Permanents in linear optical networks

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We develop an abstract look at linear optical networks from the viewpoint of combinatorics and permanents. In particular we show that calculation of matrix elements of unitarily transformed photonic multi-mode states is intimately linked to the computation of permanents. An implication of this remarkable fact is that all calculations that are based on evaluating matrix elements are generically computationally hard. Moreover, quantum mechanics provides simpler derivations of certain matrix analysis results which we exemplify by showing that the permanent of any unitary matrix takes its values across the unit disk in the complex plane.

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I. INTRODUCTION

Quantum computing promises to be able to perform certain computational tasks such as factoring of large numbers [1] exponentially faster than by using classical computing. Such statements can be made when looking into computational complexity classes of these algorithms. Recently, a wealth of proposals for physical implementation of such algorithms has emerged. The particular one we have in mind is based on qubit encoding in photons which are subjected to passive linear transformation in beam splitter networks. These so-called quantum-gate engineering protocols make use of measurement-induced nonlinearities in which effective nonlinear evolutions are achieved by conditional partial measurements in an extended Hilbert space augmented by auxiliary modes [2]. The probabilistic nature of these schemes makes it necessary to optimize linear-optical networks with respect to their probability of success. Hence, apart from the complexity of the algorithm that is supposed to run on the network there is another complexity associated with the optimization problem.

It seems to have been first noted in [3] (and somewhat more discussed in [4]) that optimization of those networks with respect to their probability of success is closely linked to maximizing a permanent under certain constraints. Permanents have found their way into quantum physics by the work of Caianiello [5, 6] who showed that generally all expectation values of bosonic field operators can be written as a permanent (or a hafnian, a matrix function related to the permanent which was actually introduced by him) of a certain coefficient matrix. At that time it seemed to be little more than a notation although he was eventually able to sum up Dyson series in field theory without the use of Feynman graphs.

In this article we will show that indeed the natural way of writing unitary transformations of multi-mode Fock states is by means of permanents of matrices closely associated with the unitary matrix of the state transformation. This immediately implies that the calculation of matrix elements of unitarily transformed multi-mode states (and all subsequent calculations based on it) is computationally expensive and cannot be performed in polynomial time (polynomial in the number of modes).

The article is organized as follows. Section II is used to introduce some definitions and notations. The main result of this paper, the unitary transformation of multi-mode states, is then presented in Sec. III. As an example for the application of these formulas, in Sec. IV we describe how the entanglement power of linear-optical networks can be calculated. The paper finishes with some concluding remarks in Sec. V.

II. SOME DEFINITIONS AND NOTATIONS

For the sake of definiteness, we introduce some notation which will be used throughout this article. To begin with, let us specify what one commonly understands by the permanent of a matrix. We will concentrate only on square matrices since this is what we will be using later.

Def.: The permanent of an $(n\times n)$-matrix $\mathbf{\Lambda}$ is a generalized matrix function defined by

$$\mathrm{per}\,\mathbf{\Lambda} = \sum_{\sigma\in S_n}\prod_{i=1}^{n}\Lambda_{i\sigma_i}\,, \tag{1}$$

where S_n is the group of permutations. For a fixed permutation $\sigma\in S_n$, the product $\prod_{i=1}^{n}\Lambda_{i\sigma_i}$ is called a diagonal. The permanent is thus the sum over all diagonals of the matrix $\mathbf{\Lambda}$ [7].

It is easy to see that the permanent is similar to the determinant of a matrix, but lacks the signs related to the order of the permutation. In fact, these signs cause the permanent to be a very special matrix function in that all usual computational rules known for the determinant fail. For example, no rules known from linear algebra can be used to simplify the computation of the permanent. Especially, rules such as $\det\mathbf{OSO}^T = \det\mathbf{O}\det\mathbf{S}\det\mathbf{O}^T$

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$= \det \mathbf{S}$ for orthogonal matrices $\mathbf{O}$ do not hold. The only known simplification for permanents can be achieved for permutation matrices $\mathbf{P}$ and diagonal matrices $\mathbf{D}$ in which case we have

$$\operatorname{per} \mathbf{P}\mathbf{\Lambda}\mathbf{D} = \operatorname{per}\mathbf{P} \operatorname{per}\mathbf{\Lambda} \operatorname{per}\mathbf{D} \,. \tag{2}$$

In general, the computation of the permanent has to be done without possible simplification in which case there are $n!$ terms in the sum (1). In fact, there exists a result by Marcus and Minc [8] that asserts that there exists no linear transformation on $n \times n$-matrices with $n \geq 3$ such that the permanent of these matrices can be converted into a determinant.

