Mathematical Proof of Feedback Decoupling and Singularity Classification in Amplification Dynamics

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Abstract

We provide a complete mathematical characterization of dynamical systems with observable amplification ratio $R_A(t)$ and time-varying feedback coupling $L(t)$. Under explicit regularity conditions, we prove that finite-time singularities cannot occur when external resources grow sublinearly, classify long-term asymptotic behavior based on the amplification ratio, and derive sharp sufficient conditions for global boundedness. We distinguish coordinate singularities—where efficiency $E(t) = 1/L(t)$ diverges as $L(t) \to 0$—from state-space singularities, where the system state $S(t)$ itself blows up. The former represent *asymptotic feedback decoupling*: points where intrinsic time halts, the system becomes purely resource-driven, and internal amplification ceases to contribute. These results formalize how bounded resources and vanishing coupling yield measurable regimes of divergent efficiency without physical blow-up, unifying amplification, stability, and singularity within a single observable framework.

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1 Introduction and Mathematical Framework

1.1 The System

We study feedback-coupled dynamical systems governed by:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S) \tag{1}$$

where:

- $S(t) \geq 0$: system state (capacity, capability, resources) with dimension $[S]$
- $t \geq 0$: time with dimension $[T]$
- $L(t) \geq 0$: time-varying feedback coupling strength with dimension $[T]^{-1}$
- $R_A(t) \geq 0$: observable dimensionless amplification ratio (empirically measurable)
- $\Sigma(S) \geq 0$: external resource input with dimension $[S]/[T]$

Dimensional consistency: All terms have dimension $[S]/[T]$.

1.2 Drift Parameter

Define:

$$\omega(t) := L(t)(R_A(t) - 1) \tag{2}$$

This determines the instantaneous feedback tendency:

- $\omega(t) > 0$: amplification (when $R_A > 1$)
- $\omega(t) < 0$: damping (when $R_A < 1$)
- $\omega(t) = 0$: neutral equilibrium (when $R_A = 1$)

1.3 Standing Assumptions

Assumption 1 (Regularity of External Resources)**.** $\Sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *is locally Lipschitz continuous.*

Assumption 2 (Non-negativity)**.** $\Sigma(s) \geq 0$ *for all* $s \geq 0$ *and* $\Sigma(0) \geq 0$*.*

Assumption 3 (Growth Control)**.** Σ *satisfies one of:*

(a) ***Sublinear:*** $\limsup_{s\to\infty} \Sigma(s)/s < \infty$

(b) ***Bounded:*** $\sup_{s\geq 0} \Sigma(s) =: M < \infty$

(c) ***Affine:*** $\Sigma(s) \leq a + bs$ *for some* $a, b \geq 0$

Assumption 4 (Observable Amplification Ratio). *$R_A : [0, \infty) \to \mathbb{R}_{\geq 0}$ is measurable and piecewise continuous.*

Assumption 5 (Local Boundedness of Amplification). *For each $T < \infty$:*

$$R_{\max}(T) := \sup_{t \in [0,T]} R_A(t) < \infty$$

Assumption 6 (Time-Varying Feedback Coupling). *$L : [0, \infty) \to \mathbb{R}_{\geq 0}$ is measurable and locally integrable:*

- *$\int_0^T L(s)\, ds < \infty$ for each $T < \infty$*
- *$L(t) \geq 0$ for all $t \geq 0$*

We distinguish:

- ***Non-degenerate:*** *$\int_0^\infty L(u)\, du = \infty$ (unbounded cumulative coupling)*
- ***Degenerate:*** *$\int_0^\infty L(u)\, du =: \varepsilon_\infty < \infty$ (finite total coupling)*

Assumption 7 (Initial Condition). *$S(0) = S_0 \geq 0$.*

1.4 Physical Interpretation

This model captures systems where:

- Current state S generates feedback at rate $L(t)(R_A(t) - 1)S$
- External resources $\Sigma(S)$ may depend on system state
- Feedback coupling strength $L(t)$ can vary, decay, or vanish over time
- Amplification ratio $R_A(t)$ is empirically observable at each time

Examples:

- **Economic growth:** S = capital, R_A = return on investment, L = reinvestment rate, Σ = external investment
- **Technological progress:** S = capability, R_A = efficiency of self-improvement, L = resource allocation to R&D
- **AI development:** S = intelligence/capability, R_A = recursive self-improvement factor, L = training efficiency

Perfect efficiency: When $L(t) \to 0$ under finite S, the efficiency parameter $E(t) = 1/L(t) \to \infty$, marking a feedback decoupling regime where external resources dominate.

2 Foundational Lemmas

Lemma 2.1 (Variation of Constants Formula)**.** *For the ODE* $\frac{dS}{dt} = \omega(t)S + \Sigma(S(t))$*, the solution satisfies:*

$$S(t) = e^{\Phi(t)} \left[S_0 + \int_0^t e^{-\Phi(\tau)} \Sigma(S(\tau))\, d\tau \right]$$

where $\Phi(t) := \int_0^t \omega(s)\, ds = \int_0^t L(s)(R_A(s) - 1)\, ds$*.*

Proof. Define $\mu(t) := e^{-\Phi(t)}$. Then:

$$\frac{d\mu}{dt} = -\omega(t)e^{-\Phi(t)} = -\omega(t)\mu(t)$$

Multiply the ODE by $\mu(t)$:

$$\mu(t)\frac{dS}{dt} = \mu(t)\omega(t)S + \mu(t)\Sigma(S)$$

$$\frac{d}{dt}[\mu(t)S(t)] = \mu(t)\Sigma(S(t))$$

Integrate from 0 to t:

$$\mu(t)S(t) - \mu(0)S_0 = \int_0^t \mu(\tau)\Sigma(S(\tau))\, d\tau$$

Since $\mu(0) = 1$:

$$e^{-\Phi(t)}S(t) = S_0 + \int_0^t e^{-\Phi(\tau)}\Sigma(S(\tau))\, d\tau$$

Multiply both sides by $e^{\Phi(t)}$. □

Lemma 2.2 (Comparison Principle)**.** *Suppose* $f(s,t)$ *is continuous in* s *and measurable in* t*. If:*

1. $\frac{dS}{dt} \leq f(S,t)$ *almost everywhere*
2. $\bar{S}(t)$ *satisfies* $\frac{d\bar{S}}{dt} = f(\bar{S},t)$ *with* $S(0) = \bar{S}(0)$

Then $S(t) \leq \bar{S}(t)$ *for all* $t \geq 0$ *in the domains of both solutions.*

Proof. Standard comparison theorem for ODEs (see Khalil, *Nonlinear Systems*, Theorem 3.4). The conditions on measurability and local Lipschitz continuity are satisfied by $F(S,t)$ under Assumptions 1–6. □

Lemma 2.3 (Blow-Up Alternative)**.** *Let* $S(t)$ *be the maximal solution on* $[0, T_{\max})$*. Then either:*

1. $T_{\max} = \infty$ *(global solution), or*
2. $\lim_{t \to T_{\max}^-} S(t) = \infty$ *(finite-time blow-up)*

Proof. Suppose $T_{\max} < \infty$ and $\limsup_{t \to T_{\max}^-} S(t) =: M < \infty$.

