Distributed Relation Algebra

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Abstract

We define some extra relation operators in Relation Algebras and examine the relationship between Relation Algebras and Relevance Logic. We then extract binary relations from Kripke models of Relation Algebras and extend those to Distributed Relation Algebras. The term *distribution* refers to using a multigraph of local logics connected by the relation operators. In the context of Kripke frames for Relation Algebras, the distribution refers to a multigraph of Stone spaces each supporting a *local* Kripke frame. There are then distributed Kripke relations connecting the local frames. Finally, we show the relationship between the relation operators and Kan extensions and lifts from category theory.

1 Introduction

This paper arose from considering applications of Distributed Logic [AH16] to Field Programmable Gate Arrays (FPGAs). On an FPGA, there are several components each with its own notion of internal state. Clearly a logic aiming to reflect this must have some notion of component as well. Thus, the map between the logic and FPGA applications is direct. An FPGA device can be thought of as a sea of small circuits which are corralled by an FPGA application to form components. Every application will then have a different collection of components. Therefore a logic reflecting the component structure must make that structure parametric to the logic. We parameterized Distributed Logic with a graph[1] of *localities* connected by arcs. The localities represent components and the arcs represent relations. This structure forms a distributed Kripke frame where each locality is a collection of points or states and the arcs are distributed Kripke relations, i.e., binary relations between point sets. The logic then introduced distributed modal operators interpreted by distributed Kripke models.

We felt that we could use a different representation of FPGAs by having a logic of distributed relations directly. One option was to formalize Relation Algebra (RA) as a logic, which we accomplished in [AHR17]. In this paper, we ignore the logic and work directly with RA. However, the semantics of [AHR17] does not yield a semantics as a collection of binary relations. This paper does provide such a semantics. We start from [CT51, Ng84]. We have been heavily influenced by Pratt's Action Logic [Pra90a] and his paper on the origins of the calculus of binary relations [Pra92]. From [ADP04], we knew how to extract a calculus of relations from a three-place Kripke relation similarly to that used in the semantics of Relevance Logic. Pratt in [Pra90b] talks about a similar construction but leaves out the details. We adopt the distributed RA of [AHR17] but then supply a new semantics using binary relations.

RA should have some connection with Relevance Logic [AB75, RMPB82]. The main composition connective is similar to Relevance Logic's fusion. The defined operators used some ideas of Pratt's [Pra90a] and the fact that they can be defined in Relevance logic from fusion and DeMorgan negation. The operators are definable using the monoid of RA and DeMorgan negation, which is manufactured from RA's converse and

[1] Warning: all authors do not agree on the definitions. Technically, we use *multigraphs* of graph theory with own identities: there can be multiple arcs between two nodes and multiple edges between two nodes are different edges; loops are permitted.

classical negation. Hence, there is a relationship to Classical Relevance Logic; we relied upon [RMPB82] for this latter. Were it not for Relevance Logic's insistence on a commutative fusion operator, i.e., a commutative monoid operator in the algebras, then the match would have been perfect. We detail the relationship here where converse, as a period-two operator on the Kripke frames, replaces the Routley-star operator and we rearrange the semantics so as not to import the commutativity; Relevance Logic's semantics builds in the commutativity.

The main issue with RAs is that in general they are not representable. In logic, this means the completeness cannot be proven. However, from [ADP04, Pra90b] we can get representations for simple RAs called simple Boolean monoids. We generate these using the device in [ADP04] called Tabularity. This is the same notion as Tabularity in [FS90], whence the name, although we apply it to the three-place relations of Kripke frames. The result yields relations as interpretations of the elements of an RA. We call the collection of relations thus obtained an *algebra of relations*.

We first present RA as in [CT51, Ng84] (without the induction axiom). After showing a few properties, we define some extra algebraic operators that are used in the sequel. Next, the relationship between RA and Relevance Logic is examined. We then show how to form Kripke models of the algebras (in the sense of Kripke models for logics where for us the algebras replace the logics). Using [ADP04], we show how to extract binary relations from the Kripke models and show how to derive the algebraic operators, now as operators on actual relations. We briefly explain the mechanisms necessary to distribute RA and how the extraction of binary relations carries over to the distributed case. This yields typed relations; each relation has a source and target. These form a category of relations with two forms of composition of relations. We used Dunn's notion of Gaggle Theory [Dun90, AD93] to produce intensional nor and nand operators and show how they integrate with algebras of relations. Finally, we show how operators relate to Kan extensions and lifts and recover the residuation rules for the relations and their operators, as adjoints. The intensional nor and nand form constructions that resemble Kan extensions and lifts, however the details are a bit different and do not fit the Kan notions precisely.

The two forms of composition could have been formalized directly using the devices of enriched category theory; we do not do so here in interest of brevity. However, a truly abstract characterization of the two forms, as well as the intensional nor and nand operations, would use enriched category theory. The category theory presents a template for thinking of the mathematics as presenting a realizability notion for the distributed algebras of relations. We expect this will have utility when discovering and proving properties about FPGAs. Another interesting feature of the categorical notion of extensions and lift is that the operators need not be defined over the entire distributed relation category. This notion comes in handy with respect to FPGAs where sometimes the relations between components may not be discoverable, say, when what is known as foreign IP (components with unknown internal structure) is used. Foreign IP is used in virtually all FPGA designs.

2 Relation Algebra (RA)

We assume the usual Boolean algebra substrate of meet, join, and Boolean negation, i.e., $\wedge$, $\vee$, $\neg$ where the Boolean $\vee$ symbol will be used in place of RA's typical $+$ symbol (which we reserve for an *intensional or* as a defined connective) and $\wedge$ is a defined operation. To this we add operators derived from RA, $\circ$, $\leftarrow$, and $\rightarrow$; the first is relational composition, and the latter two are the left and right residuals of relational composition and are analogous to Relevance Logic's entailment connectives. We also add $+$, $\rightharpoonup$, and $\leftharpoonup$ for another set of residuated connectivess. We use the terms *tensor* and *cotensor* to refer to $\circ$ and $+$ in the algebras. We also use these same terms in the algebras of sets that we extract from the Kripke Frames. Finally we add the self-residuated *intensional nor* $\downarrow$ and *intensional nand* $\uparrow$.

2.1 Axioms

Definition 2.1.1 The Relation Algebra Axioms are from Chin and Tarski and Ng [CT51, Ng84]:

M1. $(B, \vee, \neg, \bot, \top)$ is a Boolean algebra

M2. $a^{\smile\smile} = a$ for any $a \in A$

M3. $(a \circ b) \circ c = a \circ (b \circ c)$

M4. $(a \vee b) \circ c = (a \circ c) \vee (b \circ c)$

M5. $a \circ 1 = a$ for any $a \in A$

M6. $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$ for any $a, b \in A$

M7. $(a \circ b)^{\smile} = b^{\smile} \circ a^{\smile}$

M8. $(a^{\smile} \circ \neg(a \circ b)) \vee \neg b = \neg b$ for any $a, b \in A$

where $\bot$ and $\top$ are the bottom and top of the Boolean algebra.

The follow lemma is useful:

Lemma 2.1.2 *For any period 2 function f,*

$$\forall v(P(f(v))) \text{ iff } \forall z(P(z)).$$

Proof: From left to right, assume $\forall z(P(f(z)))$ holds and choose an arbitrary point y. Letting z be $f(y)$ yields $P(f(f(y)))$. Since f is assumed period-2, we have $P(y)$. Since y was arbitrary, we have $\forall z(P(z))$.

From right to left, assume $\forall z(P(z))$ holds and choose an arbitrary point y. Letting z be $f(y)$ yields $P(f(f(y)))$. Since the choice of y was arbitrary, we have $\forall z(P(f(z)))$. ∎

Using the axioms, it is straightforward to prove left associativity of $\circ$ and two-sided monotonicity:

Lemma 2.1.3

$$a \circ (b \vee c) = (a \circ b) \vee (a \circ c) \qquad a \leq b \text{ implies } c \circ a \leq c \circ b \text{ and } a \circ c \leq b \circ c$$

We generally use axiom M8 in its inequational form along with a variant:

M8. $\neg(b \circ a) \circ a^{\smile} \leq \neg b$

M8'. $\neg(b \circ a) \circ a^{\smile} \leq \neg b$

Lemma 2.1.4 *The axioms M8 and M8' can be uniformly substituted with $a^{\smile}$ and $\neg a$ for a and $b^{\smile}$ and $\neg b$ for b yielding valid instances of those axioms. All the instances are equivalent.*

Proof: The proof is simply the application of Lemma 2.1.2. As an example,

$$\begin{array}{rll} a^{\smile} \circ \neg(a \circ \neg b^{\smile}) \leq b^{\smile} & \text{iff } (a^{\smile} \circ \neg(a \circ \neg b^{\smile}))^{\smile} \leq b^{\smile\smile} & \text{application of converse} \\ & \text{iff } \neg(\neg b \circ a^{\smile}) \circ a \leq b & \text{converse is period 2} \\ & \text{iff } \neg(b \circ a) \circ a^{\smile} \leq \neg b & \text{Lemma 2.1.2.} \end{array}$$

∎

From now on we will assume any of these forms as needed without calling out the use of Lemma 2.1.2.

2.2 Converse and Entailment

The following lemma is well-known, but it is important to state it as it eliminates some of the baggage of converse:

Lemma 2.2.1

$$1^{\smile} = 1 \qquad 1 \circ a = a \qquad \neg(a^{\smile}) = (\neg a)^{\smile}.$$

We generally elide the parentheses as in the following:

Definition 2.2.2 The DeMorgan operator $\sim$ is defined with

$$\sim a = \neg a^{\smile}.$$

It is straightforwardly checked that $\sim$ is a DeMorgan negation on the Boolean lattice of an RA. Due to the distribution of $\breve{}$ over $\vee$ and subsequently over $\wedge$, the DeMorgan $\sim$ is also a classical negation. However, we will continue to call it the DeMorgan negation to distinguish it from the classical negation $\neg$.

Definition 2.2.3

$$a \rightarrow b \overset{def}{=} \sim(\sim b \circ a), \quad b \leftarrow a \overset{def}{=} \sim(a \circ \sim b).$$

We can process the axiom M8 a bit to ease its use. First,

$$\begin{aligned} a \rightarrow b &= \sim(\sim b \circ a) & b \leftarrow a &= \sim(a \circ \sim b) \\ &= \neg(\neg b^{\smile} \circ a)^{\smile} & &= \neg(a \circ \neg b^{\smile})^{\smile} \\ &= \neg(a^{\smile} \circ \neg b) & &= \neg(\neg b \circ a^{\smile}) \end{aligned}$$

Lemma 2.2.4 *The following two formulas are equivalent to axiom M8:*

altM8. $a \circ (a \rightarrow b) \leq b$ *altM8'.* $(b \leftarrow a) \circ a \leq b$

Proof: The following two statements hold by the definitions of $\rightarrow$ and $\leftarrow$ and Lemma 2.1.4:

$$a \circ (a \rightarrow b) \leq b \text{ iff } a \circ \neg(a^{\smile} \circ \neg b) \leq b \qquad (b \leftarrow a) \circ a \leq b \text{ iff } \neg(\neg b \circ a^{\smile}) \circ a \leq b$$

■

The usual residuation properties hold:

Lemma 2.2.5

$$a \leq c \leftarrow b \text{ iff } a \circ b \leq c \text{ iff } b \leq a \rightarrow c.$$

The proofs are merely expansions of the definitions and application of the properties of converse and classical negation.

Theorem 2.2.6 *The following are equivalent to axiom M8:*

M'8. $(a \rightarrow \sim b)^{\smile} \leq b^{\smile} \rightarrow \sim a^{\smile}$ *M'8'.* $(\sim b \leftarrow a)^{\smile} \leq \sim a^{\smile} \leftarrow b^{\smile}$

M"8. $b \rightarrow \sim a \leq \sim b \leftarrow a$ *M"8'.* $\sim a \leftarrow b \leq a \rightarrow \sim b$

Proof: We prove two of the statements, the rest are similar:

$$\begin{aligned} (b \rightarrow \sim a)^{\smile} \leq a^{\smile} \rightarrow \sim b^{\smile} &\text{ iff } a^{\smile} \circ (b \rightarrow \sim a)^{\smile} \leq \sim b^{\smile} \\ &\text{ iff } a^{\smile} \circ \neg(b^{\smile} \circ \neg\neg a^{\smile})^{\smile} \leq \sim b^{\smile} \\ &\text{ iff } a^{\smile} \circ \neg(a \circ b) \leq \neg b \end{aligned}$$

$$\begin{aligned} \sim a \leftarrow b \leq a \rightarrow \sim b &\text{ iff } a \circ (\sim a \leftarrow b) \leq \sim b \\ &\text{ iff } a \circ \neg(\neg \sim a \circ b^{\smile}) \leq \sim b \\ &\text{ iff } a \circ \neg(a^{\smile} \circ b^{\smile}) \leq \neg b^{\smile} \end{aligned}$$

■

It is straightforward to prove the converses:

Theorem 2.2.7

$$b^{\smile} \rightarrow \sim a^{\smile} = (a \rightarrow \sim b)^{\smile} \qquad \sim a^{\smile} \leftarrow b^{\smile} = (\sim b \leftarrow a)^{\smile} \qquad (a \rightarrow b)^{\smile} = b^{\smile} \leftarrow a^{\smile} \qquad (b \leftarrow a)^{\smile} = a^{\smile} \rightarrow b^{\smile}$$

Theorem 2.2.8 $\circ$ *is a normal operator.*

Proof: From Axion M4 and Lemma 2.1.3, $\circ$ distributes over $\vee$. From $\perp \leq a \rightarrow \perp$ and residuation, $a \circ \perp = \perp$. Similarly, $\perp \circ a = \perp$. ■

2.3 Some Other Defined Operators

There are coentailment, intensional nor, and intensional nand operators. We list all the operators of the sequel here and their distribution and residuation properties. These are all straightforwardly proved from their definitions and we elide the proofs:

Definition 2.3.1

Operator	Prescription	Name	Operator	Prescription	Name
1	1	identity	0	0	coidentity
$a \circ b$	$a \circ b$	tensor	$a + b$	$\sim(\sim b \circ \sim a)$	cotensor
$b \leftarrow a$	$\sim(a \circ \sim b)$	left entailment	$b \leftharpoonup a$	$b \circ \sim a$	left coentailment
$a \rightarrow b$	$\sim(\sim b \circ a)$	right entailment	$a \rightharpoonup b$	$\sim a \circ b$	right coentailment
$a \downarrow b$	$\sim a \circ \sim b$	intensional nor	$a \uparrow b$	$\sim(a \circ b)$	intensional nand

Table 1: Names of Intensional Operators

The lattice distribution properties are straightforwardly derived from Table 1 and we elide their proofs. We show the table using Dunn's Gaggle Operator Type:

Theorem 2.3.2

Operator Type	*Distribution Property*	*Distribution Property*
$\circ : (\vee, \vee) \rightarrow \vee$	$a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$	$(a \vee b) \circ c = (a \circ c) \vee (b \circ c)$
$\leftarrow : (\wedge, \vee) \rightarrow \wedge$	$a \leftarrow (b \vee c) = (a \leftarrow b) \wedge (a \leftarrow c)$	$(a \wedge b) \leftarrow c = (a \leftarrow c) \wedge (b \leftarrow c)$
$\rightarrow : (\vee, \wedge) \rightarrow \wedge$	$a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$	$(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$
$+ : (\wedge, \wedge) \rightarrow \wedge$	$a + (b \wedge c) = (a + b) \wedge (a + c)$	$(a \wedge b) + c = (a + c) \wedge (b + c)$
$\leftharpoonup : (\vee, \wedge) \rightarrow \vee$	$a \leftharpoonup (b \wedge c) = (a \leftharpoonup b) \vee (a \leftharpoonup c)$	$(a \vee b) \leftharpoonup c = (a \leftharpoonup c) \vee (b \leftharpoonup c)$
$\rightharpoonup : (\wedge, \vee) \rightarrow \vee$	$a \rightharpoonup (b \vee c) = (a \rightharpoonup b) \vee (a \rightharpoonup c)$	$(a \wedge b) \rightharpoonup c = (a \rightharpoonup c) \vee (b \rightharpoonup c)$
$\downarrow : (\wedge, \wedge) \rightarrow \vee$	$a \downarrow (b \wedge c) = (a \downarrow b) \vee (a \downarrow c)$	$(a \wedge b) \downarrow c = (a \downarrow b) \vee (b \downarrow c)$
$\uparrow : (\vee, \vee) \rightarrow \wedge$	$a \uparrow (b \vee c) = (a \uparrow b) \wedge (a \uparrow c)$	$(a \vee b) \uparrow c = (a \uparrow b) \wedge (b \uparrow c)$

Table 2: Distribution Properties

The left and right arrow operators are always antitone in their source and monotone in their target. Antitone equates to flipping the Boolean connective to its dual and monotone equates to preserving it.

The residuation properties also follow directly from the definitions:

Theorem 2.3.3 *The lattice residuation properties are*

Residuation Property	*Residuation Property*
$b \leq a \rightarrow c$ iff $a \circ b \leq c$ iff $a \leq c \leftarrow b$	$a \rightharpoonup c \leq b$ iff $c \leq a + b$ iff $c \leftharpoonup b \leq a$
$a \downarrow b \leq c$ iff $c \downarrow a \leq b$	$a \leq b \uparrow c$ iff $c \leq a \uparrow b$

Table 3: Residuation Properties

Theorem 2.3.4 *The $+$ operator forms a monoid with $\neg 1$ as its unit.*

Proof:

$$\begin{aligned} a + \neg 1 &= \sim(\sim \neg 1 \circ \sim a) \\ &= \sim(1^{\smile} \circ \sim a) \\ &= \sim(1 \circ \sim a) \\ &= \sim \sim a \\ &= a \end{aligned} \qquad \begin{aligned} a + (b + c) &= \sim(\sim(b + c) \circ \sim a) \\ &= \sim((\sim c \circ \sim b) \circ \sim a) \\ &= \sim(\sim c \circ (\sim b \circ \sim a)) \\ &= \sim(\sim c \circ \sim(a + b)) \\ &= (a + b) + c \end{aligned}$$

That $\neg 1 + a$ is similar. ■

The following lemma is mirrors the rules for $\sim$ in Gentzen systems for Relevance Logic:

Lemma 2.3.5

$$a \leq c + {\sim} b \text{ iff } a \circ b \leq c \text{ iff } b \leq {\sim} a + c.$$

The proof follows directly from residuation and the definition of $+$.

Theorem 2.3.6

$$a \circ (b + c) \leq (a \circ b) + c.$$

Proof:

$$\begin{aligned}
{\sim} b + {\sim} a &\leq {\sim} b + {\sim} a \\
{\sim}(a \circ b) \circ a &\leq {\sim} b \\
{\sim} c \circ {\sim}(a \circ b) \circ a &\leq {\sim} c \circ {\sim} b \\
{\sim} c \circ {\sim}(a \circ b) &\leq {\sim}(b + c) + {\sim} a \\
{\sim}((a \circ b) + c) &\leq {\sim}(a \circ (b + c)) \\
a \circ (b + c) &\leq (a \circ b) + c
\end{aligned}$$

■

Remark 2.3.7 *The operator $+$ could be axiomatized by itself using axioms similar to those involving $\circ$. If so, then the above Theorem would be taken as an axiom. We do not do so here but point it out because it features in Distributed Relation Algebras when we consider two different forms of composition involving tensor and cotensor.*

There are many other properties that mirror $\circ$ such as

$$(a \rightharpoonup b)^{\smile} = b^{\smile} \leftharpoonup a^{\smile} \qquad (a + b)^{\smile} = b^{\smile} + a^{\smile}.$$

2.4 Relation Algebra Frames

The Kripke frames for RA are exactly what one would expect given frames for Classical Relevance Logics except that the Routley-Meyer * operator has been replaced by the weaker converse $^{\smile}$ operator. That is, they are collections of points which are maximal filters when the frames arise from an RA. There is a single three-place relation which is used in evaluating the monoid operation. Also, there is a set of "zero worlds" (using the terminology of Relevance Logic) used in evaluating the unit of the monoid.

This paper will assume that an RA frame will be denoted as $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ where $\mathcal{X} \subseteq X \times X \times X$ is the three-place relation on worlds and $\mathbb{X} \subseteq X$ is the collection of zero worlds. The symbol $\mathcal{X}$ is overloaded but since the three-place relation is so central to the frame, the reader is asked to overlook this and accept the simplicity it gives to the notation. Context will distinguish the two uses of "$\mathcal{X}$". For X and $\mathbb{X}$, this same letter in the two different fonts means different things, but both are related to the same structure.

The following definition is based on [ADP04] where Boolean Monoid Frames are used. Here, those frames are augmented with a "converse" operator on points.

Definition 2.4.1 A *Relation Algebra Frame*, $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$, is a structure where X is a set of points, $\mathcal{X} \subseteq X \times X \times X$, and $\mathbb{X} \subseteq X$ and $\mathbb{X} \neq \emptyset$. The following axioms, also called frame conditions, apply:

FA1. $\mathcal{X}^2 uvyz$ iff $\mathcal{X}^2 u(vy)z$; where

$$\mathcal{X}^2 uvyz \text{ iff } \exists x(\mathcal{X}uvx \text{ and } \mathcal{X}xyz) \qquad \mathcal{X}^2 u(vy)z \text{ iff } \exists w(\mathcal{X}uwz \text{ and } \mathcal{X}vyw).$$

FA2. there is some $z \in \mathbb{X}$ such that $\mathcal{X}xzx$ and $\mathcal{X}zxx$.

