

## 0.1 Baker-Campbell-Hausdorff の公式 1

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \dots \quad (0.1.1)$$

なる式を示す.

*Proof.* 函数  $f(t)$  を,

$$f(t) := e^{t\hat{A}}\hat{B}e^{-t\hat{A}} \quad (0.1.2)$$

と定義する.  $f(t)$  を  $t=0$  の周りで展開することを考えると,

$$f(t) = f(0) + \left. \frac{df}{dx} \right|_{t=0} t + \frac{1}{2!} \left. \frac{d^2 f}{dx^2} \right|_{t=0} t^2 + \dots \quad (0.1.3)$$

と書ける. さて,

$$\frac{df}{dx} = \hat{A}e^{t\hat{A}}\hat{B}e^{-t\hat{A}} - e^{t\hat{A}}\hat{B}\hat{A}e^{-t\hat{A}} \quad (0.1.4)$$

$$= e^{t\hat{A}}\hat{A}\hat{B}e^{-t\hat{A}} - e^{t\hat{A}}\hat{B}\hat{A}e^{-t\hat{A}} \quad (0.1.5)$$

$$= e^{t\hat{A}}(\hat{A}\hat{B} - \hat{B}\hat{A})e^{-t\hat{A}} \quad (0.1.6)$$

$$= e^{t\hat{A}}[\hat{A}, \hat{B}]e^{-t\hat{A}} \quad (0.1.7)$$

である. よって,

$$\left. \frac{df}{dx} \right|_{t=0} = [\hat{A}, \hat{B}] \quad (0.1.8)$$

である. 2 階以上の微分では, 式 (0.1.7) において,  $\hat{B} \rightarrow [\hat{A}, \hat{B}]$  とすればよい. よって, 式 (0.1.3) に式 (0.1.7) を代入すると,

$$f(t) = B + [\hat{A}, \hat{B}]t + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]]t^2 + \dots \quad (0.1.9)$$

である.  $t=1$  とすれば,

$$e^{\hat{A}}\hat{B}e^{\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \dots \quad (0.1.10)$$

となる. 特に,

$$[\hat{A}, [\hat{A}, \hat{B}]] = 0 \quad (0.1.11)$$

のとき,

$$e^{\hat{A}}\hat{B}e^{\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] \quad (0.1.12)$$

である. □

## 0.2 Baker-Campbell-Hausdorff の公式 2

$$e^{\hat{A}}e^{\hat{B}} = \exp\left(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}[(\hat{A} - \hat{B}), [\hat{A}, \hat{B}]] + \dots\right) \quad (0.2.1)$$

なる式を示す.

Proof.  $f(t)$  と  $g(t)$  を,

$$\begin{cases} f(t) := e^{t\hat{A}}e^{t\hat{B}} = e^{g(t)} \\ g(t) := \log(e^{t\hat{A}}e^{t\hat{B}}) = \log(f(t)) \end{cases} \quad (0.2.2)$$

と定義する.  $f(t)$  の Taylor 展開を考える.

$$\frac{d}{dt}f(t) = \hat{A}e^{t\hat{A}}e^{t\hat{B}} + e^{t\hat{A}}\hat{B}e^{t\hat{B}} \quad (0.2.3)$$

$$= e^{t\hat{A}}(\hat{A} + \hat{B})e^{t\hat{B}} \quad (0.2.4)$$

$$\frac{d^2}{dt^2}f(t) = e^{t\hat{A}}\left\{\hat{A}(\hat{A} + \hat{B}) + (\hat{A} + \hat{B})\hat{B}\right\}e^{t\hat{B}} \quad (0.2.5)$$

$$= e^{t\hat{A}}\left\{\hat{A}^2 + 2\hat{A}\hat{B} + \hat{B}^2\right\}e^{t\hat{B}} \quad (0.2.6)$$

$$= e^{t\hat{A}}\left\{\hat{A}^2 + \hat{A}\hat{B} + \hat{B}\hat{A} + \hat{B}^2 + \hat{A}\hat{B} - \hat{B}\hat{A}\right\}e^{t\hat{B}} \quad (0.2.7)$$

$$= e^{t\hat{A}}\left\{(\hat{A} + \hat{B})^2 + [\hat{A}, \hat{B}]\right\}e^{t\hat{B}} \quad (0.2.8)$$

$$\frac{d^3}{dt^3}f(t) = e^{t\hat{A}}\left\{\hat{A}(\hat{A} + 2\hat{A}\hat{B} + \hat{B}^2) + (\hat{A} + 2\hat{A}\hat{B} + \hat{B}^2)\hat{B}\right\}e^{t\hat{B}} \quad (0.2.9)$$

$$= e^{t\hat{A}}\left\{\hat{A}^3 + 3\hat{A}^2\hat{B} + 3\hat{A}\hat{B}^2 + \hat{B}^3\right\}e^{t\hat{B}} \quad (0.2.10)$$