It will prove to be convenient to introduce some more notation. Let $\mathbf{\Lambda}[k_1, \ldots, k_m | l_1, \ldots, l_m]$ be the $(m \times m)$-matrix whose matrix elements are those of the original matrix $\mathbf{\Lambda}$ with row indices $k_1, \ldots, k_m$ and column indices $l_1, \ldots, l_m$. For example,

$$\mathbf{\Lambda}[k_1, k_2, k_3 | l_1, l_2, l_3] = \begin{pmatrix} \Lambda_{k_1 l_1} & \Lambda_{k_1 l_2} & \Lambda_{k_1 l_3} \\ \Lambda_{k_2 l_1} & \Lambda_{k_2 l_2} & \Lambda_{k_2 l_3} \\ \Lambda_{k_3 l_1} & \Lambda_{k_3 l_2} & \Lambda_{k_3 l_3} \end{pmatrix} . \tag{3}$$

The object $\mathbf{\Lambda}[(1^{m_1}, 2^{m_2}, \ldots) | (1^{n_1}, 2^{n_2}, \ldots)]$ denotes a matrix whose entries are taken from the matrix $\mathbf{\Lambda}$ and whose row index i occurs exactly m_i times and whose column index j occurs exactly n_j times, for example,

$$\mathbf{\Lambda}[(1^1, 2^1, 3^1) | (1^0, 2^2, 3^1)] = \begin{pmatrix} \Lambda_{12} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{32} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} . \tag{4}$$

Furthermore, let $G_{n,N}$ be the set of all non-decreasing integer sequences $\boldsymbol{\omega}$,

$$G_{n,N} := \{\boldsymbol{\omega} : 1 \leq \omega_1 \leq \cdots \leq \omega_n \leq N\} \,. \tag{5}$$

That is, each sequence $\boldsymbol{\omega} \in G_{n,N}$ has length n and its entries take integer values up to N.

III. UNITARY TRANSFORMATION OF MULTI-MODE FOCK STATES

With the above definitions it is now rather straightforward to write down the expression how a bosonic (photonic) Fock state $|n_1, n_2, \ldots, n_N\rangle$ transforms under the action of a unitary network specified by an operator $\hat{U}$ or a unitary matrix $\mathbf{\Lambda}$, respectively. Traditionally, this transformation is written by using the transformation rule for photonic amplitude operators [9]

$$\hat{\mathbf{a}} \mapsto \mathbf{\Lambda}^{+} \hat{\mathbf{a}} \,, \quad \hat{\mathbf{a}}^\dagger \mapsto \mathbf{\Lambda}^T \hat{\mathbf{a}}^\dagger \tag{6}$$

as

$$\hat{U} |n_1, n_2, \ldots, n_N\rangle = \prod_{i=1}^{N} \frac{1}{\sqrt{n_i!}} \left(\sum_{k_i=1}^{N} \Lambda_{k_i, i} \hat{a}_{k_i}^\dagger \right)^{n_i} |0\rangle^{\otimes N} \,. \tag{7}$$

By using the multinomial expansion theorem, this can be further rewritten as

$$\hat{U} |n_1, n_2, \ldots, n_N\rangle = \sum_{\substack{\{n_{ij}\} \\ \sum_{j=1}^{N} n_{ij} = n_i}} \frac{\prod_{i=1}^{N} (n_i!)^{1/2}}{\prod_{i,j=1}^{N} n_{ij}!} \prod_{j_1=1}^{N} (\Lambda_{j_1 1} \hat{a}_{j_1}^\dagger)^{n_{1 j_1}} \cdots \prod_{j_N=1}^{N} (\Lambda_{j_N N} \hat{a}_{j_N}^\dagger)^{n_{N j_N}} |0\rangle^{\otimes N} \,. \tag{8}$$