FA3. for all $y \in \mathbb{X}$, ($\mathcal{X}xyz$ or $\mathcal{X}yxz$) implies $x = z$.

FA4. $\mathcal{X}xyz$ implies $\mathcal{X}zy^{\smile}x$.

FA5. $\mathcal{X}xyz$ implies $\mathcal{X}x^{\smile}zy$.

These axioms can be augmented with the Tabularity Axiom

FA6. $\mathcal{X}xyz$ and $\mathcal{X}xy'z$ implies $y = y'$.

There are, of course, implicit universal quantifications given to the free variables in the frame conditions.

Lemma 2.4.2

$$\mathcal{X}xyz \text{ implies } \mathcal{X}y^{\smile}x^{\smile}z^{\smile}.$$

The proof falls right out of Frame Conditions FA4 and FA5. Canonically, we let $x^{\smile} = \{a^{\smile} \mid a \in x\}$ where x is a maximal filter. Given that $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$ and that $(\neg a)^{\smile} = \neg(a^{\smile})$, then $x^{\smile}$ is also a maximal filter. The consequence is that $a \in x^{\smile}$ iff $a^{\smile} \in x$. Also, just as in Relevance Logic,

$$\mathcal{X}xyz \text{ iff } \forall a, b(a \in x \text{ and } b \in y \text{ implies } a \circ b \in z) \qquad x \in \mathbb{X} \text{ iff } 1 \in x.$$

Also as in Relevance Logic, there are the following equivalent definitions of $\mathcal{X}xyz$

$$\mathcal{X}xyz \text{ iff } \forall a, b\ (b \leftarrow a \in x \text{ and } b \in y \text{ implies } b \in z) \qquad \mathcal{X}xyz \text{ iff } \forall a, b\ (a \in x \text{ and } a \rightarrow b \in y \text{ implies } b \in z)$$

where the commutativity of $\circ$ is not assumed. Just as in Relevance Logic, these definitions are equivalent because of residuation.

Theorem 2.4.3 *The modeling conditions FA1, FA2, FA3, FA4, and FA5 hold canonically in any Stone space arising as a dual space to an RA.*

Proof: That the frame conditions FA1, FA2, and FA3 hold is known from classical Relevance Logic's algebras (DeMorgan monoids) which shares the properties of the operator 1 and of $\circ$. Axioms FA4 and FA5 hold in the presence of the converse axioms. We show FA4:

Let $\mathcal{X}xyz$ and assume $a \in z$ and $a \rightarrow {\sim} b \in y^{\smile}$, then $(a \rightarrow {\sim} b)^{\smile} \in y$ and so $b^{\smile} \rightarrow {\sim} a^{\smile} \in y$. Towards a reductio, let ${\sim} b \notin x$, then $b \in x^{\smile}$ and so $b^{\smile} \in x$. From $\mathcal{X}xyz$, then ${\sim} a^{\smile} \in z$. Hence $a \notin z^{\smile\smile}$ and $a \notin z$, which is a contradiction. Thus, ${\sim} b \in x$ and $\mathcal{X}zy^{\smile}x$. ■

The algebra of sets can be extracted:

Definition 2.4.4 Let A and B be sets in the power set of points of a Relation Algebra Frame. The basic definitions are, where $\cdot$ is composition in the set algebra and the unit of the monoid is t:

$$z \in A \cdot B \text{ iff } \exists xy(\mathcal{X}xyz \text{ and } x \in A \text{ and } y \in B), \quad x \in \neg A \text{ iff } x \notin A, \quad x \in A^{\smile} \text{ iff } x^{\smile} \in A, \quad t = \mathbb{X}.$$

Using these definitions, it is straightforward to show

Lemma 2.4.5

$$(\neg A)^{\smile} = \neg(A^{\smile}).$$

Lemma 2.4.6 *The RA axioms are valid in Relation Algebra Frames*

Proof: We will only show one as an example, using the set algebra derived from the Relation Algebra Frames and noting that RA is equationally defined and hence has free algebras. The axioms M1, M3 through M4 and M5 are shown in [ADP04]. Axiom M2 is straightforward. Axiom M8 holds because defining the operator $\rightarrow$ (or $\leftarrow$) using the same relation as that for $\circ$ guarantees that axiom holds.

As an example of the axiom validity, let A and B be sets of points of a Relation Algebra Frame and $\cdot$ is the set theoretic correlate to $\circ$:

$$\begin{aligned} z \in (A \cdot B)^{\smile} & \text{ iff } z^{\smile} \in A \cdot B \\ & \text{ iff } \exists xy(\mathcal{X}xyz^{\smile} \text{ and } x \in A \text{ and } y \in B) \\ & \text{ iff } \exists xy(\mathcal{X}y^{\smile}x^{\smile}z \text{ and } x \in A \text{ and } y \in B) \\ & \text{ iff } \exists xy(\mathcal{X}y^{\smile}x^{\smile}z \text{ and } x^{\smile\smile} \in A \text{ and } y^{\smile\smile} \in B) \\ & \text{ iff } \exists uv(\mathcal{X}vuz \text{ and } u^{\smile} \in A \text{ and } v^{\smile} \in B) \\ & \text{ iff } \exists uv(\mathcal{X}vuz \text{ and } u \in A^{\smile} \text{ and } v \in B^{\smile}) \\ & \text{ iff } \exists uv(\mathcal{X}vuz \text{ and } v \in B^{\smile} \text{ and } u \in A^{\smile}) \\ & \text{ iff } z \in B^{\smile} \cdot A^{\smile} \end{aligned}$$

∎

A table of intensional operators can be generated by DeMorgan negation and tensor. The only condition we take for granted is that of $z \in A \cdot B$. The reason is because it is simple and it answers well to composition of relations.

Theorem 2.4.7 *The operators of the table below are all definable in terms of the composition operator* $\cdot$.

$z \in A \cdot B$	$z \in A \cdot B$	$\exists xy(\mathcal{X}xyz \text{ and } x \in A \text{ and } y \in B)$
$x \in \sim(A \cdot \sim B)$	$x \in B \leftarrow A$	$\forall yz(\mathcal{X}xyz \text{ and } y \in A \text{ implies } z \in B)$
$y \in \sim(\sim B \cdot A)$	$y \in A \rightarrow B$	$\forall xz(\mathcal{X}xyz \text{ and } x \in A \text{ implies } z \in B)$
$z \in \sim(\sim B \cdot \sim A)$	$z \in A + B$	$\forall xy(\mathcal{X}xyz \text{ implies } x \in A \text{ or } y \in B)$
$x \in B \cdot \sim A$	$x \in B \leftharpoondown A$	$\exists y,z(\mathcal{X}xyz \text{ and } y \notin A \text{ and } z \in B)$
$y \in \sim A \cdot B$	$y \in A \rightharpoondown B$	$\exists x,z(\mathcal{X}xyz \text{ and } x \notin A \text{ and } z \in B)$
$z \in \sim A \cdot \sim B$	$z \in A \downarrow B$	$\exists xy(\mathcal{X}xyz \text{ and } x^{\smile} \notin A \text{ and } y^{\smile} \notin B)$
$z \in \sim(A \cdot B)$	$z \in A \uparrow B$	$\forall xy(\mathcal{X}xyz \text{ implies } x^{\smile} \notin B \text{ or } y^{\smile} \notin A)$

Table 4: Intensional Operators of Sets

The first-order logic statements follow directly from the definitions. The definitions allow us to validate the following residuation conditions:

Theorem 2.4.8

$$\begin{aligned} & B \subseteq A \rightarrow C \text{ iff } A \cdot B \subseteq C \text{ iff } A \subseteq C \leftarrow B, \\ & C \leftharpoondown B \subseteq A \text{ iff } C \subseteq A + B \text{ iff } A \rightharpoondown C \subseteq B, \\ & A \downarrow B \subseteq C \text{ iff } B \downarrow C \subseteq A, \\ & A \subseteq B \uparrow C \text{ iff } C \subseteq A \uparrow B. \end{aligned}$$

Note that the first two residuation triads keep the position of the formulas on the left and right. That is, if B is the active formula in $A \cdot B$ or $A + B$, then it is on the right and continues to be on the right after residuation. That determines the direction of the arrows. This has to do with deriving $(a \leftarrow b) \circ b \leq a$ from $a \leftarrow b \leq a \leftarrow b$ and $a \leq (a \leftharpoondown b) + b$ from $a \leftharpoondown b \leq a \leftharpoondown b$, and these being the algebraic analogue of modus ponens. Note also that the left and right shifts using $\downarrow$ or $\uparrow$ are equivalent, i.e., two lefts yields a right, two rights yields a left.

Proof: As an example, we show $C \leftharpoondown B \subseteq A$ implies $C \subseteq A + B$. Let $C \leftharpoondown B \subseteq A$ and assume (in order) $z \in C$, $\mathcal{X}xyz$, and that $y \notin B$. This gives us $\exists uv(\mathcal{X}xuv$ and $u \notin B$ and $v \in C)$ and by definition, $x \in C \leftharpoondown B$. Since we assumed $y \notin B$ we have $y \notin B$ implies $x \in A$, which is $x \in A$ or $y \in B$. Since we assumed $\mathcal{X}xyz$, we have $\mathcal{X}xyz$ implies ($x \in A$ or $y \in B$). By definition, $z \in A + B$. Thus, $C \subseteq A + B$.

■

The following theorem shows cotensors as monoids in the set algebras. Cotensor will become another form of composition in the category theory:

Theorem 2.4.9 *Let $f = X - \mathbb{X}$, then the following formulas are sound:*

$$A + (B + C) \equiv (A + B) + C \qquad f + A \equiv A \equiv A + f.$$

2.5 Distribution of Tensor over Cotensor

The formula

$$A \cdot (B + C) \subseteq (A \cdot B) + C$$

is validated in the set algebras. The proof uses the Frame Condition FA1 similarly its use in showing associativity of $\cdot$ in Relevance Logic. However, it also requires converse given the definition of $+$ on the set algebras. To remove the use of converse requires a new frame condition:

$$\exists z(\mathcal{X}xyz \text{ and } \mathcal{X}uvz) \text{ implies } \exists w(\mathcal{X}uwx \text{ and } \mathcal{X}wyv).$$

To completely separate $\cdot$ and $+$ requires the use of two Kripke relations rather than one. This is in anticipation of the category theory below. Thus:

Theorem 2.5.1

$$A \cdot (B + C) \subseteq (A \cdot B) + C$$

holds in set algebras when the Relation Algebra Frames are augmented with a new three-place relation subject to a new frame condition

$$\mathcal{X}xyz \text{ and } \mathcal{Y}uvz \text{ implies } \exists w(\mathcal{X}uwx \text{ and } \mathcal{Y}wyv).$$

2.6 Relationship with Relevance Logic

In Relevance Logic [RMPB82], there is an axiomization of a star operator which we will denote with *. The star operator is known from the semantics where it is used on worlds to interpret DeMorgan negation $\sim$:

$$x \models \sim A \text{ iff } x^* \not\models A.$$

In the context of Relevance Logic, worlds canonically are prime filters. It turns out that $\sim x$, the DeMorgan negation applied pointwise to the elements of the prime filter x, is a a prime ideal because $\sim$ is order-reversing. The complement of a prime ideal is a prime filter. Hence canonically,

$$x^* = \mathcal{A} - \sim x.$$

where $\mathcal{A}$ is the carrier set of the DeMorgan monoid we started with and where $\sim$ is applied pointwise to every element of the filter x. There is one key axiom of *:

$$\mathcal{X}xyz \text{ implies } \mathcal{X}z^*yx^*$$

This property is unacceptable, now we go about showing precisely why.

Relevance Logic (without classical negation) has the following in its modeling axioms:

$$x^{**} = x \qquad \mathcal{R}xyz \text{ implies } \mathcal{R}xz^*y^*.$$

This works in the presence of commutativity in the first two positions to yield

$$\mathcal{X}xyz \text{ implies } \mathcal{X}z^*yx^*.$$

The set of possibilities using the Relational Algebra Frames is

$$\mathcal{X}xyz \quad \mathcal{X}zy\breve{}x \quad \mathcal{X}x\breve{}zy \quad \mathcal{X}z\breve{}xy\breve{} \quad \mathcal{X}yz\breve{}x\breve{} \quad \mathcal{X}y\breve{}x\breve{}z\breve{}$$

The set of possibilities using Relevance Logic frames is

$$\mathcal{X}xyz \quad \mathcal{X}xz^*y^* \quad \mathcal{X}yxz \quad \mathcal{X}z^*xy^* \quad \mathcal{X}yz^*x^* \quad \mathcal{X}z^*yx^*$$

Theorem 2.6.1 *Assuming the axioms for* $\breve{}$ *with* * *replacing* $\breve{}$*, then the axiom for* * *and its commutative equivalents are obtained.*

Proof:

$$\begin{array}{rlrl} \mathcal{X}xyz & \text{implies } \mathcal{X}x^*zy & \qquad \mathcal{X}xyz & \text{implies } \mathcal{X}zy^*x \\ & \text{implies } \mathcal{X}yz^*x^* & & \text{implies } \mathcal{X}z^*xy^* \end{array}$$

■

Theorem 2.6.2 *The relational formulas* $\mathcal{X}zy^*x$ *and* $\mathcal{X}x^*zy$ *which correspond to the Relational Algebra Frame conditions* $\mathcal{X}zy\breve{}x$ *and* $\mathcal{X}x\breve{}zy$ *are not obtainable from the Relevance Logic frame conditions.*

Proof: The Relevance Logic frame conditions are unable to generate any relational formulas with simply a single instance of *. Those frame conditions either commute or, modulo the period 2 property introduce two * operators. ■

Relevance logic in general conflates $\mathcal{X}xyz$ and $\mathcal{X}yxz$ because the algebras have the axiom $a \circ b = b \circ a$. This is sometimes built into the other axioms for the frames which has the knock on effect of conflating $a \rightarrow b$ with $b \leftarrow a$. Let us separate these, then one might think to use the relevance algebra axioms:

CR1. $(b \rightarrow \sim a) \rightarrow (\sim b \leftarrow a)$

CR2. $(\sim b \leftarrow a) \rightarrow (b \rightarrow \sim a)$

and the following rules from Classical Relevance Logic [RMPB82]:

CR3. $a \rightarrow b$ implies $a^* \rightarrow b^*$

CR4. $b \leftarrow a$ implies $b^* \leftarrow a^*$

where the * operator has been added to the logic and has the evaluation condition

$$x \models a^* \text{ iff } x^* \models a.$$

The rules are straightforwardly proven and do not add additional properties to negation not involving *. Incidentally, in Classical Relevance Logic, any one of $\sim$, $\neg$, and * can be defined from the other two.

In both the Relevance Algebras and Relational Algebra have the same axioms for contraposition. However, Relevance Logic also conflates $\rightarrow$ and $\leftarrow$, so in effect Relevance Algebra contains

$$a \rightarrow \sim b \rightarrow (b \rightarrow \sim a).$$

This of course leads to

$$a \circ b = b \circ a$$

which is not available in RA.

So the moral of the story is that the difference between Relevance Algebras and Relation Algebras is that $\sim$ for Relation Algebras can be defined using the Boolean $\neg$. If the Relevance Algebras are promoted to Classical Relevance Algebras, then the commutativity of $\circ$ destroys the difference between $\rightarrow$ and $\leftarrow$.

3 Relation Algebra of Sets

The collection of operators fill out a Dunn style Gaggle Theory [Dun90, AD93] for the relation operators. We extract binary relations from a Relation Algebra Frame. The extraction is in preparation for Distributed Relation Algebra of the next section. The Distributed Relation Algebra Frame will then be dropped out and the relations treated as 1-arrows in a 2-category. Throughout this section we assume an ambient Relation Algebra Frame $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$.

The paper [ADP04] shows how to extract a Boolean monoid of relations. RAs are at least Boolean monoids. We were able to extract two entailment operators residuated with relational composition of binary relations from a Boolean monoid frame. We employ a similar strategy here for the collection of RA operators. The extracted relations are either sets or multisets of pairs of points. Without Tabularity, they are generally multisets of pairs of points. Tabularity squeezes out duplicate pairs. We will use the locution *relation algebra of sets* rather than "relation algebra", the latter is use only for the abstract algebraic characterization using the algebra axioms in Section 2.

3.1 The Extracted Relations

The representation of Boolean monoids relied on the dual Stone space and a three-place relation induced by the monoid operator. Due to the two entailment operators being residuated with the monoid operator, the same three-place replace relation can be used to represent the entailment operators. Elements of an algebra of sets extracted from a frame use Roman upper case letters, i.e., A, whereas the relations we extract either use Roman upper case letters surrounded by brackets, i.e. $\langle A \rangle$.

Definition 3.1.1 Assume a Boolean monoid frame $X = \langle X, \mathcal{X}, \mathbb{X} \rangle$ (just like a Relation Algebra Frame without the $\breve{}$ frame conditions). Let A be an element of the set algebra, then

$$\langle A \rangle = \{\langle x, z \rangle \mid \exists y(\mathcal{X}xyz \text{ and } y \in A)\}.$$

Notice that the algebra of relations does not have $X \times X$ as the top element for an ambient set X. Rather, it has as a top element

$$\langle X \rangle = \{\langle x, z \rangle \mid \exists y(\mathcal{X}xyz)\}.$$

Definition 3.1.2 Elements of a Boolean monoid $(L, \wedge, \vee, \neg, \circ)$ are represented the following way with β the representation function:

$$\beta(a) = \{x \mid x \text{ is a maximal filter and } a \in x\}.$$

From [ADP04], we have the following theorem

Theorem 3.1.3 *Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame with the additional Tabularity Axiom*

$$\mathcal{X}xyz \text{ and } \mathcal{X}xy'z \text{ implies } y = y,$$

Then $\langle \ldots \rangle$ is a homomorphism from the Boolean monoid of sets to a Boolean monoid of relations. Specifically,

$$\langle A \cup B \rangle = \langle A \rangle \cup \langle B \rangle \qquad \langle X - A \rangle = \langle X \rangle - \langle A \rangle \qquad \langle A \cdot B \rangle = \langle A \rangle \circledcirc \langle B \rangle \qquad \langle \mathbb{X} \rangle = \{\langle x, x \rangle\}$$

where $\circledcirc$ is relational composition.

Proof: We show $\langle X - A \rangle = \langle X \rangle - \langle A \rangle$, the rest are much easier where $\langle A \cdot B \rangle = \langle A \rangle \circledcirc \langle B \rangle$ requires the Frame Condition FA1.

Assume $\langle x, z \rangle \in \langle X \rangle - \langle A \rangle$. Then there is some y such that $\mathcal{X}xyz$ and $\langle x, z \rangle \notin \langle A \rangle$. Therefore $y \notin A$, and hence $y \in X - A$, so $\langle x, z \rangle \in \langle X - A \rangle$.

Assume $\langle x, z \rangle \in \langle X - A \rangle$, then there is at least one y such that $\mathcal{X}xyz$ and $y \notin A$. Also, $\langle x, z \rangle \notin \langle A \rangle$ since otherwise there would be some $y' \in A$ and $\mathcal{X}xy'z$. Given Tabularity, $y = y'$ and that would be a contradiction. Hence $\langle x, z \rangle \in \langle X \rangle$ and $\langle x, z \rangle \notin \langle A \rangle$, yielding $\langle x, z \rangle \in \langle X \rangle - \langle A \rangle$. ∎

The Tabularity Axiom essentially says that $\langle X \rangle$ i.e., the largest relation, and consequently every $\langle A \rangle$ can be "tabulated" with a monic pair [FS90] of functions.

3.2 Extracted Relation Operators

The extraction is via the three-place accessibility relation of a frame. We generally assume a *Relation Algebra Frame*, $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ with Tabularity defined in Section 2.4. A relation then uses the following prescription for A a member of the set algebra, $\mathcal{P}(X)$ where $\mathcal{P}(-)$ is the powerset operator:

Definition 3.2.1

$$\langle x, z\rangle \in \langle A\rangle \text{ iff } \exists y(\mathcal{X}xyz \text{ and } y \in A) \qquad \langle A\rangle^{\smile} = \{\langle z, x\rangle \mid \langle x, z\rangle \in \langle A\rangle\} \qquad \neg\langle A\rangle = X - \langle A\rangle$$

In Theorem 3.1.3 we already have $\langle X - A\rangle = \langle X\rangle - \langle A\rangle$. We change the notation slightly:

Theorem 3.2.2

$$\neg\langle A\rangle = \langle \neg A\rangle \qquad \langle A\rangle^{\smile} = \langle A^{\smile}\rangle.$$

The proof of the first we showed above and the proof of the second is straightforward.

Corollary 3.2.3

$$\sim\langle A\rangle = \langle \sim A\rangle.$$

In Theorem 3.1.3, we had

$$\langle A \cdot B\rangle = \langle A\rangle \circledcirc \langle B\rangle.$$

Using similar proofs and Table 4, we get the following

Theorem 3.2.4 *The operators satisfy the following table:*

$\langle A\rangle \circledcirc \langle B\rangle = \langle A \cdot B\rangle$	$\langle A\rangle \oplus \langle B\rangle = \langle A + B\rangle$
$\langle B\rangle \leftarrowtriangle \langle A\rangle = \langle B \leftarrow A\rangle$	$\langle B\rangle \leftharpoonup \langle A\rangle = \langle B \leftharpoonup A\rangle$
$\langle A\rangle \rightarrowtriangle \langle B\rangle = \langle A \rightarrow B\rangle$	$\langle A\rangle \rightharpoonup \langle B\rangle = \langle A \rightharpoonup B\rangle$
$\langle A\rangle \downarrow \langle B\rangle = \langle A \downarrow B\rangle$	$\langle A\rangle \uparrow \langle B\rangle = \langle A \uparrow B\rangle$

Table 5: Relational Operators Representation

We get the following the derivation of the relational form of the intensional operators:.

