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Fuzzy Logic

Fuzzy Rules and Fuzzy Rule Based Systems

COMP4660/8420 - Neural Networks, Deep Learning and Bio-inspired Computing

Human Centred Computing

Contents

- Fuzzy Rules
  - examples
- Fuzzy Inference
  - Fuzzification of the input variables
  - Rule evaluation
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Fuzzy Rules

- In 1973, Lotfi Zadeh suggested capturing human knowledge in fuzzy rules
- A fuzzy rule can be defined as a conditional statement in the form:

IF	*x*	is *A*
THEN	*y*	is *B*

where *x* and *y* are linguistic variables; and *A* and *B* are linguistic values determined by fuzzy sets on the universe of discourses *X* and *Y*, respectively.

Classical Vs Fuzzy Rules

- A classical IF-THEN rule uses binary logic, for example,

Rule: 1	Rule: 2
IF speed > 100km/h	IF speed < 40km/h
THEN stopping_distance is >88m	THEN stopping_distance is <26m

- The variable speed can have any numerical value between 0 and 220 km/h
- Linguistic variable stopping_distance can take either value long or short.
- What about speed = 99 km/h ?

Classical Vs Fuzzy Rules

- We can also represent the stopping distance rules in a fuzzy form:

Rule: 1	Rule: 2
IF speed is fast	IF speed is slow
THEN stopping_distance is long	THEN stopping_distance is short

- In fuzzy rules, the linguistic variable speed also has the range (the universe of discourse) between 0 and 220 km/h
- Range includes fuzzy sets, such as slow, medium and fast.
- Now speed = 99 km/h, partially belongs to both fast and medium

Classical Vs Fuzzy Rules

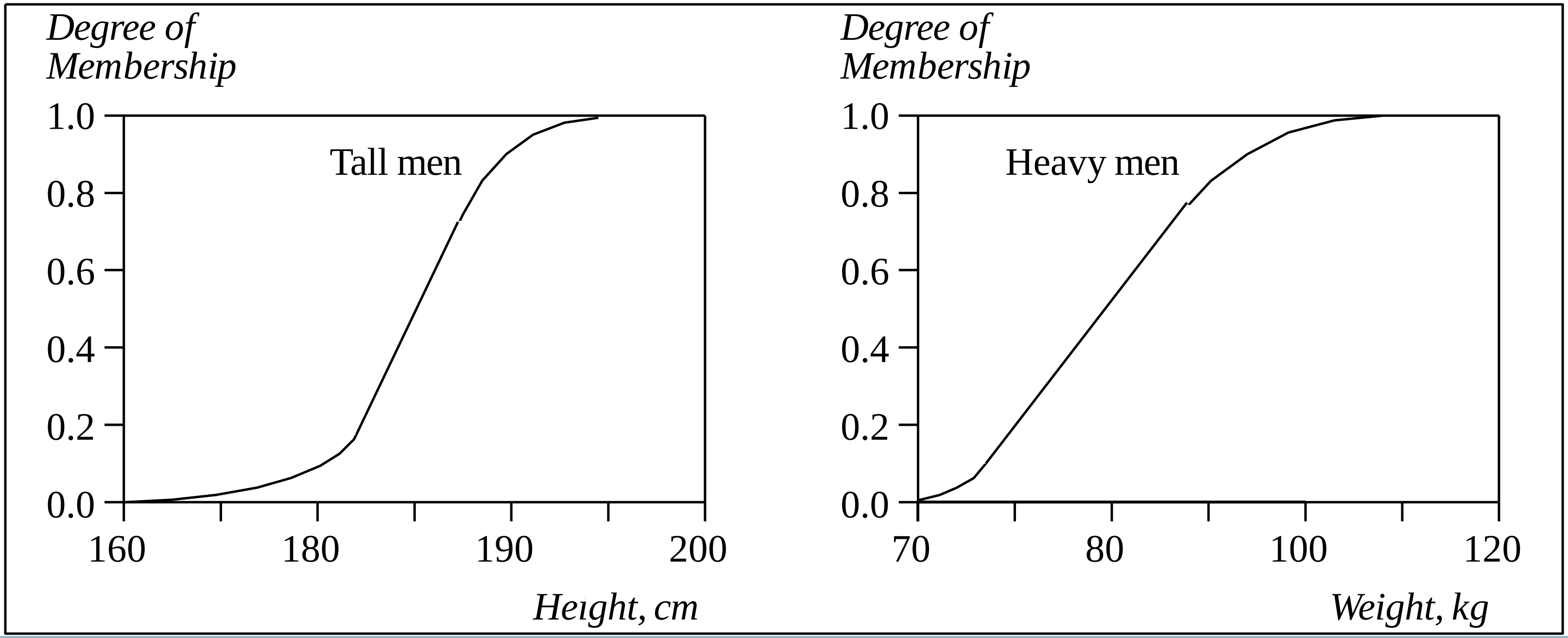
- Fuzzy rules uses fuzzy sets in inputs and/or outputs
- In a fuzzy system, all rules fire to some extent, or in other words they fire partially.
- If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

Firing Fuzzy Rules

- These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:

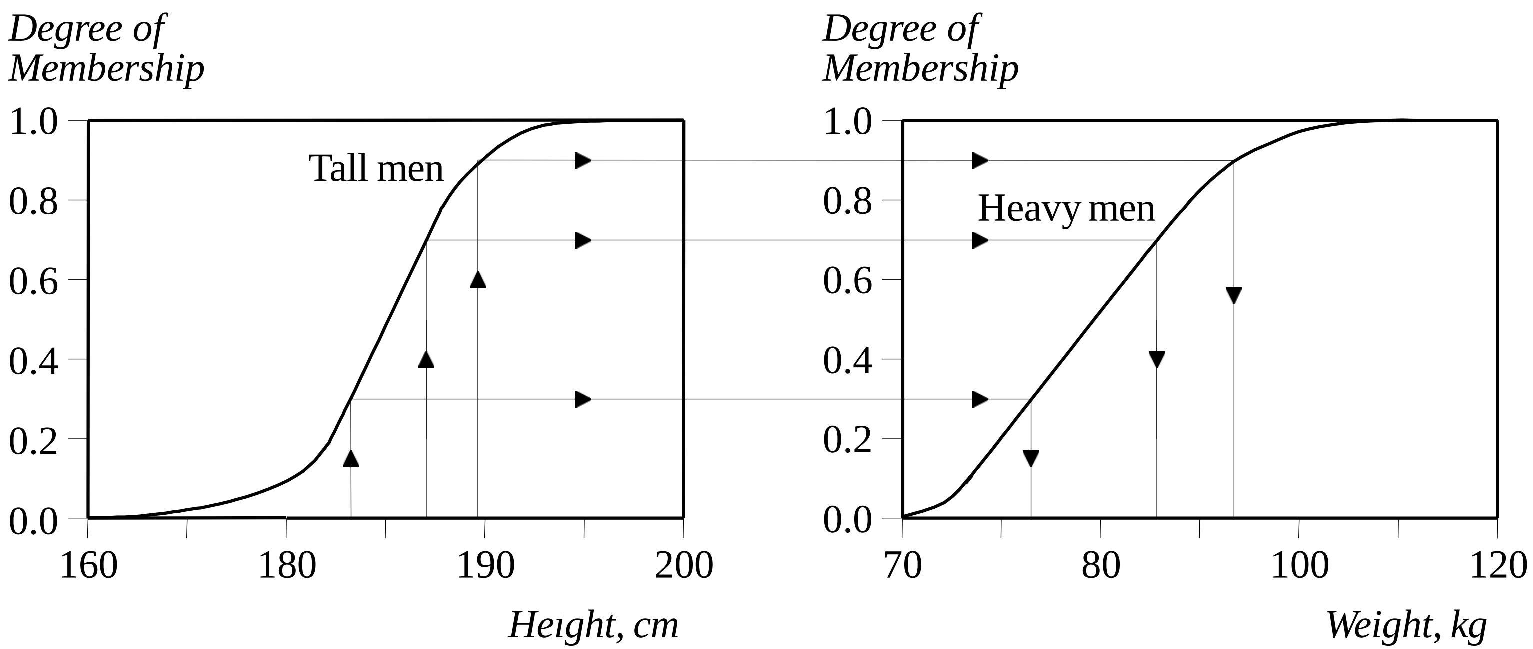
  IF height is tall

  THEN weight is heavy



Firing Fuzzy Rules

- The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called monotonic selection.



Firing Fuzzy Rules

- A fuzzy rule can have multiple antecedents, for example:

IF	project_duration is long
AND	project_staffing is large
AND	project_funding is inadequate
THEN	risk is high

IF	service is excellent
OR	food is delicious
THEN	tip is generous

- The consequent of a fuzzy rule can also include multiple parts, for instance:

IF	temperature is hot
THEN	hot_water is reduced;
	cold_water is increased

Fuzzy Sets Example

- Air-conditioning
  – under thermostatic control - generally, the amount of air being compressed is proportional to the ambient temperature.
- Consider Johnny’s air-conditioner which hasfive control switches: COLD, COOL, PLEASANT, WARM and HOT. The corresponding speeds of the motor controlling the fan on the air-conditioner has the graduations: MINIMAL, SLOW, MEDIUM, FAST and BLAST.

