0.0.1　電磁場のハミルトニアン

前節での議論により，系のハミルトニアンは，

$$\hat{H}_{\mathrm{sys}} = \int \mathrm{d}^3k \sum_{\sigma=1}^{2} \frac{\hbar\omega_{\boldsymbol{k}}}{2}\left(\hat{a}^\dagger_{\boldsymbol{k\sigma}}\hat{a}_{\boldsymbol{k}\sigma} + \hat{a}_{\boldsymbol{k}\sigma}\hat{a}^\dagger_{\boldsymbol{k}\sigma}\right) \tag{0.0.1}$$

と書けるのであった．以下では，簡単のために，1 方向成分・シングルモードの波を考える．

$$\hat{H}_{\mathrm{sys}} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) \tag{0.0.2}$$

$$= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{0.0.3}$$

と書ける．屈折率が n の物質中では[1]，

$$\hat{H}_{n,\mathrm{sys}} = \frac{\hbar\omega}{n}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{0.0.4}$$

と書ける．

0.0.2　ユニタリ行列の分解

ユニタリ行列は一般に，

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Psi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2}\end{pmatrix}\begin{pmatrix}\cos(\Theta/2) & \sin(\Theta/2)\\ -\sin(\Theta/2) & \cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Phi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2}\end{pmatrix} \tag{0.0.5}$$

と分解できる．具体的に U を計算すると，

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Psi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2}\end{pmatrix}\begin{pmatrix}\cos(\Theta/2) & \sin(\Theta/2)\\ -\sin(\Theta/2) & \cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Phi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2}\end{pmatrix} \tag{0.0.6}$$

$$= \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Psi/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\Psi/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}\Psi/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\Psi/2}\cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\Phi/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2}\end{pmatrix} \tag{0.0.7}$$

$$= \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2)\end{pmatrix} \tag{0.0.8}$$

であり，$\alpha = \Psi + \Phi$，$\beta = \Psi - \Phi$ とすると，

$$U = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2)\end{pmatrix} \tag{0.0.9}$$

$$= \begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Lambda+\alpha)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Lambda+\beta)/2}\sin(\Theta/2)\\ -\mathrm{e}^{\mathrm{i}(\Lambda-\beta)/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Lambda-\alpha)/2}\cos(\Theta/2)\end{pmatrix} \tag{0.0.10}$$

と書ける．

Proof. 任意 2×2 の行列は，実数 r_{ij} と θ_{ij} を用いて，

$$M = \begin{pmatrix}r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}}\\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}}\end{pmatrix} \tag{0.0.11}$$

と書けて，

$$M^\dagger M = \begin{pmatrix}r_{11}\mathrm{e}^{-\mathrm{i}\theta_{11}} & r_{21}\mathrm{e}^{-\mathrm{i}\theta_{21}}\\ r_{12}\mathrm{e}^{-\mathrm{i}\theta_{12}} & r_{22}\mathrm{e}^{-\mathrm{i}\theta_{22}}\end{pmatrix}\begin{pmatrix}r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}}\\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}}\end{pmatrix} \tag{0.0.12}$$

$$= \begin{pmatrix}r_{11}^2 + r_{21}^2 & r_{11}r_{12}\mathrm{e}^{-\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{-\mathrm{i}(\theta_{21}-\theta_{22})}\\ r_{11}r_{12}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} & r_{12}^2 + r_{22}^2\end{pmatrix} \tag{0.0.13}$$

[1] 謎である．屈折率により波動は変化しないはずである．

$$MM^\dagger = \begin{pmatrix} r_{11}\mathrm{e}^{\mathrm{i}\theta_{11}} & r_{12}\mathrm{e}^{\mathrm{i}\theta_{12}} \\ r_{21}\mathrm{e}^{\mathrm{i}\theta_{21}} & r_{22}\mathrm{e}^{\mathrm{i}\theta_{22}} \end{pmatrix}\begin{pmatrix} r_{11}\mathrm{e}^{-\mathrm{i}\theta_{11}} & r_{21}\mathrm{e}^{-\mathrm{i}\theta_{21}} \\ r_{12}\mathrm{e}^{-\mathrm{i}\theta_{12}} & r_{22}\mathrm{e}^{-\mathrm{i}\theta_{22}} \end{pmatrix} \tag{0.0.14}$$

$$= \begin{pmatrix} r_{11}^2 + r_{12}^2 & r_{11}r_{21}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + r_{11}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} \\ r_{11}r_{21}\mathrm{e}^{-\mathrm{i}(\theta_{11}-\theta_{21})} + r_{12}r_{22}\mathrm{e}^{-\mathrm{i}(\theta_{12}-\theta_{22})} & r_{21}^2 + r_{22}^2 \end{pmatrix} \tag{0.0.15}$$

