

Recursion

C++ Recursive Functions

In this week

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Recursion

- In mathematics and generally in computing, we say a solution to a problem is **recursive** if the solution is defined in terms of itself and satisfies the following two conditions:
 - **Stopping Criteria**: At certain stage the solution is defined explicitly
 - **Down Sizing**: Each subsequent self definition of the solution approaches the stopping criteria. **Self definition step is also known as recursive step**
- **Recursion**: The process of solving problems recursively

Recursion: Example

- In mathematics, the most common recursive solution of a problem might be the solution to the factorial problem, given by

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n * f(n-1) & \text{if } n > 0 \end{cases}$$

- As can be seen, the solution to a factorial problem is **defined in terms of the solution**
- However, the solution has a **stopping criteria which happens when n=0**. For $n > 0$, the solution is defined in terms of itself but closer to the stopping criteria

Recursion in C++

- In C++, recursion is possible because **C++ allows a function to call itself**
- As an example, we may write a function named **factorial** in order to calculate the factorial of a non-negative integer
- Looking back at the mathematical recursive definition of factorial function, we therefore see that the following C++ function implementation does the job

Recursion in C++

```
int factorial(int n)
{
    if (n == 0)
        return 1;
    else
        return n * factorial(n-1);
}

int main()
{
    int a;
    do
    {
        cout << "Enter a non-negative integer ";
        cin >> a;
    }while (a < 0);
    int x = factorial(a);
    cout << "The factorial of " << a << " is " << x << endl;
    system("pause");
    return 0;
}
```

Recursion in C++

- Pictorially, this will look like as follows for the function call factorial(4):

factorial(4)

↑
return $4 * \text{factorial}(3) = 4 * 6 = 24$

↑
return $3 * \text{factorial}(2) = 3 * 2 = 6$

↑
return $2 * \text{factorial}(1) = 2 * 1 = 2$

↑
return $1 * \text{factorial}(0) = 1 * 1 = 1$

↑
return 1

Recursion in C++

- The heart of recursion in programming, in this case in C++, lies in the fact that the **factorial** function was able to call itself
- While recursion may seem an amazingly magic power; it should however be emphasized that recursive functions are very expensive computationally
- This is because a function call is computationally expensive process as we will soon see

How and When to think recursively

- In order to appreciate the power of recursive solutions, let us consider the problem of printing non-negative integer numbers vertically
- **Problem Statement:** Write a C++ function named **printVertical** that takes a non-negative integer argument and prints the argument vertically; that is, each digit of the argument is printed vertically.
- For example, if the argument is **5728**, then the function must print the following

5
7
2
8

How and When to think recursively

- **Solution:** For this problem, we may come up with a direct non-recursive solution as follows
- We consider the specific input argument **n=5728**
- Then we ask ourselves, how can we extract the digit **5** from **n**?
- To see this we note that
$$n = 5728 = 5*1000 + 7*100 + 2*10 + 8$$
- Observe that 1000 is the largest integer power of 10 that is less than or equal to n
- But how do we get the value of $m = 1000$?
- We first calculate $k = \text{int}(\log_{10}n)$ which will give us **3**; then $m=10^k$ will give us **m = 1000**
- Therefore if we set **m = 1000** then $\text{int}(n/m)$ will yield the required digit **5**. **So we print that digit**

How and When to think recursively

- Next, we update $n = n \% m$. This will result to $n = 728$ and set $m = m/10$ to get $m = 100$
- Then $\text{int}(n/m)$ will yield the required digit 7 and we print that digit
- Next, we update $n = n \% m$. This will result to $n = 28$ and set $m = m/10$ to get $m = 10$
- We repeat the process as long as $m > 0$
- Therefore the following code does the job!

How and When to think recursively

```
#include <iostream>
#include <cmath>
using namespace std;
void printVertical(int n)
{
    int k = log10(1.0 * n);
    int m = pow(10.0, k);
    while (m > 0)
    {
        cout << n / m << endl;
        n = n % m;
        m = m / 10;
    }
}
int main()
{
    int x;
    do
    {
        cout << "Enter a non-negative integer ";
        cin >> x;
    }while (x < 0);
    cout << "The number you entered printed vertically is" << endl;
    printVertical(x);
    system("pause");
    return 0;
}
```

How and When to think recursively

- This is almost a correct algorithm
- But run the program for $n = 0$ Oops!!! Doesn't work!
- Moreover... it is simply too much of mathematical work for such easy problem.
- When faced with problems that seem to be too complicated to be solved directly, it might be a good idea to see if we can think of a recursive solution that seems natural solution
- So we ask ourselves, can we find a much neater and more natural recursive solution?