Fuzzy Sets Example

- The rules governing the air-conditioner are as follows:

RULE 1:

IF TEMP is COLD THEN SPEED is MINIMAL

RULE 2:

IF TEMP is COOL THEN SPEED is SLOW

RULE 3:

IF TEMP is PLEASANT THEN SPEED is MEDIUM

RULE 4:

IF TEMP is WARM THEN SPEED is FAST

RULE 5:

IF TEMP is HOT THEN SPEED is BLAST

Fuzzy Sets Example

The temperature graduations are related to Johnny's perception of ambient temperatures.

where:

Y: *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal (core) member of the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Temp (^{0}C).	COLD	COOL	PLEASANT	WARM	HOT
0	Y*	N	N	N	N
5	Y	Y	N	N	N
10	N	Y	N	N	N
12.5	N	Y*	N	N	N
15	N	Y	N	N	N
17.5	N	N	Y*	N	N
20	N	N	N	Y	N
22.5	N	N	N	Y*	N
25	N	N	N	Y	N
27.5	N	N	N	N	Y
30	N	N	N	N	Y*

Fuzzy Sets Example

Johnny's perception of the speed of the motor is as follows:

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Rev/sec (RPM)	MINIMAL	SLOW	MEDIUM	FAST	BLAST
0	Y*	N	N	N	N
10	Y	N	N	N	N
20	Y	Y	N	N	N
30	N	Y*	N	N	N
40	N	Y	N	N	N
50	N	N	Y*	N	N
60	N	N	N	Y	N
70	N	N	N	Y*	N
80	N	N	N	Y	Y
90	N	N	N	N	Y
100	N	N	N	N	Y*

Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the temperature are:
- COLD:

  for $0 \le t \le 10$ ${}_{COLD}(t) = -t / 10 + 1$

- COOL:

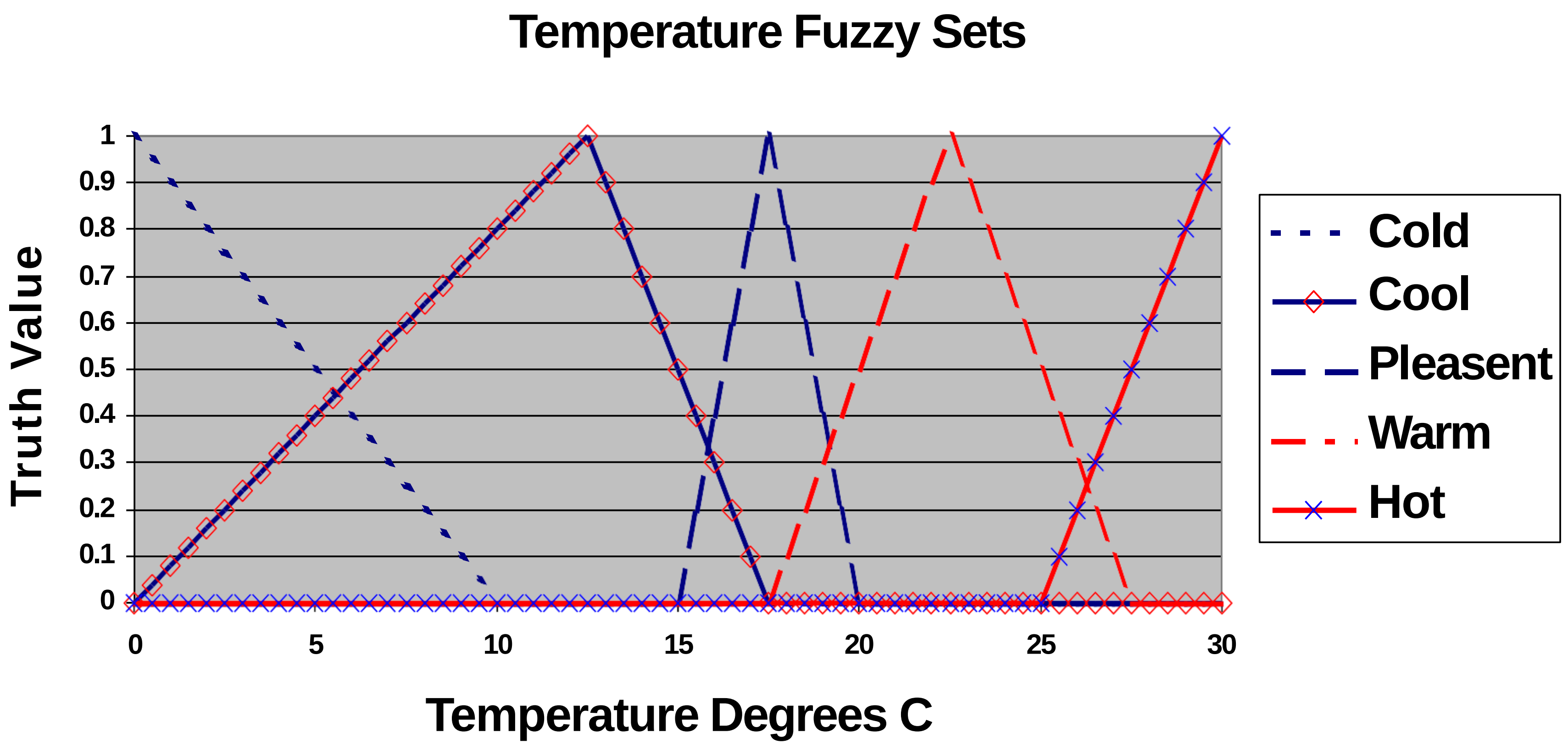
  for $0 \le t \le 12.5$ ${}_{COOL}(t) = t / 12.5$

  for $12.5 \le t \le 17.5$ ${}_{COOL}(t) = -t / 5 + 3.5$

- etc… all based on the linear equation:

  $y = ax + b$

Fuzzy Sets Example



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Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the speed are:
- MINIMAL:

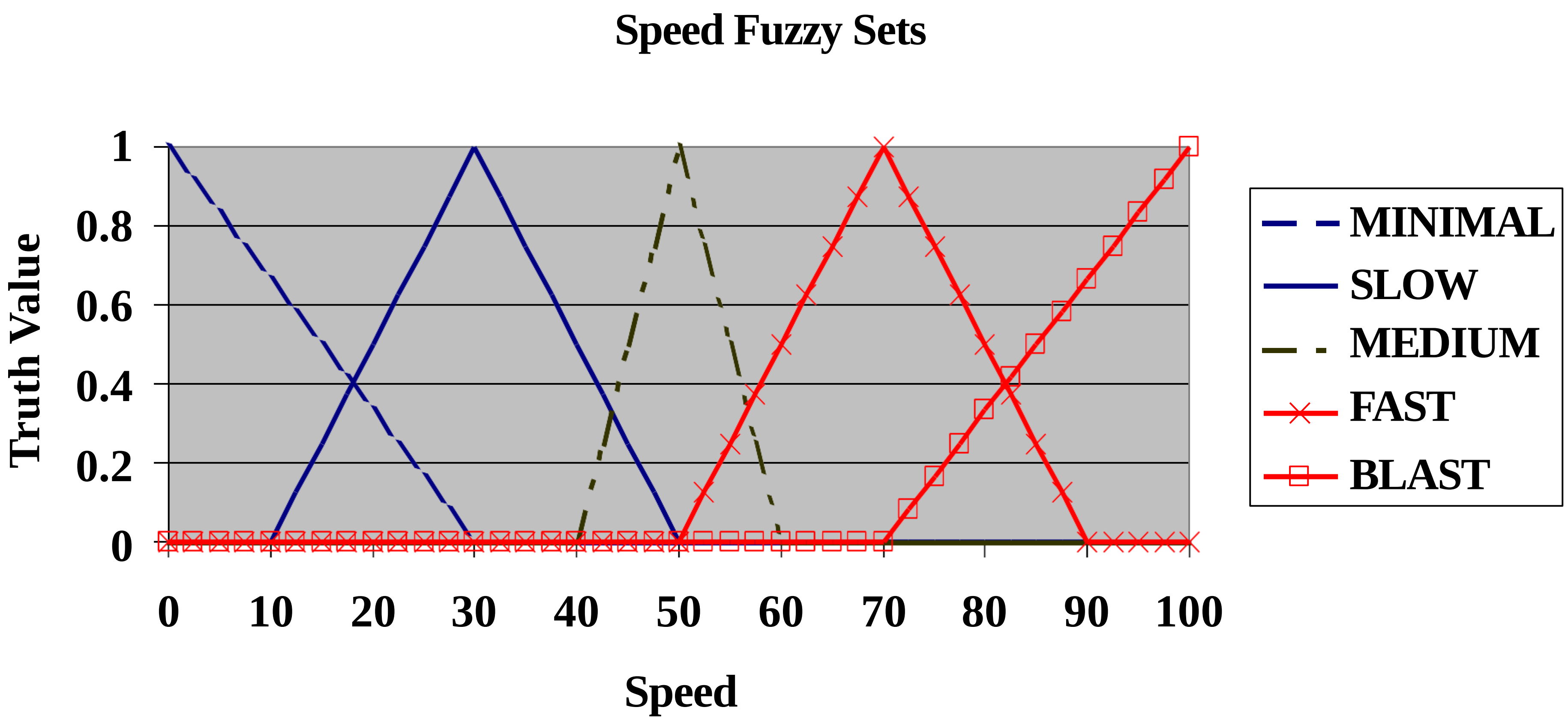
  for $0 \le v \le 30$ $\qquad {}_{MINIMAL}(t) = -v/30 + 1$
- SLOW:

  for $10 \le v \le 30$ $\qquad {}_{SLOW}(t) = v/20 - 0.5$

  for $30 \le v \le 50$ $\qquad {}_{SLOW}(t) = -v/20 + 2.5$
- etc… all based on the linear equation:

  $y = ax + b$

Fuzzy Sets Example



Fuzzy Inference

- The most commonly used fuzzy inference technique is the so-called Mamdani method.
- In 1975, Professor Ebrahim Mamdani of London University built one of the first fuzzy systems to control a steam engine and boiler combination. He applied a set of fuzzy rules supplied by experienced human operators.

Mamdani Fuzzy Inference

- The Mamdani-style fuzzy inference process is performed in four steps:
  1. Fuzzification of the input variables
  2. Rule evaluation (inference)
  3. Aggregation of the rule outputs (composition)
  4. Defuzzification.