Note that the n_{ij} form a $n \times n$-matrix whose row sums are the photon numbers n_i, hence $\sum_{j=1}^{N} n_{ij} = n_i$. Moreover, the row sums (as well as the column sums to ensure unitarity, hence conservation of the total photon number) add up to the total photon number n which makes the matrix $\{n_{ij}/n\}$ (doubly) stochastic. Denoting the column sums of $\{n_{ij}\}$ as $\sum_{i=1}^{N} n_{ij} = m_j$, we can write the second line in Eq. (8) as

$$\left(\prod_{l=1}^{N} m_l! \right)^{1/2} \left(\prod_{k,l=1}^{N} \Lambda_{lk}^{n_{kl}} \right) |m_1, m_2, \ldots, m_N\rangle \,. \tag{9}$$

Note that $\prod_{k,l=1}^{N} \Lambda_{lk}^{n_{kl}}$ is a product with exactly n factors where each row index i occurs m_i times and where each column index j occurs n_j times. Therefore, it must be a diagonal (according to the definition in Sec. II) of the matrix $\mathbf{\Lambda}[(1^{m_1}, \ldots, N^{m_N}) | (1^{n_1}, \ldots, N^{n_N})]$. If we take the permanent of this matrix, we see that out of all possible permutations of column indices, $\prod_j n_j!$ of those permutations are identical. Analogously, there are $\prod_i m_i!$ ways of distributing the row indices. Hence, not all diagonals are different from each other. However, only $(\prod_i m_i!)(\prod_j n_j!)/(\prod_{i,j} n_{ij}!)$ terms actually lead to the same diagonal. With the knowledge of these multiplicities it follows from Eq. (8) that

$$\langle m_1, m_2, \ldots, m_N | \hat{U} | n_1, n_2, \ldots, n_N\rangle = \left(\prod_i n_i! \right)^{-1/2} \left(\prod_j m_j! \right)^{-1/2} \operatorname{per} \mathbf{\Lambda}[\mathbf{\Omega}' | \mathbf{\Omega}] \tag{10}$$

where

$$\mathbf{\Omega} = (1^{n_1}, 2^{n_2}, \ldots, N^{n_N}) \,, \tag{11}$$

$$\mathbf{\Omega}' = (1^{m_1}, 2^{m_2}, \ldots, N^{m_N}) \,. \tag{12}$$

Equation (10) is proof for the intimate relation between unitary transformation of multi-mode Fock states and permanents of matrices associated with the unitary matrix $\mathbf{\Lambda}$.

An immediate consequence of Eq. (10) is that

$$\operatorname{per} \mathbf{\Lambda} = {}^{\otimes N}\langle 1 | \hat{U} | 1 \rangle^{\otimes N} \tag{13}$$

with the effect that the permanent of a unitary matrix takes its value across the unit disk in the complex plane, $|\mathrm{per}\,\mathbf{\Lambda}| \leq 1$, since the expectation value to find the state $|1\rangle^{\otimes N}$ after performing a unitary transform of a similar state is actually a probability amplitude. Although this result is usually derived from the Marcus–Newman theorem [10], the probability interpretation of quantum mechanics provides a much simpler derivation.

The description (10) of matrix elements can be straightforwardly generalized to the full unitary transformation as

$$\hat{U}|n_1, n_2, \ldots, n_N\rangle = \left(\prod_i n_i!\right)^{-1/2} \sum_{\boldsymbol{\omega}\in G_{n,N}} \frac{1}{\sqrt{\mu(\boldsymbol{\omega})}} \mathrm{per}\,\mathbf{\Lambda}[\boldsymbol{\omega}|\boldsymbol{\Omega}]|m_1(\boldsymbol{\omega}), m_2(\boldsymbol{\omega}), \ldots, m_N(\boldsymbol{\omega})\rangle \tag{14}$$

in which the $m_i(\boldsymbol{\omega})$ are the multiplicities of the occurrence of the value i in the non-decreasing integer sequence $\boldsymbol{\omega}$ and $\mu(\boldsymbol{\omega}) = \prod_i m_i(\boldsymbol{\omega})!$.

