0.0.1　電磁場のハミルトニアン

前節での議論により，系のハミルトニアンは，

$$\hat{H}_{\mathrm{sys}} = \int \mathrm{d}^3 k \sum_{\sigma=1}^{2} \frac{\hbar\omega_{\boldsymbol{k}}}{2}\left(\hat{a}^{\dagger}_{\boldsymbol{k\sigma}}\hat{a}_{\boldsymbol{k}\sigma} + \hat{a}_{\boldsymbol{k}\sigma}\hat{a}^{\dagger}_{\boldsymbol{k}\sigma}\right) \tag{0.0.1}$$

と書けるのであった．以下では，簡単のために，1 方向成分・シングルモードの波を考える．

$$\hat{H}_{\mathrm{sys}} = \frac{\hbar\omega}{2}\left(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}\right) \tag{0.0.2}$$

$$= \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{0.0.3}$$

と書ける．屈折率が n の物質中では[1]，

$$\hat{H}_{n,\mathrm{sys}} = \frac{\hbar\omega}{n}\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{0.0.4}$$

と書ける．

0.0.2　数学的準備

ユニタリ行列は一般に，

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2} \end{pmatrix}\begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2} \end{pmatrix} \tag{0.0.5}$$

と分解できる．具体的に U を計算すると，

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2} \end{pmatrix}\begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2} \end{pmatrix} \tag{0.0.6}$$

$$= \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Psi/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\Psi/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\Psi/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\Psi/2}\cos(\Theta/2) \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2} \end{pmatrix} \tag{0.0.7}$$

$$= \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}(\Psi+\Psi)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2) \end{pmatrix} \tag{0.0.8}$$

であり，$\alpha = \Psi + \Phi$，$\beta = \Psi - \Phi$ とすると，

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix} \tag{0.0.9}$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\Lambda+\alpha)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Lambda+\beta)/2}\sin(\Theta/2) \\ -\mathrm{e}^{\mathrm{i}(\Lambda-\beta)/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Lambda-\alpha)/2}\cos(\Theta/2) \end{pmatrix} \tag{0.0.10}$$

と書ける．

Proof. 任意 2×2 の行列は，実数 r_{ij} と θ_{ij} を用いて，

$$M = \begin{pmatrix} r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}} \\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}} \end{pmatrix} \tag{0.0.11}$$

と書けて，

$$M^{\dagger}M = \begin{pmatrix} r_{11}\mathrm{e}^{-\mathrm{i}\theta_{11}} & r_{21}\mathrm{e}^{-\mathrm{i}\theta_{21}} \\ r_{12}\mathrm{e}^{-\mathrm{i}\theta_{12}} & r_{22}\mathrm{e}^{-\mathrm{i}\theta_{22}} \end{pmatrix}\begin{pmatrix} r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}} \\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}} \end{pmatrix} \tag{0.0.12}$$

$$= \begin{pmatrix} r_{11}^2 + r_{21}^2 & r_{11}r_{12}\mathrm{e}^{-\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{-\mathrm{i}(\theta_{21}-\theta_{22})} \\ r_{11}r_{12}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} & r_{12}^2 + r_{22}^2 \end{pmatrix} \tag{0.0.13}$$

[1] 謎である．屈折率により波動は変化しないはずである．

$$MM^\dagger = \begin{pmatrix} r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}} \\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}} \end{pmatrix}\begin{pmatrix} r_{11}\mathrm{e}^{-\mathrm{i}\theta_{11}} & r_{21}\mathrm{e}^{-\mathrm{i}\theta_{21}} \\ r_{12}\mathrm{e}^{-\mathrm{i}\theta_{12}} & r_{22}\mathrm{e}^{-\mathrm{i}\theta_{22}} \end{pmatrix} \tag{0.0.14}$$

$$= \begin{pmatrix} r_{11}^2 + r_{12}^2 & r_{11}r_{21}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + r_{11}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} \\ r_{11}r_{21}\mathrm{e}^{-\mathrm{i}(\theta_{11}-\theta_{21})} + r_{12}r_{22}\mathrm{e}^{-\mathrm{i}(\theta_{12}-\theta_{22})} & r_{21}^2 + r_{22}^2 \end{pmatrix} \tag{0.0.15}$$

となる．M がユニタリ行列であることの必要十分条件は,

$$r_{11}^2 + r_{21}^2 = 1 \tag{0.0.16}$$

$$r_{12}^2 + r_{22}^2 = 1 \tag{0.0.17}$$

$$r_{11}^2 + r_{12}^2 = 1 \tag{0.0.18}$$

$$r_{21}^2 + r_{22}^2 = 1 \tag{0.0.19}$$

$$r_{11}r_{12}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} = 0 \tag{0.0.20}$$

$$r_{11}r_{21}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + r_{11}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} = 0 \tag{0.0.21}$$

である．$M^\dagger M$ や $MM^\dagger$ の非対角成分は複素共役になっていることに注意する．**式** (0.0.16) から**式** (0.0.19) を満たすような r_{ij} の組は，実数 Θ を用いて,

$$r_{11} = r_{22} = \cos(\Theta/2) \tag{0.0.22}$$

$$r_{12} = -r_{21} = \sin(\Theta/2) \tag{0.0.23}$$

なるものである．また，これらの r_{ij} の値を**式** (0.0.20) と**式** (0.0.21) に代入すると,

$$\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} - \mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} = 0 \tag{0.0.24}$$

$$-\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + \mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} = 0 \tag{0.0.25}$$