Theorem 3.2.5 *The intensional operators follow straightforwardly from their definition via $\cdot$ and $\sim$:*

$\langle x.z\rangle \in \langle A\rangle$	iff	$\exists y(\mathcal{X}xyz \text{ and } y \in A)$
$\langle x, z\rangle \in \langle A\rangle \circledcirc \langle B\rangle$	iff	$\exists y(\langle x, y\rangle \in \langle A\rangle \text{ and } \langle y, z\rangle \in \langle B\rangle)$
$\langle y, z\rangle \in \langle A\rangle \rightarrowtriangle \langle B\rangle$	iff	$\forall x(\langle x, y\rangle \in \langle A\rangle \text{ implies } \langle x, z\rangle \in \langle B\rangle)$
$\langle y, z\rangle \in \langle B\rangle \leftarrowtriangle \langle A\rangle$	iff	$\forall x(\langle z, x\rangle \in \langle A\rangle \text{ implies } \langle y, x\rangle \in \langle B\rangle)$
$\langle x, z\rangle \in \langle A\rangle \oplus \langle B\rangle$	iff	$\forall y(\langle x, y\rangle \in \langle A\rangle \text{ or } \langle y, z\rangle \in \langle B\rangle)$
$\langle x, z\rangle \in \langle A\rangle \rightharpoonup \langle B\rangle$	iff	$\exists y(\langle y, x\rangle \notin \langle A\rangle \text{ and } \langle y, z\rangle \in \langle B\rangle)$
$\langle x, z\rangle \in \langle B\rangle \leftharpoonup \langle A\rangle$	iff	$\exists y(\langle x, y\rangle \in \langle B\rangle \text{ and } \langle z, y\rangle \notin \langle A\rangle)$
$\langle x, z\rangle \in \langle A\rangle \downarrow \langle B\rangle$	iff	$\exists y(\langle y, x\rangle \notin \langle A\rangle \text{ and } \langle z, y\rangle \notin \langle B\rangle)$
$\langle x, z\rangle \in \langle A\rangle \uparrow \langle B\rangle$	iff	$\forall y(\langle y, x\rangle \notin \langle B\rangle \text{ or } \langle z, y\rangle \notin \langle A\rangle)$

Table 6: Relational Operators

The proofs are just follow directly from the definitions.

Theorem 3.2.6 *The intensional relational operators satisfied the distribution and residuation properties of Tables 2 and 3 in this set theoretic form, e.g.,*

$$\langle A\rangle \rightarrowtriangle (\langle B\rangle \cap \langle C\rangle) = (\langle A\rangle \rightarrowtriangle \langle B\rangle) \cap (\langle A\rangle \rightarrowtriangle \langle C\rangle) \qquad \langle C\rangle \subseteq \langle A\rangle \oplus \langle B\rangle \text{ iff } \langle C\rangle \leftharpoonup \langle B\rangle \subseteq \langle A\rangle.$$

Proof: The proofs follow directly from the definitions. We show an example of the residuation condition above. Let $\langle C\rangle \subseteq \langle A\rangle \oplus \langle B\rangle$ and assume $\langle x,z\rangle \in \langle C\rangle \leftharpoonup \langle B\rangle$. From the definition of $\leftharpoonup$, for some y, $\langle x,y\rangle \in \langle C\rangle$ and $\langle z,y\rangle \notin \langle B\rangle$, and $\langle x,y\rangle \in \langle A\rangle \oplus \langle B\rangle$. The definition of $\oplus$ gives us $\forall u(\langle x,u\rangle \in \langle A\rangle$ or $\langle u,y\rangle \in \langle B\rangle)$. Eliminating the universal quantifier with z for u yields $\langle x,z\rangle \in \langle A\rangle$ or $\langle z,y\rangle \in \langle B\rangle$. Thus, $\langle x,z\rangle \in \langle A\rangle$.

To go in the other direction, let $\langle C\rangle \subseteq \langle A\rangle \oplus \langle B\rangle$ and assume $\langle x,z\rangle \in \langle C\rangle$. Towards a reductio, let $\langle x,z\rangle \notin \langle A\rangle \oplus \langle B\rangle$. From the definition of $\oplus$, we have for some y, $\langle x,y\rangle \notin \langle A\rangle$ and $\langle y,z\rangle \notin \langle B\rangle$. Therefore, $\langle x,z\rangle \in \langle C\rangle$ and $\langle y,z\rangle \notin \langle B\rangle$. By definition, $\langle x,y\rangle \in \langle C\rangle \leftharpoonup \langle B\rangle$ and hence $\langle x,y\rangle \in \langle A\rangle$, which is a contradiction. Therefore $\langle x,z\rangle \in \langle A\rangle \oplus \langle B\rangle$.

■

The following theorem is a direction consequence of the definitions:

Theorem 3.2.7 *Tensors and cotensors satisfy the following distribution law:*

$$\langle A\rangle \circledcirc (\langle B\rangle \oplus \langle C\rangle) \subseteq (\langle A\rangle \circledcirc \langle B\rangle) \oplus \langle C\rangle.$$

3.3 Cotensors as a Monoid of Relations

Definition 3.3.1

$$\langle t\rangle \stackrel{def}{=} \{\langle x,z\rangle \mid \exists u(\mathcal{X}xuz \text{ and } u \in \mathbb{X})\}$$

where here t is under interpretation as a collection of "zero" worlds.

Given the axioms for frames,

$$\forall u \in \mathbb{X}(\mathcal{X}xuz \text{ implies } x = z) \qquad x = z \text{ implies } \exists u \in \mathbb{X}(\mathcal{X}xux).$$

From this and Tabularity, it is straightforward to see that

$$\langle t\rangle = \{\langle x,x\rangle\} = \mathcal{I}_X$$

for $\mathcal{I}_X$ the identity relation on X. Hence

$$\langle f\rangle = {\sim}\langle t\rangle = \neg\langle t\rangle^{\smile} = \neg\langle t\rangle = (X \times X) - \mathcal{I},$$

for $\mathcal{I}$ the diagonal relation on the ambient Stone space whose point set is X.

Theorem 3.3.2

$$\langle A\rangle \oplus (\langle B\rangle \oplus \langle C\rangle) = (\langle A\rangle \oplus \langle B\rangle) \oplus \langle C\rangle \qquad \langle f\rangle \oplus \langle A\rangle = \langle A\rangle = \langle A\rangle \oplus \langle f\rangle.$$

4 Distributed Relation Algebra

In [AHR17], we presented a distributed relation logic which used a three-place relation to interpret the logic. The logic had a Boolean base in that every *local logic*, i.e., the logic at a locality (node in the multigraph parameterizing the logic) had a Boolean base. The interpretation used typed three-place relations much like that used for Relevance Logics. The intuitive picture is as follows:

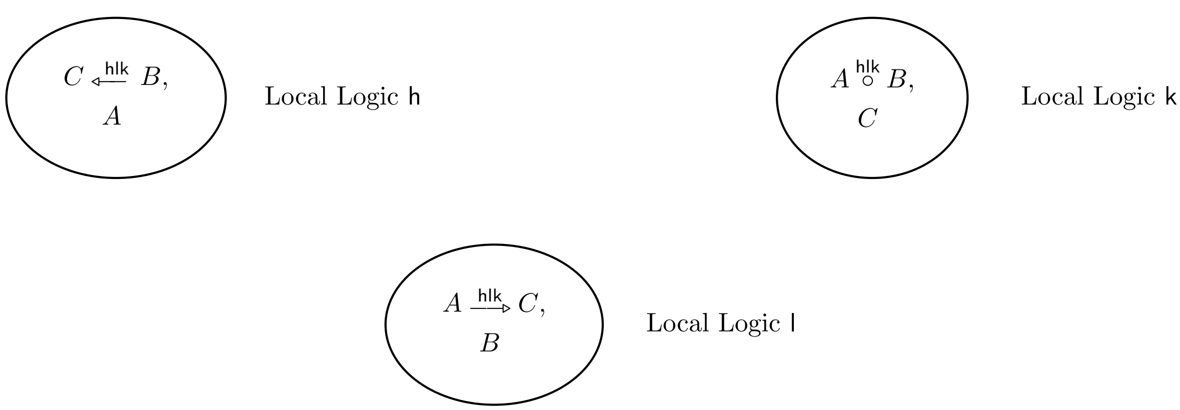

Figure 1: Distributing the Two-Place Connectives

using the convention:

Connective	Type h	Type l	Type k
$\overset{\mathsf{hlk}}{\circ}$	A	B	$A \overset{\mathsf{hlk}}{\circ} B$
$\xrightarrow{\mathsf{hlk}}$	A	$A \xrightarrow{\mathsf{hlk}} C$	C
$\xleftarrow{\mathsf{hlk}}$	$C \xleftarrow{\mathsf{hlk}} B$	B	C

Figure 2: Localities and their Logics

Each proposition A, B, and C is part of the local logic at locality. We use the terminology of *type* h to talk about a locality h. Note the positions of the input and output types of each operator shift depending upon the operator. This matches the corresponding position of the formula containing the operator in the semantic definitions using the interpreting relations. The relation $\mathcal{R}^{\mathsf{hlk}}$ means $\mathcal{R} \subseteq H \times L \times K$ for sets of points H, L, K at localities $\mathsf{h}, \mathsf{l}, \mathsf{k}$ respectively:

- $z \overset{\mathsf{k}}{\models} A \overset{\mathsf{hlk}}{\circ} B$ iff $\exists x, y(x \overset{\mathsf{h}}{\models} A$ and $y \overset{\mathsf{l}}{\models} B$ and $\mathcal{R}^{\mathsf{hlk}}xyz)$,
- $y \overset{\mathsf{l}}{\models} A \xrightarrow{\mathsf{hlk}} C$ iff $\forall x, z(x \overset{\mathsf{h}}{\models} A$ and $\mathcal{R}^{\mathsf{hlk}}xyz$ implies $z \overset{\mathsf{k}}{\models} C)$,
- $x \overset{\mathsf{h}}{\models} C \xleftarrow{\mathsf{hlk}} B$ iff $\forall y, z(\mathcal{R}^{\mathsf{hlk}}xyz$ and $y \overset{\mathsf{l}}{\models} B$ implies $z \overset{\mathsf{k}}{\models} C)$.

In the set algebras, $\in_{\mathsf{k}}$ replaces $\overset{\mathsf{k}}{\models}$ and similarly for the rest.

4.1 Distributed Relation Algebra Axioms

The distribution structure is hypergraph of a collection of nodes with certain three tuples identified as *cliques*[2] with three elements, say, hlk. The diagram on the right displays a clique as a multigraph. We need a way of abstracting over cliques in a notationally convenient way. We use a string such as hlk to refer to an arbitrary clique. For any one clique hlk, we assume h refers to a node h and similarly for the rest. When we wish to restrict reference to a clique such that the clique has all the same members, we use hhh. We will also have need to abstract over the members of an arbitrary clique, we use the variables h, l, k to range over an arbitrary clique hlk with the restriction that the variables refer to pairwise distinct positions in hlk. $h, k, l \Subset \mathsf{hlk}$ denotes this. h can take on any of the values h, k, and l and similarly with l and k, respecting the pairwise distinct restriction. When abstracting over the clique hhh, the variables h, l, and k are still respecting the condition of no two referring to the same position in the clique. The locution $A \in \mathsf{h}$ means that the formula A is a member of the local logic at node h.

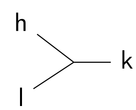

Figure 3: Three Place Relation $\mathcal{R}^{\mathsf{hlk}}$

A generic *distributed semigroup operator* has the tying structure $h \times l \twoheadrightarrow k$. Our version of heterogenous algebras is contained in the following definition:

Definition 4.1.1 A Distributed Relation Algebra (DRAlg) contains a distribution structure $\mathfrak{G}$ of nodes called *types* and *operators* with type in a multiset of cliques $\mathfrak{C}$. A type for h is a *converse algebra* $(D_{\mathsf{h}}, \wedge, \vee, \neg, \bot, \top, \breve{\ })$ (called the *local converse algebra at* h) LCAlg where $(D_{\mathsf{h}}, \wedge, \vee, \neg, \bot, \top)$ is a Boolean lattice with an extra operator $\breve{\ }$. $\bot$ and $\top$ are the bottom and top of the lattice respectively. The *distributed operators* are of the form $\overset{\mathsf{hlk}}{\circ} : D_{\mathsf{h}} \times D_{\mathsf{l}} \twoheadrightarrow D_{\mathsf{k}}$ where we use the indices h, l, and k to index the carrier sets of the types. The phyla of heterogeneous algebras are the carrier sets. A DRAlg respects the axioms below on the right. The original axioms from Chin and Tarski and Ng [CT51, Ng84] are on the left and the new typed axioms are on the right.

[2]We view three-place relations as arc in a hypergraph: arcs connect more than one node, three always in our case, and multiple arcs are permitted as well as loops.

M1	$(B, \vee, \neg, \bot, \top)$ is a Boolean algebra	$(D_h, \vee, \neg, \breve{}, \bot_h, \top_h)$ is a LCAlg for each sort h
M2	$(a \circ b) \circ c = a \circ (b \circ c)$	$(a \overset{\mathsf{hlm}}{\circ} b) \overset{\mathsf{mno}}{\circ} c \overset{\circ}{=} a \overset{\mathsf{hno}}{\circ} (b \overset{\mathsf{lkn}}{\circ} c)$, for cliques hlm, mno, hno, and lkn.
M3	$(a \vee b) \circ c = (a \circ c) \vee (b \circ c)$	$(a \vee \hat{a}) \overset{hlk}{\circ} b \overset{k}{=} (a \overset{hlk}{\circ} b) \vee (\hat{a} \overset{hlk}{\circ} b)$, $h, l, k \isinE \mathsf{hlk}$ and $\mathsf{hlk} \in \mathfrak{C}$
M4	$a \circ 1 = a$ for any $a \in A$	$a \overset{\mathsf{hhh}}{\circ} 1 \overset{\mathsf{h}}{=} a, \quad \mathsf{hhh} \in \mathfrak{C}$
M5	$a^{\breve{}\,\breve{}} = a$ for any $a \in A$	$a^{\breve{}\,\breve{}} \overset{\mathsf{h}}{=} a$
M6	$(a \vee b)^{\breve{}} = a^{\breve{}} \vee b^{\breve{}}$ for any $a, b \in A$	$(a \vee \hat{a})^{\breve{}} \overset{\mathsf{h}}{=} a^{\breve{}} \vee \hat{a}^{\breve{}}$
M7	$(a \circ b)^{\breve{}} = b^{\breve{}} \circ a^{\breve{}}$, for any $a, b \in A$	$(a \overset{hlk}{\circ} b)^{\breve{}} \overset{k}{=} b^{\breve{}} \overset{lhk}{\circ} a^{\breve{}} \qquad h, l, k \isinE \mathsf{hlk} \in \mathfrak{C}$
M8	$(a^{\breve{}} \circ \neg(a \circ c)) \vee \neg c = \neg c$ for any $a, c \in A$	$(a^{\breve{}} \overset{hlk}{\circ} \neg(a \overset{hkl}{\circ} c)) \vee \neg c \overset{k}{=} \neg c \qquad h, l, k \isinE \mathsf{hlk} \in \mathfrak{C}$

The notation $h, l, k \isinE \mathsf{hlk} \in \mathfrak{C}$ means that h, l, k refer to members of the set $\{\mathsf{h}, \mathsf{k}, \mathsf{l}\}$ and that hlk is a clique in $\mathfrak{C}$; h need not refer to h and similarly for the rest.

4.2 Derivation of Relational Operators

In Figure 2, each locality could support a three-place relation in the sense that $\mathcal{R}^{\mathsf{uhv}}$, and for sets H, I, K and sets U, V, and Z, let $u \in U$, $v \in V$ and $z \in Z$. Define the relations $\langle A \rangle$ with

$$\langle A \rangle \overset{def}{=} \{\langle u, v \rangle \mid \exists h \in H(\mathcal{R}^{\mathsf{uhv}} uhv \text{ and } h \in_{\mathsf{h}} A)\}$$
$$\langle B \rangle \overset{def}{=} \{\langle v, z \rangle \mid \exists l \in L(\mathcal{R}^{\mathsf{viz}} vlz \text{ and } i \in_{\mathsf{i}} B)\}$$
$$\langle C \rangle \overset{def}{=} \{\langle u, z \rangle \mid \exists k \in K(\mathcal{R}^{\mathsf{ukz}} ukz \text{ and } k \in_{\mathsf{k}} C)\}$$

In the locality graph of Figure 2, consider each oval to be a set of points of local Kripke frame and let H, L, K be the sets of points of the local frame at localities $\mathsf{h}, \mathsf{l}, \mathsf{k}$ respectively. Similarly, U, V, Z are the sets of points of the local frame at localities $\mathsf{u}, \mathsf{v}, \mathsf{z}$. We have the following diagram (where cliques like Figure 3 have been replaced by triangles):

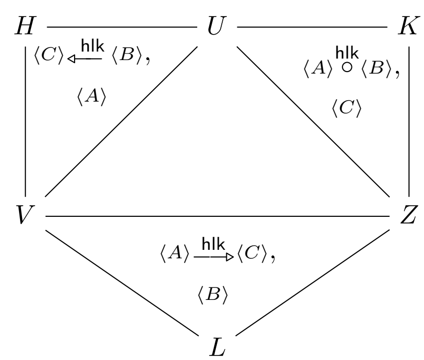

Figure 4: Diagram of 3-Place Relations

Assuming the Tabularity Axiom now on three-place typed relations, the sets H, I, and K recede into the background. We change notation because the angle brackets are now superfluous. Let $\mathcal{T} = \langle A \rangle$, $\mathcal{S} = \langle B \rangle$, and $\mathcal{R} = \langle C \rangle$. By fiat we declare these relations to be arrow with a domain and codomain matching their first and second positions. Hence $\mathcal{T} : U \to V$, $\mathcal{S} : V \to Z$, and $\mathcal{R} : U \to Z$. This yields the same collection of relation operators but now as arrows in a category of sets:

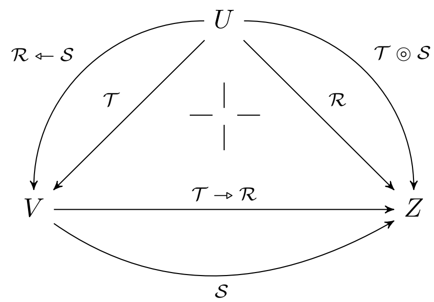

Figure 5: Diagram of Relation Operators

where the puncture mark (from [FS90] indicates the inner triangle does not commute. A similar story holds for all the operators. Hence we have the following definition:

Definition 4.2.1

$$
\begin{array}{rcl}
\langle u, z\rangle \in \mathcal{T} \circledcirc \mathcal{S} & \text{iff} & \exists v \in V(\langle u, v\rangle \in \mathcal{T} \text{ and } \langle v, z\rangle \in \mathcal{S}) \\
\langle v, z\rangle \in \mathcal{T} \multimap \mathcal{R} & \text{iff} & \forall u \in U(\langle u, v\rangle \in \mathcal{T} \text{ implies } \langle u, z\rangle \in \mathcal{R}) \\
\langle u, v\rangle \in \mathcal{R} \multimapinv \mathcal{S} & \text{iff} & \forall z \in Z(\langle v, z\rangle \in \mathcal{S} \text{ implies } \langle u, z\rangle \in \mathcal{R}) \\
\hline
\langle u, z\rangle \in \mathcal{T} \oplus \mathcal{S} & \text{iff} & \forall v \in V(\langle u, v\rangle \in \mathcal{T} \text{ or } \langle v, z\rangle \in \mathcal{S}) \\
\langle v, z\rangle \in \mathcal{T} \rightharpoonup \mathcal{R} & \text{iff} & \exists u \in U(\langle u, v\rangle \notin \mathcal{T} \text{ and } \langle u, z\rangle \in \mathcal{R}) \\
\langle u, v\rangle \in \mathcal{R} \leftharpoondown \mathcal{S} & \text{iff} & \exists z \in Z(\langle u, z\rangle \in \mathcal{R} \text{ and } \langle v, z\rangle \notin \mathcal{S}) \\
\hline
\langle u, z\rangle \in \breve{\mathcal{T}} \downarrow \breve{\mathcal{S}} & \text{iff} & \exists v \in V(\langle v, u\rangle \notin \breve{\mathcal{T}} \text{ and } \langle z, v\rangle \notin \breve{\mathcal{S}}) \\
\langle u, z\rangle \in \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}} & \text{iff} & \forall v \in V(\langle v, u\rangle \notin \breve{\mathcal{T}} \text{ or } \langle z, v\rangle \notin \breve{\mathcal{S}})
\end{array}
$$

Table 7: Relational Operators

with the warning that the relations $\breve{\mathcal{T}}$ and $\breve{\mathcal{S}}$ used in $\downarrow$ and $\uparrow$ run in the opposite directions as $\mathcal{T}$ and $\mathcal{S}$ used in the rest. This is indicated by the converse accents above $\mathcal{T}$ and $\mathcal{S}$, i.e., $\breve{\mathcal{T}}$ and $\breve{\mathcal{S}}$. To use the same letters in $\downarrow$ and $\uparrow$ as the other operations require the relations to which they refer be turned around. We could have used new letters used but this would prevent those arrows from being compared to the arrows in the previous operators. Hence we use the converse $\breve{}$ now as an accent. The reader can either treat the $\breve{}$ as an actual converse operator on the arrow or simply treat it as an accent symbol; we prefer the latter as mere notational convention.