Analogously, we obtain partial matrix elements (or rather projections) with $\sum_{j=2}^{N} m_j \leq n$ as

$$\langle m_2, \ldots, m_N|\hat{U}|n_1, n_2, \ldots, n_N\rangle = \left(\prod_{i=1}^{N} n_i!\right)^{-1/2} \left(\prod_{j=2}^{N} m_j!\right)^{-1/2} \mathrm{per}\,\mathbf{\Lambda}[\mathbf{\Omega}''|\boldsymbol{\Omega}] \left| n - \sum_{j=2}^{N} m_j \right\rangle \tag{15}$$

with

$$\mathbf{\Omega}'' = (1^{n-\sum_{j=2}^{N} m_j}, 2^{m_2}, \ldots, N^{m_N})\,. \tag{16}$$

Equations (10) and (14) constitute the main result of this paper. Either equation implies that quantum-state transformations can be written in terms of permanents of matrices associated with the unitary matrix $\mathbf{\Lambda}$. Although nothing has changed in the state transformation itself, this result nevertheless shows that the computational complexity of computing matrix elements in the Fock basis and functions depending on it is the same as computing permanents. The intrinsic interest in this result stems from the fact that it provides an argument for the affiliation of any type of optimization algorithm for passive linear optical networks or any maximization or averaging procedure over unitary operation in the Fock basis to the same complexity class, namely that of computing a permanent. The following section shall provide an example for an averaging procedure over unitary operator that can entirely be written as a function of permanents.

IV. ENTANGLEMENT POWER OF UNITARY NETWORKS

Over the recent years, entanglement and its quantification has played a major role in the discussions on the foundations of quantum mechanics. The basic principle lies in the distinction between separable and inseparable (entangled) states. Commonly, a bipartite state is called separable if its density matrix can be decomposed into a convex sum of tensor product states of the respective subsystems. On the contrary, an entangled state cannot be written in this form. Apart from separability criteria that can tell to which of the above classes a given quantum state belongs, there exist entropic or distance-based measures to quantify the amount of entanglement. We will not give any details here since they can be found in the vast literature on this subject (for a recent review, see e.g. [11]), we merely note that for pure bipartite states all of these entanglement measures collapse to the von Neumann entropy of the reduced density matrix.

A somewhat related question is to ask how much entanglement a quantum operation can generate when applied to some initial bipartite state [12]. Among all possible questions of this type we will concentrate only on the problem how much entanglement will be generated on average from a pure product state as opposed to the maximally available entanglement [13]. We will henceforth call a functional $P_E(\hat{U})$ the entanglement power of the unitary operator $\hat{U}$ with respect to the entanglement measure E if

$$P_E(\hat{U}) = \int dg(|\psi_1\rangle) dg(|\psi_2\rangle)\, E(\hat{U}|\psi_1, \psi_2\rangle)\,, \tag{17}$$

where $dg(|\psi_i\rangle)$ denotes the group measure with respect to the state $|\psi_i\rangle$. In what follows, we will restrict ourselves to the linear entropy as the entanglement measure of choice, simply because it is easiest to compute. The linear entropy, defined by

$$L(\hat{U}|\psi_1, \psi_2\rangle) = 2(1 - \mathrm{Tr}\hat{\varrho}_1^2) \tag{18}$$

where $\hat{\varrho}_1 = \mathrm{Tr}_2(\hat{U}|\psi_1, \psi_2\rangle\langle\psi_1, \psi_2|\hat{U}^\dagger)$, although not a proper entanglement measure in the strict sense, serves as an upper bound to the distillable entanglement.

A. Qubit initial states

To begin with, let us consider the simplest example in which two qubit states impinge on a beam splitter with transmittivity $T \in \mathbb{C}$. We will not use the language of permanents here but rather perform a straightforward calculation because that appears to be simpler (we leave the calculation involving permanents to the example in the next section). Let us define the input states as

$$|\psi_1\rangle = c_0|0\rangle + c_1|1\rangle\,, \tag{19}$$

$$|\psi_2\rangle = d_0|0\rangle + d_1|1\rangle\,, \tag{20}$$

where $|c_0|^2 + |c_1|^2 = |d_0|^2 + |d_1|^2 = 1$, and let the unitary matrix associated with the beam splitter be given by

$$\mathbf{\Lambda} = \begin{pmatrix} T & R \\ -R^* & T^* \end{pmatrix}. \tag{21}$$

The usual replacement rules for the bosonic amplitude operators, $\mathbf{a}^\dagger \mapsto \mathbf{\Lambda}^T\mathbf{a}^\dagger$, immediately imply that

$$\begin{aligned}&\hat{U}|\psi_1,\psi_2\rangle = c_0d_0|00\rangle \\ &+(c_0d_1R + c_1d_0T)|10\rangle + (c_0d_1T^* - c_1d_0R^*)|01\rangle \\ &+c_1d_1\left[\sqrt{2}TR|20\rangle - \sqrt{2}T^*R^*|02\rangle + (|T|^2-|R|^2)|11\rangle\right].\end{aligned} \tag{22}$$

