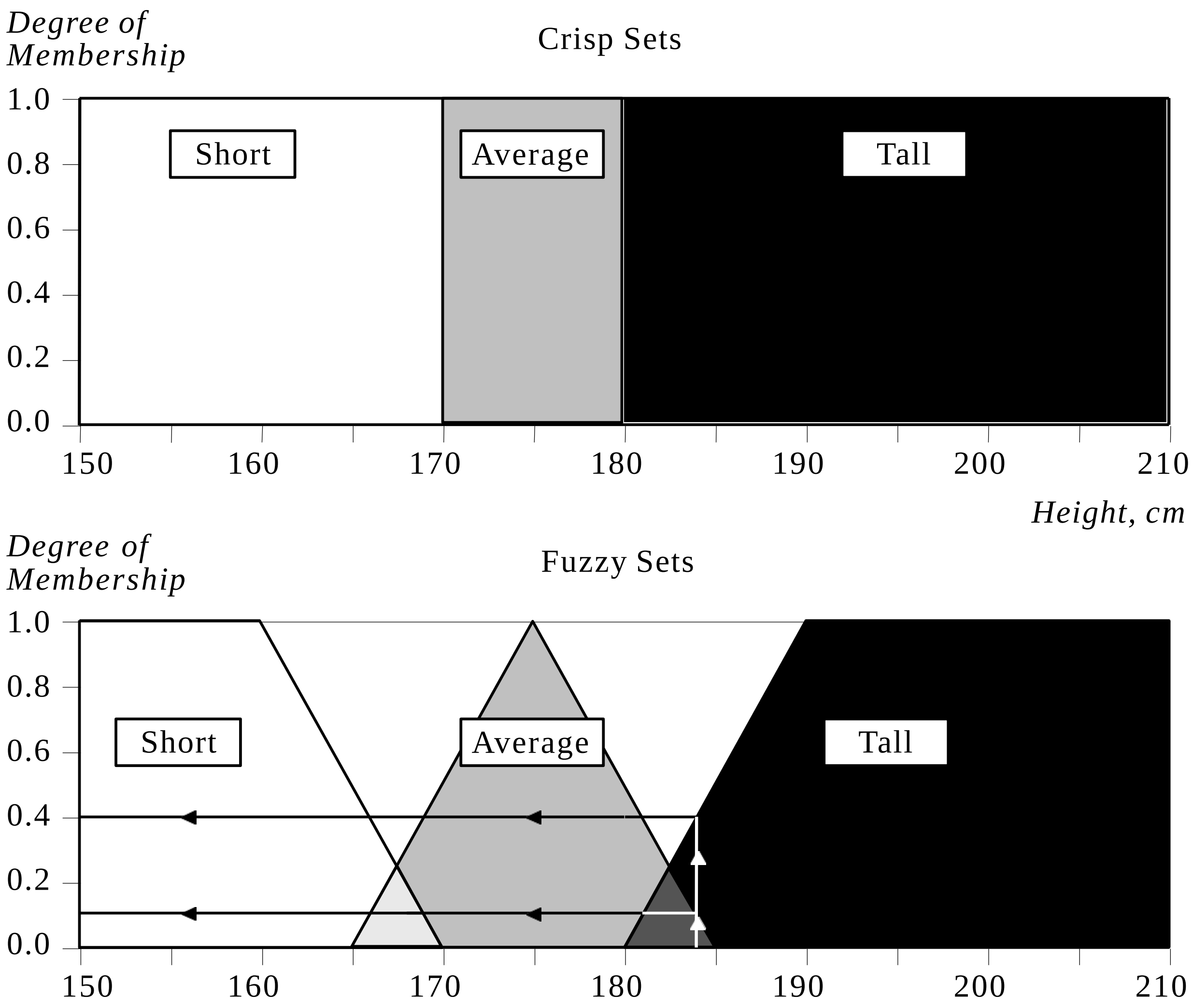
# Fuzzy Set Representation

- Represent a fuzzy set
- Natural functions
  - e.g. sigmoid, Gaussian
  - increased the time of computation
- Piecewise linear functions
  - e.g. Triangular and Trapezoidal
  - drawback is derivatives are not defined for all points.

