ユニタリ行列は一般に,

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Psi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2}\end{pmatrix}\begin{pmatrix}\cos(\Theta/2) & \sin(\Theta/2)\\ -\sin(\Theta/2) & \cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Phi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2}\end{pmatrix} \tag{0.0.1}$$

と分解できる．具体的に U を計算すると,

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Psi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2}\end{pmatrix}\begin{pmatrix}\cos(\Theta/2) & \sin(\Theta/2)\\ -\sin(\Theta/2) & \cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Phi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2}\end{pmatrix} \tag{0.0.2}$$

$$= \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Psi/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\Psi/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}\Psi/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\Psi/2}\cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Phi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2}\end{pmatrix} \tag{0.0.3}$$

$$= \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2)\end{pmatrix} \tag{0.0.4}$$

であり，$\alpha = \Psi + \Phi$，$\beta = \Psi - \Phi$ とすると,

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2)\end{pmatrix} \tag{0.0.5}$$

$$= \begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Lambda+\alpha)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Lambda+\beta)/2}\sin(\Theta/2)\\ -\mathrm{e}^{\mathrm{i}(\Lambda-\beta)/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Lambda-\alpha)/2}\cos(\Theta/2)\end{pmatrix} \tag{0.0.6}$$

と書ける．

Proof. 任意 2×2 の行列は，実数 r_{ij} と θ_{ij} を用いて,

$$M = \begin{pmatrix}r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}}\\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}}\end{pmatrix} \tag{0.0.7}$$

と書けて,

$$M^\dagger M = \begin{pmatrix}r_{11}\mathrm{e}^{-\mathrm{i}\theta_{11}} & r_{21}\mathrm{e}^{-\mathrm{i}\theta_{21}}\\ r_{12}\mathrm{e}^{-\mathrm{i}\theta_{12}} & r_{22}\mathrm{e}^{-\mathrm{i}\theta_{22}}\end{pmatrix}\begin{pmatrix}r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}}\\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}}\end{pmatrix} \tag{0.0.8}$$

$$= \begin{pmatrix}r_{11}^2 + r_{21}^2 & r_{11}r_{12}\mathrm{e}^{-\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{-\mathrm{i}(\theta_{21}-\theta_{22})}\\ r_{11}r_{12}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} & r_{12}^2 + r_{22}^2\end{pmatrix} \tag{0.0.9}$$

$$MM^\dagger = \begin{pmatrix}r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}}\\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}}\end{pmatrix}\begin{pmatrix}r_{11}\mathrm{e}^{-\mathrm{i}\theta_{11}} & r_{21}\mathrm{e}^{-\mathrm{i}\theta_{21}}\\ r_{12}\mathrm{e}^{-\mathrm{i}\theta_{12}} & r_{22}\mathrm{e}^{-\mathrm{i}\theta_{22}}\end{pmatrix} \tag{0.0.10}$$

$$= \begin{pmatrix}r_{11}^2 + r_{12}^2 & r_{11}r_{21}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + r_{11}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})}\\ r_{11}r_{21}\mathrm{e}^{-\mathrm{i}(\theta_{11}-\theta_{21})} + r_{12}r_{22}\mathrm{e}^{-\mathrm{i}(\theta_{12}-\theta_{22})} & r_{21}^2 + r_{22}^2\end{pmatrix} \tag{0.0.11}$$

となる．M がユニタリ行列であることの必要十分条件は,

$$r_{11}^2 + r_{21}^2 = 1 \tag{0.0.12}$$

$$r_{12}^2 + r_{22}^2 = 1 \tag{0.0.13}$$

$$r_{11}^2 + r_{12}^2 = 1 \tag{0.0.14}$$

$$r_{21}^2 + r_{22}^2 = 1 \tag{0.0.15}$$

$$r_{11}r_{12}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} = 0 \tag{0.0.16}$$

$$r_{11}r_{21}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + r_{11}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} = 0 \tag{0.0.17}$$

である．$M^\dagger M$ や $MM^\dagger$ の非対角成分は複素共役になっていることに注意する．**式** (0.0.12) から**式** (0.0.15) を満たすような r_{ij} の組は，実数 Θ を用いて,

$$r_{11} = r_{22} = \cos(\Theta/2) \tag{0.0.18}$$

$$r_{12} = -r_{21} = \sin(\Theta/2) \tag{0.0.19}$$

なるものである．また，これらの r_{ij} の値を**式** (0.0.16) と**式** (0.0.17) に代入すると，

$$\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} - \mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} = 0 \tag{0.0.20}$$

$$-\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + \mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} = 0 \tag{0.0.21}$$

が成立する．

$$\Phi = \theta_{11} - \theta_{12} = \theta_{21} - \theta_{22} \tag{0.0.22}$$

$$\Psi = \theta_{11} - \theta_{21} = \theta_{12} - \theta_{22} \tag{0.0.23}$$

$$\tag{0.0.24}$$

とすると，

$$\theta_{11} = \frac{\Lambda + \Psi + \Phi}{2} \tag{0.0.25}$$

$$\theta_{12} = \frac{\Lambda + \Psi - \Phi}{2} \tag{0.0.26}$$

$$\theta_{21} = \frac{\Lambda - \Psi + \Phi}{2} \tag{0.0.27}$$

$$\theta_{22} = \frac{\Lambda - \Psi - \Phi}{2} \tag{0.0.28}$$

となり，**式** (0.0.4) を得る．つまり，任意のユニタリ行列は**式** (0.0.4) で書けることが示された． □

実際に**式** (0.0.4) がユニタリ行列であることを確かめる．

$$U^\dagger U = \mathrm{e}^{-\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) & -\mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ \mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix}\mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.29}$$

$$UU^\dagger = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix}\mathrm{e}^{-\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) & -\mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ \mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.30}$$

となり，U はユニタリ行列であることが分かる．