Linear Equations

Learning Objectives

- Solve linear equations where the unknown appears once
- Solve equations involving brackets
- Solve equations involving fractions
- Solve equations where the unknown appears more than once
- Forming linear equations from word problems

Inverse Pairs

For each mathematical operation, there is an inverse operation.

$$\boxed{\times} \longleftrightarrow \boxed{\div}$$

$$\boxed{+} \longleftrightarrow \boxed{-}$$

To solve equations, we use inverse pairs to isolate the variable x for which we are trying to solve.

Example: One-Step Equations

To solve one-step equations, we simply use the relevant inverse pair:

1)

$$\begin{aligned} x + 3 &= 2 \\ x + 3 - 3 &= 2 - 3 \\ x &= -1 \end{aligned}$$

2)

$$\begin{aligned} y - 5 &= 7 \\ y - 5 + 5 &= 7 + 5 \\ y &= 12 \end{aligned}$$

3)

$$\begin{aligned} 3u &= 12 \\ u \times 3 \div 3 &= 12 \div 3 \\ u &= 4 \end{aligned}$$

4)

$$\begin{aligned} v \div 2 &= 8 \\ v \div 2 \times 2 &= 8 \times 2 \\ v &= 16 \end{aligned}$$

Example: Two-Step Equations

When there is more than one step, it's important to think about the **order of operations**. Drawing a diagram can help. For the equation

$$6x - 2 = 16$$

we can draw the following diagram:

$$x \longrightarrow \boxed{\times 6} \longrightarrow \boxed{-2} \longrightarrow 16$$

We need to work backwards using inverse pairs, so we first add 2 then divide by 6:

$$\begin{aligned} 6x - 2 + 2 &= 16 + 2 \\ 6x &= 18 \\ x \times 6 \div 6 &= 18 \div 6 \\ x &= 3 \end{aligned}$$

$$\frac{x+3}{2} = 5$$

The diagram for this equation is

$$x \longrightarrow \boxed{+3} \longrightarrow \boxed{\div 2} \longrightarrow 5$$

We need to work backwards using inverse pairs,

$$\begin{aligned}
\frac{x+3}{2} &= 5 \\
(x+3) \div 2 \times 2 &= 5 \times 2 \\
x + 3 &= 10 \\
x + 3 - 3 &= 10 - 3 \\
x &= 7
\end{aligned}$$

Test Your Understanding

Solve these equations:

1) $5x - 8 = 27$

2) $\dfrac{x}{3} + 4 = 9$

3) $\dfrac{x - 3}{4} = 1$

4) $3 - 2x = 7$

Answers

1) $5x - 8 = 27 \Rightarrow x = 7$

2) $\dfrac{x}{3} + 4 = 9 \Rightarrow x = 15$

3) $\dfrac{x-3}{4} = 1 \Rightarrow x = 7$

4) $3 - 2x = 7 \Rightarrow x = -2$

Exercise 1

Please complete the worksheet.

Example: Equations with Brackets

There are often two ways to solve these kinds of questions:

$$\begin{aligned} 3(x+7) &= 15 \\ 3x + 21 &= 15 \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

$$\begin{aligned} 3(x+7) &= 15 \\ x + 7 &= 5 \\ x &= -2 \end{aligned}$$

As you can see, expanding the brackets is not always the quickest method.

But sometimes if there is more than one bracket you need to expand the brackets and simplify first:

$$\begin{aligned}
5(x+2)-3(x-4) &= 28 \\
5x+10-3x+12 &= 28 \\
2x+22 &= 28 \\
2x &= 6 \\
x &= 3
\end{aligned}$$

Test Your Understanding

Solve these equations:

1) $6(x-3)+5=29$

2) $8(x+2)-3(2x-1)=39$

Answers

1) $6(x-3)+5=29 \Rightarrow x=7$

2) $8(x+2)-3(2x-1)=39 \Rightarrow x=10$

Exercise 2

Please complete the worksheet.