# Fuzzy Set Representation

- First, determine the membership functions. In our "tall men" example, we can obtain fuzzy sets of *tall*, *short* and *average* men.
- Universe of discourse
  - All possible values
  - the men's heights
  - consists of three sets: *short*, *average* and *tall* men.

# Fuzzy Set Representation



# Linguistic Variables and Hedges

- One advantage of  the fuzzy set theory lies the idea of linguistic variables.
- A linguistic variable is represented by a fuzzy set.
  - E.g. “John is YOUNG” implies that the linguistic variable John’s AGE takes the linguistic value YOUNG.
  - Or “John is TALL” implies that the linguistic variable John’s HEIGHT takes the linguistic value TALL

# Linguistic Variables and Hedges

- In fuzzy expert systems, linguistic variables are used in fuzzy rules.

| | | |
|---|---|---|
| IF | wind | is strong |
| THEN | sailing | is good |

| | | |
|---|---|---|
| IF | project_duration | is long |
| THEN | completion_risk | is high |

| | | |
|---|---|---|
| IF | speed | is slow |
| THEN | stopping_distance | is short |

# Linguistic Variables and Hedges

- The range of possible values of a linguistic variable represents the universe of discourse of that variable.
  - E.g. the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *slow*, *medium*, and *fast*.
- A linguistic variable carries with it the concept of fuzzy set qualifiers, called **hedges**.
- Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as *very*, *somewhat*, *quite*, *more or less* and *slightly*.

# Characteristics of Fuzzy Sets

- The classical set theory developed in the late 19th century by Georg Cantor describes how crisp sets can interact. These interactions are called **operations**.
- Also fuzzy sets have well defined **properties**.
- These properties and operations are the basis on which the fuzzy sets are used to deal with uncertainty on the one hand and to represent knowledge on the other.

# Membership Functions

- For the sake of convenience, usually a fuzzy set is denoted as:

$$A = \mu_A(x_i)/x_i + \ldots\ldots\ldots\ldots + \mu_A(x_n)/x_n$$

where $\mu_A(x_i)/x_i$ (a singleton) is a pair “grade of membership” element, that belongs to a finite universe of discourse:

$$A = \{x_1, x_2, .., x_n\}$$

Eg: Suppose that the universe of discourse
***X* ={3, 4, 5, 6, 7}**
is a set of positive integers (or natural numbers).

Consider the fuzzy set ***A*** in this discrete universe, given by the Zadeh notation:

$$A = 0.2/3 + 0.3/4 + 1.0/5 + 0.2/6 + 0.1/7$$

Here fuzzy set ***A*** represent the expression **"close to 5"**.

$$\mu_A(4) = 0.3$$

Eg: Consider the linguistic variable "*TALL*". This can assume a fuzzy value and can be represented by a fuzzy set.

$$TALL = \frac{0.2}{Paul} + \frac{0.3}{Sam} + \frac{0.3}{Yan} + \frac{0.5}{Karl} + \frac{0.9}{Ben}$$

$$TALL = \frac{0.2}{145} + \frac{0.3}{155} + \frac{0.5}{180} + \frac{0.9}{205}$$

# Properties of Fuzzy Sets

- Equality of two fuzzy sets
- Inclusion of one set into another fuzzy set
- Cardinality of a fuzzy set
- An empty fuzzy set
- $\alpha$-cuts (alpha-cuts)

# Equality

- Fuzzy set *A* is considered equal to a fuzzy set *B*, IF AND ONLY IF (iff):