Then $S(t)$ remains in the compact set $[0, M+1]$ for all $t \in [0, T_{\max})$.

Since $F(S,t) = L(t)(R_A(t) - 1)S + \Sigma(S)$ is locally Lipschitz in S and locally integrable in t, the Carathéodory theorem guarantees that the solution can be extended beyond $T_{\max}$, contradicting maximality.

Therefore, if $T_{\max} < \infty$, we must have $\lim_{t \to T_{\max}^-} S(t) = \infty$. □

Lemma 2.4 (Positivity Invariance — Nagumo Condition). *Under Assumptions 1–7, if $S_0 \geq 0$, then $S(t) \geq 0$ for all $t \in [0, T_{\max})$.*

Proof. At the boundary $S = 0$:

$$F(0,t) = L(t)(R_A(t) - 1) \cdot 0 + \Sigma(0) = \Sigma(0) \geq 0$$

The vector field points into (or tangent to) the non-negative orthant $\mathbb{R}_{\geq 0}$.

By the Nagumo tangency condition, the set $\mathbb{R}_{\geq 0}$ is positively invariant. Therefore, if $S_0 \geq 0$, then $S(t) \geq 0$ for all $t \in [0, T_{\max})$. □

3 Well-Posedness Theory

Theorem 3.1 (Existence and Uniqueness). *Under Assumptions 1–7, for any initial condition $S_0 \geq 0$, there exists a unique maximal solution $S : [0, T_{\max}) \to \mathbb{R}_{\geq 0}$ satisfying:*

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S), \quad S(0) = S_0$$

Moreover, by Lemma 2.3, either $T_{\max} = \infty$ or $\lim_{t \to T_{\max}^-} S(t) = \infty$.

Proof. Define $F(S,t) := L(t)(R_A(t) - 1)S + \Sigma(S)$.

Step 1: Carathéodory conditions

Measurability in t: For fixed S, the map $t \mapsto F(S,t) = L(t)(R_A(t) - 1)S + \Sigma(S)$ is measurable since $L(t)$ and $R_A(t)$ are measurable (Assumptions 4, 6).

Local Lipschitz in S: Fix $T < \infty$ and compact $K \subset \mathbb{R}_{\geq 0}$.

Let $M_T := \sup_{t \in [0,T]} |R_A(t)| < \infty$ (by Assumption 5) and $L_T := \sup_{t \in [0,T]} L(t) < \infty$ (by local boundedness).

For $S_1, S_2 \in K$:

$$\begin{aligned}
|F(S_1,t) - F(S_2,t)| &= |L(t)(R_A(t) - 1)(S_1 - S_2) + \Sigma(S_1) - \Sigma(S_2)| \\
&\leq L(t)|R_A(t) - 1||S_1 - S_2| + |\Sigma(S_1) - \Sigma(S_2)| \\
&\leq L_T(M_T + 1)|S_1 - S_2| + L_K|S_1 - S_2| \\
&= [L_T(M_T + 1) + L_K]|S_1 - S_2|
\end{aligned}$$

Local integrability: For $S \in K$ and $t \in [0,T]$:

$$\int_0^T |F(S,t)|\, dt \leq \int_0^T [L(t)|R_A(t) - 1|\|K\|_\infty + M_\Sigma]\, dt$$

where $M_\Sigma := \sup_{s \in K} \Sigma(s) < \infty$ (by continuity of Σ on compact sets).

Since L is locally integrable (Assumption 6) and R_A is locally bounded (Assumption 5):

$$\leq (M_T + 1)\|K\|_\infty \int_0^T L(t)\, dt + T M_\Sigma < \infty$$

Step 2: Apply Carathéodory Existence Theorem

Since F satisfies all Carathéodory conditions, there exists a unique maximal solution $S(t)$ on $[0, T_{\max})$.

Step 3: Positivity preservation

By Lemma 2.4, $S(t) \geq 0$ for all $t \in [0, T_{\max})$. □

4 Intrinsic Time Reparametrization

Lemma 4.1 (Time-Change to Intrinsic Coupling Scale)**.** *Define the cumulative coupling function:*

$$\varepsilon(t) := \int_0^t L(u)\, du$$

Assumption: *$L(t) > 0$ almost everywhere, so $\varepsilon(t)$ is strictly increasing and absolutely continuous.*

Case I (Non-degenerate: $\varepsilon(t) \to \infty$ as $t \to \infty$):

1. *$\varepsilon : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, and surjective*

2. *There exists an inverse $t : [0, \infty) \to [0, \infty)$ with $\varepsilon(t(\varepsilon)) = \varepsilon$*

3. *Define the* efficiency parameter*:*

$$E(\varepsilon) := \frac{1}{L(t(\varepsilon))} \quad \textit{(well-defined a.e.)}$$

4. *For $\tilde{S}(\varepsilon) := S(t(\varepsilon))$, the ODE transforms to:*

$$\frac{d\tilde{S}}{d\varepsilon} = (R_A(\varepsilon) - 1)\tilde{S} + E(\varepsilon)\Sigma(\tilde{S}) \tag{3}$$

where $R_A(\varepsilon) := R_A(t(\varepsilon))$

Case II (Degenerate: $\varepsilon_\infty := \int_0^\infty L(u)\, du < \infty$):

1. *$\varepsilon : [0, \infty) \to [0, \varepsilon_\infty)$ is continuous and surjective onto $[0, \varepsilon_\infty)$*

2. *The inverse $t : [0, \varepsilon_\infty) \to [0, \infty)$ exists with $t(\varepsilon) \to \infty$ as $\varepsilon \to \varepsilon_\infty^-$*

3. *The transformed ODE exists on $\varepsilon \in [0, \varepsilon_\infty)$:*

$$\frac{d\tilde{S}}{d\varepsilon} = (R_A(\varepsilon) - 1)\tilde{S} + E(\varepsilon)\Sigma(\tilde{S})$$

4. *As $t \to \infty$, we have $\varepsilon(t) \to \varepsilon_\infty < \infty$ (intrinsic time freezes)*

Proof. **Step 1: Properties of $\varepsilon(t)$**

Since $L \geq 0$ is locally integrable and positive a.e., $\varepsilon(t)$ is well-defined, continuous, strictly increasing, and differentiable a.e. with $\varepsilon'(t) = L(t) > 0$ a.e.

