$$= \int dp\, p^2 \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar} \langle p | \chi \rangle$$

$$= \left(\frac{\hbar}{i}\right)^2 \frac{d^2}{dx^2} \int dp \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar} \langle p | \chi \rangle$$

$$= \left(\frac{\hbar}{i}\right)^2 \frac{d^2}{dx^2} \int dp \langle x | p \rangle \langle p | \chi \rangle$$

$$= \left(\frac{\hbar}{i}\right)^2 \frac{d^2}{dx^2} \langle x | \chi \rangle$$

$\boxtimes$

(1.27) を解くと...

$$\varphi_n(x) = \sqrt{\frac{1}{2^n n! \sqrt{\frac{m\omega}{\pi\hbar}}}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-\frac{m\omega}{2\hbar} x^2}$$

※ $H_n(\xi)$: エルミート多項式

位置, 運動量 : 連続

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

エネ : 離散

$\hat{H}$ の量子化

$$\hat{H} = \frac{1}{2} m\omega^2 \hat{x}^2 + \frac{1}{2m} \hat{p}^2$$

$$= \hbar\omega\left(\frac{m\omega}{2\hbar} \hat{x}^2 + \frac{1}{2m\hbar\omega} \hat{p}^2\right)$$

$$= \hbar\omega\left(\underbrace{\left(\sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i\sqrt{\frac{1}{2m\hbar\omega}} \hat{p}\right)}^{\hat{a}^\dagger}\underbrace{\left(\sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i\sqrt{\frac{1}{2m\hbar\omega}} \hat{p}\right)}^{\hat{a}}\right.$$

$$-i\sqrt{\frac{m\omega}{2\hbar}}\sqrt{\frac{1}{2m\hbar\omega}}\,\underbrace{[\hat{x},\hat{p}]}_{i\hbar}\Big)$$

$$= \hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)$$

(準備)

$$\psi(x) := \langle x|\psi\rangle$$

$$|\psi\rangle = \int dx\, \psi(x)|x\rangle$$

$$\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{d}{dx}\psi(x)$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$\langle x|$ ↓

$$\underbrace{\langle x|\hat{H}|\psi\rangle}_{\psi(x)\text{で表現!}} = E\,\psi(x)$$

Step 1 $\quad \langle x|\hat{p}|x\rangle = -i\hbar\dfrac{d}{dx}\psi(x)$

Step 2 $\quad \langle x|p\rangle = \dfrac{1}{\sqrt{2\pi\hbar}}e^{\frac{ixp}{\hbar}}$

step3. $\quad \langle x|\hat{p}^2|x\rangle$

step. 0 $\quad f(x)\delta'(x) = -f'(x)\delta(x) \quad \lim_{|x|\to\infty} f(x) \to 0$

(証)

$$\int_{-\infty}^{\infty} \underline{f(x)\,\delta'(x)}\,dx = \left[f(x)\delta(x)\right]_{-\infty}^{\infty} - \int \underline{f(x)'\delta(x)}\,dx$$

$$f(x) = x \rightarrow \frac{\delta(x)}{x} = -\delta'(x)$$

step 1.

$$\langle x|\overset{i\hbar}{[\hat{x},\hat{p}]}|x'\rangle = \langle x|\hat{x}\hat{p}-\hat{p}\hat{x}|x'\rangle$$

$$i\hbar\langle x|x'\rangle \qquad = x\langle x|\hat{p}|x'\rangle - x'\langle x|\hat{p}|x'\rangle$$

$$= i\hbar\,\delta(x-x') \qquad = (x-x')\langle x|\hat{p}|x'\rangle$$

$$\langle x|\hat{p}|x'\rangle = i\hbar\frac{\delta(x-x')}{x-x'}$$

$$= i\hbar\frac{d}{dx'}\delta(x-x')$$

$$\langle x|\hat{p}|\psi\rangle = \langle x|\hat{p}\,\hat{1}|\psi\rangle$$

$$= \langle x|\hat{p}\int dx'\,|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \langle x|\hat{p}|x'\rangle\langle x'|\psi\rangle$$

$$= i\hbar \int dx' \left[\frac{d}{dx'}\delta(x-x')\right]\psi(x')$$

$$= i\hbar\left\{\left[\delta(x-x')\psi(x')\right]_{-\infty}^{\infty} - \int dx' \psi'(x')\delta(x-x')\right\}$$

$$= -i\hbar \frac{d}{dx}\psi(x)$$

Step 2.