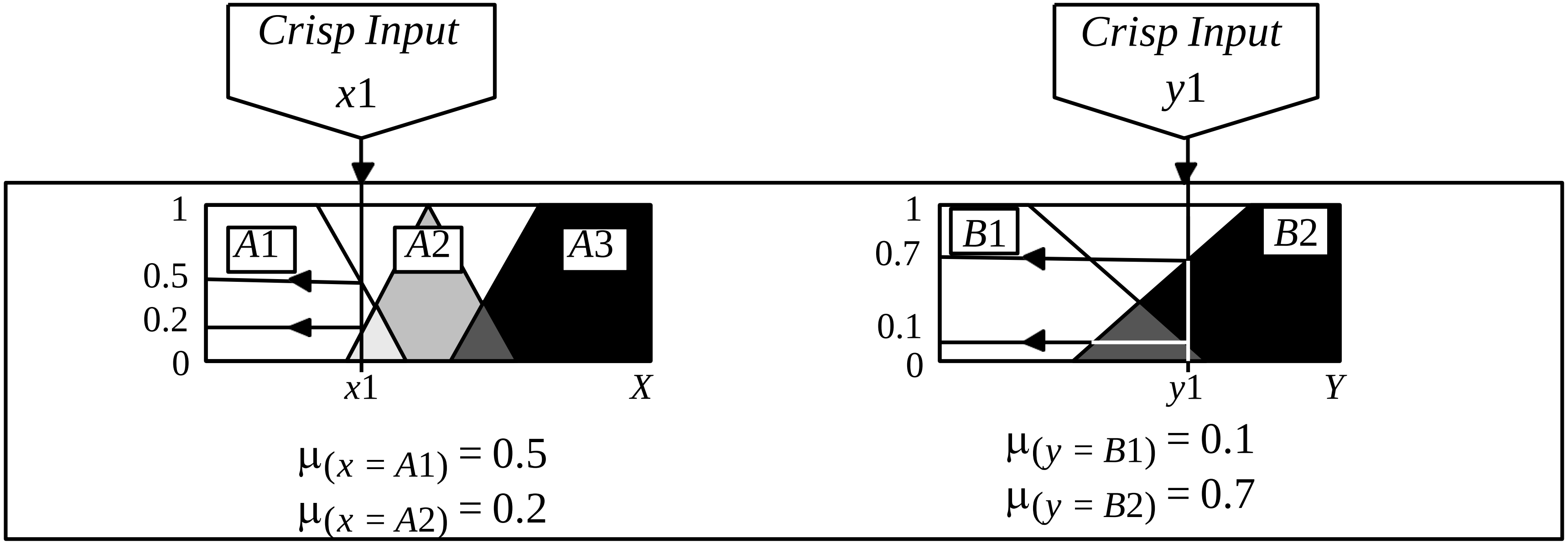
Mamdani Fuzzy Inference

We examine a simple two-input one-output problem that includes three rules:

Rule: 1		**Rule: 1**		
IF	x is A3	IF	project_funding	is adequate
OR	y is B1	OR	project_staffing	is small
THEN	z is C1	THEN	risk	is low
Rule: 2		**Rule: 2**		
IF	x is A2	IF	project_funding	is marginal
AND	y is B2	AND	project_staffing	is large
THEN	z is C2	THEN	risk	is normal
Rule: 3		**Rule: 3**		
IF	x is A1	IF	project_funding	is inadequate
THEN	z is C3	THEN	risk	is high

Step 1: Fuzzification

- The first step is to take the crisp inputs, x1 and y1 (*project funding* and *project staffing*), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



Step 2: Rule Evaluation

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules.
- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.
- This number (the truth value) is then applied to the consequent membership function.

Step 2: Rule Evaluation

RECALL:

To evaluate the disjunction of the rule antecedents, we use the OR fuzzy operation. Typically, fuzzy expert systems make use of the classical fuzzy operation union:

$$\mu_{A\cup}\mu_B(x) = \max\,[\mu_A(x), \mu_B(x)]$$

Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the AND fuzzy operation intersection:

$$\mu_{A\cup}\mu_B(x) = \min\,[\mu_A(x), \mu_B(x)]$$

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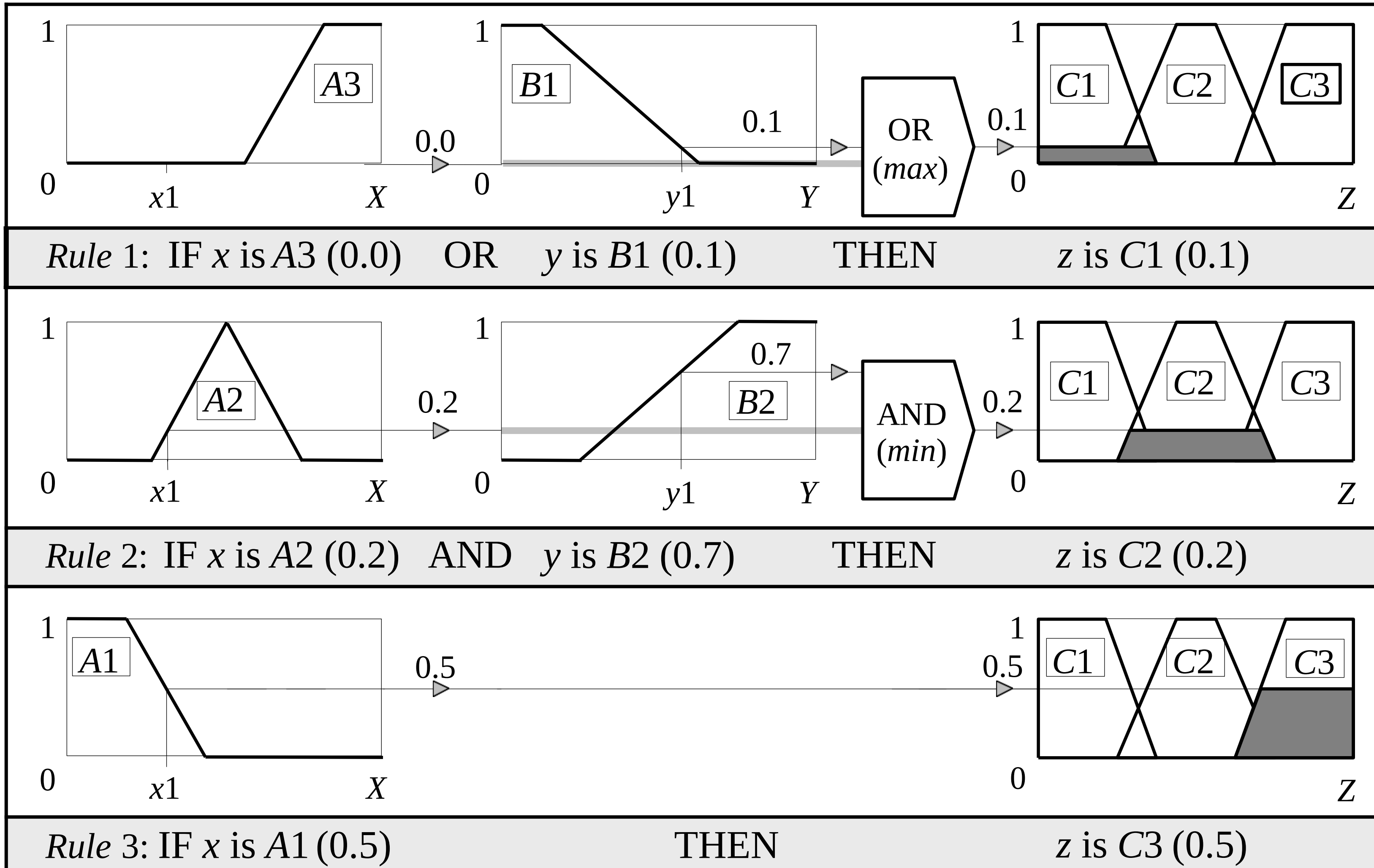
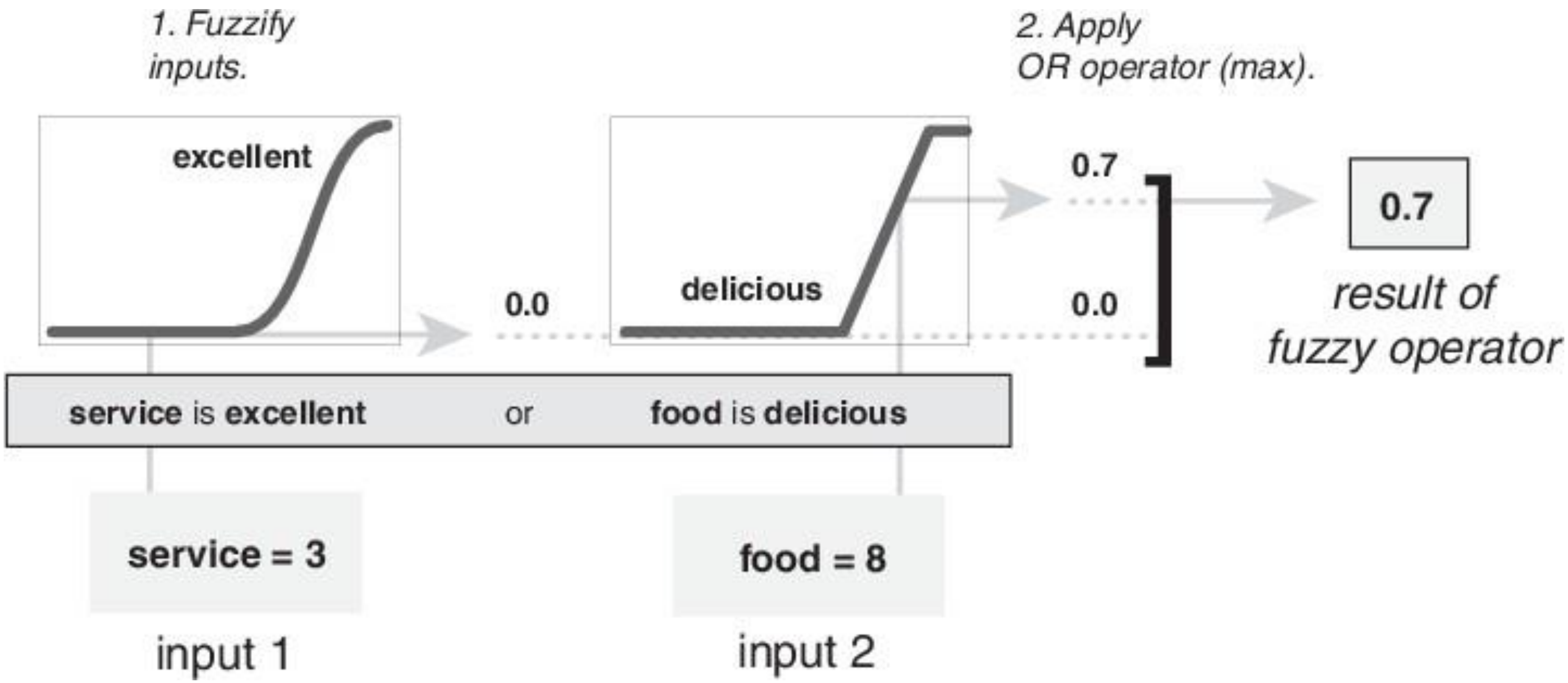
Step 2: Rule Evaluation