Example: Equations with Fractions

We can use the same method for questions like this:

$$\frac{3h}{2} + 3 = 12$$

The diagram and working:

$$h \longrightarrow \boxed{\times 3} \longrightarrow \boxed{\div 2} \longrightarrow \boxed{+3} \longrightarrow 12$$

$$\begin{aligned} \frac{3h}{2} + 3 &= 12 \\ \frac{3h}{2} &= 9 \\ h &= 6 \end{aligned}$$

There are two methods for questions like this:

Subtract 4 first:

$$\begin{aligned}\frac{x}{3}+4&=9\\ \frac{x}{3}&=5\\ x&=15\end{aligned}$$

Multiply by 3 first:

$$\begin{aligned}\frac{x}{3}+4&=9\\ x+12&=27\\ x&=15\end{aligned}$$

Warning

When you multiply both sides by 3 you must multiply **every term** by 3.

Test Your Understanding

1) $\dfrac{x}{2} - 8 = 16$

2) $\dfrac{2x}{5} + 6 = 19$

Answers

1) $\dfrac{x}{2} - 8 = 16 \Rightarrow x = 48$

2) $\dfrac{2x}{5} + 6 = 19 \Rightarrow x = 32.5$

Example: Multiplying First

$$\frac{y-5}{3} = 4$$

In some questions, you should multiply first because of the order of operations:

$$y \longrightarrow \boxed{-5} \longrightarrow \boxed{\div 3} \longrightarrow 4$$

$$\begin{aligned} \frac{y-5}{3} &= 4 \\ y-5 &= 12 \\ y &= 17 \end{aligned}$$

Example: Fractions on Both Sides

If both sides of the equation are fractions, you can mutiply by the lowest common multiple of the denominators.

$$\begin{aligned}
\frac{3y+2}{3} &= \frac{2}{5} \\
(3y+2) \div 3 \times 15 &= 2 \div 5 \times 15 \\
(3y+2) \times 5 &= 2 \times 3 \\
15y + 10 &= 6 \\
15y &= -4 \\
y &= -\frac{4}{15}
\end{aligned}$$

Test Your Understanding

1) $\dfrac{a+2}{3} = 4$

2) $\dfrac{g-4}{10} = 2$

3) $6 = \dfrac{2w+4}{3}$

4) $\dfrac{4r-3}{10} = \dfrac{1}{5}$

Answers

1) $\dfrac{a+2}{3} = 4 \Rightarrow a = 10$

2) $\dfrac{g-4}{10} = 2 \Rightarrow g = 24$

3) $6 = \dfrac{2w+4}{3} \Rightarrow w = 7$

4) $\dfrac{4r-3}{10} = \dfrac{1}{5} \Rightarrow r = \dfrac{5}{4}$

Example: Variable on the Denominator

If the variable is on the denominator, the first step is always to multiply **both sides** of the equation by whatever the denominator is.

$$\begin{aligned}\frac{7}{z} &= 42\\ 7 \div z \times z &= 42 \times z\\ 7 &= 42z\\ 7 \div 42 &= z \times 42 \div 42\\ \frac{1}{6} &= z\end{aligned}$$

This same method applies even when the denominator is more complex:

$$\begin{aligned}
\frac{7}{2v+3} &= 2 \\
7 \div (2v+3) \times (2v+3) &= 2 \times (2v+3) \\
7 &= 4v+6 \\
1 &= 4v \\
\frac{1}{4} &= v
\end{aligned}$$

Example: Comparing Methods

In some questions, there are two possible methods:

$$\begin{aligned} \frac{2}{v} + 2 &= 9 \\ \frac{2}{v} &= 7 \\ 2 &= 7v \\ \frac{2}{7} &= v \end{aligned}$$

$$\begin{aligned} \frac{2}{v} + 2 &= 9 \\ 2 + 2v &= 9v \\ 2 &= 7v \\ \frac{2}{7} &= v \end{aligned}$$

Warning

A very common error in the second method is to forget to multiply the 2 by v. When you multiply both sides of an equation, you must multiply **every term**.