となる．M がユニタリ行列であることの必要十分条件は,

$$r_{11}^2 + r_{21}^2 = 1 \tag{0.0.16}$$
$$r_{12}^2 + r_{22}^2 = 1 \tag{0.0.17}$$
$$r_{11}^2 + r_{12}^2 = 1 \tag{0.0.18}$$
$$r_{21}^2 + r_{22}^2 = 1 \tag{0.0.19}$$
$$r_{11}r_{12}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} + r_{21}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} = 0 \tag{0.0.20}$$
$$r_{11}r_{21}\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + r_{11}r_{22}\mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} = 0 \tag{0.0.21}$$

である．$M^\dagger M$ や $MM^\dagger$ の非対角成分は複素共役になっていることに注意する．**式** (0.0.16) から**式** (0.0.19) を満たすような r_{ij} の組は，実数 Θ を用いて,

$$r_{11} = r_{22} = \cos(\Theta/2) \tag{0.0.22}$$
$$r_{12} = -r_{21} = \sin(\Theta/2) \tag{0.0.23}$$

なるものである．また，これらの r_{ij} の値を**式** (0.0.20) と**式** (0.0.21) に代入すると,

$$\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{12})} - \mathrm{e}^{\mathrm{i}(\theta_{21}-\theta_{22})} = 0 \tag{0.0.24}$$
$$-\mathrm{e}^{\mathrm{i}(\theta_{11}-\theta_{21})} + \mathrm{e}^{\mathrm{i}(\theta_{12}-\theta_{22})} = 0 \tag{0.0.25}$$

が成立する．

$$\Phi = \theta_{11} - \theta_{12} = \theta_{21} - \theta_{22} \tag{0.0.26}$$
$$\Psi = \theta_{11} - \theta_{21} = \theta_{12} - \theta_{22} \tag{0.0.27}$$
$$\tag{0.0.28}$$

とすると,

$$\theta_{11} = \frac{\Lambda + \Psi + \Phi}{2} \tag{0.0.29}$$
$$\theta_{12} = \frac{\Lambda + \Psi - \Phi}{2} \tag{0.0.30}$$
$$\theta_{21} = \frac{\Lambda - \Psi + \Phi}{2} \tag{0.0.31}$$
$$\theta_{22} = \frac{\Lambda - \Psi - \Phi}{2} \tag{0.0.32}$$

となり，**式** (0.0.8) を得る．つまり，任意のユニタリ行列は**式** (0.0.8) で書けることが示された． □

実際に**式** (0.0.8) がユニタリ行列であることを確かめる．

$$U^\dagger U = \mathrm{e}^{-\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) & -\mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ \mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix}\mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.33}$$

$$UU^\dagger = \mathrm{e}^{\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ -\mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix}\mathrm{e}^{-\mathrm{i}\Lambda/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2}\cos(\Theta/2) & -\mathrm{e}^{\mathrm{i}\beta/2}\sin(\Theta/2) \\ \mathrm{e}^{-\mathrm{i}\beta/2}\sin(\Theta/2) & \mathrm{e}^{\mathrm{i}\alpha/2}\cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.34}$$

となり，U はユニタリ行列であることが分かる．

0.0.3 ビームスプリッタ行列

2 入力 2 出力のビームスプリッタを考える．E_1 と E_2 の電場が入射して，E'_1 と E'_2 が出力されるとする．古典的に考えると，

$$\begin{pmatrix} E'_1 \\ E'_2 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{0.0.35}$$

と書ける．このまま電場演算子を中心に議論を進めることはいささか冗長である．なぜならば，$\hat{a}_1$ と $\hat{a}_1^\dagger$ は複素共役の関係にあるのだから，片方が定まれば自然ともう片方が定まるからだ．よって**式** (0.0.35) を量子化して，消滅演算子 $\hat{a}_1$，$\hat{a}_2$ を用いて表せば，

$$\begin{pmatrix} \hat{a}'_1 \\ \hat{a}'_2 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{0.0.36}$$

と書ける．2 つの消滅演算子の交換関係は，

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta^i_j \tag{0.0.37}$$

$$[\hat{a}_i, \hat{a}_j] = 0 \tag{0.0.38}$$

である．B はビームスプリッタ行列という．光子数が保存することから，

$$\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 = \hat{a}_1'^\dagger \hat{a}'_1 + \hat{a}_2'^\dagger \hat{a}'_2 \tag{0.0.39}$$

$$= \left(B_{11}\hat{a}_1 + B_{12}\hat{a}_2\right)^\dagger \left(B_{11}\hat{a}_1 + B_{12}\hat{a}_2\right) + \left(B_{21}\hat{a}_1 + B_{22}\hat{a}_2\right)^\dagger \left(B_{21}\hat{a}_1 + B_{22}\hat{a}_2\right) \tag{0.0.40}$$