Step 2: Existence of inverse

In the non-degenerate case, $\varepsilon(t) \to \infty$ implies ε is surjective. Since ε is continuous and strictly increasing, it has a continuous inverse $t(\varepsilon)$.

In the degenerate case, $\varepsilon(t) \to \varepsilon_\infty < \infty$ as $t \to \infty$, so ε is surjective onto $[0, \varepsilon_\infty)$.

Step 3: Chain rule transformation (valid a.e. for absolutely continuous functions)

By the chain rule:

$$\frac{d\tilde{S}}{d\varepsilon} = \left.\frac{dS}{dt}\right|_{t=t(\varepsilon)} \cdot \frac{dt}{d\varepsilon}$$

From the original ODE:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S)$$

Since $\frac{d\varepsilon}{dt} = L(t) > 0$ a.e., we have $\frac{dt}{d\varepsilon} = \frac{1}{L(t(\varepsilon))} = E(\varepsilon)$.

Substituting:

$$\begin{aligned}
\frac{d\tilde{S}}{d\varepsilon} &= [L(t(\varepsilon))(R_A(t(\varepsilon)) - 1)S(t(\varepsilon)) + \Sigma(S(t(\varepsilon)))] \cdot E(\varepsilon) \\
&= [L(t(\varepsilon))(R_A(\varepsilon) - 1)\tilde{S} + \Sigma(\tilde{S})] \cdot \frac{1}{L(t(\varepsilon))} \\
&= (R_A(\varepsilon) - 1)\tilde{S} + E(\varepsilon)\Sigma(\tilde{S})
\end{aligned}$$

Note on measurability: When $L = 0$ on sets of positive measure, we interpret the chain rule in the sense of Lebesgue integration and absolutely continuous functions. The key is that $\varepsilon(t)$ remains absolutely continuous, allowing the transformation to be well-defined almost everywhere. □

Remark 4.2 (Physical Interpretation).

- ***Intrinsic time*** ε***:*** *Measures cumulative feedback coupling, not calendar time*
- ***Efficiency*** $E(\varepsilon)$***:*** *Ratio of external resource impact to feedback strength*
- ***When*** $L(t) \to 0$***:*** *$E(\varepsilon) \to \infty$ (coordinate singularity), but S may remain bounded*
- ***Degenerate case:*** *Total feedback coupling is finite; system asymptotically decouples*

5 Main Results: No Finite-Time Blow-Up

Theorem 5.1 (Damped Regime — No Finite-Time Blow-Up)**.** *Assume:*

1. $\sup_{t\geq 0} R_A(t) \leq 1$ *(damped regime)*

2. Σ *satisfies Assumption 3(b):* $\sup_{s\geq 0} \Sigma(s) =: M < \infty$

Then $T_{\max} = \infty$ *and* $S(t) < \infty$ *for all finite* $t < \infty$*.*

Proof. Since $R_A(t) \leq 1$:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S) \leq 0 \cdot S + M = M$$

By comparison with the ODE $\frac{d\bar{S}}{dt} = M$, $\bar{S}(0) = S_0$:

$$S(t) \leq S_0 + Mt < \infty$$

for all finite t. Therefore, $T_{\max} = \infty$ by Lemma 2.3. □

Remark 5.2. *This theorem establishes no finite-time blow-up but does NOT guarantee* $\sup_{t\geq 0} S(t) < \infty$ *(uniform boundedness). For that, see Theorem 5.3.*

Theorem 5.3 (Strict Damping — Global Boundedness)**.** *Assume:*

1. $\inf_{t\geq 0} R_A(t) \leq 1 - \delta$ *for some* $\delta > 0$ *(strict damping)*

2. Σ *satisfies Assumption 3(b):* $\sup_{s\geq 0} \Sigma(s) =: M < \infty$

3. $\inf_{t\in[0,T]} L(t) \geq \ell_T > 0$ *for each* $T < \infty$ *(non-vanishing on finite intervals)*

Then $\sup_{t\geq 0} S(t) < \infty$ *(global boundedness).*

Proof. From the ODE:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S) \leq L(t)(1 - \delta - 1)S + M = -L(t)\delta S + M$$

Since $L(t) \geq \ell_T > 0$ on $[0, T]$:

$$\frac{dS}{dt} \leq -\ell_T \delta S + M$$

This is a linear ODE. By comparison:

$$S(t) \leq \left(S_0 - \frac{M}{\ell_T\delta}\right) e^{-\ell_T\delta t} + \frac{M}{\ell_T\delta}$$

As $t \to \infty$, if $\ell_\infty := \inf_{t\geq 0} L(t) > 0$:

$$S(t) \to \frac{M}{\ell_\infty \delta}$$

For global boundedness when L may vanish, we require dissipativity (see Theorem 7.1). □

Theorem 5.4 (Amplified Regime with Sublinear Resources)**.** *Assume:*

1. $\inf_{t\geq 0} R_A(t) \geq 1+\delta$ *for some* $\delta > 0$ *(amplified regime)*

2. R_A *satisfies Assumption 5:* $R_{\max}(T) := \sup_{t\in[0,T]} R_A(t) < \infty$ *for each* $T < \infty$

3. Σ *satisfies Assumption 3(c):* $\Sigma(s) \leq a + bs$ *with* $b < \delta \inf_{t\geq 0} L(t)$ *if* $\inf L > 0$

Then $T_{\max} = \infty$ *and* $S(t) < \infty$ *for all finite* $t < \infty$.

Important: *This guarantees no finite-time blow-up but permits* $S(t) \to \infty$ *as* $t \to \infty$ *(asymptotic blow-up).*

Proof. For large S, using $\Sigma(s) \leq a + bs$:

$$\frac{dS}{dt} = L(t)(R_A(t)-1)S + \Sigma(S) \leq L(t)(R_{\max}(T)-1)S + a + bS$$

For $t \in [0, T]$:

$$\frac{dS}{dt} \leq [L(t)(R_{\max}(T)-1)+b]S + a$$

Let $\mu_T := \sup_{t\in[0,T]} L(t)(R_{\max}(T)-1) + b < \infty$.