Example


1. Fuzzify
inputs.
2. Apply
OR operator (max).
excellent
0.0
delicious
0.7
0.0
0.7
result of
fuzzy operator
service is excellent
or
food is delicious
service = 3
input 1
food = 8
input 2

Step 2: Rule Evaluation

- Now the result of the antecedent evaluation can be applied to the membership function of the consequent.
- There are two main methods for doing so:
  - Clipping
  - Scaling

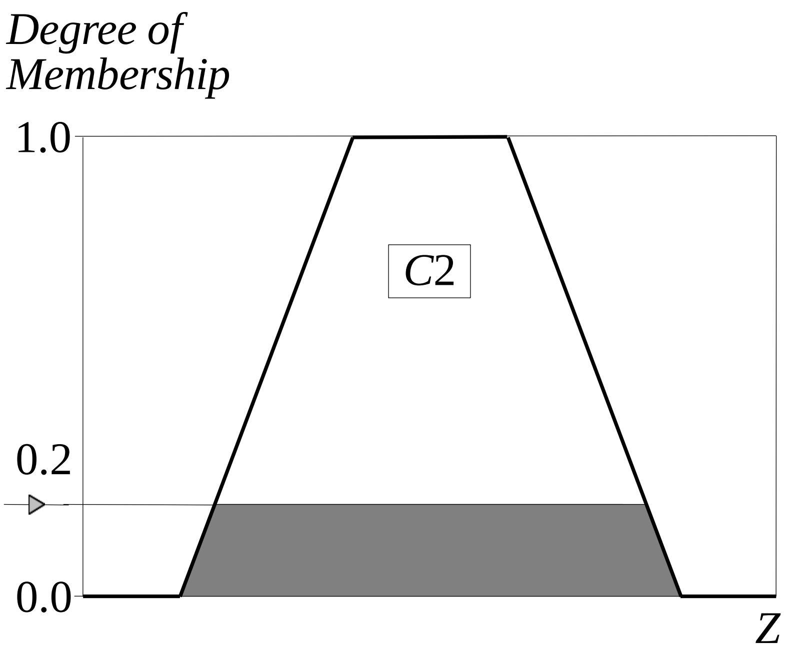
Step 2: Rule Evaluation

- clipping (alpha-cut):
  - The most common method of correlating the rule consequent with the truth value of the rule antecedent is to cut the consequent membership function at the level of the antecedent truth.
  - Since the top of the membership function is sliced, the clipped fuzzy set loses some information.
- However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.

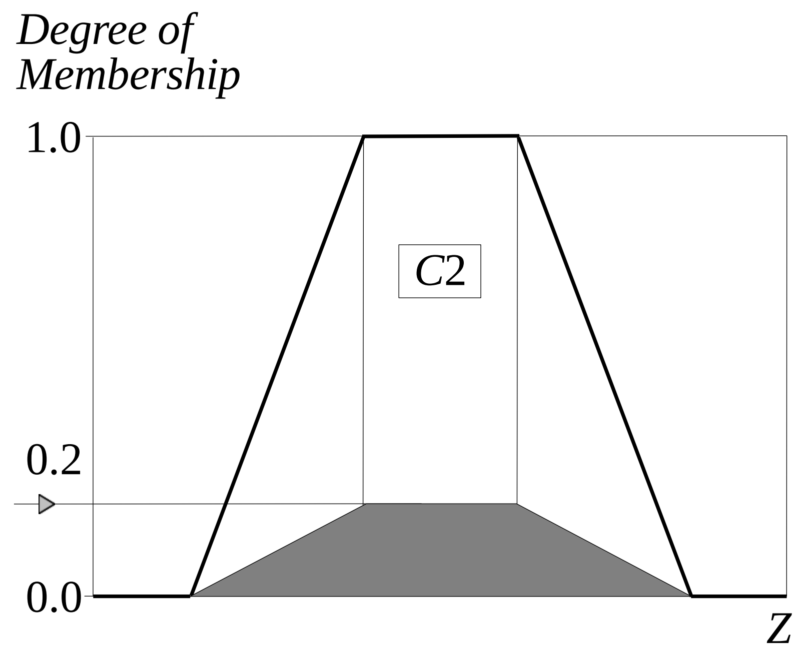
Step 2: Rule Evaluation

- While clipping is a frequently used method, scaling offers a better approach for preserving the original shape of the fuzzy set.
- The original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent.
- This method, which generally loses less information, can be very useful in fuzzy expert systems.

Step 2: Rule Evaluation



clipping

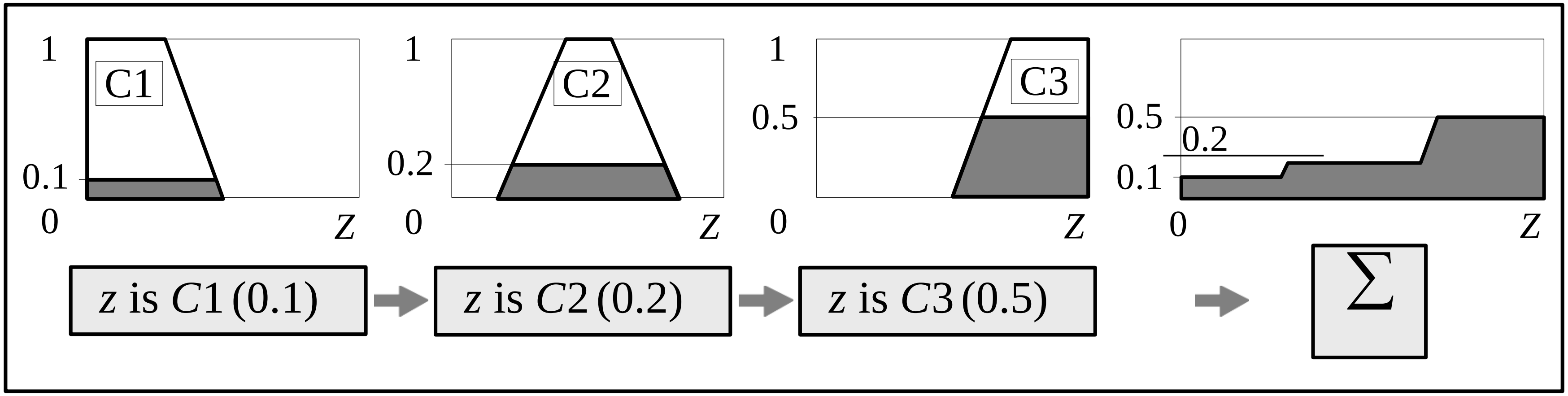


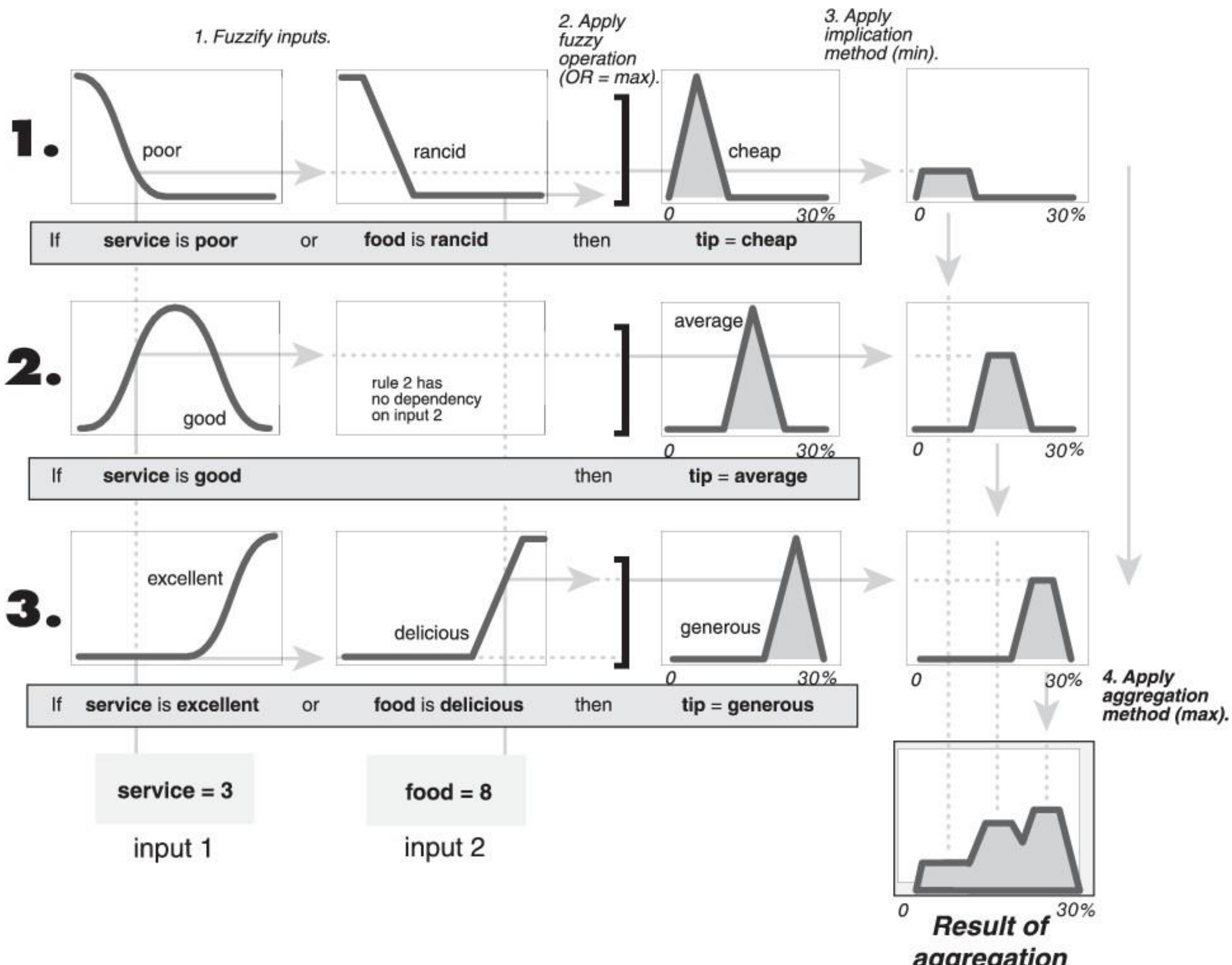
scaling

Step 3: Aggregation of the rule outputs

- Aggregation is the process of unification of the outputs of all rules.
- We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.
- The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.