Test Your Understanding

1) $3 = \dfrac{15}{y}$

2) $\dfrac{8}{3p} = 12$

3) $1 = \dfrac{5}{y + 2}$

4) $\dfrac{3}{3a - 4} = 5$

Answers

1) $3 = \dfrac{15}{y} \Rightarrow y = 5$

2) $\dfrac{8}{3p} = 12 \Rightarrow p = \dfrac{2}{9}$

3) $1 = \dfrac{5}{y+2} \Rightarrow y = 3$

4) $\dfrac{3}{3a-4} = 5 \Rightarrow a = \dfrac{23}{15}$

Exercise 3

Please complete the worksheet.

Example: Unknown on Both Sides

When the unknown appears on both sides, the goal is to combine those terms together so that the unknown appears only once. Then we can solve it in the same way as the previous examples. To do this, we can add or subtract one of the terms to both sides.

$$\begin{aligned} 11 - 2x &= 6x - 13 \\ 11 - 2x + 2x &= 6x - 13 + 2x \\ 11 &= 8x - 13 \\ 24 &= 8x \\ 3 &= x \end{aligned}$$

If there are brackets, we can expand those first:

$$\begin{aligned} 2(x-4) &= 6x-28 \\ 2x-8 &= 6x-28 \\ -8 &= 4x-28 \\ 20 &= 4x \\ 5 &= x \end{aligned}$$

Test Your Understanding

Solve these equations:

1) $8 - 3x = 4x + 22$

2) $3(x + 5) = 10x + 1$

3) $4(x + 5) = 3(3x - 5)$

Answers

1) $8 - 3x = 4x + 22 \Rightarrow x = -2$

2) $3(x + 5) = 10x + 1 \Rightarrow x = 2$

3) $4(x + 5) = 3(3x - 5) \Rightarrow x = 7$

Exercise 4

Please complete the worksheet.

Example: Fractions with Unknown on Both Sides

When the unknown appears on both side **and** there is a fraction, you should start by getting rid of the fraction by multiplying both sides by the denominator:

$$
\begin{aligned}
\frac{7p}{4} &= p + 3 \\
7p &= 4(p + 3) \\
7p &= 4p + 12 \\
7p - 4p &= 12 \\
3p &= 12 \\
p &= 4
\end{aligned}
$$

$$
\begin{aligned}
\frac{2x}{3} &= 2x - 12 \\
2x &= 3(2x - 12) \\
2x &= 6x - 36 \\
2x - 6x &= -36 \\
-4x &= -36 \\
x &= 9
\end{aligned}
$$

Test Your Understanding

Solve these equations:

1) $\dfrac{3x}{2} = x + 1$

2) $2d + 6 = \dfrac{8d}{3}$

3) $\dfrac{4t}{5} = 2t - 6$

4) $\dfrac{7a}{8} = 2a$

Answers

1) $\dfrac{3x}{2} = x + 1 \Rightarrow x = 2$

2) $2d + 6 = \dfrac{8d}{3} \Rightarrow d = 9$

3) $\dfrac{4t}{5} = 2t - 6 \Rightarrow t = 5$

4) $\dfrac{7a}{8} = 2a \Rightarrow a = 0$

Example: Comparing Methods

Solve:

$$\frac{x}{3} + 4 = x + 1$$

Subtract first:

$$\begin{aligned}
\frac{x}{3} + 4 &= x + 1 \\
\frac{x}{3} &= x - 3 \\
x &= 3(x - 3) \\
x &= 3x - 9 \\
0 &= 2x - 9 \\
9 &= 2x \\
\frac{9}{2} &= x
\end{aligned}$$

Multiply first:

$$\begin{aligned}
\frac{x}{3} + 4 &= x + 1 \\
3 \times (\frac{x}{3} + 4) &= 3 \times (x + 1) \\
x + 12 &= 3x + 3 \\
12 &= 2x + 3 \\
9 &= 2x \\
\frac{9}{2} &= x
\end{aligned}$$

$$\frac{x}{3} + 4 = x + 1$$
$$x + 12 = 3x + 3$$

Warning

Remember, when you multiply both sides, you must multiply **every term**. A very common error in this question is to forget to multiply the 4 by 3.