By Grönwall's inequality:

$$S(t) \leq S_0 e^{\mu_T t} + a\int_0^t e^{\mu_T(t-s)}\, ds = S_0 e^{\mu_T t} + \frac{a}{\mu_T}(e^{\mu_T t}-1)$$

For finite $t \leq T$:

$$S(t) \leq \left(S_0 + \frac{a}{\mu_T}\right) e^{\mu_T T} < \infty$$

Therefore, $T_{\max} = \infty$ by Lemma 2.3. □

Theorem 5.5 (Robustness to Time-Varying L)**.** *Assume:*

1. $L(t) \geq 0$ *satisfies Assumption 6 (locally integrable, may vanish)*

2. $R_A(t)$ *satisfies Assumptions 4 and 5 (measurable, locally bounded)*

3. $\Sigma(s) \leq a + bs$ *for some* $a, b \geq 0$ *(affine growth)*

4. *One of:*

   - ***Damped:*** $\sup_{t\geq 0} R_A(t) \leq 1$
   - ***Amplified with sublinear resources:*** $\inf_{t\geq 0} R_A(t) \geq 1+\delta$ *with* $\delta > b$

Then $T_{\max} = \infty$ *and* $S(t) < \infty$ *for all finite* t*, regardless of whether* $L(t)$ *is constant, vanishing, or oscillating.*

Proof. **Case A: Non-degenerate coupling ($\int_0^\infty L = \infty$)**

Step 1: Transform to ε-time

By Lemma 4.1, the system becomes:

$$\frac{d\tilde{S}}{d\varepsilon} = (R_A(\varepsilon) - 1)\tilde{S} + E(\varepsilon)[a + b\tilde{S}]$$

Rearranging:

$$\frac{d\tilde{S}}{d\varepsilon} = [(R_A(\varepsilon) - 1) + bE(\varepsilon)]\tilde{S} + aE(\varepsilon)$$

Step 2: Damped subcase

If $R_A(\varepsilon) \leq 1$:

$$\frac{d\tilde{S}}{d\varepsilon} \leq bE(\varepsilon)\tilde{S} + aE(\varepsilon)$$

By Grönwall's inequality:

$$\tilde{S}(\varepsilon) \leq \tilde{S}(0) \exp\left(\int_0^\varepsilon bE(u)\, du\right) + \int_0^\varepsilon aE(s) \exp\left(\int_s^\varepsilon bE(u)\, du\right) ds$$

Key observation:

$$\int_0^\varepsilon E(u)\, du = \int_0^\varepsilon \frac{d\varepsilon/dv}{1}\bigg|_{v=t(u)} du = \int_0^{t(\varepsilon)} 1\, dv = t(\varepsilon)$$

Therefore:

$$\exp\left(\int_0^\varepsilon bE(u)\, du\right) = e^{bt(\varepsilon)}$$

For any finite t, letting $\varepsilon = \varepsilon(t)$:

$$S(t) = \tilde{S}(\varepsilon(t)) \leq S_0 e^{bt} + \frac{a}{b}(e^{bt} - 1) < \infty$$

Step 3: Amplified subcase

Similar analysis using $R_{\max}(T) < \infty$ for each $T < \infty$.

Case B: Degenerate coupling ($\varepsilon_\infty := \int_0^\infty L < \infty$)

From Case A, $\tilde{S}(\varepsilon)$ remains bounded on $[0, \varepsilon_\infty)$ by the Grönwall bound applied on this finite interval. Specifically, since $\varepsilon_\infty < \infty$, we have:

$$\int_0^{\varepsilon_\infty} E(u)\, du = \lim_{t \to \infty} t(\varepsilon(t)) = \infty$$

but ε itself is bounded. The transformed system evolves on a bounded intrinsic time interval, so boundedness in ε implies boundedness in t. Since $\varepsilon_\infty < \infty$, $S(t) = \tilde{S}(\varepsilon(t))$ remains bounded for all $t \geq 0$. □

Corollary 5.6. *The behavior of $L(t)$—whether constant, decaying, oscillating, or vanishing—does not affect whether finite-time blow-up occurs, only the rate of evolution and whether feedback decoupling arises.*

6 Coordinate Degeneracy and Feedback Decoupling

Remark 6.1 (Classification of Singularities). *We distinguish three types of singularities:*

Singularity Type Comparison

Type 1: State-Space Singularity

- *Definition:* $\lim_{t\to T^-} S(t) = \infty$ *for some* $T < \infty$
- *Interpretation: Finite-time blow-up of the system state itself*
- *Physical meaning: System diverges in finite time*
- *Status: Ruled out by Theorems 5.1–5.5 under sublinear* Σ

Type 2: Coordinate Singularity (Generic)

- *Definition:* $\lim_{t\to t_*} L(t) = 0$, *implying* $E(t) = 1/L(t) \to \infty$
- *Interpretation: Efficiency parameter diverges; artifact of time parametrization*
- *Physical meaning: Change of variables produces apparent singularity*
- *Status:* $S(t)$ *may be bounded or unbounded independently*

Type 3: Feedback Decoupling (Special Case)

- *Definition:* $L(t) \to 0$ *AND* $S(t)$ *remains bounded, with* $\int_0^\infty L < \infty$
- *Interpretation: Intrinsic time freezes at* ε_∞*; feedback ceases*
- *Physical meaning: System transitions to purely resource-driven dynamics*
- *Observable transition: Internal amplification becomes negligible; evolution proceeds only through* $\Sigma(S)$

Definition 6.2 (Feedback-Decoupling Point). *A* feedback-decoupling point *occurs when* $L(t) \to 0$ *and* $S(t)$ *remains finite, so that intrinsic time* $\varepsilon(t) = \int_0^t L(u)\,du$ *converges to a finite limit* $\varepsilon_\infty < \infty$ *and feedback ceases to contribute to evolution.*

In this regime, internal amplification no longer contributes; evolution proceeds only through external forcing $\Sigma(S)$*.*

Remark 6.3 (Physical Interpretation). *If feedback coupling vanishes* ($L \to 0$)*, the system decouples and evolves purely under external resources* $\Sigma(S)$*. If* Σ *is bounded or sublinear,* S *remains bounded even as efficiency* $E \to \infty$*.*

Feedback decoupling is a coordinate singularity with specific physical interpretation: it marks the analytical boundary between internally self-amplifying and externally driven dynamics, not a new singularity class.

Example 6.4 (Feedback Decoupling with Finite State)**.** *Let $L(t) = \frac{1}{1+t^2}$, $R_A = 2$, $\Sigma(S) = 1$ (constant external forcing).*

Then:

$$\varepsilon(t) = \int_0^t \frac{du}{1+u^2} = \arctan(t) \to \frac{\pi}{2} \quad \text{as } t \to \infty$$

So $\varepsilon_\infty = \pi/2 < \infty$ (degenerate case).

The efficiency parameter:

$$E(t) = \frac{1}{L(t)} = 1 + t^2 \to \infty \quad \text{as } t \to \infty$$

This is a coordinate singularity. However, the state $S(t)$ satisfies:

$$\frac{dS}{dt} = \frac{1}{1+t^2} \cdot (2-1) \cdot S + 1 = \frac{S}{1+t^2} + 1$$

For large t, $\frac{dS}{dt} \approx 1$, so $S(t) \sim t$ grows linearly, remaining finite for all finite t.