$$\langle x|\hat{p}|p\rangle = p\langle x|p\rangle$$

$$= p \cdot p(x)$$

$$\langle x|\hat{p}|p\rangle = -i\hbar\frac{d}{dx}p(x) \quad (\because \text{Step } 1)$$

$$p(x) = c\exp\left(\frac{ixp}{\hbar}\right)$$

$$\delta(x-x') = \langle x|x'\rangle$$

$$= \langle x\int dp\,|p\rangle\langle p|x'\rangle = \int dp\; p(x)\,p^*(x')$$

$$= |c|^2 \int \exp\left(\frac{i(x-x')p}{\hbar}\right) dp$$

$$1 = \int_{-\infty}^{\infty} dt\, \delta(t)\, e^{-i\omega t}$$

$$\delta(t) = \frac{1}{2\pi}\int d\omega\ 1\cdot e^{i\omega t}$$

$$\omega \to \frac{p}{\hbar}$$

$$t \to x - x'$$

$$\delta(x-x') = \frac{1}{2\pi\hbar}\int dp\ e^{i\frac{x-x'}{\hbar}p}$$

$$|c|^2 = \frac{1}{2\pi\hbar} \qquad \therefore \quad c = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\therefore\ \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ixp}{\hbar}\right)$$

Step 3.

$$\langle x|\hat{p}^2|\psi\rangle$$

$$= \langle x|\hat{p}^2 \int dp\, |p\rangle\langle p|\psi\rangle$$

$$= \int dp\ p^2 \langle x|p\rangle\langle p|\psi\rangle$$

$$= \int dp\ \underbrace{p^2 \cdot \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ixp}{\hbar}\right)}\ \langle p|\psi\rangle$$

$$\left(\frac{d^2}{dx^2}\langle x|p\rangle \cdot \left(\frac{\hbar}{i}\right)^2 = p^2\langle x|p\rangle\right)$$

$$= \left(\frac{\hbar}{i}\right)^2 \int dp \ \frac{d^2}{dx^2} \langle x | p \rangle \langle p | \psi \rangle$$

$$= \underbrace{\left(\frac{\hbar}{i}\right)^2 \langle x |}_{\frac{d^2}{dx^2}} \left[\int dp \ | p \rangle \langle p | \right] | \psi \rangle = \left(\frac{\hbar}{i}\right)^2 \frac{d^2}{dx^2} \psi(x)$$

$$\left(\frac{1}{2} m\omega^2 \hat{x}^2 + \frac{1}{2m} \hat{p}^2 \right) | \psi \rangle = E | \psi \rangle$$

$\langle x|$ かける

$$\frac{1}{2} m\omega^2 x^2 \psi(x) + \frac{1}{2m} \left(\frac{\hbar}{i}\right)^2 \frac{d}{dx^2} \psi(x) = E \psi(x)$$

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right) \psi(x) = E \psi(x)$$

解く

準備

$$\left(\frac{d^2}{ds^2} - 2s \frac{d}{ds} + 2n \right) H_n(s) = 0$$ ← エルミート多項式

$$\int_{-\infty}^{\infty} ds \ \underbrace{H_m(s) H_n(s) e^{-s^2}}_{H_m e^{-\frac{s^2}{2}} \ H_n e^{-\frac{s^2}{2}}} = \sqrt{\pi} \ 2^n n! \ \delta_n^m$$