Step 3: Aggregation of the rule outputs




1. Fuzzify inputs.
2. Apply fuzzy operation (OR = max).
3. Apply implication method (min).
1.
poor
rancid
cheap
0
30%
0
30%
If service is poor or food is rancid then tip = cheap
2.
good
rule 2 has no dependency on input 2
average
0
30%
0
30%
If service is good then tip = average
3.
excellent
delicious
generous
0
30%
0
30%
4. Apply aggregation method (max).
If service is excellent or food is delicious then tip = generous
service = 3
input 1
food = 8
input 2
0
30%
Result of aggregation

Step 4: Defuzzification

- The last step in the fuzzy inference process is defuzzification.
- Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.
- The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.

Step 4: Defuzzification

- There are several defuzzification methods, but probably the most popular one is the centroid technique. It finds the point where a vertical line would slice the aggregate set into two equal masses. Mathematically this centre of gravity (COG) can be expressed as:

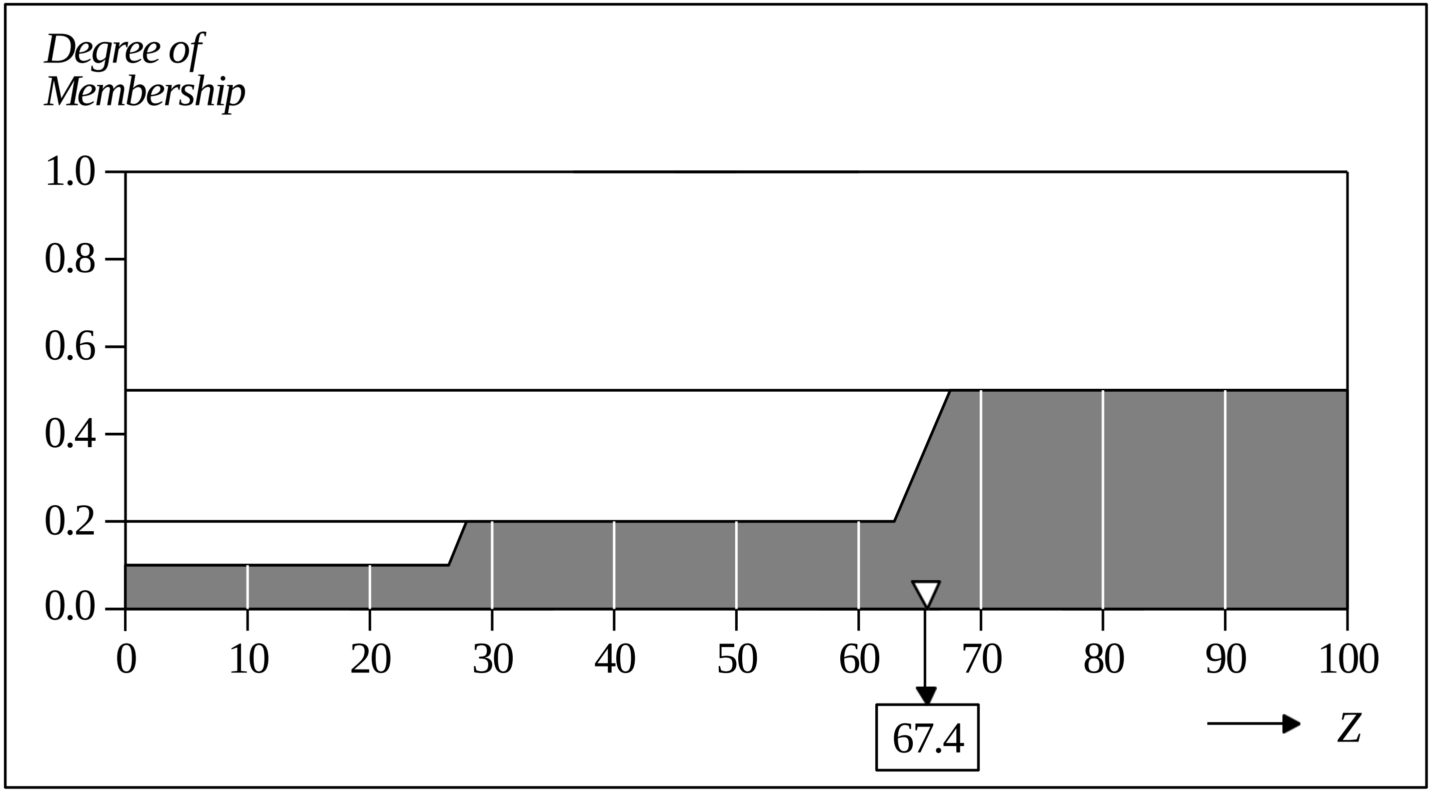
$$COG = \frac{\int_a^b \mu_A(x)\, x\, dx}{\int_a^b \mu_A(x)\, dx}$$

Step 4: Defuzzification

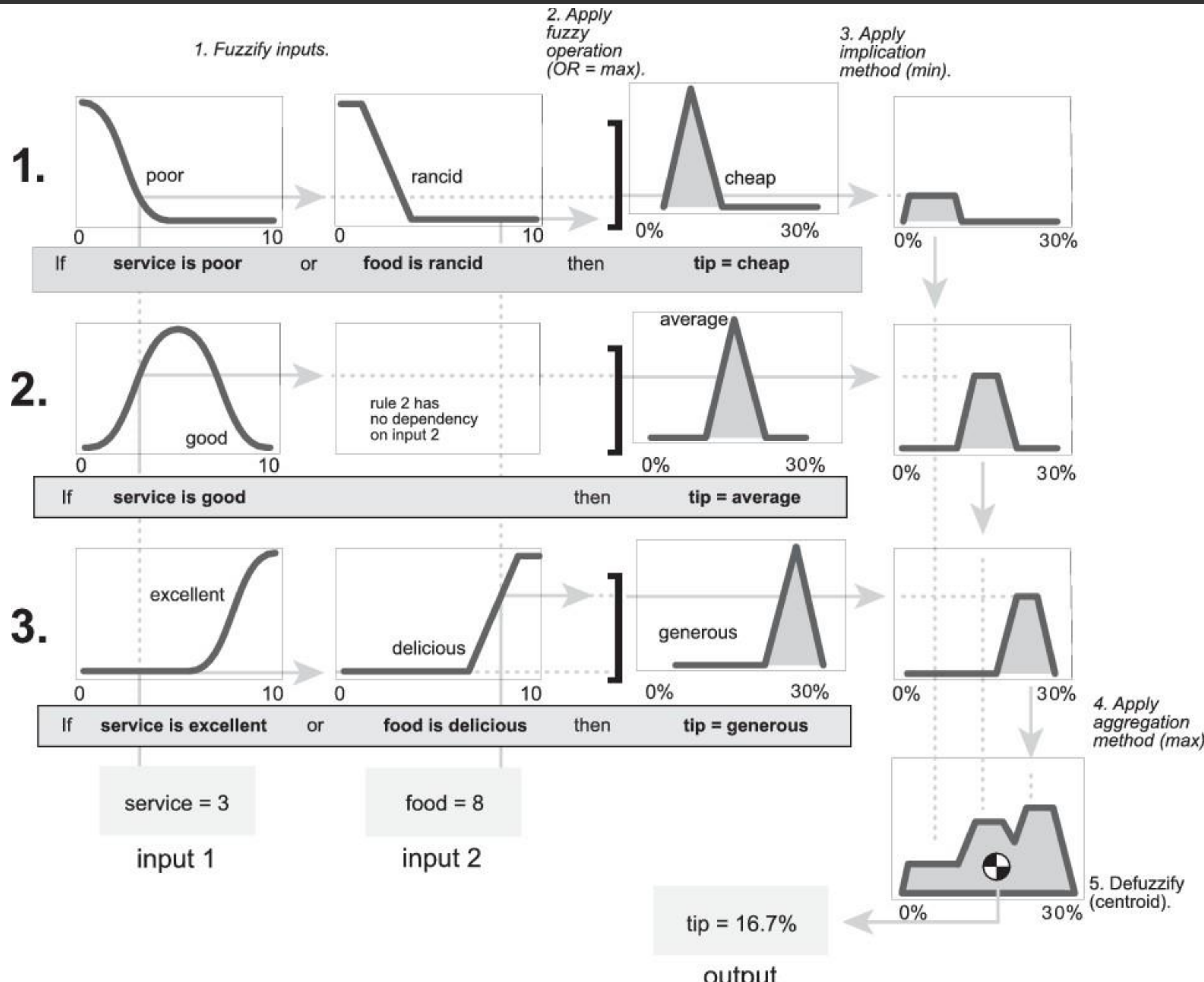
- Centroid defuzzification method finds a point representing the centre of gravity of the fuzzy set, *A*, on the interval, *ab*.
- A reasonable estimate can be obtained by calculating it over a sample of points.

$$COG = \frac{\sum_{x=a}^{b} \mu_A(x) \cdot x}{\sum_{x=a}^{b} \mu_A(x)}$$

Step 4: Defuzzification



$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$


1. Fuzzify inputs.
2. Apply fuzzy operation (OR = max).
3. Apply implication method (min).
1.
poor
0
10
rancid
0
10
cheap
0%
30%
0%
30%
If service is poor or food is rancid then tip = cheap
2.
good
0
10
rule 2 has no dependency on input 2
average
0%
30%
0%
30%
If service is good then tip = average
3.
excellent
0
10
delicious
0
10
generous
0%
30%
0%
30%
If service is excellent or food is delicious then tip = generous
4. Apply aggregation method (max).
service = 3
input 1
food = 8
input 2
5. Defuzzify (centroid).
0%
30%
tip = 16.7%
output

Sugeno Fuzzy Inference

- Mamdani-style inference requires the centroid of a two-dimensional shape by integrating across a continuously varying function.
  - In general, this process is not computationally efficient.
- Michio Sugeno suggested to use a single spike, a singleton, as the membership function of the rule consequent.
  - A singleton, or more precisely a fuzzy singleton, is a fuzzy set with a membership function that is unity at a single particular point on the universe of discourse and zero everywhere else.