Interpretation: *This is a* feedback decoupling regime*. Intrinsic time freezes at $\varepsilon_\infty = \pi/2$, efficiency diverges ($E \to \infty$), yet the state remains bounded (grows only linearly). The system transitions from feedback-dominated (early times) to resource-dominated (late times) evolution.*

7 Global Boundedness via Dissipativity

Theorem 7.1 (Dissipativity Condition)**.** *Suppose there exist $S_* < \infty$ and $\kappa_0 > 0$ (dimensionless) such that:*

$$L(t)(R_A(t) - 1)S + \Sigma(S) \le -\kappa_0 L(t) S \quad \forall S \ge S_*, \forall t \ge 0 \tag{D}$$

Equivalently:

$$R_A(t) \le 1 - \kappa_0 - \frac{\Sigma(S)}{L(t)S} \quad \forall S \ge S_*$$

Assume additionally that $L(t) > 0$ on a set of positive measure within each bounded interval.

Then:

1. *$\sup_{t \ge 0} S(t) \le \max\{S_0, S_*\}$ (global boundedness)*
2. *If $S_0 > S_*$, then $S(t)$ enters $[0, S_*]$ in finite time*
3. *$[0, S_*]$ is positively invariant*

Proof. **Step 1: Trajectories decrease beyond S_***

For $S(t) > S_*$, condition (D) gives:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S) \le -\kappa_0 L(t) S < 0$$

(assuming $L(t) > 0$ on a set of positive measure).

Therefore, $S(t)$ is strictly decreasing whenever $S(t) > S_*$ and $L(t) > 0$.

Step 2: Upper bound

If $S_0 \leq S_*$: By Step 1, S cannot increase beyond S_*. Thus $S(t) \leq S_*$ for all $t \geq 0$.

If $S_0 > S_*$: Since $\frac{dS}{dt} < 0$ for $S > S_*$, we have $S(t) \leq S_0$ for all $t \geq 0$.

Step 3: Finite-time entry

Suppose $S_0 = S(0) > S_*$. For $S(t) \in [S_*, S(0)]$:

$$\frac{dS}{dt} \leq -\kappa_0 L(t) S \leq -\kappa_0 L(t) S_*$$

Integrating from 0 to t:

$$S(t) - S_0 \leq -\kappa_0 S_* \int_0^t L(u)\, du = -\kappa_0 S_* \varepsilon(t)$$

The trajectory reaches S_* when:

$$\varepsilon(t_{\text{entry}}) = \frac{S(0) - S_*}{\kappa_0 S_*}$$

If $\int_0^\infty L = \infty$ (non-degenerate), there exists finite t_{entry} such that $\varepsilon(t_{\text{entry}})$ equals the above value.

Step 4: Positive invariance

By Step 1, any trajectory starting in $[0, S_*]$ cannot exit, since $\frac{dS}{dt} < 0$ at $S = S_*$ prevents upward crossing. □

Corollary 7.2 (Constructive Examples). *Condition* (D) *holds under:*

1. ***Resource depletion with maintenance:***

$$\Sigma(S) = \Sigma_0 \left(1 - \frac{S}{S_{phys}}\right)_+ - \alpha S$$

For $S \geq S_* := \Sigma_0/\alpha$*:*

$$\Sigma(S) \leq -\alpha S$$

If $R_A(t) \leq 1$*, taking* $\kappa_0 = \alpha / L_{\min}$ *with* $L_{\min} := \inf_t L(t)$ *satisfies* (D).

2. ***Negative feedback domination:***

$$\Sigma(S) = -\varsigma S \quad \textit{for } S \geq S_*$$

with $\varsigma > L_{\max}(R_{\max} - 1)$ *where* $L_{\max} := \sup_t L(t)$.

8 Asymptotic Convergence

Theorem 8.1 (Convergence in Strictly Damped Regime)**.** *Assume:*

1. $R_A(t) \leq 1 - \delta$ *for some* $\delta > 0$ *and all* $t \geq 0$ *(strict damping)*
2. $L(t)$ *is Lipschitz continuous:* $|L(t_1) - L(t_2)| \leq M_L|t_1 - t_2|$
3. $R_A(t)$ *is Lipschitz continuous:* $|R_A(t_1) - R_A(t_2)| \leq M_R|t_1 - t_2|$
4. Σ *is bounded:* $\Sigma(s) \leq M$ *for all* $s \geq 0$
5. $\Sigma(s) \to 0$ *as* $s \to \infty$ *(eventually vanishing external input)*
6. $\inf_{t \geq 0} L(t) \geq \ell_\infty > 0$ *(non-vanishing coupling)*

Then $\lim_{t\to\infty} S(t)$ *exists and equals* 0.

Proof. **Via Lyapunov Function:**

Step 1: Let $V(S) := \frac{1}{2}S^2$. Then:

$$\frac{dV}{dt} = S\frac{dS}{dt} = S[L(t)(R_A(t) - 1)S + \Sigma(S)]$$

Since $R_A(t) \leq 1 - \delta$:

$$\frac{dV}{dt} \leq S[-L(t)\delta S + \Sigma(S)] = -L(t)\delta S^2 + S\Sigma(S)$$

Step 2: For large S, $\Sigma(S) \to 0$. For any $\epsilon > 0$, there exists S_ϵ such that $\Sigma(S) \leq \epsilon S$ for $S \geq S_\epsilon$.

For $S \geq S_\epsilon$:

$$\frac{dV}{dt} \leq -L(t)\delta S^2 + \epsilon S^2 = -(L(t)\delta - \epsilon)S^2$$

Choose $\epsilon < \delta\ell_\infty$ (where $\ell_\infty := \inf_{t\geq 0} L(t) > 0$). Then:

$$\frac{dV}{dt} \leq -cS^2 = -2cV$$

for some $c > 0$, giving exponential decay: $V(t) \leq V(0)e^{-2ct} \to 0$.

Therefore, $S(t) \to 0$ as $t \to \infty$. □

9 Counterexample: Superlinear Resources

Theorem 9.1 (Superlinear Resources Cause Finite-Time Blow-Up)**.** *Assume:*

1. $\Sigma(S) \geq \phi S^{1+\varsigma}$ *for some* $\phi, \varsigma > 0$ *and all* $S \geq S_1 > 0$ *(superlinear growth for large* S*)*
2. $L(t) \geq 0$ *arbitrary (possibly constant, vanishing, or oscillating)*

3. $R_A(t) \geq R_{\min} > -\infty$ *(bounded below)*

Then finite-time blow-up occurs: $T_{\max} < \infty$ *and* $\lim_{t \to T_{\max}^-} S(t) = \infty$.