$$= e^{t\hat{A}}\left\{\hat{A}^3 + \hat{A}^2\hat{B} + \hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}^2 + \hat{B}^2\hat{A} + \hat{B}\hat{A}\hat{B} + \hat{A}\hat{B}^2 + \hat{B}^3 + 2\hat{A}^2\hat{B} - \hat{A}\hat{B}\hat{A} - \hat{B}\hat{A}^2 - \hat{B}^2\hat{A} - \hat{B}\hat{A}\hat{B} + 2\hat{A}\hat{B}^2\right\}e^{t\hat{B}} \quad (0.2.11)$$

$$= e^{t\hat{A}}\left\{(\hat{A} + \hat{B})^3 + \hat{A}^2\hat{B} - \hat{A}\hat{B}\hat{A} + \hat{A}^2\hat{B} - \hat{B}\hat{A}^2 + \hat{A}\hat{B}^2 - \hat{B}^2\hat{A} + \hat{A}\hat{B}^2 - \hat{B}\hat{A}\hat{B}\right\}e^{t\hat{B}} \quad (0.2.12)$$

$$= e^{t\hat{A}}\left\{(\hat{A} + \hat{B})^3 + \hat{A}[\hat{A}, \hat{B}] + [\hat{A}^2, \hat{B}] + [\hat{A}, \hat{B}^2] + [\hat{A}, \hat{B}]\hat{B}\right\}e^{t\hat{B}} \quad (0.2.13)$$

となる.  $t = 0$  周りで  $f(t)$  を Taylor 展開すると,

$$f(t) = f(0) + \frac{1}{1!}\left(\frac{d}{dt}f(t)\Big|_{t=0}\right)t + \frac{1}{2!}\left(\frac{d^2}{dt^2}f(t)\Big|_{t=0}\right)t^2 + \frac{1}{3!}\left(\frac{d^3}{dt^3}f(t)\Big|_{t=0}\right)t^3 + \dots \quad (0.2.14)$$

$$= 1 + \frac{1}{1!}(\hat{A} + \hat{B})t + \frac{1}{2!}\left\{(\hat{A} + \hat{B})^2 + [\hat{A}, \hat{B}]\right\}t^2 + \frac{1}{3!}\left\{(\hat{A} + \hat{B})^3 + \hat{A}[\hat{A}, \hat{B}] + [\hat{A}^2, \hat{B}] + [\hat{A}, \hat{B}^2] + [\hat{A}, \hat{B}]\hat{B}\right\}t^3 + \dots \quad (0.2.15)$$

$$= 1 + \hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3 + \dots \quad (0.2.16)$$

$$(0.2.17)$$

と書ける. ただし,

$$\hat{P}_1 := \frac{1}{1!}(\hat{A} + \hat{B}) \quad (0.2.18)$$

$$\hat{P}_2 := \frac{1}{2!}\left\{(\hat{A} + \hat{B})^2 + [\hat{A}, \hat{B}]\right\} \quad (0.2.19)$$

$$\hat{P}_3 := \frac{1}{3!}\left\{(\hat{A} + \hat{B})^3 + \hat{A}[\hat{A}, \hat{B}] + [\hat{A}^2, \hat{B}] + [\hat{A}, \hat{B}^2] + [\hat{A}, \hat{B}]\hat{B}\right\} \quad (0.2.20)$$

と定義した.  $F(t)$  を,

$$F(t) := f(t) - 1 \quad (0.2.21)$$

$$= \hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3 + \dots \quad (0.2.22)$$

と定義する． $h(x) := \log(1+x)$  は  $x=0$  のまわりで、

$$h(x) = h(0) + \frac{1}{1!} \left( \frac{d}{dx} h(x) \Big|_{x=0} \right) x + \frac{1}{2!} \left( \frac{d^2}{dx^2} h(x) \Big|_{x=0} \right) x^2 + \frac{1}{3!} \left( \frac{d^3}{dx^3} h(x) \Big|_{x=0} \right) x^3 + \cdots \quad (0.2.23)$$

$$= 0 + \frac{1}{1!} \frac{1}{1+0} x + \frac{1}{2!} \left( -\frac{1}{(1+0)^2} \right) x^2 + \frac{1}{3!} \left( \frac{2}{(1+0)^3} \right) x^3 + \cdots \quad (0.2.24)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \quad (0.2.25)$$

であることと、 $t \rightarrow 0$  で  $F(t) \rightarrow 0$  であることを用いて  $g(t)$  を  $t=0$  の周りで Taylor 展開すると、

$$g(t) = \log(f(t)) \quad (0.2.26)$$

$$= \log(1 + F(t)) \quad (0.2.27)$$

$$= F(t) - \frac{F(t)^2}{2} + \frac{F(t)^3}{3} - \cdots \quad (0.2.28)$$

$$= (\hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3) - \frac{1}{2} (\hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3)^2 + \frac{1}{3} (\hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3)^3 - \cdots \quad (0.2.29)$$

$$= \hat{P}_1 t + \left( \hat{P}_2 - \frac{1}{2} \hat{P}_1^2 \right) t^2 + \left\{ \hat{P}_3 - \frac{1}{2} (\hat{P}_1 \hat{P}_2 + \hat{P}_2 \hat{P}_1) + \frac{1}{3} \hat{P}_1^3 \right\} t^3 - \cdots \quad (0.2.30)$$