Test Your Understanding

Solve these equations:

1) $\dfrac{a}{2} + 3 = a + 1$

2) $\dfrac{3e}{4} - 1 = e + 2$

3) $\dfrac{3p}{5} + 6 = 2p - 3$

4) $7w - 3 = \dfrac{2w}{3} - 4$

Answers

1) $\dfrac{a}{2} + 3 = a + 1 \Rightarrow a = 4$

2) $\dfrac{3e}{4} - 1 = e + 2 \Rightarrow e = -12$

3) $\dfrac{3p}{5} + 6 = 2p - 3 \Rightarrow p = \dfrac{45}{7}$

4) $7w - 3 = \dfrac{2w}{3} - 4 \Rightarrow w = -\dfrac{3}{19}$

Example: Equations with Fractions on Both Sides

When there are fractions on both sides, you can multiply by both denominators at once, or the lowest common multiple of both denominators if they have a common factor.

$$\begin{aligned}
\frac{4y-3}{4} &= \frac{2y-4}{5} \\
5(4y-3) &= 4(2y-4) \\
20y-15 &= 8y-16 \\
20y-8y &= -16+15 \\
12y &= -1 \\
y &= -\frac{1}{12}
\end{aligned}$$

$$\begin{aligned}
\frac{3x+5}{3} &= \frac{x-1}{5} \\
5(3x+5) &= 3(x-1) \\
15x+25 &= 3x-3 \\
15x-3x &= -3-25 \\
12x &= -28 \\
x &= -\frac{7}{3}
\end{aligned}$$

Test Your Understanding

Solve these equations:

1) $\dfrac{h+1}{2} = h - 5$

2) $2m + 3 = \dfrac{3m+2}{4}$

3) $\dfrac{2x+3}{5} = \dfrac{3x+2}{3}$

4) $\dfrac{4t-2}{3} = \dfrac{5t-2}{5}$

Answers

1) $\dfrac{h+1}{2} = h - 5 \Rightarrow h = 11$

2) $2m + 3 = \dfrac{3m+2}{4} \Rightarrow m = -2$

3) $\dfrac{2x+3}{5} = \dfrac{3x+2}{3} \Rightarrow x = -\dfrac{1}{9}$

4) $\dfrac{4t-2}{3} = \dfrac{5t-2}{5} \Rightarrow t = \dfrac{4}{5}$

Example: Equations with the Unknown on the Denominator

In these cases, we still begin by multiplying both sides by the denominator.

$$\begin{aligned}
\frac{3e+2}{e-4} &= 5 \\
3e+2 &= 5(e-4) \\
3e+2 &= 5e-20 \\
2 &= 2e-20 \\
22 &= 2e \\
11 &= e
\end{aligned}$$

$$\begin{aligned}
\frac{7p-2}{3p+2} &= 2 \\
7p-2 &= 2(3p+2) \\
7p-2 &= 6p+4 \\
7p-6p &= 4+2 \\
p &= 6
\end{aligned}$$

Test Your Understanding

Solve these equations:

1) $\dfrac{x+5}{3x} = 4$

2) $\dfrac{7y+2}{6y} = \dfrac{2}{3}$

3) $\dfrac{5t+4}{t+5} = 3$

4) $\dfrac{2x+5}{3x+2} = 3$

Answers

1) $\dfrac{x+5}{3x} = 4 \Rightarrow x = \dfrac{5}{11}$

2) $\dfrac{7y+2}{6y} = \dfrac{2}{3} \Rightarrow y = -\dfrac{2}{3}$

3) $\dfrac{5t+4}{t+5} = 3 \Rightarrow t = \dfrac{11}{2}$

4) $\dfrac{2x+5}{3x+2} = 3 \Rightarrow x = -\dfrac{1}{7}$

Test Your Understanding

Solve these equations:

1) $\dfrac{x+5}{3x} = 4$

2) $\dfrac{7y+2}{6y} = \dfrac{2}{3}$

3) $\dfrac{5t+4}{t+5} = 3$

4) $\dfrac{2x+5}{3x+2} = 3$

Answers

1) $\dfrac{x+5}{3x} = 4 \Rightarrow x = \dfrac{5}{11}$

2) $\dfrac{7y+2}{6y} = \dfrac{2}{3} \Rightarrow y = -\dfrac{2}{3}$

3) $\dfrac{5t+4}{t+5} = 3 \Rightarrow t = \dfrac{11}{2}$

4) $\dfrac{2x+5}{3x+2} = 3 \Rightarrow x = -\dfrac{1}{7}$

Example: When the Denominators have a Common Factor

To solve these questions, we first identify the lowest common multiple of the two denominators, and then multiply both sides by this.

$$\frac{5}{3a} + \frac{2}{5a} = 3$$

Find the LCD: $15a$

$$\frac{5}{6b} - \frac{7}{15b} = \frac{1}{2}$$

Find the LCD: $30b$

$$\begin{aligned} \frac{5}{3a} + \frac{2}{5a} &= 3 \\ 15a \times (\frac{5}{3a} + \frac{2}{5a}) &= 15a \times 3 \\ 25 + 6 &= 45a \\ 45a &= 31 \\ a &= \frac{31}{45} \end{aligned}$$

$$\begin{aligned} \frac{5}{6b} - \frac{7}{15b} &= \frac{1}{2} \\ 5 \times 5 - 7 \times 2 &= \frac{1}{2} \times 30b \\ 25 - 14 &= 15b \\ 15b &= 11 \\ b &= \frac{11}{15} \end{aligned}$$

Test Your Understanding

Solve these equations:

1) $\dfrac{a}{3} + \dfrac{2}{3} = 7$

2) $\dfrac{b}{2} - \dfrac{b}{8} = 9$

3) $\dfrac{5}{t} + \dfrac{1}{t} = 3$

4) $\dfrac{2}{3w} - \dfrac{1}{9w} = 15$

Answers

1) $\dfrac{a}{3} + \dfrac{2}{3} = 7 \Rightarrow a = 19$

2) $\dfrac{b}{2} - \dfrac{b}{8} = 9 \Rightarrow b = 24$

3) $\dfrac{5}{t} + \dfrac{1}{t} = 3 \Rightarrow t = 2$

4) $\dfrac{2}{3w} - \dfrac{1}{9w} = 15 \Rightarrow w = 27$

Example: Fractions Leading to Brackets

In questions where there is more than one term within the numerator or denominator, you need to remember that there are effectively 'invisible brackets'. As before, we multiply by the **lowest common multiple** of the denominators.

$$
\begin{aligned}
\frac{k-12}{4}+\frac{2k+1}{2} &= k-2 \\
(k-12)\div 4+(2k+1)\div 2 &= k-2 \\
4(k-12)+2(2k+1) &= 4(k-2) \\
4k-48+4k+2 &= 4k-8 \\
8k-46 &= 4k-8 \\
8k-4k &= -8+46 \\
4k &= 38 \\
k &= \frac{38}{4} = 2
\end{aligned}
$$

$$
\begin{aligned}
\frac{5p+3}{2} - \frac{3p+6}{3} &= 2p - 3 \\
(5p+3) \div 2 - (3p+6) \div 3 &= 2p - 3 \\
3(5p+3) - 2(3p+6) &= 6p - 9 \\
15p + 9 - 6p - 12 &= 6p - 9 \\
9p - 3 &= 6p - 9 \\
9p - 6p &= -9 + 3 \\
3p &= -6 \\
p &= -2
\end{aligned}
$$

Exercise 5

Please complete the worksheet.

$$\frac{h}{2} = h - 4$$

Example: Forming Simple Expressions

Phrase	Algebraic Expression
3 less than a	$a - 3$
b divided by 4	$\frac{b}{4}$
11 more than c	$c + 11$
d lots of e	de

Example: Expressions From Context

Ben has x chocolates.
April has twice as many chocolates as Ben.
Jonny 2 fewer chocolates than April.
How many chocolates do they have all together?

$$x + 2x + 2x - 2 = 5x - 2$$

The weight of a pineapple is a and the weight of an orange is b. Find the difference between the weight of 3 pineapples and 2 oranges.

$$3a - 2b$$

Test Your Understanding

Solve the following:

1) A chair costs £c and a table costs 4 times as much. What is the cost of 4 chairs?