How and When to think recursively

- In order to think recursively, let us first think of the simplest case: **single digit number n**
- In that case, we just print n and we are done. This looks perfectly stopping criteria
- Next, we ask ourselves can we break the solution to printing the number 5728 in terms of the solution itself but with a smaller number?
- **Wait a minute! Why don't we first print the number 572 vertically and then we print 8!!!**

How and When to think recursively

- When we then try to print 572, we will say let us first print **57** and then print **2**. Next, we will say let us first **5** and then print **7**
- Now, we see that printing **5** will not need to be broken in two parts for it is single digit which is our stopping criteria
- Therefore all we need to observe now is the relationships among the subsequent sub solutions and the digits to be printed

How and When to think recursively

- It is easy to see that given $n=5728$, the number **572** is $n/10$. Moreover the digit **8** is $n\%10$
- Therefore, all we need is the following: If n is single digit, then print it and you are done; otherwise first print $n/10$ **VERTICALLY** and then print $n \% 10$
- This is perfectly fitting, neat, natural recursive solution to the problem and the following function does the job

How and When to think recursively

```
void printVertical(int n)
{
    if (n < 10)
    {
        cout << n << endl;
        return;
    }
    else
    {
        printVertical(n / 10);
        cout << n % 10 << endl;
    }
}

int main()
{
    int x;
    do
    {
        cout << "Enter a non-negative integer ";
        cin >> x;
    }while (x < 0);
    cout << "The number you entered printed vertically is" << endl;
    printVertical(x);
    system("pause");
    return 0;
}
```


How and When to think recursively

- Observe that unlike the direct solution, this approach seems more natural way of thinking
- Most importantly, this is a correct solution that works well also for $n=0$
- Recursive functions need careful thinking though! Make sure a subsequent function call is closer to the stopping criteria; and that the solution is correct

Recursive Search Algorithms

- Searching is a very common process in computing
- While searching can easily be performed using non-recursive algorithms, it is a problem that has a natural recursive solution as well
- For the sake of exploring recursion and how to think recursively, we now look at sequential and binary recursive search algorithms

Sequential Search

- **Problem Statement:** Given an array of n elements (say integers), we would like to search the array if a search value is found in the array. If the search value is found, we would like to return the index of the element matching the search value; otherwise we would like to return -1 to mean the search value is not found in the array.
- Let us name the array **A** and the value to be searched **searchValue**

Sequential Search

- **Solution**: A recursive solution can be given as follows: Check if the **searchValue** is equal to the first element of the array. If yes, we return the index of the first element; otherwise we search in the remaining elements of the array. We stop the search when there are no more elements to search in the array
- **Programming**: In order to keep track of the first element of the array we need to have the knowledge of the **start index** at each step. Moreover, in order not to exceed the last element we will also need to have the knowledge of the **last index** at each step

Sequential Search

- Thus the following function does the job

```
int sequentialSearch(const int A[], const int startIndex, const int lastIndex, const int searchValue)
{
    if (startIndex > lastIndex)
        return -1;
    else if (A[startIndex] == searchValue)
        return startIndex;
    else
        return sequentialSearch(A, startIndex+1, lastIndex, searchValue);
}

int main()
{
    const int size = 8;
    int A[size] = {2, 8, 3, 7, 9, 0, 1, 6};
    int x;
    cout << "Enter an integer number to search in the array ";
    cin >> x;
    int index = sequentialSearch(A, 0, size-1, x);
    if (index == -1)
        cout << x << " is not found in the array." << endl;
    else
        cout << x << " is found in the array at index " << index << endl;
    system("pause");
    return 0;
}
```

Binary Search

- In binary search algorithm, we assume that the array is already sorted (say in increasing order) and we don't have to search sequentially
- Instead, we check the middle element of the array if it is equal to the **searchValue**; if yes we return the middle index; otherwise we search one half of the array depending which side the **searchValue** lies
- The following function implements the recursive binary search algorithm

Binary Search

```
int binarySearch(const int A[], const int startIndex, const int lastIndex, const int searchValue)
{
    if (startIndex > lastIndex)
        return -1;
    else
    {
        int m = (startIndex + lastIndex) / 2;
        if (A[m] == searchValue)
            return m;
        else if (A[m] > searchValue)
            return binarySearch(A, startIndex, m-1, searchValue);
        else
            return binarySearch(A, m+1, lastIndex, searchValue);
    }
}
```

```
int main()
{
    const int size = 8;
    int A[size] = {3, 7, 12, 17, 21, 25, 30, 35};
    int x;
    cout << "Enter an integer number to search in the array ";
    cin >> x;
    int index = binarySearch(A, 0, size-1, x);
    if (index == -1)
        cout << x << " is not found in the array." << endl;
    else
        cout << x << " is found in the array at index " << index << endl;
    system("pause");
    return 0;
}
```