$\mu$ $\mu_A(x) = \mu_B(x), \forall x \in X$

$A = 0.3/1 + 0.5/2 + 1/3$

$B = 0.3/1 + 0.5/2 + 1/3$

therefore $A = B$

# Inclusion

- Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:

$$\mu_A(x) \leq {}_B(x), \ \forall x \in X$$

Consider $X = \{1, 2, 3\}$ and sets $A$ and $B$

$$A = 0.3/1 + 0.5/2 + 1/3$$

$$B = 0.5/1 + 0.55/2 + 1/3$$

then $A$ is a subset of $B$, or $A \subseteq B$

# Cardinality

- Cardinality of a non-fuzzy set, Z, is the number of elements in Z
- Cardinality of a fuzzy set A is the sum of the values of the membership function of A, $\mu_A(x)$:
  - – also called SIGMACOUNT

$$card_A = \mu_A(x_1) + \mu_A(x_2) + \ldots \mu_A(x_n) = \Sigma\mu_A(x_i), \text{ for } i=1..n$$

Consider $X = \{1, 2, 3\}$ and sets $A$ and $B$

$A = 0.3/1 + 0.5/2 + 1/3$ $\quad card_A = 1.8$

$B = 0.5/1 + 0.55/2 + 1/3$ $\quad card_B = 2.05$

# Empty Fuzzy Set

- A fuzzy set *A* is empty, IF AND ONLY IF:

$\mu_A(x) = 0, \forall x \in X$

Consider $X = \{1, 2, 3\}$ and set *A*

$$A = 0/1 + 0/2 + 0/3$$

then *A* is *empty*

# Alpha-cut

- An $\alpha$-cut or $\alpha$-level set of a fuzzy set $A \subseteq X$ is an ORDINARY SET $A_\alpha \subseteq X$, such that:

$$A_{\alpha\alpha} = \{x \mid \mu_A(x) \geq \alpha,\ \forall x \in X\}$$

Consider $X = \{1, 2, 3\}$ and set $A$

$$A = 0.3/1 + 0.5/2 + 1/3$$

Then $\quad A_{0.5} = \{2, 3\}$,

$A_{0.1} = \{1, 2, 3\}$,

$A_1 = \{3\}$

# Strong Alpha-cut

- Strong $\alpha$-cut of a fuzzy set $A \subseteq X$ is an ORDINARY SET such that:

$$A_{\overline{\alpha}} = \{x \mid \mu_A(x) > \alpha\}$$

Consider $X = \{1, 2, 3\}$ and set $A$

$A = 0.3/1 + 0.5/2 + 1/3$

then

$$A_{\overline{0.5}} = \{3\}$$
$$A_{\overline{0.1}} = \{1,2,3\}$$
$$A_{\overline{1}} = \Phi$$

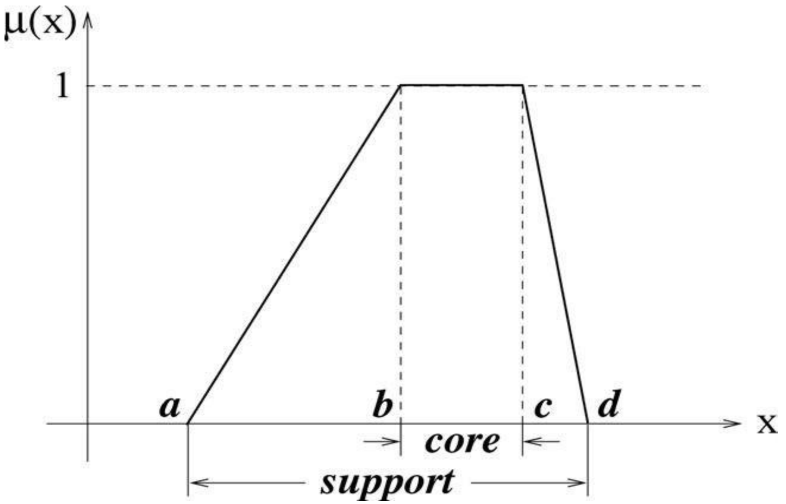
# Fuzzy Set Normality

- A fuzzy subset of X is called **normal** if there exists at least one element $x \in X$ such that $\mu_A(x) = 1$.
- A fuzzy subset that is not normal is called **subnormal**.
- The **height** of a fuzzy subset *A* is the largest membership grade of an element in *A*

$$height(A) = \max_x(\mu_A(x))$$

# Fuzzy Sets Core and Support

- *A* is a fuzzy subset of *X*:
- the **support** of *A* is the crisp subset of *X* consis ting of all elements with membership grade:




$$supp(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\}$$

- the **core** of *A* is the crisp subset of *X* consisting of all elements with membership grade:

$$core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\}$$

# Fuzzy Set Math Operations

- $aA = \{a\mu_A(x), \forall x \in X\}$

  Let $a$ =0.5, and

  $A$ = {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  $aA$ = {0.25/a, 0.15/b, 0.1/c, 0.5/d}