5 Kan Extensions and Lifts

The definition of a right Kan lift (see below) was taken from [Str83] and the notion of 2-category from [Bén67]. The subsequent definitions of Kan extensions and right Kan lifts were extrapolations. Kan extensions exist in MacLane [Mac98] but for functors and natural transformations, they provide the objects and arrows for the 2-categories that Bénabou uses.

We make a subtle point that can be confusing if not brought out. In the extensions and lifts, we never require the underlying graph of 0-objects and 1-arrows to actually be a category. For the Kan extensions and lifts this point is moot because the underlying graph is a category. However for intensional nor and nand this is not the case. We only require a graph with 0-arrows (objects) and 1-arrows. The 2-category has as objects the 1-arrows and as arrows the 2-arrows. We compose 1-arrows, but do not require any equations for the composition unless they involve 2-arrows. This does not affect the mathematics. We will use the locution "2-category qua category" for structure of a 1-graph with a 2-category on top when necessary. Put another way, the lifts and extensions are statements about the 2-categories qua categories.

The upper diagrams in the extensions and lifts, e.g. Figure 11, using the diagrammatic style of [FS90], do not have their large triangles commuting. Putting in the puncture marks of [FS90] would make the diagrams even noisier than they are currently so we elide them with the understanding they are there. The commuting conditions are actually part of the lower diagrams on the 2-categories qua categories. This becomes more noticeable in the diagrams for ↓ and ↑ given the directions of the arrows. This is brought out in the statements of the lifts and extensions which do not involve of commuting diagrams of the 1-arrows. The commutation stated is for the 2-arrows.

The proofs for the adjunctions were taken from [Mac98] but adapted to our use of 2-categories qua categories.

5.1 2-Diagrams

The following diagram of right whiskering comes from the data for Right Kan Lifts (see below) The objects in the following diagrams are called 0-cells, the single arrows are 1-cells, and the double are 2-cells. The 2-diagram on the left as equivalent to the two diagrams on the right in Figure 6 where $Z^{\mathcal{T}}$ is a functor on the 2-category qua category and merely a mapping on the 1-arrows; using the notation of 2-categories, $Z^{\mathcal{T}}\sigma = \sigma \triangleleft \mathcal{T}$. This reads just as in the usual category theory: first $\mathcal{T}$ then σ, or if you like, σ applied to $\mathcal{T}$:

$$
\begin{tikzcd}
U \arrow[r, "\mathcal{T}"] \arrow[rr, bend left=60, "\mathcal{R}"] & V \arrow[u, phantom, "\Uparrow \varepsilon_{\mathcal{R}}"] \arrow[r, bend left, "\mathcal{X}"] \arrow[r, bend right, "\mathcal{S}"'] \arrow[r, phantom, "\Uparrow \sigma"] & Z
\end{tikzcd}
=
\begin{tikzcd}
U \arrow[r, bend left=70, "\mathcal{R}"] \arrow[r, bend left, "Z^{\mathcal{T}}\mathcal{X}"] \arrow[r, bend right, "Z^{\mathcal{T}}\mathcal{S}"'] \arrow[r, phantom, "\Uparrow Z^{\mathcal{T}}\sigma"] & Z
\end{tikzcd}
=
\begin{tikzcd}
U \arrow[r, bend left, "\mathcal{R}"] \arrow[r, bend right, "Z^{\mathcal{T}}\mathcal{S}"'] \arrow[r, phantom, "\Uparrow \varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma)"] & Z
\end{tikzcd}
$$

Figure 6: Right Whiskering of 2-Diagrams

The diagram is telling us that Z^U is a functor on the 2-category qua category, i.e. for $\tau : \mathcal{S} \Rightarrow \mathcal{X}$ and $\rho : \mathcal{X} \Rightarrow \mathcal{Q}$,

$$Z^{\mathcal{T}}(\rho \circ \tau) = (Z^{\mathcal{T}}\rho) \circ (Z^{\mathcal{T}}\tau).$$

A similar right whiskering diagram used by Left Kan Extension with $\eta_{\mathcal{R}}$ in the opposite direction of and replacing $\varepsilon_{\mathcal{R}}$ is had by turning all the 2-arrows around, i.e., $\rho : \mathcal{Q} \Rightarrow \mathcal{X}$ and $\tau : \mathcal{X} \Rightarrow \mathcal{S}$, and

$$Z^{\mathcal{T}}(\tau \circ \rho) = (Z^{\mathcal{T}}\tau) \circ (Z^{\check{\mathcal{T}}}\rho).$$

Similarly, the following diagram of left whiskering comes from the data for Right Kan Lifts (see below). Similar to right whiskering, $U^{\mathcal{S}}$ is a functor on the 2-category qua category and merely a mapping on the 1-arrows; using the notation of 2-categories, $U^{\mathcal{S}}\sigma = \mathcal{S} \triangleright \sigma$, and reads: first σ then $\mathcal{T}$, or if you like, which is still σ applied to $\mathcal{T}$ except now from the right:

$$
\begin{tikzcd}
U \arrow[r, bend left, "\mathcal{X}"] \arrow[r, bend right, "\mathcal{T}"'] \arrow[r, phantom, "\Uparrow \sigma"] \arrow[rr, bend left=60, "\mathcal{R}"] & V \arrow[u, phantom, "\Uparrow \varepsilon_{\mathcal{R}}"] \arrow[r, "\mathcal{S}"] & Z
\end{tikzcd}
=
\begin{tikzcd}
U \arrow[r, bend left=70, "\mathcal{R}"] \arrow[r, bend left, "U^{\mathcal{S}}\mathcal{X}"] \arrow[r, bend right, "U^{\mathcal{S}}\mathcal{T}"'] \arrow[r, phantom, "\Uparrow U^{\mathcal{S}}\sigma"] & Z
\end{tikzcd}
=
\begin{tikzcd}
U \arrow[r, bend left, "\mathcal{R}"] \arrow[r, bend right, "U^{\mathcal{S}}\mathcal{T}"'] \arrow[r, phantom, "\Uparrow \varepsilon_{\mathcal{R}} \circ (U^{\mathcal{S}}\sigma)"] & Z
\end{tikzcd}
$$

Figure 7: Left Whiskering of 2-Diagrams

The diagram is telling us that $U^{\mathcal{S}}$ is a functor on the 2-category qua category, i.e. for $\tau : \mathcal{T} \Rightarrow \mathcal{X}$ and $\rho : \mathcal{X} \Rightarrow \mathcal{Q}$,

$$U^{\mathcal{S}}(\rho \circ \tau) = (U^{\mathcal{S}}\rho) \circ (U^{\mathcal{S}}\tau).$$

A similar left whiskering diagram used by Left Kan Lift with $\eta_{\mathcal{R}}$ in the opposite direction of $\varepsilon_{\mathcal{R}}$ and replacing $\varepsilon_{\mathcal{R}}$ is had by turning all the 2-arrows around, i.e., $\rho : \mathcal{Q} \Rightarrow \mathcal{X}$ and $\tau : \mathcal{X} \Rightarrow \mathcal{T}$, and

$$U^{\mathcal{S}}(\tau \circ \rho) = (U^{\mathcal{S}}\tau) \circ (U^{\mathcal{S}}\rho).$$

5.2 Right Kan Extension

Definition 5.2.1 The diagram

$$
\begin{tikzcd}
& U \arrow[dl, "\mathcal{T}"'] \arrow[dr, "\mathcal{R}"] \arrow[dr, bend left=50, "Z^{\mathcal{T}}\mathcal{S}"] \arrow[dr, bend right=40, "Z^{\mathcal{T}}\mathcal{X}"'] \arrow[dr, phantom, bend left=25, "\theta"] \arrow[dr, phantom, bend right=20, "\varepsilon_{\mathcal{R}}"'] & \\
V \arrow[rr, bend right=10, "\mathcal{X} = \mathrm{Ran}_{\mathcal{T}}\mathcal{R}"] \arrow[rr, bend right=50, "\mathcal{S}"'] \arrow[rr, phantom, bend right=30, "\sigma"] & & Z
\end{tikzcd}
$$

Figure 8: Right Kan Extension

is said to exhibit $\mathcal{X}$, denoted $\mathrm{Ran}_{\mathcal{T}}\mathcal{R}$, as a *right extension of* $\mathcal{R}$ *along* $\mathcal{T}$ when each 2-cell $\theta : Z^{\mathcal{T}}\mathcal{S} \Rightarrow \mathcal{R}$ factors as $\varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma)$ for a unique 2-cell $\sigma : \mathcal{S} \Rightarrow \mathcal{X}$. $\mathrm{Ran}_{\mathcal{T}}\mathcal{R}$ is a particular choice of right extension of $\mathcal{R}$ along $\mathcal{T}$.

Right Kan extension defines a universal arrow from a covariant functor to an object.

Definition 5.2.2 A *universal arrow* (i.e., couniversal arrow) from $Z^{\mathcal{T}}$ to $\mathcal{R}$ is a pair $\langle \mathrm{Ran}_{\mathcal{T}}\mathcal{R}, \varepsilon\rangle$ consisting of an object $\mathrm{Ran}_{\mathcal{T}}\mathcal{R} : V \Rightarrow Z$ and an arrow $\varepsilon_{\mathcal{R}} : Z^{\mathcal{T}}(\mathrm{Ran}_{\mathcal{T}}\mathcal{R}) \Rightarrow \mathcal{R}$ such that to every pair $\langle \mathcal{S}, \theta\rangle$ with $\mathcal{S}$ an object of Z^V and $\theta : Z^{\mathcal{T}}\mathcal{S} \Rightarrow \mathcal{R}$, there is a unique $\sigma : \mathcal{S} \Rightarrow (\mathrm{Ran}_{\mathcal{T}}\mathcal{R})$ with $\theta = \varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma)$ (where $Z^{\mathcal{T}}\sigma = \sigma \lhd \mathcal{T}$). In other words, every arrow θ factors uniquely through the universal arrow $\varepsilon_{\mathcal{R}}$ as in the commutative diagram

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, dotted, "\sigma"'] & Z^{\mathcal{T}}\mathcal{S} \arrow[d, dotted, "Z^{\mathcal{T}}\sigma"'] \arrow[dr, Rightarrow, "\theta"] & \\
\mathrm{Ran}_{\mathcal{T}}\mathcal{R} & Z^{\mathcal{T}}(\mathrm{Ran}_{\mathcal{T}}\mathcal{R}) \arrow[r, Rightarrow, "\varepsilon_{\mathcal{R}}"'] & \mathcal{R}
\end{tikzcd}
$$

Figure 9: Universal Arrow from $Z^{\mathcal{T}}$ to $\mathcal{R}$

This sets up an adjunction:

Theorem 5.2.3

$$\varphi_{\mathcal{S},\mathcal{R}} : Z^U(Z^{\mathcal{T}}\mathcal{S}, \mathcal{R}) \cong Z^V(\mathcal{S}, \mathrm{Ran}_{\mathcal{T}}\mathcal{R}),$$

where Z^U is a functor that returns the set of all arrows from U to Z and Z^V is a functor that returns all arrows from Z to V.

Proof: We let $G_0 = \mathrm{Ran}_{\mathcal{T}}$ and will extend G_0 to arrows to achieve

$$(\rho : \mathcal{R} \Rightarrow \mathcal{R}') \mapsto G\rho : G_0\mathcal{R}' \Rightarrow G_0\mathcal{R}.$$

We are given a universal arrow $\langle G_0\mathcal{R}, \varepsilon_{\mathcal{R}}\rangle$ (to $\mathcal{R}$ from $Z^{\mathcal{T}}$) for every object $\mathcal{R} \in Z^U$; we shall show that there is exactly one way to make G_0 the object function of a functor G for which $\varepsilon : Z^{\mathcal{T}}G \Rightarrow I_{Z^U}$ will be natural. Specifically, for each $\rho : \mathcal{R}' \Rightarrow \mathcal{R}$, the universality of $\varepsilon_{\mathcal{R}}$ states that there is exactly one arrow (dotted)which can make the diagram commute. Choose this arrow as $G\rho : G_0\mathcal{R}' \to G_0\mathcal{R}$; the commutativity states that ε is now natural, and it is straightforward to check that this choice of $G\rho$ makes G a functor.

$$
\begin{tikzcd}
G_0\mathcal{R}' \arrow[d, dotted] & Z^{\mathcal{T}}G_0\mathcal{R}' \arrow[d, dotted] \arrow[r, Rightarrow, "\varepsilon_{\mathcal{R}'}"] & \mathcal{R}' \arrow[d, Rightarrow, "\rho"] \\
G_0\mathcal{R} & Z^{\mathcal{T}}G_0\mathcal{R} \arrow[r, Rightarrow, "\varepsilon_{\mathcal{R}}"'] & \mathcal{R}
\end{tikzcd}
$$

Figure 10: Extension of G_0 to be a Functor G

That is, it is obvious that G preserves identity arrows because identity arrows are unique. Next, let $\tau : \mathcal{P} \Rightarrow \mathcal{R}'$ and $\rho : \mathcal{R}' \Rightarrow \mathcal{R}$, then we get $G\tau : G\mathcal{P} \Rightarrow G\mathcal{R}'$ and $G\rho : G\mathcal{R}' \Rightarrow G\mathcal{R}$. Now choose $\nu = \rho \circ \tau : \mathcal{P} \Rightarrow \mathcal{R}$. There is a unique arrow $G_0\mathcal{P} \Rightarrow G_0\mathcal{R}$, which we designate $G\nu$, that causes the resulting diagram involving $\varepsilon_{\mathcal{P}}$, $\varepsilon_{\mathcal{R}'}$, and $\varepsilon_{\mathcal{R}}$ to commute. However, $G\rho \circ G\tau$ also causes the diagram to commute, so $G\nu = G(\rho \circ \tau) = G\rho \circ G\tau$. Thus G is a functor.

The statement that $\varepsilon_{\mathcal{R}}$ is universal means that for each $\theta : Z^{\mathcal{T}}\mathcal{S} \Rightarrow \mathcal{R}$ there is exactly one σ as in the commutative Diagram 9. This states that $\psi(\sigma) = \varepsilon_{\mathcal{R}} \circ Z^{\mathcal{T}}\sigma$ defines a bijection

$$\psi_{\mathcal{S},\mathcal{R}} : Z^V(\mathcal{S}, G\mathcal{R}) \rightarrow Z^U(Z^{\mathcal{T}}\mathcal{S}, \mathcal{R})$$

Expanding this out a bit, by virtue of the uniqueness criteria, there is a function ψ^{-1} from $Z^U(Z^{\mathcal{T}}\mathcal{S}, \mathcal{R})$ to $Z^V(\mathcal{S}, G\mathcal{R})$. The universal condition states that ψ^{-1} is a function.

To see that $\psi^{-1}_{\mathcal{S},\mathcal{R}}$ is 1-1, select $\theta', \theta \in Z^U(Z^{\mathcal{T}}\mathcal{S}, \mathcal{R})$ such that $\theta' \neq \theta$. Towards a reductio, assume that $\psi^{-1}_{\mathcal{S},\mathcal{R}}(\theta) = \sigma$, $\theta = \varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma)$, $\psi^{-1}_{\mathcal{R},\check{\mathcal{S}}}(\theta') = \sigma'$, and $\theta' = \varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma')$ and that $\sigma = \sigma'$. From this latter condition, $\varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma) = \varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma')$, and since the preimages of θ' and θ (under ψ) must be unique, $\theta = \theta'$, a contradiction. So $\psi^{-1}_{\mathcal{S},\mathcal{R}}$ is 1-1.

To show that $\psi^{-1}_{\mathcal{S}\mathcal{R}}$ is surjective, let $\sigma \in V^Z(\mathcal{S}, G\mathcal{R})$, then $\psi\sigma = \varepsilon_{\mathcal{R}} \circ (Z^{\mathcal{T}}\sigma) : Z^{\mathcal{T}}\mathcal{S} \Rightarrow \mathcal{R}$. Therefore there is a unique σ' such that $\sigma' \in V^Z(\mathcal{S}, G\mathcal{R})$ such that $\psi\sigma' = \varepsilon_{\mathcal{R}} \circ Z^{\mathcal{T}}\sigma$. Since σ' is unique, then $\sigma = \sigma'$.

This bijection is natural in $\mathcal{R}$ because η is natural, and natural in $\mathcal{S}$ because $Z^{\mathcal{T}}$ is a functor, hence gives an adjunction $\langle Z^{\mathcal{T}}, G, \varphi\rangle$. In this case, ε was the unit obtained from the adjunction $\langle Z^{\mathcal{T}}, G, \varphi\rangle$.

■

For our situation with $\circledcirc$ and $\multimap$, we use the following definition:

Definition 5.2.4 Given an arrow $\mathcal{T} : U \rightarrow V$ and an object Z we consider the arrow category Z^V with objects the arrows $\mathcal{S} : V \rightarrow Z$ and 2-arrows $\sigma : \mathcal{S} \Rightarrow \mathcal{X}$, then we define the covariant functor $Z^{\mathcal{T}} : Z^V \rightarrow Z^U$ and the object map $\mathrm{Ran}_{\mathcal{T}} : Z^U \rightarrow Z^V$ on objects; let $\mathcal{P}, \mathcal{Q} : V \rightarrow Z$ and $\mathcal{Y} : U \rightarrow Z$:

$$Z^{\mathcal{T}} \stackrel{\mathit{def}}{=} \lambda\mathcal{Q} \,.\, \mathcal{T} \circledcirc \mathcal{Q} \qquad \mathrm{Ran}_{\mathcal{T}} \stackrel{\mathit{def}}{=} \lambda\mathcal{Y} \,.\, \mathcal{T} \multimap \mathcal{Y},$$

and 2-arrows by

$$(\nu : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (Z^{\mathcal{T}}\nu : \mathcal{T} \circledcirc \mathcal{P} \Rightarrow \mathcal{T} \circledcirc \mathcal{Q})$$

Lemma 5.2.5 *Assuming Definition 5.2.4, $Z^{\mathcal{T}}$ is a covariant functor on the category Z^V whose objects are 1-arrows and arrows are the 2-arrows.*

Proof: The proof is simple and merely relies on the fact that arrows in the 2-category qua category compose. Assume $\nu : \mathcal{P} \Rightarrow \mathcal{Q}$ and $\mu : \mathcal{Q} \Rightarrow \mathcal{N}$, then $Z^{\mathcal{T}}\nu : \mathcal{T} \circledcirc \mathcal{P} \Rightarrow \mathcal{T} \circledcirc \mathcal{Q}$ and $Z^{\mathcal{T}}\mu : \mathcal{T} \circledcirc \mathcal{Q} \Rightarrow \mathcal{T} \circledcirc \mathcal{N}$. Hence $(Z^{\mathcal{T}}\mu) \circ (Z^{\mathcal{T}}\nu) = Z^{\mathcal{T}}(\mu \circ \nu)$ where the latter follows from the properties of 2-categories qua categories from the right whiskering above. Similarly, that $Z^{\mathcal{T}}$ preserves the identity arrows (i.e., the equality relation) also follows from right whiskering in the 2-category qua categories.

In our context of typed relations, that $Z^{\mathcal{T}}$ is a covariant functor (where $Z^{\mathcal{T}}\sigma = \sigma \triangleleft \mathcal{T}$) follows from

$$\frac{\mathcal{P} \subseteq_{\sigma} \mathcal{Q}}{\mathcal{T} \circledcirc \mathcal{P} \subseteq_{\sigma \triangleleft \mathcal{T}} \mathcal{T} \circledcirc \mathcal{Q}}$$

and the definition of $\circledcirc$. This says that

$$(\sigma : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (\sigma \triangleleft \mathcal{T} : \mathcal{T} \circledcirc \mathcal{P} \Rightarrow \mathcal{T} \circledcirc \mathcal{Q}).$$

■

Theorem 5.2.6 *For our situation using Definition 5.2.4,*

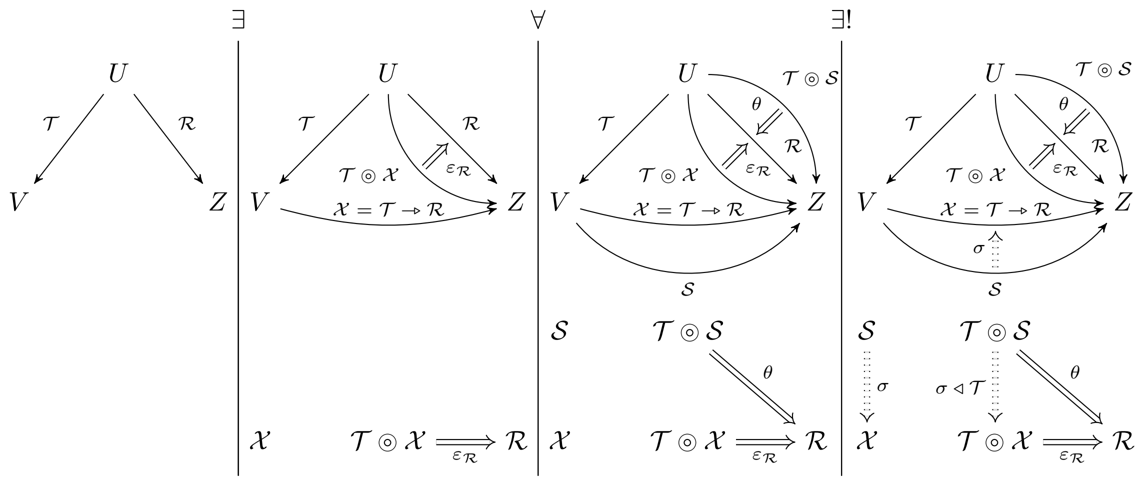

Figure 11: Right Kan Extension as a Right Residual

Proof: We first show that σ exists. Let $\langle v,z\rangle \in \mathcal{S}$. Towards a reductio, assume $\langle v,z\rangle \notin \mathcal{X}$. In that case, given $\mathcal{X} = \mathcal{T} \rightarrowtriangle \mathcal{R}$, there must be some $u \in U$ such that $\langle u,v\rangle \in \mathcal{T}$ and $\langle u,z\rangle \notin \mathcal{R}$. Since $\theta : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$ and $\langle u,z\rangle \in \mathcal{T} \circledcirc \mathcal{S}$, then $\langle u,z\rangle \in \mathcal{R}$ and we have a contradiction. Therefore $\langle v,z\rangle \in \mathcal{T} \rightarrowtriangle \mathcal{R}$. Hence, $\mathcal{S} \subseteq \mathcal{T} \rightarrowtriangle \mathcal{R}$ and there exists σ such that $\sigma : \mathcal{S} \Rightarrow \mathcal{X}$. Also, σ is unique since inclusions are unique in set theory.