Sugeno Fuzzy Inference

- Sugeno-style fuzzy inference is very similar to the Mamdani method. Sugeno changed only a rule consequent. Instead of a fuzzy set, he used a mathematical function of the input variable. The format of the Sugeno-style fuzzy rule is

  IF x is A

  AND y is B

  THEN z is $f(x, y)$

  where x, y and z are linguistic variables; A and B are fuzzy sets on universe of discourses X and Y, respectively; and $f(x, y)$ is a mathematical function.

Sugeno Fuzzy Inference

- The most commonly used zero-order Sugeno fuzzy model applies fuzzy rules in the following form:

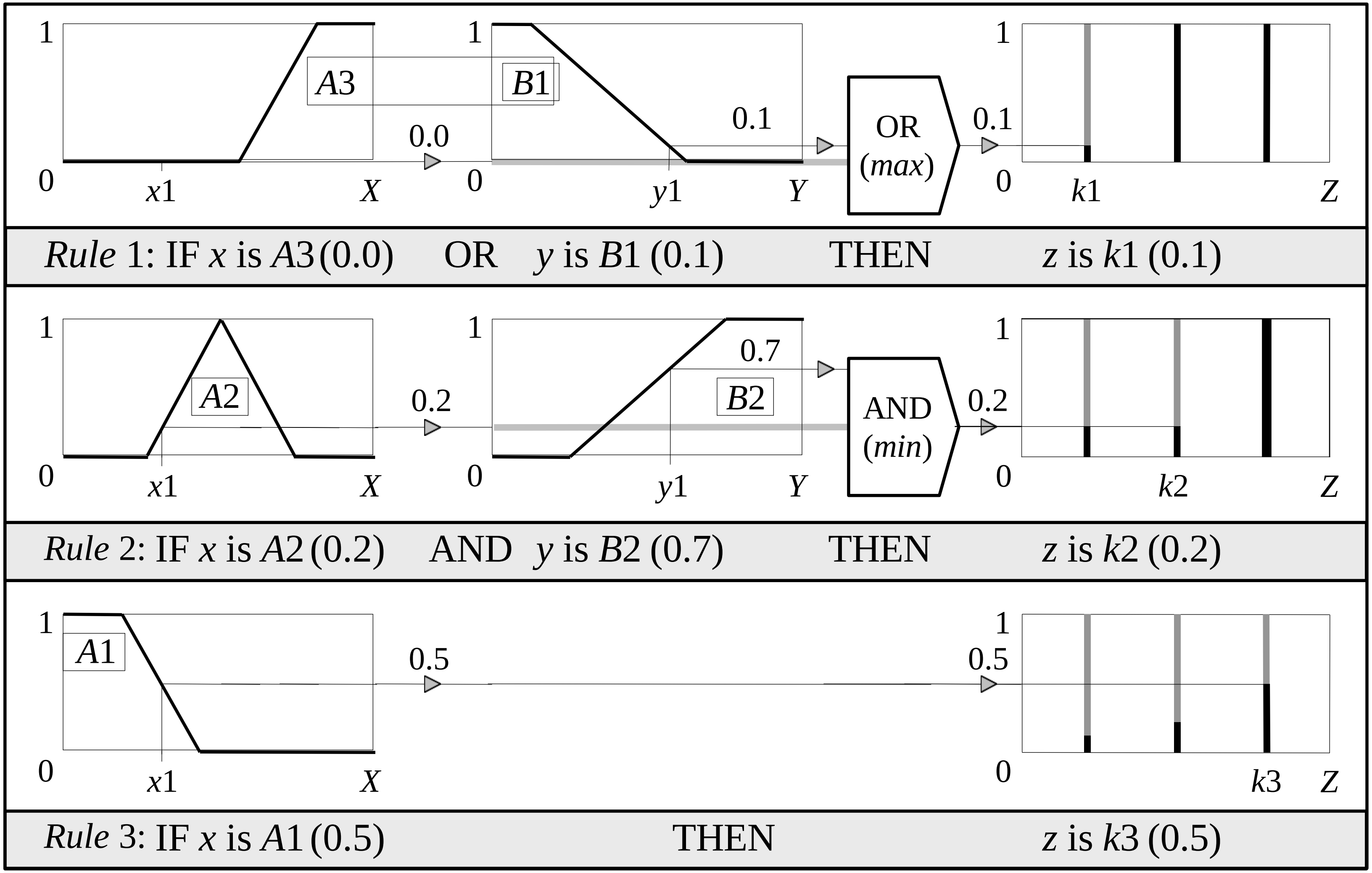
  IF x is A

  AND y is B

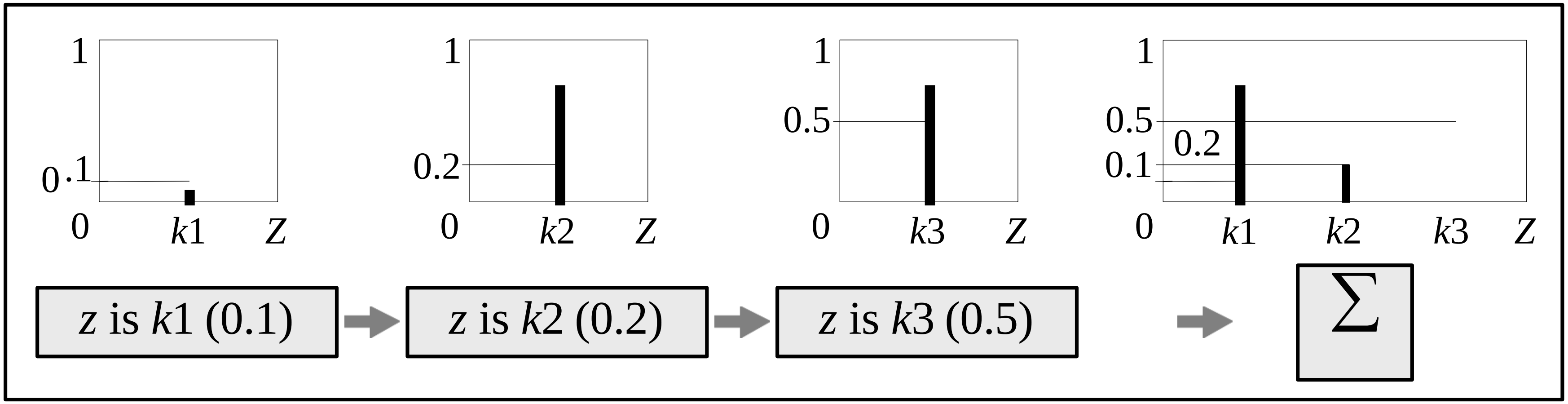
  THEN z is k

  where k is a constant.
- In this case, the output of each fuzzy rule is constant. All consequent membership functions are represented by singleton spikes.

Sugeno Rule Evaluation



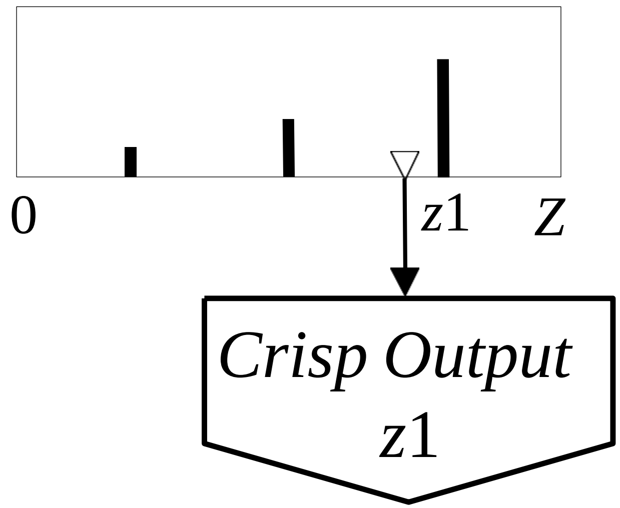
Sugeno Aggregation of the Rule Outputs



Sugeno Defuzzification

Weighted Average (WA)

$$WA = \frac{\mu(k1)\times k1 + \mu(k2)\times k2 + \mu(k3)\times k3}{\mu(k1)+\mu(k2)+\mu(k3)} = \frac{0.1\times 20 + 0.2\times 50 + 0.5\times 80}{0.1+0.2+0.5} = 65$$



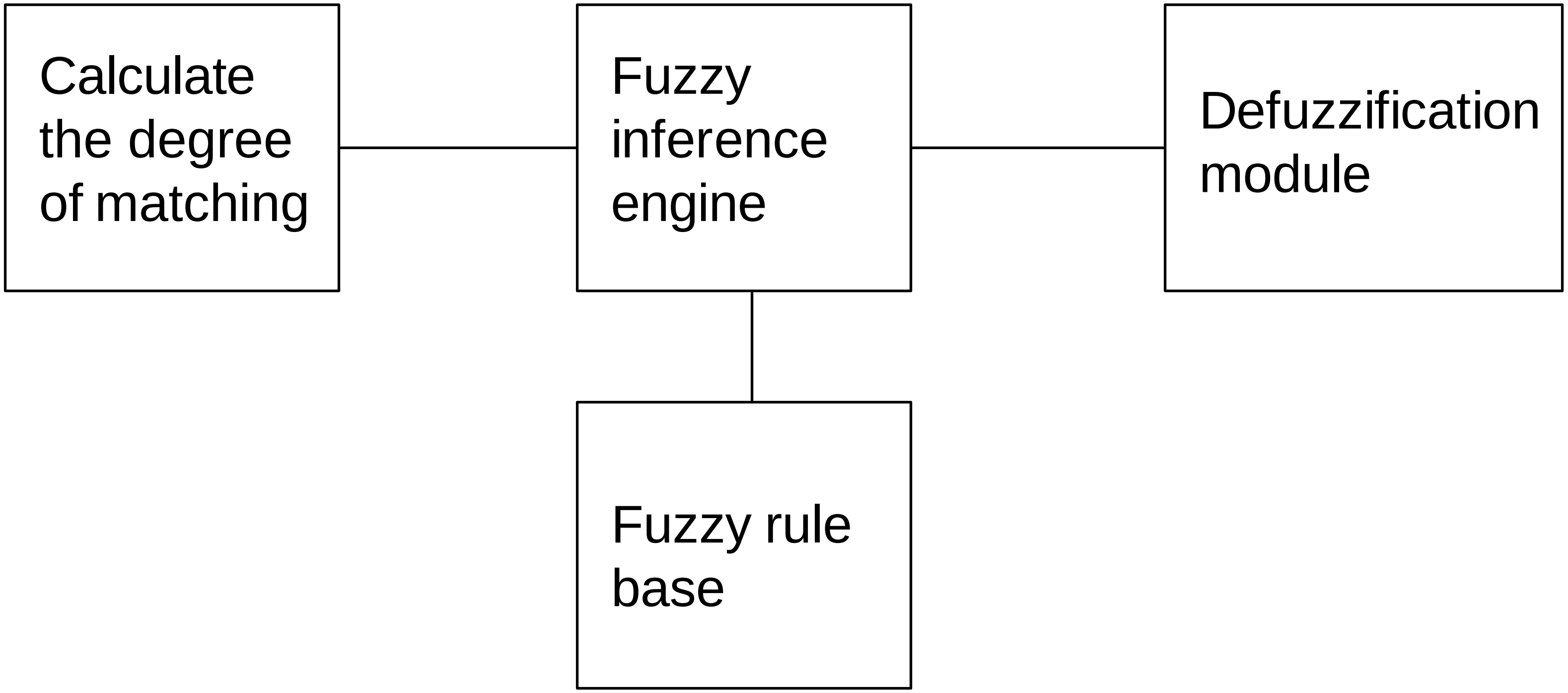
Mamdani or Sugeno?