- $A^a = \{\mu_A(x)^a, \forall x \in X\}$

  Let $a$ =2, and

  $A$ = {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  $A^a$ = {0.25/a, 0.09/b, 0.04/c, 1/d}

# Fuzzy Sets Examples

- Consider two fuzzy subsets of the set *X*,

$$X = \{ a, b, c, d, e \}$$

referred to as *A* and *B:*

$$A = \{1/a, 0.3/b, 0.2/c\ 0.8/d, 0/e\}$$

and

$$B = \{0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e\}$$

# Fuzzy Sets Examples

$A$ = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}
and
$B$ = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

- **Support:**

  $supp(A)$ = {a, b, c, d }

  $supp(B)$ = {a, b, c, d, e }

- **Core:**

  $core(A)$ = {a}

  $core(B) = \emptyset$

- **Cardinality:**

  $card(A) = 1+0.3+0.2+0.8+0 = 2.3$

  $card(B) = 0.6+0.9+0.1+0.3+0.2 = 2.1$

# Fuzzy Sets Examples

- <u>*aA*</u>:

  for *a*=0.5

  $aA$ = {0.5/a, 0.15/b, 0.1/c, 0.4/d, 0/e}

- <u>$A^a$</u>:

  for *a*=2

  $A^a$ = {1/a, 0.09/b, 0.04/c, 0.64/d, 0/e}

- $\alpha$-<u>cut</u>:

  $A_{0.2}$ = {a, b, c, d}

  $A_{0.3}$ = {a, b, d}

  $A_{0.8}$ = {a, d}

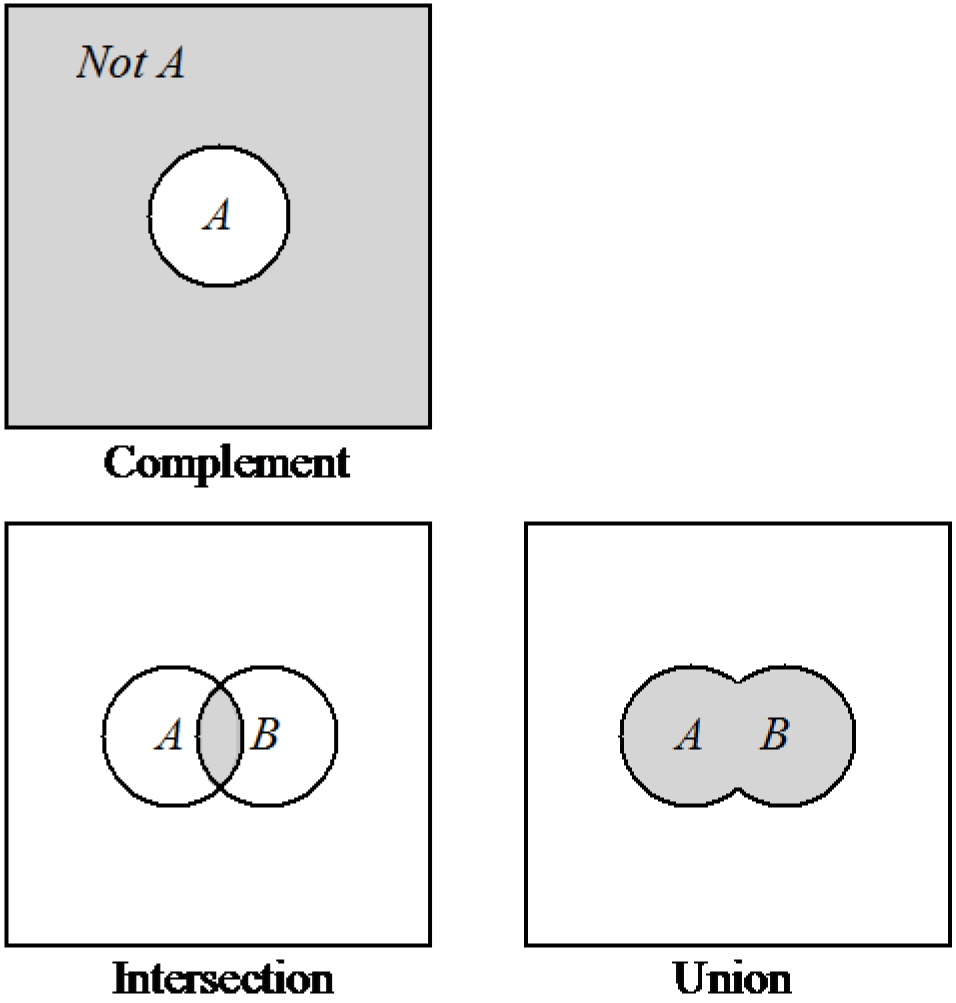
  $A_1$ = {a}

$A$ = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

and

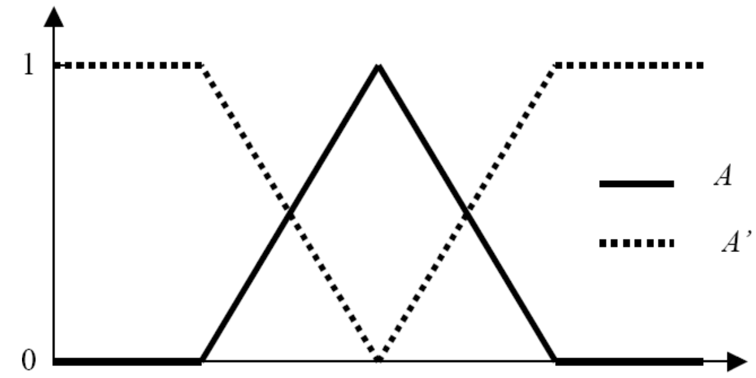
$B$ = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

# Operations of Fuzzy Sets



# Complement

<u>Crisp Sets</u>: Which does not belong to the set?
<u>Fuzzy Sets</u>: How much do elements not belong to the set?

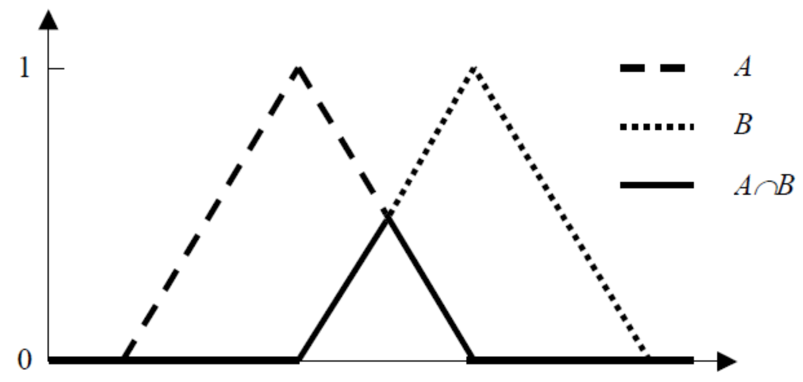



If A is the fuzzy set, its complement $\neg A$ can be found as follows:
$\mu_{\neg A}(x) = 1 - \mu_A(x)$

E.g. If we have the set of ***AVERAGE*** men, its complement is the set of NOT average men. When we remove completely all average men who were in the ***AVERAGE*** set, we obtain the complement.

# Intersection

Crisp Sets: Which element belongs to both sets?
Fuzzy Sets: How much of the element is in both sets?