Next we must show that $\theta = \varepsilon_{\mathcal{R}} \circ (\sigma \lhd \mathcal{T})$. Since $\sigma \lhd \mathcal{T} : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{T} \circledcirc \mathcal{X}$ and $\varepsilon_{\mathcal{R}} : \mathcal{T} \circledcirc \mathcal{X} \Rightarrow \mathcal{R}$, then $\varepsilon_{\mathcal{R}} \circ (\sigma \lhd \mathcal{T}) : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$. Since $\theta : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$ and inclusions are unique in set theory, $\theta = \varepsilon_{\mathcal{R}} \circ (\sigma \lhd \mathcal{T})$.

■

Our situation corresponds to the following adjunction:

$$\varphi_{\mathcal{S},\mathcal{R}} : Z^U(\mathcal{T} \circledcirc \mathcal{S}, \mathcal{R}) \cong Z^V(\mathcal{S}, \mathcal{T} \rightarrowtriangle \mathcal{R}).$$

We have the following logical rules:

$$\frac{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\theta} \mathcal{R} \qquad \mathcal{T} \circledcirc (\mathcal{T} \rightarrowtriangle \mathcal{R}) \subseteq_{\varepsilon_{\mathcal{R}}} \mathcal{R}}{\mathcal{S} \subseteq_{\sigma} \mathcal{T} \rightarrowtriangle \mathcal{R}} \text{ Right Kan Extension} \qquad \frac{\mathcal{S} \subseteq_{\sigma} \mathcal{T} \rightarrowtriangle \mathcal{R}}{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\sigma \lhd \mathcal{T}} \mathcal{T} \circledcirc (\mathcal{T} \rightarrowtriangle \mathcal{R})} \text{ Monotonicity}$$

The left rule is equivalent to

$$\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\theta} \mathcal{R} \text{ implies } \mathcal{S} \subseteq_{\sigma} \mathcal{T} \rightarrowtriangle \mathcal{R}$$

because the second premise of the first rule always holds in our situation. This premise holds via one half of the residuation condition

$$\frac{\mathcal{T} \rightarrowtriangle \mathcal{R} \subseteq_{\iota} \mathcal{T} \rightarrowtriangle \mathcal{R}}{\mathcal{T} \circledcirc (\mathcal{T} \rightarrowtriangle \mathcal{R}) \subseteq_{\varepsilon_{\mathcal{R}}} \mathcal{R}}$$

where ι is the equality relation and stands for the identity in the 2-category. In the Kan extension view, residuation is not available and hence that premise must be explicitly stated. The other direction is

$$\frac{\dfrac{\mathcal{S} \subseteq_{\sigma} \mathcal{T} \rightarrowtriangle \mathcal{R}}{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\sigma \lhd \mathcal{T}} \mathcal{T} \circledcirc (\mathcal{T} \rightarrowtriangle \mathcal{R})} \qquad \mathcal{T} \circledcirc (\mathcal{T} \rightarrowtriangle \mathcal{R}) \subseteq_{\varepsilon_{\mathcal{R}}} \mathcal{R}}{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\varepsilon_{\mathcal{R}} \circ (\sigma \lhd \mathcal{T})} \mathcal{R}}$$

The extra noise in the right premise of the Right Kan Extension rule, i.e., $\mathcal{T} \circledcirc (\mathcal{T} \rightarrowtriangle \mathcal{R}) \subseteq_{\varepsilon_{\mathcal{R}}} \mathcal{R}$, is related to the ability for some right Kan extensions to be definable without necessarily having a pair of adjoint functors. In other words, propositions are considered universally quantified in logic, where here there is no such quantification. So this premise must be explicitly stated. The statement of right Kan extension says that $\theta = \varepsilon_{\mathcal{R}} \circ (\sigma \lhd \mathcal{T})$. This holds for us because there is only a single subset relation such that $\mathcal{T} \circledcirc \mathcal{S} \subseteq \mathcal{R}$ whereas for Kan this must be explicitly declared, i.e., that σ is unique and hence $\varepsilon_{\mathcal{R}} \circ (\sigma \lhd \mathcal{T})$ is unique which implies that $\theta = \varepsilon_{\mathcal{R}} \circ (\sigma \lhd \mathcal{T})$.

5.3 Right Kan Lift

Definition 5.3.1 ([Str83]) The diagram

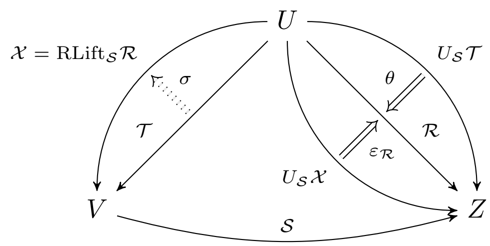

Figure 12: Right Kan Left

is said to exhibit $\mathcal{X}$, denoted $\mathrm{RLift}_{\mathcal{S}}\mathcal{R}$, as a *right lifting of* $\mathcal{R}$ *through* $\mathcal{S}$ when each 2-cell $\theta : U^{\mathcal{S}}\mathcal{T} \Rightarrow \mathcal{R}$ factors as $\varepsilon_{\mathcal{R}} \circ (U^{\mathcal{S}}\sigma)$ for a unique 2-cell $\sigma : \mathcal{T} \Rightarrow \mathcal{X}$. $\mathrm{RLift}_{\mathcal{S}}\mathcal{R}$ (Street used $\mathcal{S}\pitchfork\mathcal{R}$) is a particular choice of right Kan lift of $\mathcal{R}$ through $\mathcal{S}$.

Just as in right Kan extension, right Kan lift defines a universal arrow from a covariant functor to an object.

Definition 5.3.2 A *universal arrow* (i.e., couniversal arrow) from $U^{\mathcal{S}}$ to $\mathcal{R}$ is a pair $\langle \mathrm{RLift}_{\mathcal{S}}\mathcal{R}, \varepsilon_{\mathcal{R}}\rangle$ consisting of an object $\mathrm{RLift}_{\mathcal{S}}\mathcal{R} : U \Rightarrow V$ and an arrow $\varepsilon_{\mathcal{R}} : \mathcal{U}^{\mathcal{S}}(\mathrm{RLift}_{\mathcal{S}}\mathcal{R}) \Rightarrow \mathcal{R}$ such that to every pair $\langle \mathcal{T}, \theta\rangle$ with $\mathcal{T}$ and object of V^U and $\theta : \mathcal{U}^{\mathcal{S}}\mathcal{T} \Rightarrow \mathcal{R}$, there is a unique $\sigma : \mathcal{T} \Rightarrow (\mathrm{RLift}_{\mathcal{S}}\mathcal{R})$ with $\theta = (U^{\mathcal{S}}\sigma) \circ \varepsilon_{\mathcal{R}}$ (where $U^{\mathcal{S}}\sigma = \mathcal{S} \triangleright \sigma$). In other words, every arrow θ factors uniquely through the universal arrow $\varepsilon_{\mathcal{R}}$ as in the commutative diagram

$$\begin{tikzcd}
\mathcal{T} \arrow[d, dotted, "\sigma"'] & \mathcal{U}_{\mathcal{S}}\mathcal{T} \arrow[d, dotted, "U^{\mathcal{S}}\sigma"'] \arrow[dr, Rightarrow, "\theta"] & \\
\mathrm{RLift}_{\mathcal{S}}\mathcal{R} & \mathcal{U}^{\mathcal{S}}(\mathrm{RLift}_{\mathcal{S}}\mathcal{R}) \arrow[r, Rightarrow, "\varepsilon_{\mathcal{R}}"'] & \mathcal{R}
\end{tikzcd}$$

Figure 13: Universal Arrow from $U^{\mathcal{S}}$ to $\mathcal{R}$

As in Right Kan Extension, this sets up an adjunction:

Theorem 5.3.3

$$\varphi_{\mathcal{T},\mathcal{R}} : Z^U(\mathcal{U}^{\mathcal{S}}\mathcal{T}, \mathcal{R}) \cong V^U(\mathcal{T}, \mathrm{RLift}_{\mathcal{S}}\mathcal{R}).$$

The proof is similar to the adjunction proof for right Kan extension. For our situation with $\circledcirc$ and $\leftarrow$, we use the following definition:

Definition 5.3.4 Given an arrow $\mathcal{S} : V \to Z$ and an object U we consider the arrow category V^U with objects the arrows $\mathcal{T} : U \to V$ and 2-arrows $\sigma : \mathcal{T} \Rightarrow \mathcal{X}$, then we define the covariant functor $U^{\mathcal{S}} : V^U \to Z^U$ and the object map $\mathrm{RLift}_{\mathcal{S}} : Z^U \to V^U$; let $\mathcal{P}, \mathcal{Q} : U \to V$ and $\mathcal{Y} : U \to Z$:

$$U^{\mathcal{S}} \stackrel{def}{=} \lambda\mathcal{Q}\ .\ \mathcal{Q} \circledcirc \mathcal{S} \qquad \mathrm{RLift}_{\mathcal{S}} \stackrel{def}{=} \lambda\mathcal{Y}\ .\ \mathcal{Y} \leftarrow \mathcal{S},$$

and 2-arrows by

$$(\nu : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (U^{\mathcal{S}}\nu : \mathcal{P} \circledcirc \mathcal{S} \Rightarrow \mathcal{Q} \circledcirc \mathcal{S})$$

Lemma 5.3.5 *Assuming Definition 5.3.4, $U^{\mathcal{S}}$ is a covariant functor on the category V^U whose objects are 1-arrows and arrows are the 2-arrows.*

Proof: The proof is simple and merely relies on the fact that arrows in the 2-category qua category compose. Assume $\nu : \mathcal{P} \Rightarrow \mathcal{Q}$ and $\mu : \mathcal{Q} \Rightarrow \mathcal{N}$, then $U^{\mathcal{S}}\nu : \mathcal{P} \circledcirc \mathcal{S} \Rightarrow \mathcal{Q} \circledcirc \mathcal{S}$ and $U^{\mathcal{S}}\mu : \mathcal{Q} \circledcirc \mathcal{S} \Rightarrow \mathcal{N} \circledcirc \mathcal{S}$. Hence $(U^{\mathcal{S}}\mu) \circ (U^{\mathcal{S}}\nu) = U^{\mathcal{S}}(\mu \circ \nu)$ where the latter follows from the properties of 2-categories qua categories from the left whiskering above. Similarly, that $U^{\mathcal{S}}$ preserves the identity arrows (i.e., the equality relation) also follows from left whiskering in the 2-category qua categories.

In our context of typed relations, that $U^{\mathcal{S}}$ is a covariant functor (where $U^{\mathcal{S}}\sigma = \mathcal{S} \triangleright \sigma$) follows from

$$\frac{\mathcal{P} \subseteq_{\sigma} \mathcal{Q}}{\mathcal{P} \circledcirc \mathcal{S} \subseteq_{\mathcal{S} \triangleright \sigma} \mathcal{Q} \circledcirc \mathcal{S}}$$

and the definition of $\circledcirc$. This says that

$$(\sigma : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (\mathcal{S} \triangleright \sigma : \mathcal{P} \circledcirc \mathcal{S} \Rightarrow \mathcal{Q} \circledcirc \mathcal{S}).$$

■

Theorem 5.3.6 *For our situation using Definition 5.3.4 and where* $\mathcal{X} = \mathrm{RLift}_{\mathcal{S}}\mathcal{R}$*,*

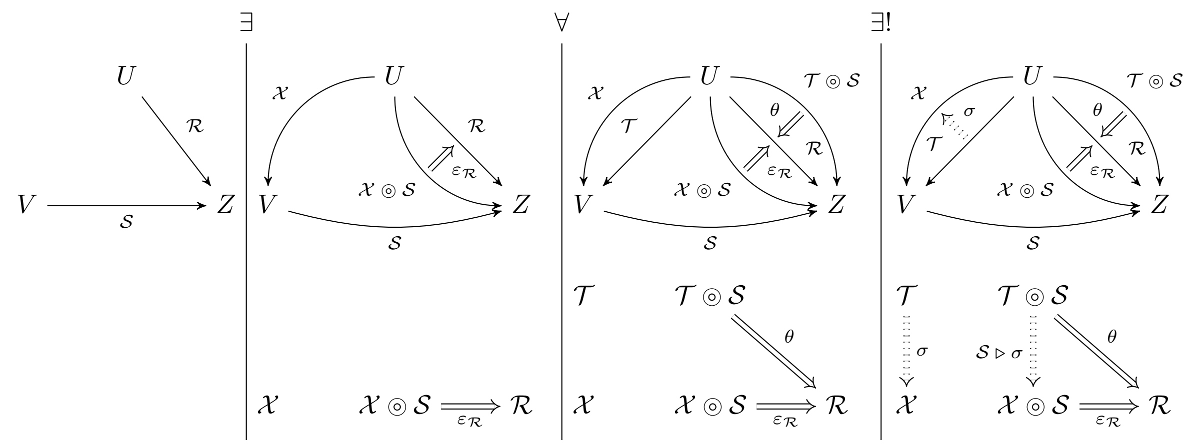

Figure 14: Right Kan Lift as a Left Residual

Proof: We first show that σ exists. Let $\langle u, v \rangle \in \mathcal{T}$. Either $\langle u, v \rangle \in \mathcal{X}$ or $\langle u, v \rangle \notin \mathcal{X}$. Towards a reductio, assume $\langle u, v \rangle \notin \mathcal{X}$. In that case, given $\mathcal{X} = \mathcal{R} \leftharpoonup \mathcal{S}$, there must be some $z \in Z$ such that $\langle v, z \rangle \in \mathcal{S}$ and $\langle u, z \rangle \notin \mathcal{R}$. Since $\theta : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$ and $\langle u, z \rangle \in \mathcal{T} \circledcirc \mathcal{S}$, then $\langle u, z \rangle \in \mathcal{R}$ and we have a contradiction. Therefore $\langle u, v \rangle \in \mathcal{R} \leftharpoonup \mathcal{S}$. Hence, $\mathcal{T} \subseteq \mathcal{R} \leftharpoonup \mathcal{S}$ and there exists σ such that $\sigma : \mathcal{T} \Rightarrow \mathcal{X}$. Also, σ is unique since inclusions are unique in set theory.

Next we must show that $\theta = \varepsilon_{\mathcal{R}} \circ (\mathcal{S} \triangleright \sigma)$. Since $\mathcal{S} \triangleright \sigma : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{X} \circledcirc \mathcal{S}$ and $\varepsilon_{\mathcal{R}} : \mathcal{X} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$, then $\varepsilon_{\mathcal{R}} \circ (\mathcal{S} \triangleright \sigma) : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$. Since $\theta : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$ and inclusions are unique in set theory, $\theta = \varepsilon_{\mathcal{R}} \circ (\mathcal{S} \triangleright \sigma)$.

■

Our situation corresponds to the following adjunction:

$$Z^U(\mathcal{T} \circledcirc \mathcal{S}, \mathcal{R}) \cong V^U(\mathcal{T}, \mathcal{R} \leftharpoonup \mathcal{S}).$$

We have the following logical rules:

$$\frac{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\theta} \mathcal{R} \qquad (\mathcal{R} \leftarrow \mathcal{S}) \circledcirc \mathcal{S} \subseteq_{\varepsilon_{\mathcal{R}}} \mathcal{R}}{\mathcal{S} \subseteq_{\sigma} \mathcal{R} \leftharpoonup \mathcal{S}} \text{ Right Kan Lift} \qquad \frac{\mathcal{T} \subseteq_{\sigma} \mathcal{R} \leftharpoonup \mathcal{S}}{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\mathcal{S} \triangleright \sigma} (\mathcal{R} \leftharpoonup \mathcal{S}) \circledcirc \mathcal{S}} \text{ Montonicity}$$

This is equivalent to

$$\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\theta} \mathcal{R} \text{ implies } \mathcal{T} \subseteq_{\sigma} \mathcal{R} \rightharpoonup \mathcal{S}$$

because the second premise of the first rule always holds in our situation. This premise holds via one half of the residuation condition

$$\dfrac{\mathcal{R} \leftarrowtriangle \mathcal{S} \subseteq_{\iota} \mathcal{R} \leftarrowtriangle \mathcal{S}}{(\mathcal{R} \leftarrowtriangle \mathcal{S}) \circledcirc \mathcal{S} \subseteq_{\varepsilon_{\mathcal{R}}} \mathcal{R}}$$

where again ι is the equality relation and stands for the identity in the 2-category. The other direction is

$$\dfrac{\dfrac{\mathcal{T} \subseteq_{\sigma} \mathcal{R} \leftarrowtriangle \mathcal{S}}{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\mathcal{S} \rhd \sigma} (\mathcal{R} \leftarrowtriangle \mathcal{S}) \circledcirc \mathcal{S}} \qquad (\mathcal{R} \leftarrowtriangle \mathcal{S}) \circledcirc \mathcal{S} \subseteq_{\varepsilon_{\mathcal{R}}} \mathcal{R}}{\mathcal{T} \circledcirc \mathcal{S} \subseteq_{\varepsilon_{\mathcal{R}} \circ (\mathcal{S} \rhd \sigma)} \mathcal{R}}$$

Similar comment apply to the extra noise in the right premise of the Right Kan Lift rule as applied to the analogous rule for Right Kan Extension.

5.4 Left Kan Extension

Definition 5.4.1 The diagram

$$\begin{tikzcd}
& U \arrow[dl, "\mathcal{T}"'] \arrow[dr, "\mathcal{R}"] \arrow[dr, bend left=50, "Z^{\mathcal{T}}\mathcal{S}"] \arrow[dr, bend right=40, "Z^{\mathcal{T}}\mathcal{X}"'] \arrow[dr, phantom, shift left=4, "\Uparrow\theta"] \arrow[dr, phantom, shift right=4, "\Downarrow\eta_{\mathcal{R}}"] & \\
V \arrow[rr, "\mathcal{X} = \mathrm{Lan}_{\mathcal{T}}\mathcal{R}"] \arrow[rr, bend right=40, "\mathcal{S}"'] \arrow[rr, phantom, shift right=4, "\Downarrow\sigma"] & & Z
\end{tikzcd}$$

Figure 15: Left Kan Extension

is said to exhibit $\mathcal{X}$, denoted $\mathrm{Lan}_{\mathcal{T}}\mathcal{R}$, as a *left extension of* $\mathcal{R}$ *along* $\mathcal{T}$ when each 2-cell $\theta : \mathcal{R} \Rightarrow Z^{\mathcal{T}}\mathcal{S}$ factors as $(\sigma \lhd \mathcal{T}) \circ \eta_{\mathcal{R}}$ for a unique 2-cell $\sigma : \mathcal{X} \Rightarrow \mathcal{S}$. $\mathcal{X}$ is a particular choice of left extension of $\mathcal{R}$ along $\mathcal{T}$.

Notice that the 2-arrows are reversed with respect to Right Kan Extension. As a result, Left Kan Extension defines a universal arrow from an object to a covariant functor.