Moreover, the blow-up time satisfies:

$$T_{\max} \leq T_{\textit{blow-up}} := \frac{1}{\varsigma \phi S_1^{\varsigma}}$$

Crucially: *This bound is independent of* $L(t)$ *and* $R_A(t)$*—the superlinear external resources dominate all feedback dynamics.*

Proof. **Step 1:** For $S(t) \geq S_1$:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S)$$

For sufficiently large S, the superlinear term dominates. There exists $S_2 \geq S_1$ such that for $S \geq S_2$:

$$\frac{dS}{dt} \geq \frac{\phi}{2} S^{1+\varsigma}$$

Step 2: Once $S(t) \geq S_2$, separating variables:

$$\int_{S_2}^{S(t)} s^{-(1+\varsigma)}\, ds \leq \int_{t_2}^{t} \frac{\phi}{2}\, du$$

$$-\frac{1}{\varsigma}\left[\frac{1}{S(t)^{\varsigma}} - \frac{1}{S_2^{\varsigma}}\right] \geq \frac{\phi}{2}(t - t_2)$$

$$\frac{1}{S(t)^{\varsigma}} \leq \frac{1}{S_2^{\varsigma}} - \frac{\varsigma\phi}{2}(t - t_2)$$

Step 3: The RHS vanishes when:

$$t - t_2 = \frac{2}{\varsigma \phi S_2^{\varsigma}}$$

Therefore:

$$T_{\max} \leq t_2 + \frac{2}{\varsigma \phi S_2^{\varsigma}} < \infty$$

The bound depends only on Σ's growth rate, not on L or R_A. □

10 Complete Classification Theorem

Theorem 10.1 (Master Classification)**.** *Consider the system:*

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S)$$

with observable $R_A(t)$*, time-varying* $L(t) \geq 0$*, and locally Lipschitz* $\Sigma \geq 0$ *satisfying Assumptions 1–7.*

Part I: Finite-Time Behavior

No finite-time blow-up ($T_{\max} = \infty$) *occurs if:*

1. $R_A(t) \le 1$ *and* Σ *bounded (Theorem 5.1), or*
2. $R_A(t) \ge 1 + \delta$ *with tail sublinearity and* R_A *locally bounded (Theorem 5.4), or*
3. $\Sigma(s) \le a + bs$ *(affine) with either damped or amplified regime (Theorem 5.5)*

Finite-time blow-up does occur if:

- $\Sigma(S) \ge \phi S^{1+\varsigma}$ *for large* S *(Theorem 9.1)*

Part II: Global Boundedness
$\sup_{t\ge 0} S(t) < \infty$ *if:*

- *Strict damping* $R_A \le 1 - \delta$ *with bounded* Σ *and* $\inf L > 0$ *(Theorem 5.3), or*
- *Dissipativity* (D) *holds (Theorem 7.1)*

Part III: Asymptotic Convergence
$\lim_{t\to\infty} S(t)$ *exists and is finite if:*

- *Strict damping* $R_A \le 1 - \delta$, *Lipschitz* L *and* R_A, *bounded* $\Sigma \to 0$, *and* $\inf L > 0$ *(Theorem 8.1)*

Part IV: Feedback Decoupling
If $L(t) \to 0$ *(efficiency* $E = 1/L \to \infty$*):*

- ***Non-degenerate (*$\int L = \infty$*):*** *Intrinsic time extends to infinity; coordinate singularity without state blow-up (Remark 6.1)*
- ***Degenerate (*$\int L < \infty$*):*** *Intrinsic time freezes at* ε_∞*; system undergoes feedback decoupling (Definition 6.2)*

11 Empirical Implications of Observable R_A

Proposition 11.1 (Real-Time Regime Detection)**.** *If* $R_A(t)$ *is measured empirically at each time* t*:*

1. ***Instantaneous growth tendency:***

$$sgn\left(\frac{dS}{dt}\right) = sgn(L(t)(R_A(t) - 1)S + \Sigma(S))$$

2. ***Critical transition:*** *System crosses from expansion to contraction when:*

$$R_A(t) = 1 - \frac{\Sigma(S(t))}{L(t)S(t)}$$

3. ***Cumulative amplification factor:***

$$\Phi(t) = \int_0^t L(s)(R_A(s) - 1)\,ds$$

determines exponential scaling: $S(t) \sim e^{\Phi(t)}$ *(when* Σ *negligible)*

4. ***Efficiency monitoring:***

$$E(t) = \frac{1}{L(t)}$$

When $E \to \infty$*, external resources dominate feedback*

Proposition 11.2 (Stability Monitoring)**.** *For equilibrium* S^* *satisfying* $L(R_A^* - 1)S^* + \Sigma(S^*) = 0$*, the stability margin is:*

$$\lambda(t) := L(t)(1 - R_A(t))S^* - \Sigma'(S^*)$$

- $\lambda(t) > 0 \iff$ *stable near* S^*
- $\lambda(t) < 0 \iff$ *unstable near* S^*
- *Directly computable from observable* $R_A(t)$

Note: Requires Σ *to be monotonic for equilibrium uniqueness.*

Proposition 11.3 (Predictive Power)**.** *Given* R_A *measurements on* $[0, t_0]$*:*
Damped regime ($R_A \leq 1$***):*** *Upper bound on future state:*

$$S(t) \leq S(t_0) + M(t - t_0)$$

Amplified regime ($R_A \geq 1 + \delta$***):*** *Exponential growth rate:*

$$S(t) \geq S(t_0)e^{L\delta(t-t_0)}$$

These bounds enable early warning systems for runaway growth or collapse.

12 Extensions and Open Questions

12.1 Stochastic Observable R_A

If $R_A(t)$ is observed with noise:

$$dS = [L(t)(R_A(t) - 1)S + \Sigma(S)]dt + \sigma(S)dW_t$$

Open questions:

- Almost-sure boundedness conditions
- Moment bounds and tail probabilities
- Optimal control under uncertainty in R_A

12.2 Adaptive Amplification

If $R_A = R_A(S, t)$ depends on state (e.g., diminishing returns):

$$R_A(S, t) = R_0(t) \cdot \frac{K}{K + S}$$

The system becomes fully nonlinear. Analysis requires:

- Phase-plane methods
- Bifurcation theory for varying K
- Multiple equilibria and stability switching

12.3 Multi-Dimensional Systems

Coupled subsystems with different amplification ratios:

$$\frac{dS_i}{dt} = \sum_j L_{ij}(t)(R_{A,j}(t) - 1)S_j + \Sigma_i(S_1, \ldots, S_n)$$

Challenges:

- Cross-coupling stability
- Emergent collective behavior
- Network effects in amplification

12.4 Delay Effects

Memory in feedback:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S(t - \tau) + \Sigma(S(t))$$

Known results:

- Delay can destabilize otherwise stable systems
- Hopf bifurcations and oscillations
- Requires functional analysis (state space $= C([-\tau, 0])$)

13 Summary

13.1 Key Results Proven

Tier 1: Finite-Time Behavior

1. No finite-time blow-up under damping or sublinear resources (Theorems 5.1–5.5)
2. Finite-time blow-up under superlinear resources (Theorem 9.1)
3. Robustness to time-varying $L(t)$, including vanishing coupling (Theorem 5.5)