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right) \psi(x) = E \psi(x)$$

$$\left(\frac{d^2}{dx^2} - \frac{m^2\omega^2}{\hbar^2}x^2 + \frac{2mE}{\hbar^2}\right)\psi(x) = 0$$

(仮定)

$$\psi(x) = f(s)\, e^{-\frac{s^2}{2}}$$

$\hookrightarrow$ $H_n(s)$ だといいな…。

$s = \underbrace{\sqrt{\frac{m\omega}{\hbar}}}\, x$ と変数変換

$\sqrt{\frac{kg\ s^{-1}}{kg\ m^2\ s^{-1}}}\ m = 1 \Rightarrow$ 無次元化!

$$\frac{d}{dx} = \frac{ds}{dx}\frac{d}{ds}$$

$$\left(\frac{m\omega}{\hbar}\frac{d^2}{ds^2} - \frac{m^2\omega^2}{\hbar^2}\frac{\hbar}{m\omega}s^2 + \frac{2mE}{\hbar^2}\right)\psi(x) = 0$$

$$\left(\frac{d^2}{ds^2} - s^2 + \frac{2E}{\hbar\omega}\right)\psi(x) = 0 \leftarrow f(s)\,e^{-\frac{s^2}{2}} \text{ 代入してみる}$$

$$\frac{d^2}{ds^2}\left(f(s)e^{-\frac{s^2}{2}}\right) = \frac{d}{ds}\left(\frac{df}{ds}e^{-\frac{s^2}{2}} - sf(s)e^{-\frac{s^2}{2}}\right)$$

$$= \frac{d^2f}{ds^2}e^{-\frac{s^2}{2}} + 2\frac{df}{ds}(-s)e^{-\frac{s^2}{2}} + f(s)(s^2-1)e^{-\frac{s^2}{2}}$$

$$= e^{-\frac{s^2}{2}}\left(\frac{d^2}{ds^2}f - 2s\frac{df}{ds} + (s^2-1)f(s)\right)$$

$$e^{-\frac{s^2}{2}}\left(\frac{d^2}{ds^2} f - 2s \frac{df}{ds} + \underline{(s^2-1)} f(s) \right)$$

$$\underline{- e^{-\frac{s^2}{2}} s^2 f} + e^{-\frac{s^2}{2}} \frac{2E}{\hbar\omega} f = 0$$

$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$

$$\Leftrightarrow \quad e^{-\frac{s^2}{2}}\left(\frac{d^2}{ds^2} - 2s \frac{d}{ds} + \underbrace{\frac{2E}{\hbar\omega} - 1}_{2n+\cancel{1}-\cancel{1}} \right) f(s) = 0$$

$\therefore$ $H_n(s)$ でいいじゃん!

$$\psi_n(x) \propto H_n(s)\, e^{-\frac{s^2}{2}}$$

規格化すると、

$$1 = \int dx\, |\psi(x)|^2 \qquad x = \sqrt{\frac{\hbar}{m\omega}}\, s$$

$$= |C|^2 \int dx\, H_n(s) H_n(s)\, e^{-s^2}$$

$$= |C|^2 \sqrt{\frac{\hbar}{m\omega}} \int ds\, H_n(s) H_n(s)\, e^{-s^2}$$

$$= |C|^2 \sqrt{\frac{\hbar}{m\omega}} \sqrt{\pi}\, 2^n n!$$

$$\therefore \quad C = \sqrt{\frac{1}{2^n n!} \sqrt{\frac{m\omega}{\pi\hbar}}}$$

$$\therefore \quad \psi(x) = \sqrt{\frac{1}{2^n n!} \sqrt{\frac{m\omega}{\pi\hbar}}}\; H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) \exp\left(-\frac{m\omega}{2\hbar} x^2\right)$$

//

$H_0 = 1,\ H_1 = 2s\ ,\ H_2 = 4s^2 - 2$

1.3 電磁場の量子化

準備

$$\mathbb{B} = \nabla \times \mathbb{A}$$

$$\mathbb{E} = -\frac{\partial \mathbb{A}}{\partial t}$$