Definition 5.4.2 A *universal arrow* from $\mathcal{R}$ to $Z^{\mathcal{T}}$ is a pair $\langle \mathrm{Lan}_{\mathcal{T}}\mathcal{R}, \eta_{\mathcal{R}} \rangle$ consisting of an object $\mathrm{Lan}_{\mathcal{T}}\mathcal{R} : V \Rightarrow Z$ and an arrow $\eta_{\mathcal{R}} : \mathcal{R} \Rightarrow Z^{\mathcal{T}}(\mathrm{Lan}_{\mathcal{T}}\mathcal{R})$ such that to every pair $\langle \mathcal{S}, \theta \rangle$ with $\mathcal{S}$ an object of Z^V and $\theta : \mathcal{R} \Rightarrow Z^{\mathcal{T}}\mathcal{S}$, there is a unique arrow $\sigma : (\mathrm{Lan}_{\mathcal{T}}\mathcal{R}) \Rightarrow \mathcal{S}$ with $\theta = (Z^{\mathcal{T}}\sigma) \circ \eta_{\mathcal{R}}$ (where $Z^{\mathcal{T}}\sigma = \sigma \lhd \mathcal{T}$). In other words, every arrow θ factors uniquely through the universal arrow $\eta_{\mathcal{R}}$, as in the commutative diagram

$$\begin{tikzcd}
\mathrm{Lan}_{\mathcal{T}}\mathcal{R} \arrow[d, dotted, Rightarrow, "\sigma"'] & Z^{\mathcal{T}}(\mathrm{Lan}_{\mathcal{T}}\mathcal{R}) \arrow[d, dotted, Rightarrow, "Z^{\mathcal{T}}\sigma"'] & \mathcal{R} \arrow[l, Rightarrow, "\eta_{\mathcal{R}}"'] \arrow[dl, Rightarrow, "\theta"] \\
\mathcal{S} & Z^{\mathcal{T}}\mathcal{S} &
\end{tikzcd}$$

Figure 16: Universal Arrow from $\mathcal{R}$ to the Covariant Functor $Z^{\check{\mathcal{T}}}$

This sets up an adjunction:

Theorem 5.4.3

$$\varphi_{\mathcal{R},\mathcal{S}} : Z^V(\mathrm{Lan}_{\mathcal{T}}\mathcal{R}, \mathcal{S}) \cong Z^U(\mathcal{R}, Z^{\mathcal{T}}\mathcal{S}).$$

Proof: We let $F_0 = \mathrm{Lan}_{\mathcal{T}}$ and will extend F_0 to arrows to achieve

$$(\rho : \mathcal{R} \Rightarrow \mathcal{R}') \mapsto F\rho : F_0\mathcal{R} \Rightarrow F_0\mathcal{R}'.$$

We are given a universal arrow $\langle F_0\mathcal{R}, \eta_{\mathcal{R}}\rangle$ (from $\mathcal{R}$ to $Z^{\mathcal{T}}$) consisting of an arrow $\eta_{\mathcal{R}} : \mathcal{R} \Rightarrow Z^{\check{\mathcal{T}}} F_0\mathcal{R}$ for every object $\mathcal{R} \in Z^U$; we shall show that there is exactly one way to make F_0 the object function of a functor for which $\eta_{\mathcal{R}} : I_{Z_U} \dashrightarrow Z^{\mathcal{T}}F$ will be natural. Specifically, for each $\rho : \mathcal{R} \to \mathcal{R}'$ the universality of $\eta_{\mathcal{R}}$ states that there is exactly one arrow (dotted) which can make the diagram commute. Choose this arrow as $F\rho : F_0\mathcal{R} \to F_0\mathcal{R}'$; the commutativity states that $\eta_{\mathcal{R}}$ is now natural, and it is straightforward to check that this choice of $F\rho$ makes F a functor.

$$\begin{tikzcd}
F_0\mathcal{R} \arrow[d, dotted] & \rho \arrow[r, Rightarrow, "\eta_{\mathcal{R}}"] \arrow[d, Rightarrow, "\rho"'] & Z^{\mathcal{T}}F_0\mathcal{R} \arrow[d, dotted] \\
F_0\mathcal{R}' & \rho' \arrow[r, Rightarrow, "\eta_{\mathcal{R}'}"'] & Z^{\mathcal{T}}F_0\mathcal{R}'
\end{tikzcd}$$

Figure 17: Extension of F_0 to be a Functor F

That is, it is obvious that F preserves identity arrows because the identity arrows are unique. Next, let $\rho : \mathcal{R} \Rightarrow \mathcal{R}'$ and $\tau : \mathcal{R}' \Rightarrow \mathcal{P}$, then we get $F\rho : F\mathcal{R} \Rightarrow F\mathcal{R}'$ and $F\tau : F\mathcal{R}' \Rightarrow F\mathcal{P}$. Now choose $\nu = \tau \circ \rho : \mathcal{R} \Rightarrow \mathcal{P}$. There is a unique arrow $F_0\mathcal{R} \Rightarrow F_0\mathcal{P}$, which we designate $F\nu$, that causes the resulting diagram involving $\eta_{\mathcal{R}}$, $\eta_{\mathcal{R}'}$, and $\eta_{\mathcal{P}}$ to commute. However, $F\tau \circ F\rho$ also causes the diagram to commute, so $F\nu = F(\tau \circ \rho) = F\tau \circ F\rho$. Thus F is a functor.

The statement that $\eta_{\mathcal{R}}$ is universal means that for each $\theta : \mathcal{R} \Rightarrow Z^{\mathcal{T}}\mathcal{S}$ there is exactly one σ as in the commutative Diagram 16. This states that $\psi(\sigma) = (Z^{\mathcal{T}}\sigma) \circ \eta_{\mathcal{R}}$ defines a bijection

$$\psi_{\mathcal{R},\mathcal{S}} : Z^V(F\mathcal{R}, \mathcal{S}) \to Z^U(\mathcal{R}, Z^{\mathcal{T}}\mathcal{S}).$$

Expanding this out a bit, by virtue of the uniqueness criteria, there is a function ψ^{-1} from $Z^U(\mathcal{R}, Z^{\mathcal{T}}\mathcal{S})$ to $Z^V(G\mathcal{R}, \check{\mathcal{S}})$. The universal condition states that ψ^{-1} is a function.

To see that $\psi^{-1}_{\mathcal{R},\mathcal{S}}$ is 1-1, select $\theta', \theta \in Z^U(\mathcal{R}, Z^{\mathcal{T}}\mathcal{S})$ such that $\theta' \neq \theta$. Towards a reductio, assume that $\psi^{-1}_{\mathcal{R},\check{\mathcal{S}}}(\theta) = \sigma$, $\theta = (Z^{\mathcal{T}}\sigma) \circ \eta_{\mathcal{R}}$, $\psi^{-1}_{\mathcal{R},\check{\mathcal{S}}}(\theta') = \sigma'$, and $\theta' = (Z^{\mathcal{T}}\sigma') \circ \eta_{\mathcal{R}}$ and that $\sigma = \sigma'$. From this latter condition, $(Z^{\mathcal{T}}\sigma) \circ \eta_{\mathcal{R}} = (Z^{\mathcal{T}}\sigma') \circ \eta_{\mathcal{R}}$, and since the preimages of θ' and θ (under ψ) must be unique, $\theta = \theta'$, a contradiction. So $\psi^{-1}_{\mathcal{R},\check{\mathcal{S}}}$ is 1-1.

To show that $\psi^{-1}_{\mathcal{R},\check{\mathcal{S}}}$ is surjective, let $\sigma \in Z^V(F\mathcal{R}, \mathcal{S})$, then $\psi\sigma = (Z^{\mathcal{T}}\sigma) \circ \eta_{\mathcal{R}} : \mathcal{R} \Rightarrow Z^{\mathcal{T}}\mathcal{S}$. Therefore there is a unique σ' such that $\sigma' \in Z^V(F\mathcal{R}, \mathcal{S})$ such that $\psi\sigma' = (Z^{\mathcal{T}}\sigma') \circ \eta_{\mathcal{R}}$. Since σ' is unique, then $\sigma = \sigma'$.

Now let $\varphi_{\mathcal{R},\mathcal{S}} = \psi_{\mathcal{R},\mathcal{S}}$. This bijection $\varphi_{\mathcal{R},\mathcal{S}}$ is natural in $\mathcal{R}$ because η is natural, and natural in $\mathcal{S}$ because $Z^{\mathcal{T}}$ is a functor, hence gives an adjunction $\langle F, Z^{\mathcal{T}}, \varphi\rangle$. In this case, η was the unit obtained from the adjunction $\langle F, Z^{\mathcal{T}}, \varphi\rangle$.

■

For our situation with $\oplus$ and $\rightharpoonup$, we use the following definition:

Definition 5.4.4 Given an arrow $\mathcal{T} : U \to V$ and an object Z we consider the arrow category Z^V with objects the arrows $\mathcal{S} : V \to Z$ and 2-arrows $\sigma : \mathcal{X} \Rightarrow \mathcal{S}$, then we define the covariant functor $Z^{\mathcal{T}} : Z^V \to Z^U$ and the object map $\mathrm{Lan}_{\mathcal{T}} : Z^U \to Z^V$ on objects; let $\mathcal{P}, \mathcal{Q} : V \to Z$ and $\mathcal{Y} : U \to Z$:

$$Z^{\mathcal{T}} \stackrel{def}{=} \lambda\mathcal{Q}\,.\,\mathcal{T} \oplus \mathcal{Q} \qquad \mathrm{Lan}_{\mathcal{T}} \stackrel{def}{=} \lambda\mathcal{Y}\,.\,\mathcal{T} \rightharpoonup \mathcal{Y},$$

and 2-arrows by

$$(\nu : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (Z^{\mathcal{T}}\nu : \mathcal{T} \oplus \mathcal{P} \Rightarrow \mathcal{T} \oplus \mathcal{Q})$$

Lemma 5.4.5 *Assuming Definition 5.4.4, $Z^{\mathcal{T}}$ is a covariant functor on the category Z^V whose objects are 1-arrows and arrows are the 2-arrows.*

Proof: The proof is simple and merely relies on the fact that arrows in the 2-category qua category compose. Assume $\nu : \mathcal{P} \Rightarrow \mathcal{Q}$ and $\mu : \mathcal{Q} \Rightarrow \mathcal{N}$, then $Z^{\mathcal{T}}\nu : \mathcal{T} \oplus \mathcal{P} \Rightarrow \mathcal{T} \oplus \mathcal{Q}$ and $Z^{\mathcal{T}}\mu : \mathcal{T} \oplus \mathcal{Q} \Rightarrow \mathcal{T} \oplus \mathcal{N}$. Hence $(Z^{\mathcal{T}}\mu) \circ (Z^{\mathcal{T}}\nu) = Z^{\mathcal{T}}(\mu \circ \nu)$ where the latter follows from the properties of 2-categories qua categories from the right whiskering above. Similarly, that $Z^{\mathcal{T}}$ preserves the identity arrows (i.e., the equality relation) also follows from right whiskering in the 2-category qua categories.

In our context of typed relations, that $Z^{\mathcal{T}}$ is a covariant functor (where $Z^{\mathcal{T}}\sigma = \sigma \triangleleft \mathcal{T}$) follows from

$$\frac{\mathcal{P} \subseteq_{\sigma} \mathcal{Q}}{\mathcal{T} \oplus \mathcal{P} \subseteq_{\sigma \triangleleft \mathcal{T}} \mathcal{T} \oplus \mathcal{Q}}$$

and the definition of $\oplus$. This says that

$$(\sigma : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (\sigma \triangleleft \mathcal{T} : \mathcal{T} \oplus \mathcal{P} \Rightarrow \mathcal{T} \oplus \mathcal{Q}).$$

■

Theorem 5.4.6 *For our situation using Definition 5.4.4,*

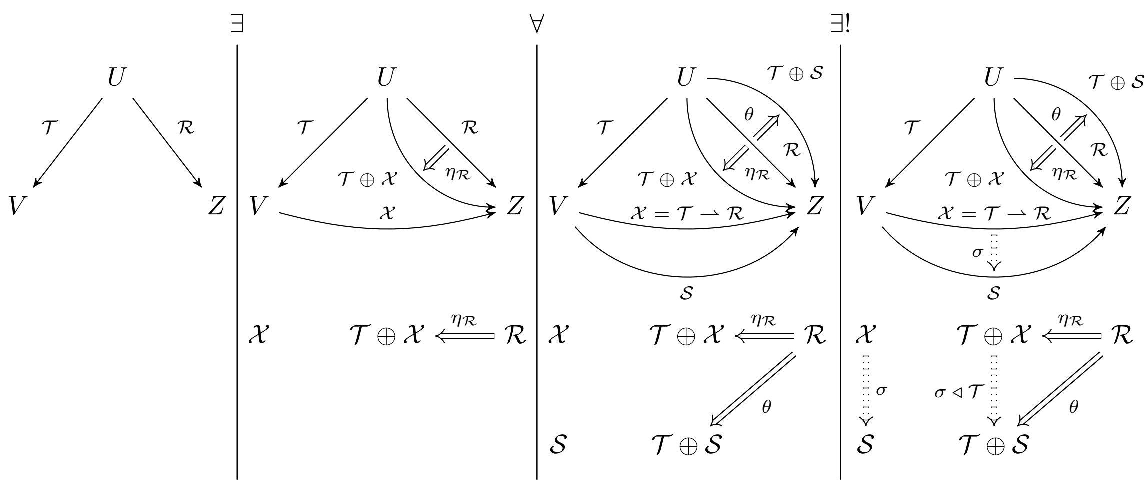

Figure 18: Left Kan Extension as a Right Residual

Proof: We first show that σ exists. Let $\langle v, z\rangle \in \mathcal{T} \rightharpoonup \mathcal{R}$. There is some $u \in U$ such that $\langle u, v\rangle \notin \mathcal{T}$ and $\langle u, z\rangle \in \mathcal{R}$. Since $\theta : \mathcal{R} \Rightarrow \mathcal{T} \oplus \mathcal{S}$, then $\langle u, z\rangle \in \mathcal{T} \oplus \mathcal{S}$. By the definition of $\mathcal{T} \oplus \mathcal{S}$, then $\langle v, z\rangle \in \mathcal{S}$. Therefore, there is a σ such that $\sigma : \mathcal{T} \rightharpoonup \mathcal{R} \Rightarrow \mathcal{S}$, and σ is unique because inclusions are unique in set theory.

Next we must show that $\theta = (\sigma \triangleleft \mathcal{T}) \circ \eta_{\mathcal{R}}$. Since $\sigma \triangleleft \mathcal{T} : \mathcal{T} \oplus \mathcal{X} \Rightarrow \mathcal{T} \oplus \mathcal{S}$ and $\eta_{\mathcal{R}} : \mathcal{R} \Rightarrow \mathcal{T} \oplus \mathcal{X}$, then $(\sigma \triangleleft \mathcal{T}) \circ \eta_{\mathcal{R}} : \mathcal{R} \Rightarrow \mathcal{T} \oplus \mathcal{S}$. Since $\theta : \mathcal{R} \Rightarrow \mathcal{T} \oplus \mathcal{S}$ and inclusions are unique in set theory, $\theta = (\sigma \triangleleft \mathcal{T}) \circ \eta_{\mathcal{R}}$.

■

Our situation corresponds to the following adjunction:

$$\varphi_{\mathcal{R},\mathcal{S}} : Z^V(\mathcal{T} \rightharpoonup \mathcal{R}, \mathcal{S}) \cong Z^U(\mathcal{R}, \mathcal{T} \oplus \mathcal{S}).$$

We have the following logical rules:

$$\frac{\mathcal{R} \subseteq_{\theta} \mathcal{T} \oplus \mathcal{S} \qquad \mathcal{R} \subseteq_{\eta_{\mathcal{R}}} \mathcal{T} \oplus (\mathcal{T} \rightharpoonup \mathcal{R})}{\mathcal{T} \rightharpoonup \mathcal{R} \subseteq_{\sigma} \mathcal{S}} \text{ Left Kan Extension} \qquad \frac{\mathcal{T} \rightharpoonup \mathcal{R} \subseteq_{\sigma} \mathcal{S}}{\mathcal{T} \oplus (\mathcal{T} \rightharpoonup \mathcal{R}) \subseteq_{\sigma \triangleleft \mathcal{T}} \mathcal{T} \oplus \mathcal{S}} \text{ Montonicity}$$

The left rule is equivalent to

$$\mathcal{R} \subseteq_{\theta} \mathcal{T} \oplus \mathcal{S} \text{ implies } \mathcal{T} \rightharpoonup \mathcal{R} \subseteq_{\sigma} \mathcal{S}$$

because the second premise of the first rule always holds in our situation. This premise holds via one half of the residuation condition

$$\frac{\mathcal{T} \rightharpoonup \mathcal{R} \subseteq_{\iota} \mathcal{T} \rightharpoonup \mathcal{R}}{\mathcal{R} \subseteq_{\eta_{\mathcal{R}}} \mathcal{T} \oplus (\mathcal{T} \rightharpoonup \mathcal{R})}$$

In the Kan extension view, residuation is not available and hence that premise must be explicitly stated. The other direction is

$$\dfrac{\mathcal{R} \subseteq^{\eta_{\mathcal{R}}} \mathcal{T} \oplus (\mathcal{T} \rightharpoonup \mathcal{R}) \qquad \dfrac{\mathcal{T} \rightharpoonup \mathcal{R} \subseteq^{\sigma} \mathcal{S}}{\mathcal{T} \oplus (\mathcal{T} \rightharpoonup \mathcal{R}) \subseteq^{\sigma \triangleleft \mathcal{T}} \mathcal{T} \oplus \mathcal{S}}}{\mathcal{R} \subseteq^{(\sigma \triangleleft \mathcal{T}) \circ \eta_{\mathcal{R}}} \mathcal{T} \oplus \mathcal{S}}$$

As in right Kan extensions, the extra noise in the right premise of the Left Kan Extension rule, i.e., $\mathcal{R} \subseteq^{\varepsilon} \mathcal{T} \circledcirc (\mathcal{T} \multimap \mathcal{R})$, is related to the ability for some left Kan extensions to be definable without necessarily having a pair of adjoint functors.

5.5 Left Kan Lift

Definition 5.5.1 The diagram

$$\begin{tikzcd}
& U \arrow[dl, "\mathcal{T}"] \arrow[dl, bend right=60, "\mathcal{X} = \mathrm{LLift}_{\mathcal{S}}\mathcal{R}"'] \arrow[dl, phantom, bend right=30, "\overset{\sigma}{\Rightarrow}"] \arrow[dr, "\mathcal{R}"] \arrow[dr, bend left=60, "\mathcal{T} \oplus \mathcal{S}"] \arrow[dr, bend right=40, "\mathcal{X} \oplus \mathcal{S}"'] \arrow[dr, phantom, bend left=30, "\Uparrow \theta"] \arrow[dr, phantom, bend right=15, "\Downarrow \eta_{\mathcal{R}}"] & \\
V \arrow[rr, bend right=15, "\mathcal{S}"'] & & Z
\end{tikzcd}$$

Figure 19: Left Kan Lift

is said to exhibit $\mathcal{X}$, denoted $\mathrm{LLift}_{\mathcal{S}}\mathcal{R}$, as a *left lifting of* $\mathcal{R}$ *through* $\mathcal{S}$ when each 2-cell $\theta : \mathcal{R} \Rightarrow U^{\mathcal{S}}\mathcal{T}$ factors as $(U^{\mathcal{S}}\sigma) \circ \eta_{\mathcal{R}}$ for a unique 2-cell $\sigma : \mathcal{X} \Rightarrow \mathcal{T}$. We write $\mathcal{X} = \mathrm{LLift}_{\mathcal{S}}\mathcal{R}$ for a particular choice of left lifting of $\mathcal{R}$ through $\mathcal{S}$.

Notice that the 2-arrows are reversed with respect to Left Kan Extension. As a result, Right Kan Lift defines a universal arrow from an object to a covariant functor.

Definition 5.5.2 A *universal arrow* from $\mathcal{R}$ to $Z^{\mathcal{T}}$ is a pair $\langle \mathrm{LLift}_{\mathcal{S}}\mathcal{R}, \eta_{\mathcal{R}} \rangle$ consisting of an object $\mathrm{LLift}_{\mathcal{S}}\mathcal{R} : U \Rightarrow V$ and an arrow $\eta_{\mathcal{R}} : \mathcal{R} \Rightarrow U^{\mathcal{S}}(\mathrm{LLift}_{\mathcal{S}}\mathcal{R})$ such that to every pair $\langle \mathcal{T}, \theta \rangle$ with $\mathcal{T}$ an object of V^U and $\theta : \mathcal{R} \Rightarrow U^{\mathcal{S}}\mathcal{T}$, there is a unique arrow $\sigma : (\mathrm{LLift}_{\mathcal{S}}\mathcal{R}) \Rightarrow \mathcal{T}$ with $\theta = (Z^{\mathcal{T}}\sigma) \circ \eta_{\mathcal{R}}$ (where $Z^{\mathcal{T}}\sigma = \mathcal{S} \triangleright \sigma$). In other words, every arrow θ factors uniquely through the universal arrow $\eta_{\mathcal{R}}$ as in the commutative diagram

$$\begin{tikzcd}
\mathrm{LLift}_{\mathcal{S}}\mathcal{R} \arrow[d, dotted, "\sigma"'] & U^{\mathcal{S}}(\mathrm{LLift}_{\mathcal{S}}\mathcal{R}) \arrow[d, dotted, "U^{\mathcal{S}}\sigma"'] & \mathcal{R} \arrow[l, Rightarrow, "\eta_{\mathcal{R}}"'] \arrow[dl, Rightarrow, "\theta"] \\
\mathcal{T} & U^{\mathcal{S}}\mathcal{T} &
\end{tikzcd}$$

Figure 20: Universal Arrow from$\mathcal{R}$ to $U^{\mathcal{S}}$

As in Left Kan Extension, this sets up an adjunction:

Theorem 5.5.3

$$\varphi_{\mathcal{R},\mathcal{S}} : V^Z(\mathrm{LLift}_{\mathcal{S}}\mathcal{R}, \mathcal{T}) \cong Z^U(\mathcal{R}, U^{\mathcal{S}}\mathcal{T}).$$

The proof is similar to the adjunction proof for Left Kan extension. For our situation with $\oplus$ and $\leftharpoonup$, we use the following definition:

Definition 5.5.4 Given an arrow $\mathcal{S} : V \twoheadrightarrow Z$ and an object U we consider the arrow category V^U with objects the arrows $\mathcal{T} : U \twoheadrightarrow V$ and 2-arrows $\sigma : \mathcal{T} \Rightarrow \mathcal{X}$, then we define the covariant functor $U^{\mathcal{S}} : V^U \twoheadrightarrow Z^U$ and the object map $\mathrm{LLift}_{\mathcal{S}} : Z^U \twoheadrightarrow V^U$; let $\mathcal{P}, \mathcal{Q} : U \twoheadrightarrow V$ and $\mathcal{Y} : U \twoheadrightarrow Z$:

$$U^{\mathcal{S}} \overset{def}{=} \lambda\mathcal{Q} \,.\, \mathcal{Q} \oplus \mathcal{S} \qquad \mathrm{LLift}_{\mathcal{S}} \overset{def}{=} \lambda\mathcal{Y} \,.\, \mathcal{Y} \leftharpoonup \mathcal{S},$$

and 2-arrows by

$$(\nu : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (U^{\mathcal{S}}\nu : \mathcal{P} \oplus \mathcal{S} \Rightarrow \mathcal{Q} \oplus \mathcal{S})$$

Lemma 5.5.5 *Assuming Definition 5.5.4, $U^{\mathcal{S}}$ is a covariant functor on the category V^U whose objects are 1-arrows and arrows are the 2-arrows.*

Proof: The proof is simple and merely relies on the fact that arrows in the 2-category qua category compose. Assume $\nu : \mathcal{P} \Rightarrow \mathcal{Q}$ and $\mu : \mathcal{Q} \Rightarrow \mathcal{N}$, then $U^{\mathcal{S}}\nu : \mathcal{P} \oplus \mathcal{S} \Rightarrow \mathcal{Q} \oplus \mathcal{S}$ and $U^{\mathcal{S}}\mu : \mathcal{Q} \oplus \mathcal{S} \Rightarrow \mathcal{N} \oplus \mathcal{S}$. Hence $(U^{\mathcal{S}}\mu) \circ (U^{\mathcal{S}}\nu) = U^{\mathcal{S}}(\mu \circ \nu)$ where the latter follows from the properties of 2-categories qua categories from the left whiskering above. Similarly, that $U^{\mathcal{S}}$ preserves the identity arrows (i.e., the equality relation) also follows from left whiskering in the 2-category qua categories.