Towers of Hanoi

- In this problem, you are given three pegs named **A**, **B** and **C**
- There are **n** discs in peg **A**
- The discs are arranged in increasing size top to bottom
- The aim is to **move the discs from peg A to peg C** such that
 - **Only one TOP disc can be moved at a time** and
 - **No bigger disc is allowed to sit on top of a smaller disc**
- You may use the peg **B** as temporary disc holder

Towers of Hanoi

- Attempting to come up with a direct solution to this problem is next to impossible especially at the CMPT130 level. Try it!
- However, one can easily see the solution can be stated as follows:
- Consider the case **$n = 1$** . That is there is only one disc in the peg **A**
- Then the solution is **trivial**: **move the disc from the peg A directly to peg C. This is a stopping criteria!**

Towers of Hanoi

- How about when there are $n > 1$ discs on peg **A**?
- It is easy to see that we need to first move the top **$n-1$** discs from peg **A** to peg **B**.... using peg **C** as temporary place
- Next we move the bottom disc from peg **A** to peg **C** directly
- Finally, we move the **$n-1$** discs in peg **B** to peg **C**.... this time using peg **A** as temporary place

Towers of Hanoi

- But how do we move the top **$n-1$** discs from peg **A** to peg **B**?
- Well, it is a sub-problem of the original problem that requires the exact same solution; hence recursion!
- Importantly, **moving $n-1$ discs is closer to the stopping criteria than the original problem of moving n discs**
- Hence we clearly have recursion and a recursive solution is the way to go and the following function implementation does the job

Towers of Hanoi

```
void toh(int n, char A, char C, char B)
{
    //This function moves n discs from Peg A to Peg C using Peg B as a temporary place holder
    if (n == 1)
    {
        cout << "Move top disc from " << A << " to " << C << endl;
        return;
    }
    else
    {
        toh(n - 1, A, B, C);
        toh(1, A, C, B);
        toh(n - 1, B, C, A);
    }
}

int main()
{
    int n;
    do
    {
        cout << "Enter a positive number of discs ";
        cin >> n;
    }while (n <= 0);

    toh(n, 'A', 'C', 'B');

    system("pause");
    return 0;
}
```

Stack Memory for Recursion

- C++ uses a stack memory to process function calls
- A stack memory is a data structure that allows addition of new item on top of it and only one item at a time can be extracted from the top of the stack
- Whenever a C++ function is called in a program, the function call is added on top of the stack
- Processing of a series of function calls is performed by adding the function calls on top of the stack in the order they were called and then processing a function on top of the stack
- Once processed a function is cleared from the stack and then the next function on top of the stack is processed
- When we first call a function from the main program, we start with an empty stack and whenever we return to the main program the stack memory becomes empty

Stack Memory for Recursion

- Considering the factorial(4) function call from the main program, we see that factorial(4) is placed onto the empty stack
- Then factorial(4) calls factorial(3) which places function call factorial(3) on top of the stack
- Next, factorial(3) calls factorial(2) which places function call factorial(2) on top of the stack
- Next , factorial(2) calls factorial(1) which places function call factorial(1) on top of the stack
- Next, factorial(1) calls factorial(0) which places function call factorial(0) on top of the stack
- Finally, factorial(0) returns 1 without adding anything on top of the stack **and is cleared from the stack**

Stack Memory for Recursion

- Although, our aim is to get factorial(4); by its design, the stack will not allow factorial(4) to be accessed before factorial(3) is accessed; which won't be accessed before factorial(2) is accessed; which again won't be accessed before factorial(1) is accessed; which once again won't be accessed before factorial(0) is processed
- Since factorial(0) doesn't add a new function call to the stack, now factorial(1) will be processed using the result of factorial(0). Once processed, factorial(1) is cleared from the stack
- This leads for factorial(2) to be processed and cleared from the stack; which leads for factorial(3) to be processed and cleared from the stack and finally for factorial(4) to be processed and cleared from the stack and its result returned to the main program

Stack Memory for Recursion

- Since computer's stack has finite memory size, the number of recursive calls that can be made is limited by the memory size
- If during a series of recursive calls the stack becomes full, then a stack overflow happens
- Stack overflow crashes a program
- Extra care must be placed to avoid infinite recursion which will surely result to stack overflow and crashes the program
- This is why you have got www.stackoverflow.com to help you whenever you get to such and other problems in programming...