が成立する．

$$\Phi = \theta_{11} - \theta_{12} = \theta_{21} - \theta_{22} \tag{0.0.26}$$

$$\Psi = \theta_{11} - \theta_{21} = \theta_{12} - \theta_{22} \tag{0.0.27}$$

$$\tag{0.0.28}$$

とすると,

$$\theta_{11} = \frac{\Lambda + \Psi + \Phi}{2} \tag{0.0.29}$$

$$\theta_{12} = \frac{\Lambda + \Psi - \Phi}{2} \tag{0.0.30}$$

$$\theta_{21} = \frac{\Lambda - \Psi + \Phi}{2} \tag{0.0.31}$$

$$\theta_{22} = \frac{\Lambda - \Psi - \Phi}{2} \tag{0.0.32}$$

となり，**式** (0.0.8) を得る．つまり，任意のユニタリ行列は**式** (0.0.8) で書けることが示された． □

実際に**式** (0.0.8) がユニタリ行列であることを確かめる．

$$U^\dagger U = \mathrm{e}^{-\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) & -\mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ \mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix}\mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.33}$$

$$UU^\dagger = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix}\mathrm{e}^{-\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) & -\mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ \mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.34}$$

となり，U はユニタリ行列であることが分かる．

0.0.3 ビームスプリッタ行列

2 入力 2 出力のビームスプリッタを考える．E_1 と E_2 の電場が入射して，E_1' と E_2' が出力されるとする．古典的に考えると，

$$\begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{0.0.35}$$

と書ける．このまま電場演算子を中心に議論を進めることはいささか冗長である．なぜならば，$\hat{a}_1$ と $\hat{a}_1^\dagger$ は複素共役の関係にあるのだから，片方が定まれば自然ともう片方が定まるからだ．よって，

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{0.0.36}$$

と書ける．B はビームスプリッタ行列という．光子数が保存することから，

$$\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 = \hat{a}_1'^\dagger \hat{a}_1' + \hat{a}_2'^\dagger \hat{a}_2' \tag{0.0.37}$$

$$= \left(B_{11}\hat{a}_1 + B_{12}\hat{a}_2\right)^\dagger \left(B_{11}\hat{a}_1 + B_{12}\hat{a}_2\right) + \left(B_{21}\hat{a}_1 + B_{22}\hat{a}_2\right)^\dagger \left(B_{21}\hat{a}_1 + B_{22}\hat{a}_2\right) \tag{0.0.38}$$

$$= \left(B_{11}^*\hat{a}_1^\dagger + B_{12}^*\hat{a}_2^\dagger\right)\left(B_{11}\hat{a}_1 + B_{12}\hat{a}_2\right) + \left(B_{21}^*\hat{a}_1^\dagger + B_{22}^*\hat{a}_2^\dagger\right)\left(B_{21}\hat{a}_1 + B_{22}\hat{a}_2\right) \tag{0.0.39}$$

$$= \left(|B_{11}|^2 + |B_{21}|^2\right)\hat{a}_1^\dagger\hat{a}_1 + \left(|B_{12}|^2 + |B_{22}|^2\right)\hat{a}_2^\dagger\hat{a}_2 + \left(B_{11}^*B_{12} + B_{21}^*B_{22}\right)\hat{a}_1^\dagger\hat{a}_2 + \left(B_{12}^*B_{11} + B_{21}^*B_{21}\right)\hat{a}_2^\dagger\hat{a}_1 \tag{0.0.40}$$

$$= \left(|B_{11}|^2 + |B_{21}|^2\right)\hat{a}_1^\dagger\hat{a}_1 + \left(|B_{12}|^2 + |B_{22}|^2\right)\hat{a}_2^\dagger\hat{a}_2 + \left(B_{11}^*B_{12} + B_{21}^*B_{22}\right)\hat{a}_1^\dagger\hat{a}_2 + \left(B_{11}^*B_{12} + B_{21}^*B_{22}\right)^*\hat{a}_2^\dagger\hat{a}_1 \tag{0.0.41}$$

となり，

$$\begin{cases} |B_{11}|^2 + |B_{21}|^2 = |B_{12}|^2 + |B_{22}|^2 = 1 \\ B_{11}^*B_{12} + B_{21}^*B_{22} = 0 \end{cases} \tag{0.0.42}$$

$$\Leftrightarrow B^\dagger B = \begin{pmatrix} B_{11}^* & B_{21}^* \\ B_{12}^* & B_{22}^* \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.43}$$

となればよい．つまり，ビームスプリッタ行列 B がユニタリ行列であれば良い．0.0.2 での議論において，

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2} \end{pmatrix} \tag{0.0.44}$$

は 2 つの入力電場 E_1，E_2 に位相差をかけること，

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2} \end{pmatrix} \tag{0.0.45}$$

は 2 つの出力電場 E_1'，E_2' に位相差をかけること，

$$\mathrm{e}^{\mathrm{i}\Lambda/2} \tag{0.0.46}$$

は 2 つの出力電場場 E_1'，E_2' に共通するグローバル位相を書けることに対応するから，実験のセットアップとして，

$$\Lambda = \Psi = \Phi = 0 \tag{0.0.47}$$

とすることができる．また，透過率 T と反射率 R を，

$$\sqrt{T} \coloneqq \cos(\Theta/2) \tag{0.0.48}$$

$$\sqrt{R} \coloneqq -\sin(\Theta/2) \tag{0.0.49}$$

と定義すれば，ビームスプリッタ行列 B は，

$$B = \begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix} \tag{0.0.50}$$

$$= \begin{pmatrix} \sqrt{T} & -\sqrt{R} \\ \sqrt{R} & \sqrt{T} \end{pmatrix} \tag{0.0.51}$$

と書ける．

0.0.4 ビームスプリッタハミルトニアン