- The Mamdani method is widely accepted for capturing expert knowledge. It allows us to describe expertise in a more intuitive, more human-like manner.
  - However, entails a substantial computational burden
- The Sugeno method is computationally effective and works well with optimisation and adaptive techniques, which makes it very attractive in control problems, particularly for dynamic nonlinear systems.

Developing a fuzzy expert system:

- The Mamdani method is widely accepted for capturing expert knowledge. It allows us to describe expertise in a more intuitive, more human-like manner.
  - However, entails a substantial computational burden
- The Sugeno method is computationally effective and works well with optimisation and adaptive techniques, which makes it very attractive in control problems, particularly for dynamic nonlinear systems.

General scheme of a fuzzy system



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Operation of a fuzzy expert system:

- Fuzzification: definition of fuzzy sets, and determination of the degree of membership of crisp inputs in appropriate fuzzy sets.
- Inference: evaluation of fuzzy rules to produce an output for each rule.
- Composition: aggregation or combination of the outputs of all rules.
- Defuzzification: computation of crisp output

Example: Air Conditioner

1a. Specify the problem

Air-conditioning involves the delivery of air, which can be warmed or cooled and have its humidity raised or lowered.

An air-conditioner is an apparatus for controlling, especially lowering, the temperature and humidity of an enclosed space. An air-conditioner typically has a fan which blows/cools/circulates fresh air and has a cooler. The cooler is controlled by a thermostat. Generally, the amount of air being compressed is proportional to the ambient temperature.

1b. Define linguistic variables

- Ambient *Temperature*
- Air-conditioner Fan *Speed*

Example: Air Conditioner

2. Determine Fuzzy Sets: Temperature

Temp (⁰C).	COLD	COOL	PLEASANT	WARM	HOT
0	Y*	N	N	N	N
5	Y	Y	N	N	N
10	N	Y	N	N	N
12.5	N	Y*	N	N	N
15	N	Y	N	N	N
17.5	N	N	Y*	N	N
20	N	N	N	Y	N
22.5	N	N	N	Y*	N
25	N	N	N	Y	N
27.5	N	N	N	N	Y
30	N	N	N	N	Y*