The coefficient in the last term in Eq. (22) is proportional to the permanent of the beam splitter matrix $\mathrm{per}\,\mathbf{\Lambda} = |T|^2 - |R|^2$, as one expects from the general theory presented above (it is just the probability amplitude of finding the state $|11\rangle$ from a $|11\rangle$ input state). The reduced density matrix of subsystem 1 is then

$$\begin{aligned}\hat{\varrho}_1 = &-\left[\sqrt{2}c_1d_1T^*R^*|0\rangle\right]\otimes\text{h.c.} \\ &+\left[(c_0d_1T^* - c_1d_0R^*)|0\rangle + c_1d_1(|T|^2-|R|^2)|1\rangle\right]\otimes\text{h.c.} \\ &+\left[c_0d_0|0\rangle + (c_0d_1R + c_1d_0T)|1\rangle + \sqrt{2}c_1d_1|2\rangle\right]\otimes\text{h.c.}\,.\end{aligned} \tag{23}$$

The remaining steps are to compute the linear entropy and to integrate over the group measures with respect to the dynamical groups associated with the initial states $|\psi_i\rangle$. The crucial observation at this point is that the group integration removes all phase dependencies from the linear entropy, an effect that can be attributed to Schur's Lemma. The remaining amplitude dependence (either on $|T|$ or on $|R|$) can be uniquely rewritten in terms of the permanent, since $|T|^2 = (1+\mathrm{per}\,\mathbf{\Lambda})/2$ and $|R|^2 = (1-\mathrm{per}\,\mathbf{\Lambda})/2$. A straightforward calculation then leads to the result that

$$P_L(\hat{U}) = \frac{3}{64}\left[1-(\mathrm{per}\,\mathbf{\Lambda})^2\right]\left[13+9(\mathrm{per}\,\mathbf{\Lambda})^2\right] \tag{24}$$

which only depends on the permanent of the beam splitter matrix. Obviously, if the beam splitter acts only as a phase shifter (in which case $\mathrm{per}\,\mathbf{\Lambda} = 1$) or as a two-sided mirror ($\mathrm{per}\,\mathbf{\Lambda} = -1$) the input beams do not mix and no entanglement is created whatsoever. The maximum of Eq. (24) is reached when $\mathrm{per}\,\mathbf{\Lambda} = 0$, i.e. when the beam splitter is balanced. The entanglement power at this point is 39/64 and monotonically decreasing with increasing $|\mathrm{per}\,\mathbf{\Lambda}|$.

B. Higher-dimensional initial states

Let us now look at the situation in which the initial states that impinge on a single beam splitter are $N+1$-dimensional, hence can be written as

$$|\psi_1\rangle = \sum_{n_1=0}^{N} c_{n_1}|n_1\rangle\,,\quad |\psi_2\rangle = \sum_{n_2=0}^{N} d_{n_2}|n_2\rangle\,, \tag{25}$$

such that the combined state reads

$$|\psi_1,\psi_2\rangle = \sum_{n_1,n_2=0}^{N} c_{n_1}d_{n_2}|n_1,n_2\rangle\,. \tag{26}$$

Now we refrain from using the straightforward calculation since that becomes messy very quickly. Instead, we use the permanent language developed in the previous section. Applying the transformation rule (14) to this state yields

$$\begin{aligned}\hat{U}|\psi_1,\psi_2\rangle = &\sum_{n_1,n_2=0}^{N}\frac{c_{n_1}d_{n_2}}{\sqrt{n_1!n_2!}} \\ &\sum_{\boldsymbol{\omega}\in G_{n_1+n_2,2}}\frac{\mathrm{per}\,\mathbf{\Lambda}[\boldsymbol{\omega}|(1^{n_1},2^{n_2})]}{\sqrt{m_1(\boldsymbol{\omega})!m_2(\boldsymbol{\omega})!}}|m_1(\boldsymbol{\omega}),m_2(\boldsymbol{\omega})\rangle\end{aligned} \tag{27}$$

where we used the notation $(1^{n_1},2^{n_2})$ to indicate the multiplicities of the occurrence of the indices 1 and 2 in the columns of the matrix $\mathbf{\Lambda}[\boldsymbol{\omega}|(1^{n_1},2^{n_2})]$, as described in Sec. II. The important object to compute is now the linear entropy. It is straightforward to see that one can write the trace over the square of the reduced density matrix as