となる． $\hat{P}_1$ ,  $\hat{P}_2$ ,  $\hat{P}_3$  の定義を思い出せば、

$$(t^1 \text{ の係数}) = \hat{P}_1 = \hat{A} + \hat{B} \quad (0.2.31)$$

$$(t^2 \text{ の係数}) = \hat{P}_2 - \frac{1}{2} \hat{P}_1^2 \quad (0.2.32)$$

$$= \frac{1}{2} \left\{ (\hat{A} + \hat{B})^2 + [\hat{A}, \hat{B}] \right\} - \frac{1}{2} (\hat{A} + \hat{B})^2 \quad (0.2.33)$$

$$= \frac{1}{2} [\hat{A}, \hat{B}] \quad (0.2.34)$$

$$(t^3 \text{ の係数}) = \hat{P}_3 - \frac{1}{2} (\hat{P}_1 \hat{P}_2 + \hat{P}_2 \hat{P}_1) + \frac{1}{3} \hat{P}_1^3 \quad (0.2.35)$$

$$\begin{aligned} &= \frac{1}{3!} \left\{ (\hat{A} + \hat{B})^3 + \hat{A} [\hat{A}, \hat{B}] + [\hat{A}^2, \hat{B}] + [\hat{A}, \hat{B}^2] + [\hat{A}, \hat{B}] \hat{B} \right\} \\ &\quad - \frac{1}{2} \left[ (\hat{A} + \hat{B}) \frac{1}{2!} \left\{ (\hat{A} + \hat{B})^2 + [\hat{A}, \hat{B}] \right\} + \frac{1}{2!} \left\{ (\hat{A} + \hat{B})^2 + [\hat{A}, \hat{B}] \right\} (\hat{A} + \hat{B}) \right] \\ &\quad + \frac{1}{3} (\hat{A} + \hat{B})^3 \end{aligned} \quad (0.2.36)$$

$$\begin{aligned} &= \frac{1}{6} (\hat{A} + \hat{B})^3 + \frac{1}{6} \hat{A} [\hat{A}, \hat{B}] + \frac{1}{6} \left\{ \hat{A} [\hat{A}, \hat{B}] + [\hat{A}, \hat{B}] \hat{A} \right\} + \frac{1}{6} \left\{ \hat{B} [\hat{A}, \hat{B}] + [\hat{A}, \hat{B}] \hat{B} \right\} + \frac{1}{6} [\hat{A}, \hat{B}] \hat{B} \\ &\quad - \frac{1}{2} (\hat{A} + \hat{B})^3 - \frac{1}{4} \hat{A} [\hat{A}, \hat{B}] - \frac{1}{4} \hat{B} [\hat{A}, \hat{B}] - \frac{1}{4} [\hat{A}, \hat{B}] \hat{A} - \frac{1}{4} [\hat{A}, \hat{B}] \hat{B} \\ &\quad + \frac{1}{3} (\hat{A} + \hat{B})^3 \end{aligned} \quad (0.2.37)$$

$$= \frac{1}{12} \hat{A} [\hat{A}, \hat{B}] - \frac{1}{12} \hat{B} [\hat{A}, \hat{B}] + \frac{1}{12} [\hat{A}, \hat{B}] \hat{A} - \frac{1}{12} [\hat{A}, \hat{B}] \hat{B} \quad (0.2.38)$$

$$= \frac{1}{12} (\hat{A} - \hat{B}) [\hat{A}, \hat{B}] - \frac{1}{12} [\hat{A}, \hat{B}] (\hat{A} - \hat{B}) \quad (0.2.39)$$

$$= \frac{1}{12} [(\hat{A} - \hat{B}), [\hat{A}, \hat{B}]] \quad (0.2.40)$$

よって、

$$e^{t\hat{A}} e^{t\hat{B}} = f(t) = e^{g(t)} \quad (0.2.41)$$

$$= \exp \left\{ (\hat{A} + \hat{B}) t + \frac{1}{2} [\hat{A}, \hat{B}] t^2 + \frac{1}{12} [(\hat{A} - \hat{B}), [\hat{A}, \hat{B}]] t^3 + \cdots \right\} \quad (0.2.42)$$

(0.2.43)

となるから,  $t = 0$  とすれば,

$$e^{\hat{A}}e^{\hat{B}} = \exp \left\{ \left( \hat{A} + \hat{B} \right) + \frac{1}{2} [\hat{A}, \hat{B}] + \frac{1}{12} \left[ \left( \hat{A} - \hat{B} \right), [\hat{A}, \hat{B}] \right] + \cdots \right\} \quad (0.2.44)$$

となる. 特に,

$$\left[ \left( \hat{A} - \hat{B} \right), [\hat{A}, \hat{B}] \right] = 0 \quad (0.2.45)$$

$$\Longleftrightarrow [\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0 \quad (0.2.46)$$

のときは,

$$e^{\hat{A}}e^{\hat{B}} = \exp \left\{ \left( \hat{A} + \hat{B} \right) + \frac{1}{2} [\hat{A}, \hat{B}] \right\} \quad (0.2.47)$$

となる.

□