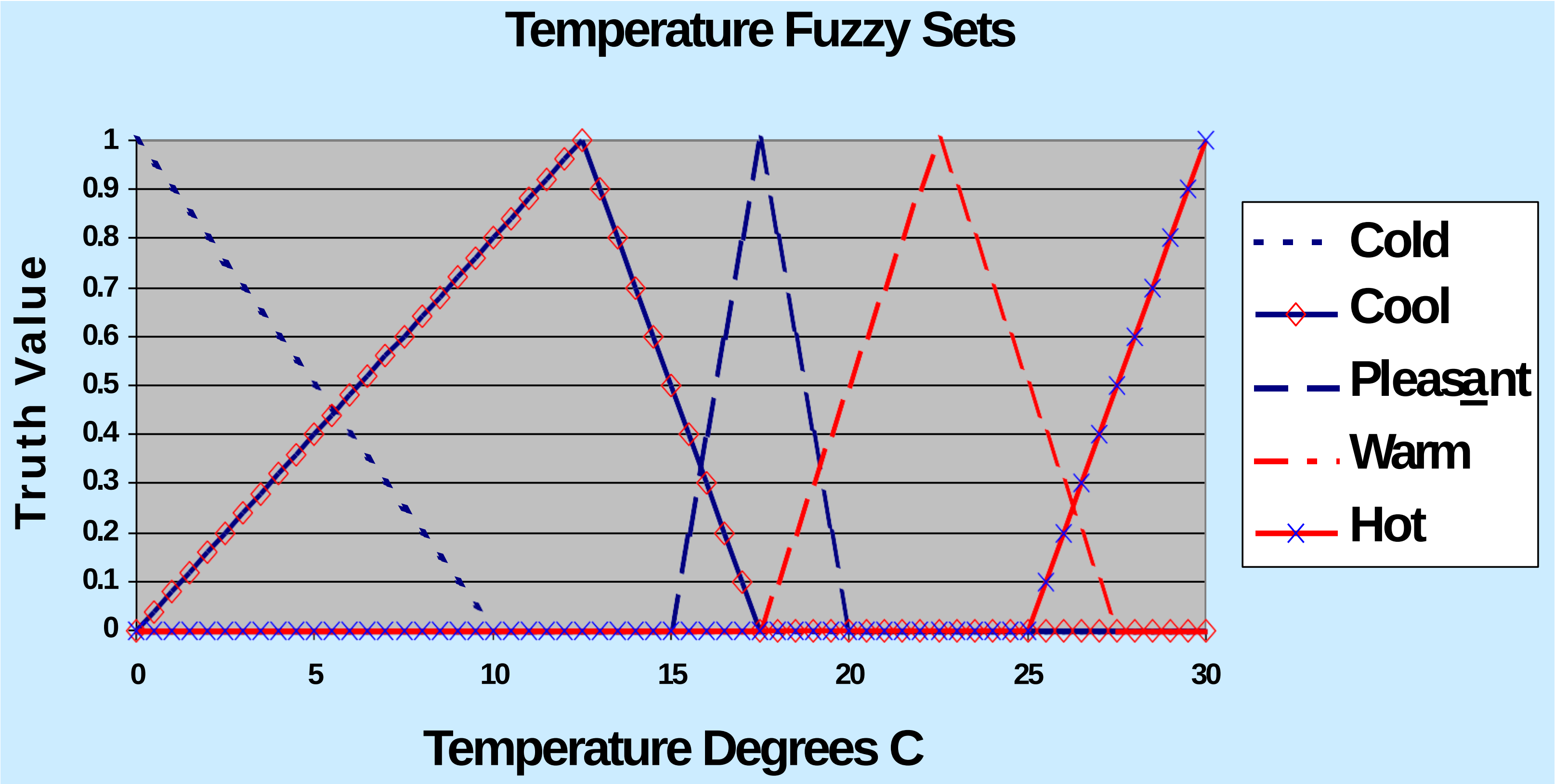
$0<\mu(T)<1$

$\mu(T)=1$

$\mu(T)=0$

Example: Air Conditioner

2. Determine Fuzzy Sets: Temperature



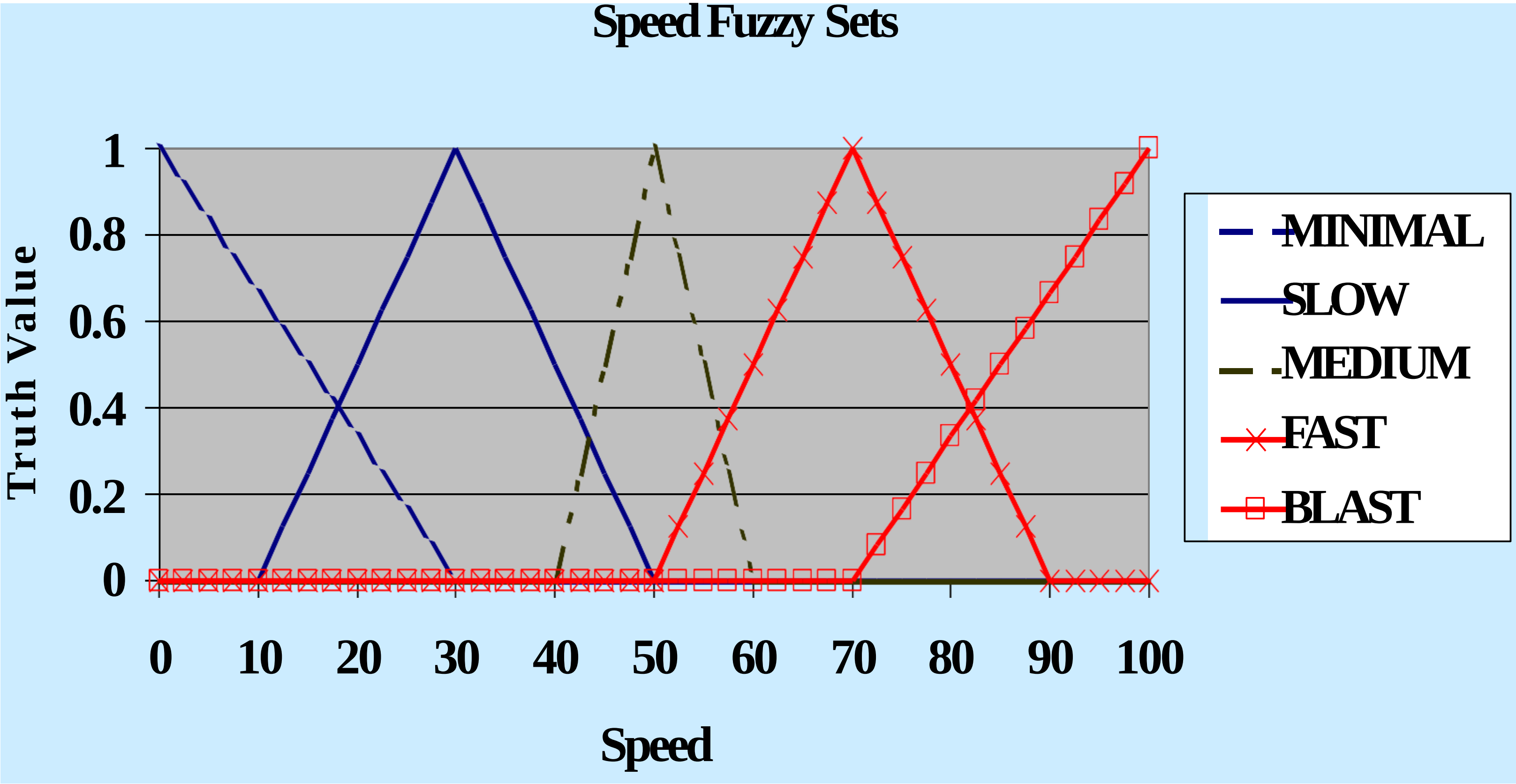
Example: Air Conditioner

2. Determine Fuzzy Sets: Fan Speed

RPM	MINIMAL	SLOW	MEDIUM	FAST	BLAST
0	Y*	N	N	N	N
10	Y	N	N	N	N
20	Y	Y	N	N	N
30	N	Y*	N	N	N
40	N	Y	N	N	N
50	N	N	Y*	N	N
60	N	N	N	Y	N
70	N	N	N	Y*	N
80	N	N	N	Y	Y
90	N	N	N	N	Y
100	N	N	N	N	Y*

Example: Air Conditioner

2. Determine Fuzzy Sets: Fan Speed



Example: Air Conditioner

3. Elicit and construct fuzzy rules

RULE 1:	IF	*temp*	is	*cold*	THEN	*speed*	is	*minimal*
RULE 2:	IF	*temp*	is	*cool*	THEN	*speed*	is	*slow*
RULE 3:	IF	*temp*	is	*pleasant*	THEN	*speed*	is	*medium*
RULE 4:	IF	*temp*	is	*warm*	THEN	*speed*	is	*fast*
RULE 5:	IF	*temp*	is	*hot*	THEN	*speed*	is	*blast*

Example: Air Conditioner

3. Encode into an Expert System

4. Evaluate and tune the system

Consider a temperature of 16^{o}C, use the system to compute the optimal fan speed.

⇒ Operation of a Fuzzy Expert System

- Fuzzification
- Inference
- Composition
- Defuzzification

Example: Air Conditioner

- Fuzzification

  → Affected fuzzy sets: COOL and PLEASANT

$$_{COOL}(T) = -T/\ 5 + 3.5$$
$$= -16/\ 5 + 3.5$$
$$= 0.3$$

$$_{PLSNT}(T) = T/2.5 - 6$$
$$= 16/2.5 - 6$$
$$= 0.4$$

Temp=16	μCOLD	μCOOL	μPLEASANT	μWARM	μHOT
	0	0.3	0.4	0	0

Example: Air Conditioner

- Inference

RULE 1: IF *temp* is *cold* THEN *speed* is *minimal*

RULE 2: IF *temp* is *cool* THEN *speed* is *slow*

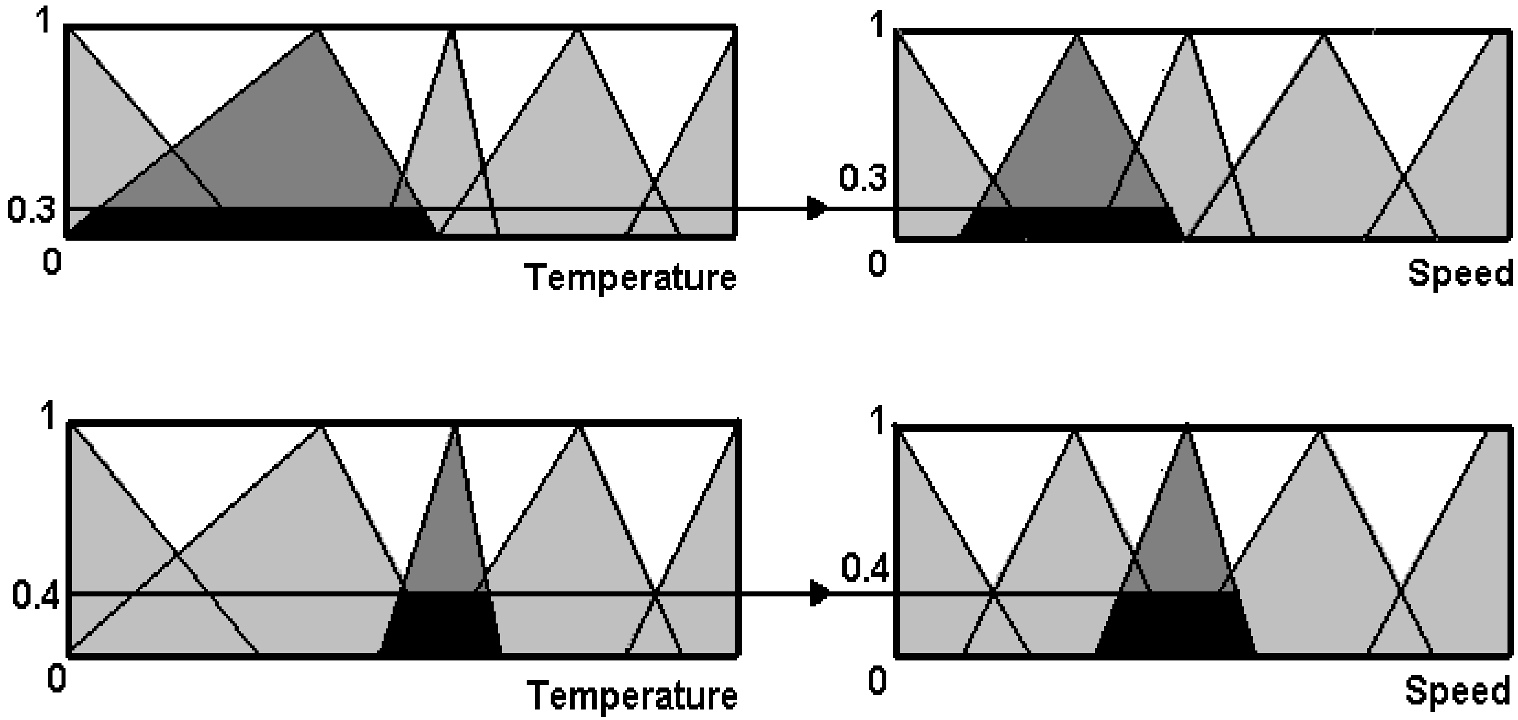
RULE 3: IF *temp* is *pleasant* THEN *speed* is *medium*

RULE 4: IF *temp* is *warm* THEN *speed* is *fast*

RULE 5: IF *temp* is *hot* THEN *speed* is *blast*

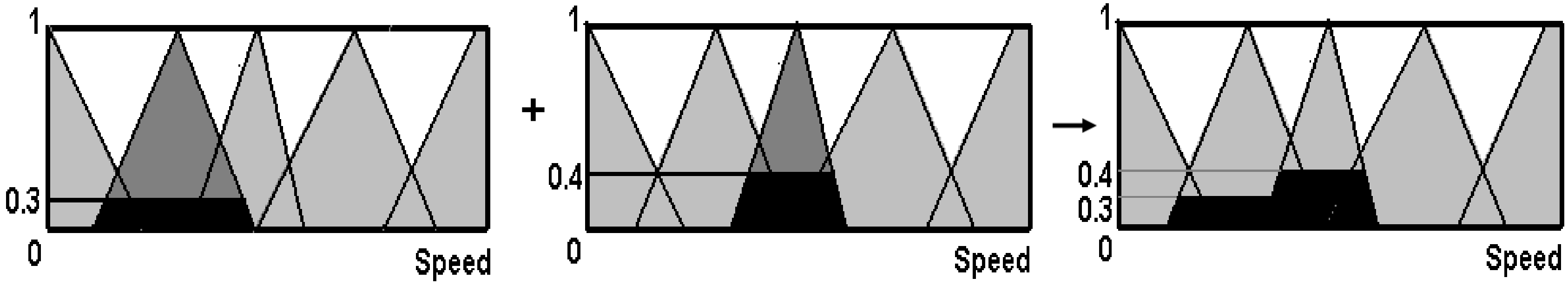
Example: Air Conditioner

- Inference



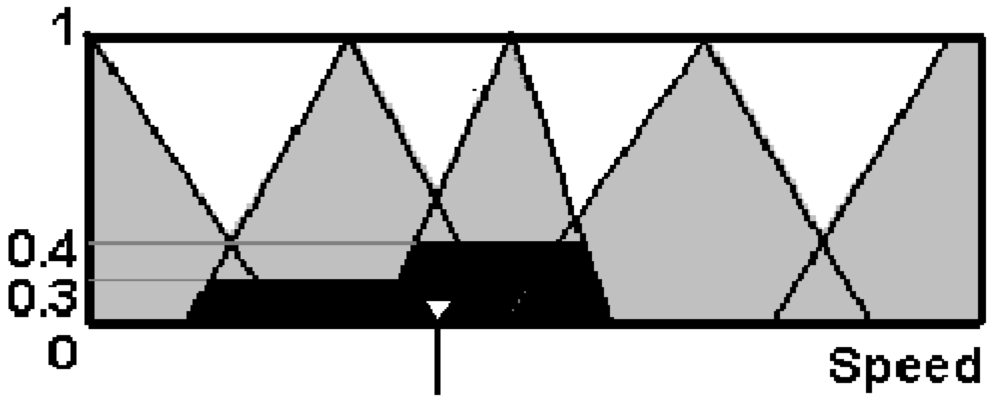
Example: Air Conditioner

- Aggregation



Example: Air Conditioner

- Defuzzification



37rpm (using centroid defuzzification)

Other defuzzification methods:

- COG
- bisector of area
- mean value of maximum
- smallest (absolute) value of maximum
- largest (absolute) value of maximum