Recurring Decimals

Example: Long Division

To turn fractions into decimals, we can use long division. For example, to calculate $\frac{17}{8}$:

$$
\begin{array}{r}
2.125 \\
8 \overline{)17.000} \\
\underline{16} \\
1.0 \\
\underline{8} \\
20 \\
\underline{16} \\
40 \\
\underline{40} \\
0
\end{array}
$$

Notice that we stop when we get a remainder of zero.

Example: Fraction to Recurring Decimal

We can also do this to find a recurring decimal. For example, to find $\frac{3}{11}$, we can do:

$$
\begin{array}{rl}
 & 0.\overline{27} \\
11 & \hspace{-0.5em}\big)\overline{3.00} \\
 & \underline{2.2} \\
 & 80 \\
 & \underline{77} \\
 & 3
\end{array}
$$

As soon as we get a repeated digit, we know we will be stuck in a loop, so we mark the decimal as recurring.

Sometimes there is a non-recurring decimal part before the recurrence begins:

$$
\begin{array}{r}
1.1\overline{3} \\
15\,\big)\overline{17.00} \\
\underline{15} \\
2.0 \\
\underline{1.5} \\
50 \\
\underline{45} \\
5
\end{array}
$$

Notice that the vertical bar only extends over the repeating part.

And sometimes it takes a long time to repeat:

$$
\begin{array}{rl}
 & 0.\overline{571428} \\
7 & \!\!\big)\overline{4.000000} \\
 & \underline{3.5} \\
 & 50 \\
 & \underline{49} \\
 & 10 \\
 & \underline{7} \\
 & 30 \\
 & \underline{28} \\
 & 20 \\
 & \underline{14} \\
 & 60 \\
 & \underline{56} \\
 & 4
\end{array}
$$

Example: Recurring Decimal Notation

There are actually two ways to represent recurring decimals, the bar over the repeating part or a dot above the first and last digit of the repeating part, so for example, the recurring decimal

$$0.3456456456456456\ldots$$

Can be represented either as

$$0.3\overline{456}$$

or as

$$0.3\dot{4}5\dot{6}$$

Test Your Understanding

Write out what each of these recurring decimals looks like:

1) $0.\dot{3}$

2) $0.4\overline{3}$

3) $0.\dot{4}\dot{3}$

4) $0.1\overline{23}$

5) $0.\dot{1}2\dot{3}$

6) $0.12\overline{3}$

7) $0.4\dot{3}0\dot{3}$

Answers

1) $0.\dot{3} = 0.333\ldots$

2) $0.4\overline{3} = 0.4333\ldots$

3) $0.\dot{4}\dot{3} = 0.434343\ldots$

4) $0.1\overline{23} = 0.1232323\ldots$

5) $0.\dot{1}2\dot{3} = 0.123123123\ldots$

6) $0.12\overline{3} = 0.12333\ldots$

7) $0.4\dot{3}0\dot{3} = 0.4303303303\ldots$

Exercise 1

Please complete the worksheet.

Example: Recurring Decimals to Fractions

To convert $0.\overline{54}$ to a fraction, we can use algebra. We let x equal the recurring decimal, then work out the value of x as fraction.

$$
\begin{aligned}
x &= 0.545454\ldots && \text{Write out the recurring decimal} \\
100x &= 54.545454\ldots && \times 100 \text{ because two recurring digits} \\
99x &= 54 && \text{Subtract the first line from the second} \\
x &= \frac{54}{99} = \frac{6}{11} && \text{Divide and simplify}
\end{aligned}
$$

The trick here is that the recurring decimal parts cancel out when they are subtracted.

$$
\begin{aligned}
x &= 0.133333\ldots && \text{Write out the recurring decimal} \\
10x &= 1.333333\ldots && \times 10 \text{ because one recurring digit} \\
9x &= 1.2 && \text{Subtract the first line from the second} \\
x &= \frac{1.2}{9} = \frac{12}{90} = \frac{2}{15} && \text{Divide and simplify}
\end{aligned}
$$

Note

Note here that when we subtract the first digit after the decimal point does not match, so we are effectively doing $1.3 - 0.1$. The decimal digits after this *do* cancel because they are identical.

$$\begin{aligned} x &= 3.0868686\ldots \\ 100x &= 308.6868686\ldots \\ 99x &= 305.6 \\ x &= \frac{305.6}{99} = \frac{3056}{990} = \frac{1528}{495} \end{aligned}$$

Note

The calculation this time is $308.6 - 3$.

$$
\begin{aligned}
x &= 0.5401401401\ldots \\
1000x &= 540.1401401401\ldots \\
999x &= 539.6 \\
x &= \frac{539.6}{999} = \frac{5396}{9990} = \frac{2698}{4995}
\end{aligned}
$$

Warning

Be careful here. 5 is greater than 1 so the calculation we need to do is $540.1 - 0.5$.

Exercise 2

Please complete the worksheet.