A fuzzy intersection is the **lower membership** in both sets of each element. The fuzzy intersection of two fuzzy sets *A* and *B* on universe of discourse X:

$\mu_A \cap \mu_B(x) = \min(\mu_A(x), \mu_B(x))$ , where $x \in X$

E.g. Consider a universe representing the driving speeds on a highway, in km/h. Suppose that the fuzzy set “FAST” is given by:

$$F = 0.6/80 + 0.8/90 + 1.0/100 + 1.0/110 + 1.0/120$$

and the fuzzy state "Medium" is given by

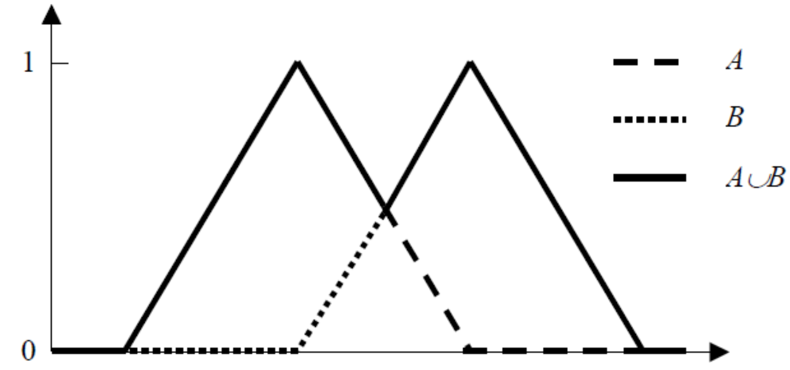
$$M = 0.6/50 + 0.8/60 + 1.0/70 + 1.0/80 + 0.8/90 + 0.4/100$$

Then the combined fuzzy condition "Fast AND Medium" is given by the fuzzy set:

$$F \cap M = 0/50 + 0/60 + 0/70 + 0.6/80 + 0.8/90 + 0.4/100 + 0/110 + 0/120$$

# Union

<u>Crisp Sets</u>: Which element belongs to either set?
<u>Fuzzy Sets</u>: How much of the element is in either set?




- The union of two crisp sets consists of every element that falls into either set.
- In fuzzy sets, the union is the reverse of the intersection. That is, the union is the **largest membership** value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets A and B on universe X can be given as:

$$\mu A \cup \mu B(x) = \max\{\mu A(x), \mu B(x)\}, \text{ where } x \in X$$

Previous example continued:

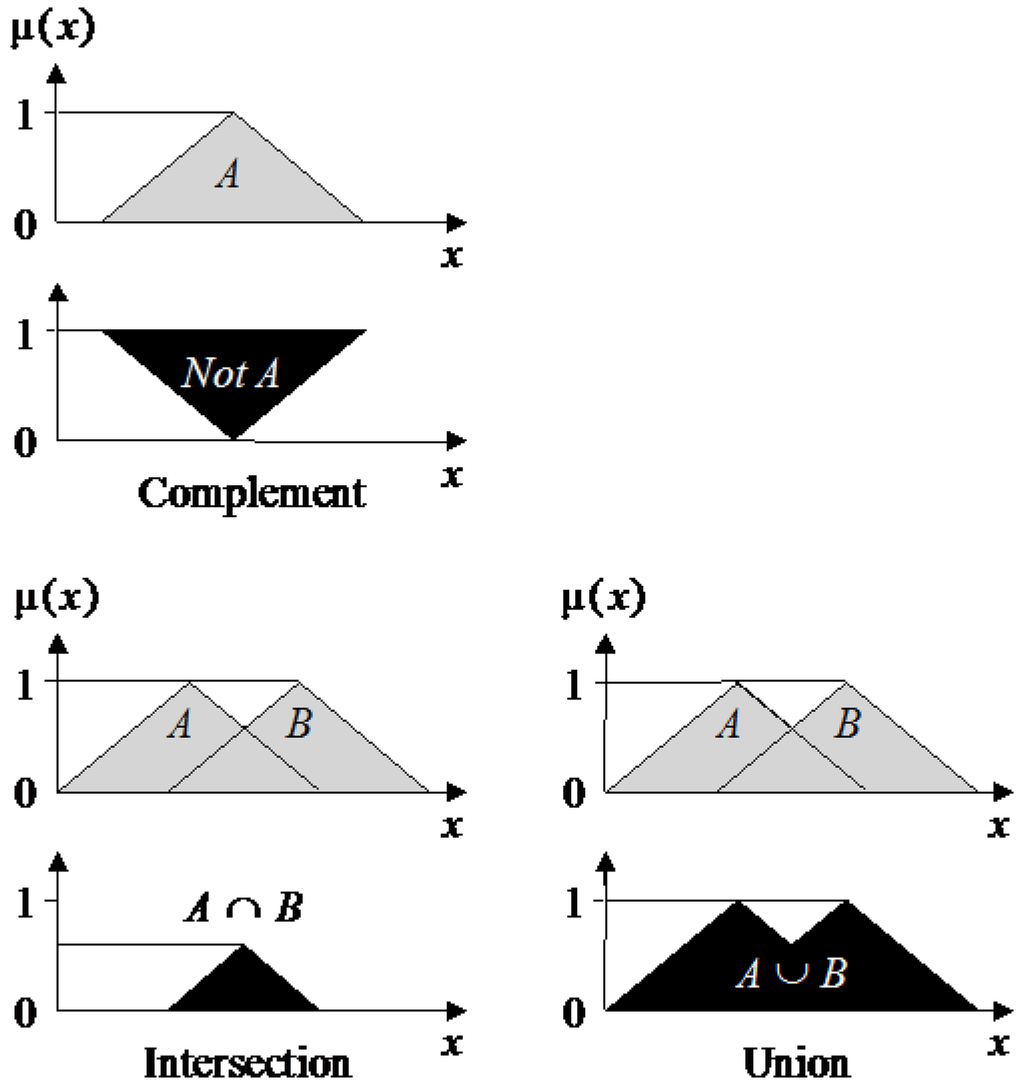
$$F = 0.6/80 + 0.8/90 + 1.0/100 + 1.0/110 + 1.0/120$$