$$\begin{aligned}\mathrm{Tr}\,\hat{\varrho}_1^2 = &\sum_{k_1,k_2}\langle k_1,k_2|\hat{U}|\psi_1,\psi_2\rangle\langle\psi_1,\psi_2|\hat{U}^\dagger|k_1,k_2\rangle \\ &\langle\psi_1,\psi_2|\hat{U}^\dagger|k_2\rangle\langle k_2|\hat{U}|\psi_1,\psi_2\rangle\,.\end{aligned} \tag{28}$$

This shows that we need all ingredients from the previous section, that is, both matrix elements of a unitary operator and projections or partial matrix elements. It is now easy to see that the matrix elements $\langle k_1,k_2|\hat{U}|\psi_1,\psi_2\rangle$ can be written as

$$\begin{aligned}\langle k_1,k_2|\hat{U}|\psi_1,\psi_2\rangle = &\sum_{n_1,n_2=0}^{N}\frac{c_{n_1}d_{n_2}}{\sqrt{k_1!k_2!n_1!n_2!}} \\ &\mathrm{per}\,\mathbf{\Lambda}[(1^{k_1},2^{k_2})|(1^{n_1},2^{n_2})]\delta_{n_1+n_2,k_1+k_2}\end{aligned} \tag{29}$$

In complete analogy, we can cast the matrix elements $\langle\psi_1,\psi_2|\hat{U}^\dagger|k_2\rangle\ \langle k_2|\hat{U}|\psi_1,\psi_2\rangle$ in the form

$$\begin{aligned}&\langle\psi_1,\psi_2|\hat{U}^\dagger|k_2\rangle\langle k_2|\hat{U}|\psi_1,\psi_2\rangle = \\ &\sum_{m_1,m_2=0}^{N}\sum_{n_1,n_2=0}^{N}\frac{c^*_{m_1}d^*_{m_2}c_{n_1}d_{n_2}}{k_2!(n_1+n_2-k_2)!\sqrt{m_1!m_2!n_1!n_2!}} \\ &\mathrm{per}\,\mathbf{\Lambda}^*[(1^{m_1+m_2-k_2},2^{k_2})|(1^{m_1},2^{m_2})] \\ &\mathrm{per}\,\mathbf{\Lambda}\,[(1^{n_1+n_2-k_2},2^{k_2})|(1^{n_1},2^{n_2})]\,\delta_{m_1+m_2,n_1+n_2}\,.\end{aligned} \tag{30}$$

Thus, Eq. (28) can be written in functional form, by using Eqs. (29) and (30), in terms of permanents associated with the beam splitter matrix $\mathbf{\Lambda}$.

Although the number of sums to be performed to compute the trace of the square of the reduced density matrix counts up to ten, there are already three Kronecker δ functions that reduce the number of sums to seven. Written out explicitly, Eq. (28) reads

$$\begin{aligned}\mathrm{Tr}\hat{\varrho}_1^2 \;=\; & \sum_{k_1,k_2}\sum_{p_1,p_2=0}^{N}\sum_{q_1,q_2=0}^{N}\sum_{m_1,m_2=0}^{N}\sum_{n_1,n_2=0}^{N}\frac{c_{p_1}d_{p_2}c^*_{q_1}d^*_{q_2}c^*_{m_1}d^*_{m_2}c_{n_1}d_{n_2}}{k_1!k_2!\sqrt{p_1!p_2!q_1!q_2!}k_2!(n_1+n_2-k_2)!\sqrt{m_1!m_2!n_1!n_2!}}\\ & \mathrm{per}\,\mathbf{\Lambda}[(1^{k_1},2^{k_2})|(1^{p_1},2^{p_2})]\mathrm{per}\,\mathbf{\Lambda}^*[(1^{k_1},2^{k_2})|(1^{q_1},2^{q_2})\\ & \mathrm{per}\,\mathbf{\Lambda}^*[(1^{m_1+m_2-k_2},2^{k_2})|(1^{m_1},2^{m_2})]\mathrm{per}\,\mathbf{\Lambda}[(1^{n_1+n_2-k_2},2^{k_2})|(1^{n_1},2^{n_2})]\\ & \delta_{p_1+p_2,k_1+k_2}\delta_{q_1+q_2,k_1+k_2}\delta_{m_1+m_2,n_1+n_2}\end{aligned} \tag{31}$$