In our context of typed relations, that $U^{\mathcal{S}}$ is a covariant functor (where $U^{\mathcal{S}}\sigma = \mathcal{S} \triangleright \sigma$) follows from

$$\frac{\mathcal{P} \subseteq_{\sigma} \mathcal{Q}}{\mathcal{P} \oplus \mathcal{S} \subseteq_{\mathcal{S} \triangleright \sigma} \mathcal{Q} \oplus \mathcal{S}}$$

and the definition of $\oplus$. This says that

$$(\sigma : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (\mathcal{S} \triangleright \sigma : \mathcal{P} \oplus \mathcal{S} \Rightarrow \mathcal{Q} \oplus \mathcal{S}).$$

■

Theorem 5.5.6 *For our situation using Definition 5.5.4 and where $\mathcal{X} = \mathrm{LLift}_{\mathcal{S}}\mathcal{R}$,*

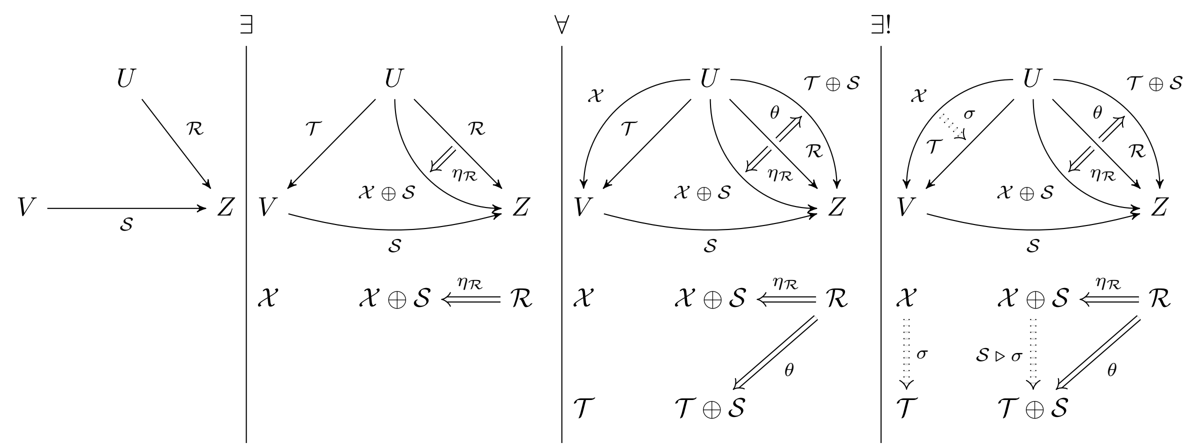

Figure 21: Left Kan Lift as a Left Residual

Proof: We first show that σ exists. Let $\langle v, z\rangle \in \mathcal{R} \leftharpoondown \mathcal{S}$. There is some $z \in Z$ such that $\langle u, z\rangle \in \mathcal{R}$ and $\langle v, z\rangle \notin \mathcal{S}$. Since $\theta : \mathcal{R} \Rightarrow \mathcal{T} \oplus \mathcal{S}$, then $\langle u, z\rangle \in \mathcal{T} \oplus \mathcal{S}$. By the definition of $\mathcal{T} \circledcirc \mathcal{S}$, then $\langle u, v\rangle \in \mathcal{T}$. Therefore, there is a σ such that $\sigma : \mathcal{R} \leftharpoondown \mathcal{S} \Rightarrow \mathcal{T}$, and σ is unique because inclusions are unique in set theory.

Next we must show that $\theta = (\mathcal{S} \triangleright \sigma) \circ \eta_{\mathcal{R}}$. Since $\mathcal{S} \triangleright \sigma : \mathcal{X} \oplus \mathcal{S} \Rightarrow \mathcal{T} \oplus \mathcal{S}$ and $\eta_{\mathcal{R}} : \mathcal{R} \Rightarrow \mathcal{X} \oplus \mathcal{S}$, then $(\mathcal{S} \triangleright \sigma) \circ \eta_{\mathcal{R}} : \mathcal{R} \Rightarrow \mathcal{T} \oplus \mathcal{S}$. Since $\theta : \mathcal{T} \circledcirc \mathcal{S} \Rightarrow \mathcal{R}$ and inclusions are unique in set theory, $\theta = (\mathcal{S} \triangleright \sigma) \circ \eta_{\mathcal{R}}$.

■

Our situation corresponds to the following adjunction:

$$V^U(\mathcal{R} \leftharpoondown \mathcal{S}, \mathcal{T}) \cong Z^U(\mathcal{R}, \mathcal{T} \oplus \mathcal{S}).$$

We have the following logical rules:

$$\frac{\mathcal{R} \subseteq_\theta \mathcal{T} \oplus \mathcal{S} \qquad \mathcal{R} \subseteq_{\eta_\mathcal{R}} (\mathcal{R} \leftharpoonup \mathcal{S}) \oplus \mathcal{S}}{\mathcal{R} \leftharpoonup \mathcal{S} \subseteq_\sigma \mathcal{S}}\ \text{Left Kan Lift} \qquad \frac{\mathcal{R} \leftharpoonup \mathcal{S} \subseteq_\sigma \mathcal{T}}{(\mathcal{R} \leftharpoonup \mathcal{S}) \oplus \mathcal{T} \subseteq_{\mathcal{S} \triangleright \sigma} \mathcal{T} \oplus \mathcal{S}}\ \text{Montonicity}$$

The left rule is equivalent to

$$\mathcal{R} \subseteq_\theta \mathcal{T} \oplus \mathcal{S} \text{ implies } \mathcal{R} \leftharpoonup \mathcal{S} \subseteq_\sigma \mathcal{T}$$

because the second premise of the first rule always holds in our situation. This premise holds via one half of the residuation condition

$$\frac{\mathcal{R} \leftharpoonup \mathcal{S} \subseteq_\iota \mathcal{R} \leftharpoonup \mathcal{S}}{\mathcal{R} \subseteq_{\eta_\mathcal{R}} (\mathcal{R} \leftharpoonup \mathcal{S}) \oplus \mathcal{S}}$$

In the Kan extension view, residuation is not available and hence that premise must be explicitly stated. The other direction is

$$\frac{\mathcal{R} \subseteq_{\eta_\mathcal{R}} (\mathcal{R} \leftharpoonup \mathcal{S}) \oplus \mathcal{S} \qquad \dfrac{\mathcal{R} \leftharpoonup \mathcal{S} \subseteq_\sigma \mathcal{T}}{(\mathcal{R} \leftharpoonup \mathcal{S}) \oplus \mathcal{S} \subseteq_{\mathcal{S} \triangleright \sigma} \mathcal{T} \oplus \mathcal{S}}}{\mathcal{R} \subseteq_{(\mathcal{S} \triangleright \sigma) \circ \eta_\mathcal{R}} \mathcal{T} \oplus \mathcal{S}}$$

5.6 Intensional Nor

We use different left whiskering diagrams than previously.

Definition 5.6.1

$$\begin{tikzcd}
Z \arrow[r, bend left, "\mathcal{X}"] \arrow[r, bend right, "\check{\mathcal{S}}"'] \arrow[r, phantom, "\Downarrow \sigma"] \arrow[rr, bend left=60, "\mathcal{R}"] & V \arrow[r, "\check{\mathcal{T}}"] \arrow[u, phantom, "\Uparrow \varepsilon_\mathcal{R}"] & U
\end{tikzcd}
=
\begin{tikzcd}
U \arrow[r, bend left=60, "\mathcal{R}"] \arrow[r, bend left, "Z^{\check{\mathcal{T}}}\mathcal{X}"] \arrow[r, bend right, "Z^{\check{\mathcal{T}}}\check{\mathcal{S}}"'] \arrow[r, phantom, "\Uparrow \check{\mathcal{T}} \triangleright \sigma"] & Z
\end{tikzcd}
=
\begin{tikzcd}
U \arrow[r, bend left, "\mathcal{R}"] \arrow[r, bend right, "Z^{\check{\mathcal{T}}}\check{\mathcal{S}}"'] \arrow[r, phantom, "\Uparrow \varepsilon_\mathcal{R} \circ (\check{\mathcal{T}} \triangleright \sigma)"] & Z
\end{tikzcd}$$

Figure 22: Left Whiskering

where by letting $Z^{\check{\mathcal{T}}}\sigma = \check{\mathcal{T}} \triangleright \sigma$, then $Z^{\check{\mathcal{T}}}$ is a contravariant functor on the 2-category qua category, the 0-objects and 1-arrows need not constitute a category. Note that the 2-arrow σ is in the opposite direction as its transform $\check{\mathcal{T}} \triangleright \sigma$. Also note that the direction of the 1-arrows changes from the first diagram to the second and third.

A simple diagram chase will show that for $\tau : \mathcal{X} \Rightarrow \check{\mathcal{S}}$ and $\rho : \check{\mathcal{S}} \Rightarrow \mathcal{Q}$,

$$Z^{\check{\mathcal{T}}}(\rho \circ \tau) = (Z^{\check{\mathcal{T}}}\tau) \circ (Z^{\check{\mathcal{T}}}\rho).$$

A similar left whiskering diagram used by Extension (see next section) with $\eta_\mathcal{R}$ in the opposite direction of $\varepsilon_\mathcal{R}$ and replacing $\varepsilon_\mathcal{R}$ is had by turning all the 2-arrows around, i.e., $\rho : \mathcal{Q} \Rightarrow \check{\mathcal{S}}$ and $\tau : \check{\mathcal{S}} \Rightarrow \mathcal{X}$, and the following holds by a diagram chase:

$$Z^{\check{\mathcal{T}}}(\tau \circ \rho) = (Z^{\check{\mathcal{T}}}\tau) \circ (Z^{\check{\mathcal{T}}}\rho).$$

Definition 5.6.2 The diagram

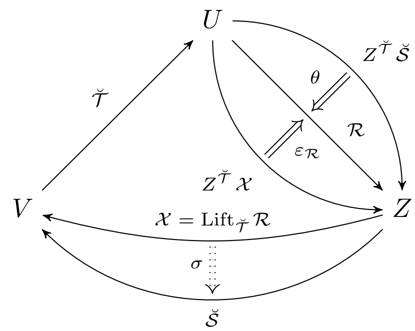

Figure 23: Lift

is said to exhibit $\mathcal{X}$, denoted $\mathcal{X} = \mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R}$, as a *lift of* $\breve{\mathcal{T}}$ *through* $\mathcal{R}$ when each 2-cell $\theta : Z^{\breve{\mathcal{T}}}\breve{\mathcal{S}} \Rightarrow \mathcal{R}$ factors as $\varepsilon_{\mathcal{R}} \circ (Z^{\breve{\mathcal{T}}}\sigma)$ for a unique 2-cell $\sigma : \mathcal{X} \Rightarrow \breve{\mathcal{S}}$. $\mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R}$ is a particular choice of lift of $\breve{\mathcal{T}}$ along $\mathcal{R}$.

The Lift defines a universal arrow from a contravariant functor to an object. The reason for the contravariant functor in contrast to Right Kan Extension is that the arrow σ is reversed:

Definition 5.6.3 A *universal arrow* (i.e., couniversal arrow) from $Z^{\breve{\mathcal{T}}}$ to $\mathcal{R}$ is a pair $\langle \mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R}, \varepsilon_{\mathcal{R}}\rangle$ consisting of an object $\mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R} : Z \Rightarrow V$ and an arrow $\varepsilon_{\mathcal{R}} : Z^{\breve{\mathcal{T}}}(\mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R}) \Rightarrow \mathcal{R}$ such that to every pair $\langle \breve{\mathcal{S}}, \theta\rangle$ with $\breve{\mathcal{S}}$ an object of V^Z and $\theta : Z^{\breve{\mathcal{T}}}\breve{\mathcal{S}} \Rightarrow \mathcal{R}$, there is a unique $\sigma : (\mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R}) \Rightarrow \breve{\mathcal{S}}$ with $\theta = \varepsilon_{\mathcal{R}} \circ (Z^{\breve{\mathcal{T}}}\sigma)$ (where $Z^{\breve{\mathcal{T}}}\sigma = \breve{\mathcal{T}} \triangleright \sigma$). In other words, every arrow θ factors uniquely through the universal arrow $\varepsilon_{\mathcal{R}}$ as in the commutative diagram

$$\begin{tikzcd}
\breve{\mathcal{S}} & & Z^{\breve{\mathcal{T}}}\breve{\mathcal{S}} \arrow[d, dotted, "Z^{\breve{\mathcal{T}}}\sigma"'] \arrow[dr, Rightarrow, "\theta"] & \\
\mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R} \arrow[u, dotted, "\sigma"] & & Z^{\breve{\mathcal{T}}}(\mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R}) \arrow[r, Rightarrow, "\varepsilon_{\mathcal{R}}"'] & \mathcal{R}
\end{tikzcd}$$

Figure 24: Universal from $Z^{\breve{\mathcal{T}}}$ to $\mathcal{R}$

This sets up an adjunction:

Theorem 5.6.4

$$\varphi_{\breve{\mathcal{S}},\mathcal{R}} : Z^U(Z^{\breve{\mathcal{T}}}\breve{\mathcal{S}}, \mathcal{R}) \cong V^Z(\mathrm{Lift}_{\breve{\mathcal{T}}}\mathcal{R}, \breve{\mathcal{S}})$$

where Z^U is a functor that returns the set of all arrows from U to Z and V^Z is a functor that returns all arrows from Z to V.

The proof of the Theorem is similar to that for Right Kan Extension although now the $U^{\mathcal{T}}$ is a contravariant functor. This presents no difficulties. For our situation with $\downarrow$, we use the following definition:

Definition 5.6.5 Given an arrow $\breve{\mathcal{T}} : V \to U$ and an object Z we consider the arrow category V^Z with objects the arrows $\breve{\mathcal{S}} : Z \to V$ and 2-arrows $\sigma : \breve{\mathcal{S}} \Rightarrow \mathcal{X}$, then we define the contravariant functor $Z^{\breve{\mathcal{T}}} : V^Z \to Z^U$ and the object map $\mathrm{Lift}_{\breve{\mathcal{T}}} : Z^U \to V^Z$; let $\mathcal{P}, \mathcal{Q} : Z \to V$ and $\mathcal{Y} : U \to Z$:

$$Z^{\breve{\mathcal{T}}} \stackrel{\mathit{def}}{=} \lambda\mathcal{Q} \,.\, \breve{\mathcal{T}} \downarrow \mathcal{Q} \qquad \mathrm{Lift}_{\breve{\mathcal{T}}} \stackrel{\mathit{def}}{=} \lambda\mathcal{Y} \,.\, \mathcal{Y} \downarrow \breve{\mathcal{T}},$$

and 2-arrows by

$$(\nu : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (Z^{\breve{\mathcal{T}}}\nu : \breve{\mathcal{T}} \downarrow \mathcal{Q} \Rightarrow \breve{\mathcal{T}} \downarrow \mathcal{P})$$

Lemma 5.6.6 *Assuming Definition 5.6.5, $Z^{\breve{\mathcal{T}}}$ is a contravariant functor on the category V^Z whose objects are 1-arrows and arrows are the 2-arrows.*

Proof: The proof is simple and merely relies on the fact that arrows in the 2-category qua category compose. Assume $\nu : \mathcal{P} \Rightarrow \mathcal{Q}$ and $\mu : \mathcal{Q} \Rightarrow \mathcal{N}$, then $Z^{\breve{\mathcal{T}}}\mu : \breve{\mathcal{T}} \downarrow \mathcal{N} \Rightarrow \breve{\mathcal{T}} \downarrow \mathcal{Q}$ and $Z^{\breve{\mathcal{T}}}\nu : \breve{\mathcal{T}} \downarrow \mathcal{Q} \Rightarrow \breve{\mathcal{T}} \downarrow \mathcal{P}$. Hence $(Z^{\breve{\mathcal{T}}}\nu) \circ (Z^{\breve{\mathcal{T}}}\mu) = Z^{\breve{\mathcal{T}}}(\mu \circ \nu)$ where the latter follows from the properties of 2-categories qua categories from the left lift whiskering above. Similarly, that $Z^{\breve{\mathcal{T}}}$ preserves the identity arrows (i.e., the equality relation) also follows from left lift whiskering in the 2-category qua category.

In our context of typed relations, that $Z^{\breve{\mathcal{T}}}$ (where $Z^{\breve{\mathcal{T}}}\sigma = \breve{\mathcal{T}} \rhd \sigma$) is a contravariant functor follows from

$$\frac{\mathcal{Q} \subseteq_\sigma \mathcal{P}}{\breve{\mathcal{T}} \downarrow \mathcal{P} \subseteq_{\breve{\mathcal{T}} \rhd \sigma} \breve{\mathcal{T}} \downarrow \mathcal{Q}}$$

and the definition of $\downarrow$. This says that

$$(\sigma : \mathcal{Q} \Rightarrow \mathcal{P}) \mapsto (\breve{\mathcal{T}} \rhd \sigma : \breve{\mathcal{T}} \downarrow \mathcal{Q} \Rightarrow \breve{\mathcal{T}} \downarrow \mathcal{P}).$$

■

The following theorem holds:

Theorem 5.6.7 *For our situation using Definition 5.6.5,*

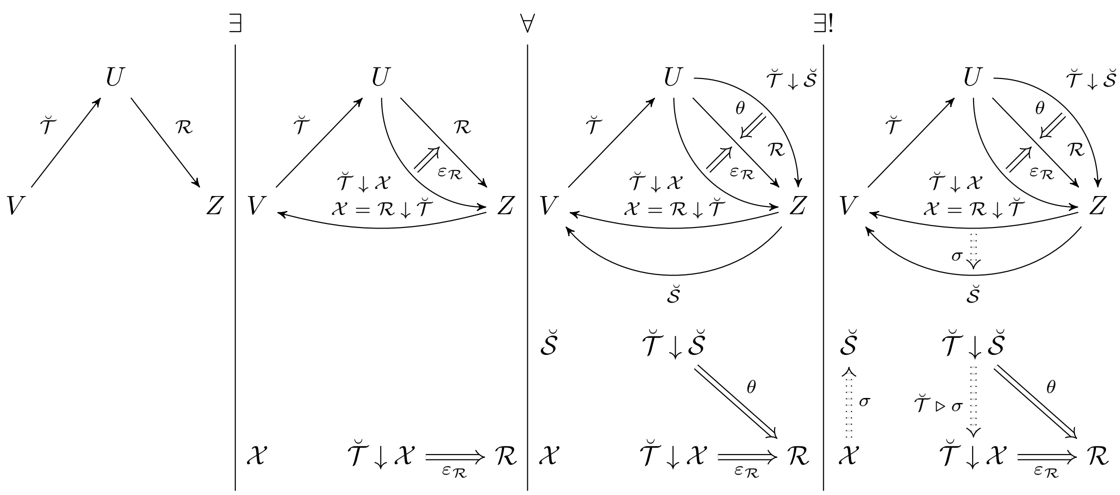

Figure 25: $\downarrow$ Lift of $\breve{\mathcal{T}}$ Through $\mathcal{R}$

Proof: We first show that σ exists. Assume $\langle z, v\rangle \in \mathcal{X}$, then $\langle z, v\rangle \in \mathcal{R} \downarrow \breve{\mathcal{T}}$ and so there is some $u \in U$ such that $\langle u, z\rangle \notin \mathcal{R}$ and $\langle v, u\rangle \notin \breve{\mathcal{T}}$. From $\theta : \breve{\mathcal{T}} \downarrow \breve{\mathcal{S}} \Rightarrow \mathcal{R}$ then $\langle u, z\rangle \notin \breve{\mathcal{T}} \downarrow \breve{\mathcal{S}}$. Hence for all v', $\langle v', u\rangle \notin \breve{\mathcal{T}}$ implies $\langle z, v'\rangle \in \breve{\mathcal{S}}$. Letting $v' = v$, we have $\langle v, u\rangle \notin \breve{\mathcal{T}}$ implies $\langle z, v\rangle \in \breve{\mathcal{S}}$ and therefore $\langle z, v\rangle \in \breve{\mathcal{S}}$. Hence $\mathcal{R} \downarrow \breve{\mathcal{T}} \subseteq \breve{\mathcal{S}}$ and there exists σ such that $\sigma : \mathcal{R} \downarrow \breve{\mathcal{T}} \Rightarrow \breve{\mathcal{S}}$. Also, σ is unique since inclusions are unique in set theory.