$$M = 0.6/50 + 0.8/60 + 1.0/70 + 1.0/80 + 0.8/90 + 0.4/100$$

Then the combined fuzzy condition "Fast OR Medium" is given by the fuzzy set:

$$F \cap M = 0.6/50 + 0.8/60 + 1.0/70 + 1.0/80 + 0.8/90 + 1.0/100 + 1.0/110 + 1.0/120$$

# Operations of Fuzzy Sets



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# Fuzzy Sets Examples

- Complement:

$$\neg A = \{0/a,\ 0.7/b,\ 0.8/c\ 0.2/d,\ 1/e\}$$

$$\neg B = \{0.4/a,\ 0.1/b,\ 0.9/c\ 0.7/d,\ 0.8/e\}$$

- Union:

$$A \cup B = \{1/a,\ 0.9/b,\ 0.2/c,\ 0.8/d,\ 0.2/e\}$$

- Intersection:

$$A \cap B = \{0.6/a,\ 0.3/b,\ 0.1/c,\ 0.3/d,\ 0/e\}$$

$$A = \{1/a,\ 0.3/b,\ 0.2/c\ 0.8/d,\ 0/e\}$$

and

$$B = \{0.6/a,\ 0.9/b,\ 0.1/c,\ 0.3/d,\ 0.2/e\}$$

# Exercises

For

$A = \{0.2/a, 0.4/b, 1/c, 0.8/d, 0/e\}$

$B = \{0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e\}$

<u>Calculate the following:</u>

-Support, Core, Cardinality, and Complement for *A* and *B* independently

- Union and Intersection of A and B
- the new set *C*, if $C = A^2$
- the new set *D*, if $D = 0.5B$
- the new set *E*, for an alpha cut at $A_{0.5}$

# Solutions

*A = {0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}*
*B = {0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}*

Support

Supp(A) = {a, b, c, d}
Supp(B) = {b, c, d, e}

Core

Core(A) = {c}
Core(B) = ∅

Cardinality

Card(A) = 0.2 + 0.4 + 1 + 0.8 + 0 = 2.4
Card(B) = 0 + 0.9 + 0.3 + 0.2 + 0.1 = 1.5