At this point we need to have a look at the restrictions the group integration imposes. The coefficients c_i, $i = 1,\ldots,N$, are complex and their squared moduli add up to unity. Hence, they can be represented by hyperspherical coordinates and some phases as

$$c_i = e^{i\varphi_i}\cos\Theta_{i+1}\prod_{j=1}^{i}\sin\Theta_j\,,\quad \Theta_{N+1}=0\,. \tag{32}$$

With the additional restriction that $\varphi_0 = 0$ (because an overall phase can be absorbed into the definition of the states) we are left with $2N$ integration variables per initial state. Going back to Eq. (28) one notes that the integration has to be done over four coefficients of the form $c_i^*c_j^*c_kc_l$ (and similarly for the coefficients of the second initial state). Averaging over the phase angles then yields

$$\begin{aligned}&\frac{1}{(2\pi)^N}\int_0^{2\pi}\left(\prod_{n=1}^{N}d\varphi_n\right)e^{i(\varphi_i+\varphi_j-\varphi_k-\varphi_l)}\\ &\quad=(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})(1-\tfrac{1}{2}\delta_{ij})\,.\end{aligned} \tag{33}$$

Since this type of integration has to be performed for both initial states, there are only few sums remaining in Eq. (28).

The integration over the angles Θ_i is done by noting that the coordinates on an N-sphere S^N induce a surface integration measure

$$d\mu(S^N)=\left(\prod_{k=1}^{N-1}\sin^{N-k}\Theta_k\right)\left(\prod_{l=1}^{N}d\Theta_l\right)\,. \tag{34}$$

Therefore, we find that

$$\frac{1}{V}\int d\mu(S^N)|c_i|^4 \;=\; \frac{3}{(N+1)(N+3)}\,, \tag{35}$$

$$\frac{1}{V}\int d\mu(S^N)|c_i|^2|c_j|^2 \;=\; \frac{1}{(N+1)(N+3)} \tag{36}$$

where V is the surface area of the N-sphere. Eventually, we are left with

$$\overline{c_i^*c_j^*c_kc_l}=\frac{(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})(1+\frac{1}{2}\delta_{ij})}{(N+1)(N+3)}\,. \tag{37}$$

Inserting this result into Eq. (28), we finally obtain something which depends only at most quartically on permanents of the type $|\mathrm{per}\,\mathbf{\Lambda}[(1^{m_1+m_2-k_2},2^{k_2})|(1^{m_1},2^{m_2})]$, i.e. on absolute values of certain permanents associated with the beam splitter matrix $\mathbf{\Lambda}$.

C. Multi-mode networks

A final remark should be made about multi-mode networks. In principle, the calculations can be repeated in this case as well and the matrix elements be expressed in terms of permanents. However, since it is not entirely clear how to define entanglement measures (or even monotones) in a multipartite setup, we are not attempting to use any partial results in these cases. Obviously, once a measure for multipartite states has been established, it must be expressible in terms of permanents, as we have seen.

V. CONCLUSIONS

In this article we have shown that calculation of matrix elements of unitary operators in the Fock basis are intimately connected with the computation of permanents. This at first surprising result becomes entirely obvious if one realizes that the (re-)distribution of photons in a linear-optical network is a purely combinatorial problem. As a matter of fact, one can think of these networks as being represented by a bipartite graph whose adjacency matrix has complex entries.

With the knowledge about the relation between matrix elements of unitary operators and permanents, it becomes immediately obvious that the computational complexity of calculating functions that depend on permanents is the same as that for permanents. Examples for such algorithms that appear in quantum information theory include maximizations of success probabilities on linear-optical networks [3] or averaging procedures involving unitary operators as in the entanglement power of such networks.

The advantages of relating unitary transformations to permanents are twofold. On one hand, unitary transformations of multi-mode photon states can be written in a rather compact form. On the other hand, we think it may be possible of gaining more insights into the theory of permanents of unitary matrices from a quantum-mechanical viewpoint. The example we gave that the permanent of a unitary matrix takes its value across the unit disk in the

complex plane shows that quantum-mechanical reasoning can simplify the derivation of such propositions. We believe that this combinatorial view on linear-optical networks in quantum information processing can stimulate the emergence of more intricate results.

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