2) A chair costs £c and a table costs 4 times as much. What is the cost of 3 tables?

3) Viktor has y apples. Nathan has twice as many apples as Viktor. Maria has three more apples than Nathan. How many apples do they have all together?

4) An adult ticket costs £10. A child ticket costs £4. Write down an expression for the total cost of a adult tickets and c child tickets.

Answers

1) $4c$

2) $12c$

3) $5y + 3$

4) $10a + 4c$

Example: Forming & Solving Equations from Context

I think of a number, I divide it by 5, then add 9 to it. The result is 42. Let the number be m. This gives:

$$\frac{m}{5} + 9 = 42$$

To solve:

$$\begin{aligned}
\frac{m}{5} + 9 &= 42 \\
\frac{m}{5} &= 42 - 9 \\
\frac{m}{5} &= 33 \\
m &= 33 \times 5 \\
m &= 165
\end{aligned}$$

I think of a number, I subtract 3 from it, then I triple it. The result is 60. Let the number be n. This gives:

$$3(n-3) = 60$$

To solve:

$$\begin{aligned} 3(n-3) &= 60 \\ n-3 &= 60 \div 3 \\ n-3 &= 20 \\ n &= 20+3 \\ n &= 23 \end{aligned}$$

Test Your Understanding

Form and solve an equation from each of these examples.

1) I think of a number, I then add 5 to it. The result is 12.

2) I think of a number, I then multiply it by 6. The result is 54.

3) I think of a number, I double it, then add 3. The result is 19.

4) I think of a number, I divide it by 3, and then subtract 11 from it. The result is -2.

5) I think of a number, I add 3 to it, then I double it. The result is 19.

Answers

1) $a + 5 = 12 \;\Rightarrow\; a = 7$

2) $6b = 54 \;\Rightarrow\; b = 9$

3) $2c + 3 = 19 \;\Rightarrow\; 2c = 16 \;\Rightarrow\; c = 8$

4) $\dfrac{d}{3} - 11 = -2 \;\Rightarrow\; \dfrac{d}{3} = 9 \;\Rightarrow\; d = 27$

5) $2(e + 3) = 19 \;\Rightarrow\; e = \dfrac{13}{2}$

Example: More Advanced Forming

- John has x sweets.
- Henri has 4 more sweets than John.
- Sam has 2 times as many sweets as John.
  Altogether they have 32 sweets. Find how many sweets John has.

Let John's number of sweets be x. Then:

- John: x
- Henri: $x + 4$
- Sam: $2x$

Since the total is 32, we form:

$$x + (x + 4) + 2x = 32$$

Combine like terms and solve:

$$\begin{aligned}4x + 4 &= 32\\ 4x &= 32 - 4\\ 4x &= 28\\ x &= 28 \div 4\\ x &= 7\end{aligned}$$

Therefore, John has 7 sweets.

Test Your Understanding

Peter has y loaves.
Andrew has 2 fewer loaves than Peter.
John has 4 times as many loaves as Peter.
Altogether they have 16 loaves. Find how many loaves Andrew has.

Answers

Let:

- Peter: y
- Andrew: $y - 2$
- John: $4y$

Since the total is 16:

$$y + (y - 2) + 4y = 16$$

Combine like terms and solve:

$$\begin{aligned} 6y - 2 &= 16 \\ 6y &= 16 + 2 \\ 6y &= 18 \\ y &= 3 \end{aligned}$$

As Andrew has $y - 2$ loaves:

$$3 - 2 = 1$$

Therefore, Andrew has 1 loaf.

> **Be careful**
>
> The value of y is not the final answer.

Exercise 6

Please complete the worksheet.