Next we must show that $\theta = \varepsilon_{\mathcal{R}} \circ (\breve{\mathcal{T}} \rhd \sigma)$. Since $\breve{\mathcal{T}} \rhd \sigma : \breve{\mathcal{T}} \downarrow \breve{\mathcal{S}} \Rightarrow \breve{\mathcal{T}} \downarrow \mathcal{X}$ and $\varepsilon_{\mathcal{R}} : \breve{\mathcal{T}} \downarrow \mathcal{X} \Rightarrow \mathcal{R}$, then $\varepsilon_{\mathcal{R}} \circ (\breve{\mathcal{T}} \rhd \sigma) : \breve{\mathcal{T}} \downarrow \breve{\mathcal{S}} \Rightarrow \mathcal{R}$. Since $\theta : \breve{\mathcal{T}} \downarrow \breve{\mathcal{S}} \Rightarrow \mathcal{R}$ and inclusions are unique in set theory, $\theta = \varepsilon_{\mathcal{R}} \circ (\breve{\mathcal{T}} \rhd \sigma)$.

■

Our situation corresponds to the following adjunction:

$$\varphi_{\breve{\mathcal{S}},\mathcal{R}} : Z^U(\breve{\mathcal{T}} \downarrow \breve{\mathcal{S}}, \mathcal{R}) \cong V^Z(\mathcal{R} \downarrow \breve{\mathcal{T}}, \breve{\mathcal{S}}).$$

We have the following logical rules:

$$\frac{\check{\mathcal{T}} \downarrow \check{\mathcal{S}} \subseteq_\theta \mathcal{R} \qquad \check{\mathcal{T}} \downarrow (\mathcal{R} \downarrow \check{\mathcal{T}}) \subseteq_{\varepsilon_\mathcal{R}} \mathcal{R}}{\mathcal{R} \downarrow \check{\mathcal{T}} \subseteq_\sigma \check{\mathcal{S}}} \text{ Right } \downarrow \text{ Extension} \qquad \frac{\mathcal{R} \downarrow \check{\mathcal{T}} \subseteq_\sigma \check{\mathcal{S}}}{\check{\mathcal{T}} \downarrow \check{\mathcal{S}} \subseteq_{\sigma \downarrow \check{\mathcal{T}}} \check{\mathcal{T}} \downarrow (\mathcal{R} \downarrow \check{\mathcal{T}})} \text{ Antitonicity}$$

The left rule is equivalent to

$$\check{\mathcal{T}} \downarrow \check{\mathcal{S}} \subseteq_\theta \mathcal{R} \text{ implies } \mathcal{R} \downarrow \check{\mathcal{T}} \subseteq_\sigma \check{\mathcal{S}}$$

because the right premise always holds in our situation. This premise holds via one half of the residuation condition

$$\frac{\mathcal{R} \downarrow \check{\mathcal{T}} \subseteq_\iota \mathcal{R} \downarrow \check{\mathcal{T}}}{\check{\mathcal{T}} \downarrow (\mathcal{R} \downarrow \check{\mathcal{T}}) \subseteq_{\varepsilon_\mathcal{R}} \mathcal{R}}$$

where ι is the equality relation and stands for the identity in the 2-category. In the right $\downarrow$ extension view, residuation is not available and hence that premise must be explicitly stated. The other direction is

$$\frac{\dfrac{\mathcal{R} \downarrow \check{\mathcal{T}} \subseteq_\sigma \check{\mathcal{S}}}{\check{\mathcal{T}} \downarrow \check{\mathcal{S}} \subseteq_{\sigma \downarrow \check{\mathcal{T}}} \check{\mathcal{T}} \downarrow (\mathcal{R} \downarrow \check{\mathcal{T}})} \qquad \check{\mathcal{T}} \downarrow (\mathcal{R} \downarrow \check{\mathcal{T}}) \subseteq_{\varepsilon_\mathcal{R}} \mathcal{R}}{\check{\mathcal{T}} \downarrow \check{\mathcal{S}} \subseteq_\theta \mathcal{R}}$$

5.7 Intensional Nand

Definition 5.7.1 The diagram

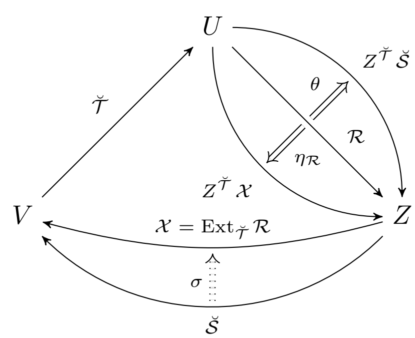

Figure 26: Extension

is said to exhibit $\mathcal{X}$, denoted $\mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R}$, is an *extension of* $\check{\mathcal{T}}$ *along* $\mathcal{R}$ when each 2-cell $\theta : \mathcal{R} \Rightarrow Z^{\check{\mathcal{T}}}\check{\mathcal{S}}$ factors as $(Z^{\check{\mathcal{T}}}\sigma) \circ \eta_\mathcal{R}$ for a unique 2-cell $\sigma : \check{\mathcal{S}} \Rightarrow \mathcal{X}$. $\mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R}$ is a particular choice of extension of $\check{\mathcal{T}}$ along $\mathcal{R}$.

The Extension defines a universal arrow from an object to a contravariant functor. The reason for the contravariant functor in contrast to Left Kan Extension is that the arrow σ is reversed:

Definition 5.7.2 A *universal arrow* from $\mathcal{R}$ to $Z^{\check{\mathcal{T}}}$ is a pair $\langle \mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R}, \eta_\mathcal{R} \rangle$ consisting of an object $\mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R} : Z \to V$ and an arrow $\eta_\mathcal{R} : \mathcal{R} \Rightarrow Z^{\check{\mathcal{T}}}(\mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R})$ such that to every pair $\langle \check{\mathcal{S}}, \theta \rangle$ with $\check{\mathcal{S}}$ an object of V^Z and $\theta : \mathcal{R} \Rightarrow Z^{\check{\mathcal{T}}}\check{\mathcal{S}}$, there is a unique $\sigma : \check{\mathcal{S}} \Rightarrow (\mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R})$ with $\theta = (Z^{\check{\mathcal{T}}}\sigma) \circ \eta_\mathcal{R}$ (where $Z^{\check{\mathcal{T}}}\sigma = \mathcal{T} \triangleright \sigma$). In other words, every arrow θ factors uniquely through the universal arrow $\eta_\mathcal{R}$ as in the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R} & Z^{\check{\mathcal{T}}}(\mathrm{Ext}_{\check{\mathcal{T}}}\mathcal{R}) \arrow[d, dotted, Rightarrow, "Z^{\check{\mathcal{T}}}\sigma"'] & \mathcal{R} \arrow[l, Rightarrow, "\eta_\mathcal{R}"'] \arrow[dl, Rightarrow, "\theta"] \\
\check{\mathcal{S}} \arrow[u, dotted, Rightarrow, "\sigma"] & Z^{\check{\mathcal{T}}}\check{\mathcal{S}} &
\end{tikzcd}$$

Figure 27: Universal from $\mathcal{R}$ to $Z^{\check{\mathcal{T}}}$

This sets up an adjunction:

Theorem 5.7.3

$$\varphi_{\mathcal{R},\breve{\mathcal{S}}} : Z^U(\mathcal{R}, Z^{\breve{\mathcal{T}}}\breve{\mathcal{S}}) \cong V^Z(\breve{\mathcal{S}}, \mathrm{Ext}_{\breve{\mathcal{T}}}\mathcal{R})$$

where Z^U is a functor that returns the set of all arrows from U to Z and V^Z is a functor that returns all arrows from Z to V.

The proof of the Theorem is similar to those analogous theorem for Left Kan Extension taking into account that $Z^{\breve{\mathcal{T}}}$ is a contravariant functor as in the analogous theorem for Intensional Nor. For our situation with $\uparrow$, we use the following definition:

Definition 5.7.4 Given an arrow $\breve{\mathcal{T}} : V \to U$ and an object Z we consider the arrow category V^Z with objects the arrows $\breve{\mathcal{S}} : Z \to V$ and 2-arrows $\sigma : \breve{\mathcal{S}} \Rightarrow \mathcal{X}$, then we define the contravariant functor $Z^{\breve{\mathcal{T}}} : V^Z \to Z^U$ and the object map $\mathrm{Ext}_{\breve{\mathcal{T}}} : Z^U \to V^Z$; let $\mathcal{P}, \mathcal{Q} : Z \to V$ and $\mathcal{Y} : U \to Z$:

$$Z^{\breve{\mathcal{T}}} \stackrel{\mathit{def}}{=} \lambda\mathcal{Q}\,.\,\mathcal{Q}\uparrow\breve{\mathcal{T}} \qquad \mathrm{Ext}_{\breve{\mathcal{T}}} \stackrel{\mathit{def}}{=} \lambda\mathcal{Y}\,.\,\breve{\mathcal{T}}\uparrow\mathcal{Y},$$

and 2-arrows by

$$(\nu : \mathcal{P} \Rightarrow \mathcal{Q}) \mapsto (Z^{\breve{\mathcal{T}}}\nu : \mathcal{Q}\uparrow\breve{\mathcal{T}} \Rightarrow \mathcal{P}\uparrow\breve{\mathcal{T}}).$$

Lemma 5.7.5 *Assuming Definition 5.7.4, $Z^{\breve{\mathcal{T}}}$ is a contravariant functor on the category V^Z whose objects are 1-arrows and arrows are the 2-arrows.*

Proof: The proof relies on the fact that arrows in the 2-category compose. Assume $\nu : \mathcal{P} \Rightarrow \mathcal{Q}$ and $\mu : \mathcal{Q} \Rightarrow \mathcal{N}$, then $Z^{\breve{\mathcal{T}}}\mu : \mathcal{N}\uparrow\breve{\mathcal{T}} \Rightarrow \mathcal{Q}\uparrow\breve{\mathcal{T}}$ and $Z^{\breve{\mathcal{T}}}\nu : \mathcal{Q}\uparrow\breve{\mathcal{T}} \Rightarrow \mathcal{P}\uparrow\breve{\mathcal{T}}$. Hence $(Z^{\breve{\mathcal{T}}}\nu)\circ(Z^{\breve{\mathcal{T}}}\mu) = Z^{\breve{\mathcal{T}}}(\mu\circ\nu)$ where the latter follows from the properties of 2-categories from the right extension whiskering above. Similarly, that $Z^{\breve{\mathcal{T}}}$ preserves the identity arrows (i.e., the equality relation) also follows from right extension whiskering in the 2-category.

In our context of typed relations, that $Z^{\breve{\mathcal{T}}}$ (where $Z^{\breve{\mathcal{T}}}\sigma = \sigma \lhd \breve{\mathcal{T}}$) is a contravariant functor follows from

$$\frac{\mathcal{Q} \subseteq_\sigma \mathcal{P}}{\breve{\mathcal{T}}\uparrow\mathcal{P} \subseteq_{\breve{\mathcal{T}}\rhd\sigma} \breve{\mathcal{T}}\uparrow\mathcal{Q}}$$

and the definition of $\uparrow$. This says that

$$(\sigma : \mathcal{Q} \Rightarrow \mathcal{P}) \mapsto (\breve{\mathcal{T}}\lhd\sigma : \mathcal{Q}\uparrow\breve{\mathcal{T}} \Rightarrow \mathcal{P}\uparrow\breve{\mathcal{T}}).$$

■

The following theorem then holds:

Theorem 5.7.6

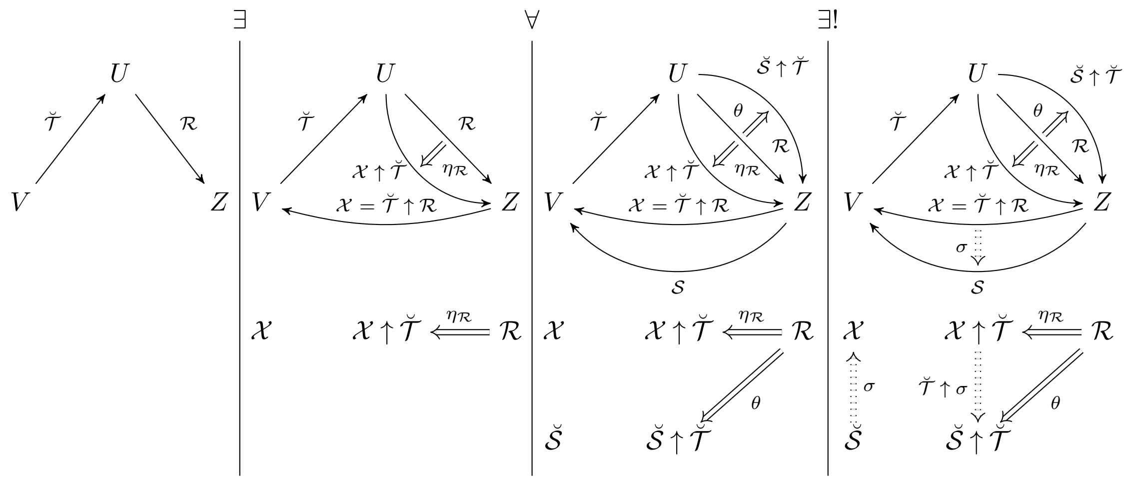

Figure 28: Left $\uparrow$ Extension of $\breve{\mathcal{T}}$ Through $\mathcal{R}$

Proof: We first show that σ exists. Let $\langle z, v\rangle \notin \mathcal{X}$. Since $\mathcal{X} = \breve{\mathcal{T}} \uparrow \mathcal{R}$, there is some u such that $\langle u, z\rangle \in \mathcal{R}$ and $\langle v, u\rangle \in \breve{\mathcal{T}}$. Since $\theta : \mathcal{R} \Rightarrow \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}$, then $\langle u, z\rangle \in \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}$. So for all $v' \in V$, it is the case that $\langle v', u\rangle \notin \breve{\mathcal{T}}$ or $\langle z, v'\rangle \notin \breve{\mathcal{S}}$. Letting $v' = v$, we have $\langle v, u\rangle \notin \breve{\mathcal{T}}$ or $\langle z, v\rangle \notin \breve{\mathcal{S}}$. Since $\langle v, u\rangle \in \breve{\mathcal{T}}$, thus $\langle z, v\rangle \notin \breve{\mathcal{S}}$. Taking the contrapositive, there is a σ such that $\sigma : \breve{\mathcal{S}} \Rightarrow \breve{\mathcal{X}}$. Since $\breve{\mathcal{S}} \subseteq \mathcal{X}$, then $\mathcal{X} \uparrow \breve{\mathcal{T}} \subseteq \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}$, therefore $\breve{\mathcal{T}} \uparrow \sigma : \mathcal{X} \uparrow \breve{\mathcal{T}} \Rightarrow \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}$.

Next we must show that $\theta = (\breve{\mathcal{T}} \uparrow \sigma) \circ \eta_{\mathcal{R}}$. Since $\breve{\mathcal{T}} \uparrow \sigma : \mathcal{X} \uparrow \breve{\mathcal{T}} \Rightarrow \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}$, and $\eta_{\mathcal{R}} : \mathcal{R} \Rightarrow \mathcal{X} \uparrow \breve{\mathcal{T}}$, then $(\breve{\mathcal{T}} \uparrow \sigma) \circ \eta_{\mathcal{R}} : \mathcal{R} \Rightarrow \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}$. Since $\theta : \mathcal{R} \Rightarrow \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}$ and inclusions are unique in set theory, $\theta = (\breve{\mathcal{T}} \uparrow \sigma) \circ \eta_{\mathcal{R}}$.

$\blacksquare$

Our situation corresponds to the following adjunction:

$$\varphi_{\breve{\mathcal{S}},\mathcal{R}} : Z^U(\mathcal{R}, \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}) \cong V^Z(\breve{\mathcal{S}}, \breve{\mathcal{T}} \uparrow \mathcal{R}).$$

We have the following logical rules:

$$\frac{\mathcal{R} \subseteq_\theta \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}} \qquad \mathcal{R} \subseteq_{\eta_{\mathcal{R}}} (\breve{\mathcal{T}} \uparrow \mathcal{R}) \uparrow \breve{\mathcal{T}}}{\breve{\mathcal{S}} \subseteq_\sigma \breve{\mathcal{T}} \uparrow \mathcal{R}} \text{ Left} \uparrow \text{Extension} \qquad \frac{\breve{\mathcal{S}} \subseteq_\sigma \breve{\mathcal{T}} \uparrow \mathcal{R}}{(\breve{\mathcal{T}} \uparrow \mathcal{R}) \uparrow \breve{\mathcal{T}} \subseteq_{\sigma \uparrow \breve{\mathcal{T}}} \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}} \text{ Antitonicity}$$

The left rule is equivalent to

$$\mathcal{R} \subseteq_\theta \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}} \text{ implies } \breve{\mathcal{S}} \subseteq_\sigma \breve{\mathcal{T}} \uparrow \mathcal{R}$$

because the right premise always holds in our situation. This premise holds via one half of the residuation condition

$$\frac{\breve{\mathcal{T}} \uparrow \mathcal{R} \subseteq_\iota \breve{\mathcal{T}} \uparrow \mathcal{R}}{\mathcal{R} \subseteq_{\eta_{\mathcal{R}}} (\breve{\mathcal{T}} \uparrow \mathcal{R}) \uparrow \breve{\mathcal{T}}}$$

In the left $\uparrow$ extension view, residuation is not available and hence that premise must be explicitly stated. The other direction is

$$\frac{\dfrac{\breve{\mathcal{S}} \subseteq_\sigma \mathcal{R} \uparrow \breve{\mathcal{T}}}{(\breve{\mathcal{T}} \uparrow \mathcal{R}) \uparrow \breve{\mathcal{T}} \subseteq_{\breve{\mathcal{T}} \uparrow \sigma} \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}} \qquad \mathcal{R} \subseteq_{\eta_{\mathcal{R}}} \breve{\mathcal{T}} \uparrow (\mathcal{R} \uparrow \breve{\mathcal{T}})}{\mathcal{R} \subseteq_\theta \breve{\mathcal{S}} \uparrow \breve{\mathcal{T}}}$$

6 Conclusion

We started with Relation Algebras as in [CT51, Ng84] and made some definitions for extra operators. We showed how to evaluate these algebras (construing them as logics) via Kripke frames using Relevance Logic

[AB75, RMPB82] as a template. The relationship between Relevance Logic in its classical version (from [RMPB82]) was then dissected. We showed how to retrieve an algebra of relations from the Kripke frames using the Tabularity Axiom. To get from there to a graph of relations, we showed some of our previous work in [AHR17] and showed how that led directly to a collection of 1-arrows where there are two categorical composition operators, tensor and cotensor. This formed a 2-category with inclusions as the 2-arrows. By not requiring the 1-category structure, we showed that intensional nor and intensional nand also could be fit into the framework by treating the 1-arrows as merely a graph of 1-arrows and the 2-arrows forming a 2-category qua category.

This paper represents the logical evolution of the work we did in [AHR17]. In that paper, the semantics relied upon the three-place Kripke relations that we borrowed from Relevance Logic (modulo a few restrictions to prevent commutativity of tensor from creeping in). However, having a semantics for a distributed relation algebra that did not use binary relations could be considered a drawback of that work. We improved on that work in this paper by showing how it can be done successfully.

The extensions and lifts were used to neatly characterize the framework from a categorical perspective. We view the extensions and lifts as prescribing those features that a realizability interpretation must meet. The extensions and lifts characterize the relational operators in a partial manner in that all the adjunctions, or residuations if you like, need not exist. This is an important aspect for us because it allows the logic to capture a crucial aspect of FPGA applications, namely that foreign IP as black boxes exist in almost all FPGA applications thus depriving a high assurance analysis of all the relations governing the operation of the FPGA application.

The intensional nor and nand forced us to consider 2-categories qua categories and made us realize the Kan extensions and lifts do not rely upon a 1-category, only 1-arrows. As a consequence, composition was replaced by covariant functors on the 2-category qua category in the case of tensor and cotensor and contravariant functors in the case of intensional nor and nand.

The Kan extensions and lifts pointed out that a fully abstract algebra of relations supports two compositions, tensor and cotensor. It is possible to abstract this further into two different hom functors. The distribution of tensor over cotensor (distribution not in the sense of distributed algebra) could be formalized by homming into a double monoidal category. The distribution then takes the form of certain natural transformations and their rules in the double monoidal category. One can go further and treat the extension of intensional nor and the lift of intensional nand as functors into that same double monoidal category. The algebraic relationships among the operators are then realized by natural transformations and rules on that double monoidal category.

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