Tier 2: Global Boundedness

1. Strict damping with bounded Σ (Theorem 5.3)
2. Dissipativity condition (Theorem 7.1)

Tier 3: Asymptotic Convergence

1. Convergence in strictly damped regime with Lipschitz conditions (Theorem 8.1)

Tier 4: Feedback Decoupling

1. Classification of state-space vs coordinate singularities (Remark 6.1)
2. Time reparametrization and efficiency parameter (Lemma 4.1)
3. Feedback decoupling as analytical boundary between self-amplifying and externally driven dynamics (Definition 6.2)

13.2 Physical Insights

Observable R_A provides complete phase-space diagnostic:

- System fate encoded in time-integrated amplification $\Phi(t) = \int L(R_A - 1)$
- Coordinate singularities ($E \to \infty$) distinct from state blow-up ($S \to \infty$)
- Resource boundedness prevents finite-time catastrophe even with $R_A > 1$

Critical thresholds:

- $R_A = 1$: Transition between amplification and damping
- Σ growth rate: Determines whether finite-time blow-up possible
- Dissipativity: Ensures long-term boundedness beyond short-term amplification

Time-varying coupling $L(t)$:

- Does not affect finite-time blow-up existence (only rate)
- Vanishing $L \to 0$ creates feedback decoupling without state singularity
- Intrinsic time $\varepsilon = \int L$ is natural timescale for dynamics

Feedback decoupling marks the analytical boundary between internally self-amplifying and externally driven dynamics, not a new singularity class.

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latex

A Notation Summary

Symbol	Meaning	Dimension
$S(t)$	System state (capacity/capability)	$[S]$
t	Calendar time	$[T]$
ε	Intrinsic time (cumulative coupling)	dimensionless
$L(t)$	Feedback coupling strength	$[T]^{-1}$
$R_A(t)$	Amplification ratio (observable)	dimensionless
$\omega(t)$	Drift parameter $= L(R_A - 1)$	$[T]^{-1}$
$\Phi(t)$	Cumulative drift $= \int_0^t \omega$	dimensionless
$\Sigma(S)$	External resource input	$[S]/[T]$
$E(\varepsilon)$	Efficiency parameter $= 1/L$	$[T]$
S_*	Dissipativity threshold	$[S]$
ε_∞	Maximum intrinsic time (degenerate)	dimensionless
$T_{\max}$	Maximal existence time	$[T]$
$R_{\max}(T)$	$\sup_{t\in[0,T]} R_A(t)$	dimensionless
δ	Damping/amplification margin	dimensionless
κ_0	Dissipativity rate constant	dimensionless

Table 1: Summary of notation and dimensional analysis

B Key Assumptions Reference

For quick reference, we restate the standing assumptions:

1. $\Sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is locally Lipschitz continuous
2. $\Sigma(s) \geq 0$ for all $s \geq 0$ and $\Sigma(0) \geq 0$
3. Σ satisfies sublinear, bounded, or affine growth
4. $R_A : [0, \infty) \to \mathbb{R}_{\geq 0}$ is measurable and piecewise continuous
5. For each $T < \infty$: $R_{\max}(T) := \sup_{t\in[0,T]} R_A(t) < \infty$
6. $L : [0, \infty) \to \mathbb{R}_{\geq 0}$ is measurable and locally integrable
7. $S(0) = S_0 \geq 0$

C Proof Techniques Summary

This paper employs several standard techniques from dynamical systems theory:

C.1 Carathéodory Theory

For ODEs with measurable time dependence:

- Local Lipschitz continuity in state variable
- Measurability in time
- Local integrability ensures existence and uniqueness

C.2 Comparison Principles

Establish bounds via differential inequalities:

- Upper solutions provide upper bounds
- Lower solutions provide lower bounds
- Essential for proving non-blow-up

C.3 Grönwall's Inequality

Standard form:

$$u(t) \leq a(t) + \int_0^t b(s)u(s)\,ds \implies u(t) \leq a(t) + \int_0^t a(s)b(s)e^{\int_s^t b(r)\,dr}\,ds$$

Linear form:

$$\frac{du}{dt} \leq \alpha(t)u + \beta(t) \implies u(t) \leq e^{\int_0^t \alpha(s)\,ds}\left[u_0 + \int_0^t \beta(s)e^{-\int_0^s \alpha(r)\,dr}\,ds\right]$$

C.4 Lyapunov Methods

For stability and convergence:

- Choose $V(S) = \frac{1}{2}S^2$ or similar
- Show $\frac{dV}{dt} \leq -cV$ for some $c > 0$
- Implies exponential decay to zero

C.5 Time Reparametrization

Change of variables to intrinsic time:

- $\varepsilon(t) = \int_0^t L(u)\,du$
- Transforms system to eliminate time-varying coupling
- Reveals structure of feedback dynamics

D Examples and Applications

D.1 Economic Growth Model

Consider an economy with capital stock $S(t)$:

- $R_A(t)$ = return on investment ratio
- $L(t)$ = fraction of output reinvested
- $\Sigma(S)$ = external investment/aid

The model becomes:

$$\frac{dS}{dt} = L(t)(R_A(t) - 1)S + \Sigma(S)$$

Scenario 1: Declining reinvestment
Let $L(t) = \frac{L_0}{1+t}$ (declining reinvestment rate as economy matures).
If $R_A = 1.5$ (constant 50% returns) and $\Sigma(S) = \Sigma_0$ (constant aid):

- $\varepsilon_\infty = L_0 \log(1 + \infty) = \infty$ (non-degenerate)
- No finite-time blow-up by Theorem 5.5
- But $S(t)$ can grow without bound (asymptotic growth)

Scenario 2: Complete decoupling
Let $L(t) = \frac{1}{1+t^2}$ (rapid decline in reinvestment).
Then $\varepsilon_\infty = \arctan(\infty) = \pi/2$ (degenerate).

Interpretation: After finite intrinsic time, the economy stops reinvesting its own output and grows only through external aid.

D.2 Technology Development

Let $S(t)$ = technological capability level:

- $R_A(t)$ = efficiency of capability improving itself
- $L(t)$ = resources allocated to self-improvement
- $\Sigma(S)$ = external innovations

Scenario: Diminishing returns with constant innovation
Let:

$$\begin{aligned} R_A(S) &= 1 + \frac{\delta K}{K + S} \quad \text{(diminishing returns)} \\ L(t) &= L_0 \quad \text{(constant allocation)} \\ \Sigma(S) &= \sigma_0 \quad \text{(constant innovation rate)} \end{aligned}$$

The system becomes:

$$\frac{dS}{dt} = L_0 \frac{\delta K}{K + S} S + \sigma_0$$

For large S:

$$\frac{dS}{dt} \approx \sigma_0 \quad \text{(linear growth)}$$

Analysis:

- Initially ($S \ll K$): $R_A \approx 1 + \delta$, exponential growth
- Eventually ($S \gg K$): $R_A \approx 1$, linear growth from external innovation
- Feedback becomes negligible but remains positive
- No blow-up by Theorem 5.1

D.3 AI Development with Safety Constraints

Let $S(t)$ = AI capability:

- $R_A(t)$ = recursive self-improvement factor
- $L(t)$ = compute/resources allocated to improvement
- $\Sigma(S)$ = human research contributions

Scenario: Safety-constrained development
Implement constraint that reduces $L(t)$ as capability increases:

$$L(t) = L_0 e^{-\alpha S(t)}$$

This creates a state-dependent decoupling where high capability automatically reduces self-improvement allocation.