Complement

¬A = {0.8/a, 0.6/b, 0/c, 0.2/d, 1/e}
¬B = {1/a, 0.1/b, 0.7/c, 0.8/d, 0.9/e}

# Solutions

*A = {0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}*
*B = {0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}*

Union

$A \cup B = \{0.2/a, 0.9/b, 1/c, 0.8/d, 0.1/e\}$

Intersection

$A \cap B = \{0/a, 0.4/b, 0.3/c, 0.2/d, 0/e\}$

$C = A^2$

$C = \{0.04/a, 0.16/b, 1/c, 0.64/d, 0/e\}$

$D = 0.5B$

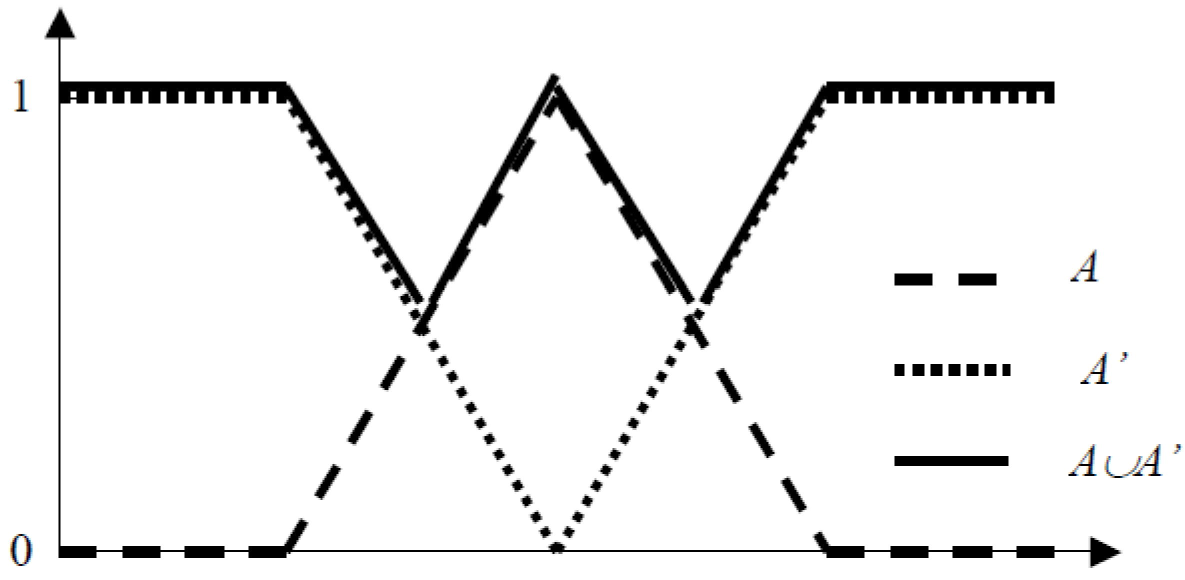
$D = \{0/a, 0.45/b, 0.15/c, 0.1/d, 0.05/e\}$

$E = A_{0.5}$

$E = \{c, d\}$

| | |
|---|---|
| Commutativity | $A \cup B = B \cup A$, $A \cap B = B \cap A$ |
| Associativity | $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$,<br>$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$ |
| Distributivity | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,<br>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ |
| Idempotence | $A \cup A = A$, $A \cap A = A$ |
| Absorption | $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$ |
| Absorption of complement | $A \cup (\overline{A} \cap B) = A \cup B$<br>$A \cap (A \cup B) = A \cap B$ |
| Abs. by X and $\emptyset$ | $A \cap X = X$, $A \cap \emptyset = \emptyset$ |
| Identity | $A \cup \emptyset = A$, $A \cup X = A$ |
| Law of contradiction | $A \cap \overline{A} \neq \emptyset$ — Only difference to Classical 2 valued logic |
| Law of excl. middle | $A \cup \overline{A} \neq X$ |
| DeMorgan's laws | $\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\qquad$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$ |

Eg: To illustrate how the law of excluded middle and contradiction are violated by fuzzy sets




An example of excluded middle in fuzzy sets