If $R_A = 2$ (doubling) and $\Sigma(S) = \sigma_0$ (constant research):

$$\frac{dS}{dt} = L_0 e^{-\alpha S} \cdot S + \sigma_0$$

For large S:

$$\frac{dS}{dt} \approx \sigma_0 \quad \text{(safety limit)}$$

Interpretation: The system self-regulates to prevent runaway growth while maintaining progress through external contributions.

E Numerical Methods

For practical implementation, we recommend:

E.1 Standard Solvers

Use adaptive Runge-Kutta methods (e.g., RK45, Dormand-Prince):

- Handle time-varying $L(t)$ and $R_A(t)$ naturally
- Adaptive step size handles rapid changes
- Error control ensures accuracy

E.2 Intrinsic Time Integration

For degenerate systems ($\int L < \infty$), integrate in ε-coordinates:

```
1. Compute (t) =  L(u) du numerically
2. Solve dS/d = (R_A() - 1)S + E()(S)
3. Map back to physical time via t()
```

E.3 Detecting Regime Transitions

Monitor these quantities in real-time:

- $R_A(t) - 1$: sign change indicates amplification/damping transition
- $\Phi(t) = \int_0^t L(R_A - 1)$: cumulative amplification
- $E(t) = 1/L(t)$: efficiency (watch for divergence)
- $\varepsilon(t) = \int_0^t L$: intrinsic time (watch for saturation)

E.4 Sample Code Structure

```
def system(t, S, L_func, R_A_func, Sigma_func):
    L = L_func(t)
    R_A = R_A_func(t)
    Sigma = Sigma_func(S)

    dS_dt = L * (R_A - 1) * S + Sigma

    # Monitor diagnostics
    efficiency = 1/L if L > 1e-10 else np.inf
    drift = L * (R_A - 1)

    return dS_dt

# Solve
from scipy.integrate import solve_ivp
```

```
sol = solve_ivp(
    lambda t, S: system(t, S, L_func, R_A_func, Sigma_func),
    t_span=[0, T_max],
    y0=[S_0],
    method='RK45',
    dense_output=True,
    events=lambda t, S: S - S_threshold  # Stop at threshold
)
```

F Relationship to Existing Literature

F.1 Classical Dynamical Systems

Our results extend classical ODE theory to systems with:

- Time-varying parameters ($L(t)$, $R_A(t)$)
- Observable amplification ratio
- Coordinate singularities vs state singularities

F.2 Growth Theory

Connections to:

- Solow-Swan model (exogenous growth)
- Romer model (endogenous growth)
- Distinction: Observable $R_A(t)$ makes amplification measurable

F.3 Control Theory

The parameter $L(t)$ can be viewed as a control variable:

- Optimal control: Choose $L(t)$ to maximize objective
- Safety constraints: Force $L(t) \to 0$ to prevent blow-up
- Adaptive control: Adjust $L(t)$ based on observed $R_A(t)$

F.4 Blow-Up Theory

Our Theorem 9.1 aligns with classical blow-up results:

- Superlinear forcing causes finite-time blow-up
- Blow-up time independent of linear dynamics
- Sublinear forcing cannot cause blow-up

G Open Problems

G.1 Sharp Thresholds

Question: What is the sharp boundary between Σ growth rates that permit vs prevent finite-time blow-up?

We know:

- $\Sigma(S) = O(S)$ (linear): No blow-up
- $\Sigma(S) = S^{1+\epsilon}$ (superlinear): Blow-up

Conjecture: $\Sigma(S) = S \log(S)$ is the critical case.

G.2 Optimal Resource Allocation

Problem: Given target trajectory $S^*(t)$ and control over $L(t)$, find:

$$L^*(t) = \arg\min_{L(\cdot)} \int_0^T [S(t) - S^*(t)]^2 + \lambda L(t)^2 \, dt$$

subject to the dynamics constraint.

G.3 Stochastic Amplification

Problem: For $R_A(t) = \bar{R}_A + \sigma\xi(t)$ where ξ is white noise:

- Characterize almost-sure behavior
- Moment explosions vs sample path boundedness
- Optimal filtering from noisy R_A measurements

G.4 Network Amplification

Problem: For n coupled subsystems:

$$\frac{dS_i}{dt} = \sum_{j=1}^{n} L_{ij}(t)[R_{A,j}(t) - \delta_{ij}]S_j + \Sigma_i(\mathbf{S})$$

- Network topology effects on stability
- Spectral analysis of coupling matrix
- Emergent collective blow-up vs individual stability

G.5 Partial Observability

Problem: If $R_A(t)$ is only partially observed:

- State estimation from noisy measurements
- Confidence regions for blow-up time
- Adaptive control under uncertainty

H Concluding Remarks

This paper provides a complete mathematical characterization of feedback-amplification systems with time-varying coupling. The key contributions are:

1. **Observable framework:** By making $R_A(t)$ directly measurable, we provide operational definitions that enable empirical testing
2. **Singularity classification:** We rigorously distinguish state-space singularities (physical blow-up) from coordinate singularities (feedback decoupling)
3. **Resource criticality:** The growth rate of external resources $\Sigma(S)$ determines finite-time behavior, independent of feedback dynamics
4. **Coupling invariance:** Time-variation in $L(t)$—including vanishing coupling—affects evolution rate but not blow-up existence
5. **Complete classification:** Theorem 10.1 provides sharp conditions for finite-time blow-up, global boundedness, and asymptotic convergence

Practical implications:

- Real-time monitoring via $R_A(t)$ enables early warning systems
- Safety constraints can be implemented through $L(t)$ control
- Feedback decoupling provides a natural stabilization mechanism

Theoretical implications:

- Intrinsic time $\varepsilon = \int L$ is the natural coordinate for amplification dynamics
- Efficiency parameter $E = 1/L$ reveals regime transitions
- The dichotomy between sublinear (safe) and superlinear (dangerous) resources is fundamental

We hope this framework proves useful for analyzing self-amplifying systems across economics, technology, and other domains where feedback dynamics and resource constraints interact.

END OF PAPER

Repository: `github.com/harrisondfletcher/amplification-dynamics`
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