$$= \left(B_{11}^*\hat{a}_1^\dagger + B_{12}^*\hat{a}_2^\dagger\right)\left(B_{11}\hat{a}_1 + B_{12}\hat{a}_2\right) + \left(B_{21}^*\hat{a}_1^\dagger + B_{22}^*\hat{a}_2^\dagger\right)\left(B_{21}\hat{a}_1 + B_{22}\hat{a}_2\right) \tag{0.0.41}$$

$$= \left(|B_{11}|^2 + |B_{21}|^2\right)\hat{a}_1^\dagger\hat{a}_1 + \left(|B_{12}|^2 + |B_{22}|^2\right)\hat{a}_2^\dagger\hat{a}_2 + \left(B_{11}^*B_{12} + B_{21}^*B_{22}\right)\hat{a}_1^\dagger\hat{a}_2 + \left(B_{12}^*B_{11} + B_{21}^*B_{21}\right)\hat{a}_2^\dagger\hat{a}_1 \tag{0.0.42}$$

$$= \left(|B_{11}|^2 + |B_{21}|^2\right)\hat{a}_1^\dagger\hat{a}_1 + \left(|B_{12}|^2 + |B_{22}|^2\right)\hat{a}_2^\dagger\hat{a}_2 + \left(B_{11}^*B_{12} + B_{21}^*B_{22}\right)\hat{a}_1^\dagger\hat{a}_2 + \left(B_{11}^*B_{12} + B_{21}^*B_{22}\right)^*\hat{a}_2^\dagger\hat{a}_1 \tag{0.0.43}$$

となり，

$$\begin{cases} |B_{11}|^2 + |B_{21}|^2 = |B_{12}|^2 + |B_{22}|^2 = 1 \\ B_{11}^*B_{12} + B_{21}^*B_{22} = 0 \end{cases} \tag{0.0.44}$$

$$\Leftrightarrow B^\dagger B = \begin{pmatrix} B_{11}^* & B_{21}^* \\ B_{12}^* & B_{22}^* \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.45}$$

となればよい．つまり，ビームスプリッタ行列 B がユニタリ行列であれば良い．0.0.2 での議論によりビームスプリッタ演算子は，

$$B = \mathrm{e}^{\mathrm{i}\Lambda/2} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2} \end{pmatrix} \begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2} \end{pmatrix} \tag{0.0.46}$$

と書ける．ところが，

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Psi/2} \end{pmatrix} \tag{0.0.47}$$

は 2 つの入力電場 E_1，E_2 に位相差をかけること，

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Phi/2} \end{pmatrix} \tag{0.0.48}$$

は 2 つの出力電場 E'_1，E'_2 に位相差をかけること，

$$\mathrm{e}^{\mathrm{i}\Lambda/2} \tag{0.0.49}$$

は 2 つの出力電場場 E_1'，E_2' に共通するグローバル位相を書けることに対応するから，実験のセットアップとして，

$$\Lambda = \Psi = \Phi = 0 \tag{0.0.50}$$

とすることができる．また，透過率 T と反射率 R を，

$$\sqrt{T} \coloneqq \cos(\Theta/2) \tag{0.0.51}$$
$$\sqrt{R} \coloneqq -\sin(\Theta/2) \tag{0.0.52}$$

と定義すれば，ビームスプリッタ行列 B は，

$$B = \begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix} \tag{0.0.53}$$
$$= \begin{pmatrix} \sqrt{T} & -\sqrt{R} \\ \sqrt{R} & \sqrt{T} \end{pmatrix} \tag{0.0.54}$$

と書ける．

$$T + R = 1 \tag{0.0.55}$$

が成立することに注意する．

0.0.4 Baker-Campbell-Hausdorff の公式 1

$$\mathrm{e}^{\hat{A}} \hat{B} \mathrm{e}^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \cdots \tag{0.0.56}$$

なる式を示す．

Proof. 函数 $f(t)$ を，

$$f(t) \coloneqq \mathrm{e}^{t\hat{A}} \hat{B} \mathrm{e}^{-t\hat{A}} \tag{0.0.57}$$

と定義する．$f(t)$ を $t = 0$ の周りで展開することを考えると，

$$f(t) = f(0) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{t=0} t + \frac{1}{2!} \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{t=0} t^2 + \cdots \tag{0.0.58}$$

と書ける．さて，

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \hat{A}\mathrm{e}^{t\hat{A}} \hat{B} \mathrm{e}^{-t\hat{A}} - \mathrm{e}^{t\hat{A}} \hat{B} \hat{A} \mathrm{e}^{-t\hat{A}} \tag{0.0.59}$$
$$= \mathrm{e}^{t\hat{A}} \hat{A} \hat{B} \mathrm{e}^{-t\hat{A}} - \mathrm{e}^{t\hat{A}} \hat{B} \hat{A} \mathrm{e}^{-t\hat{A}} \tag{0.0.60}$$
$$= \mathrm{e}^{t\hat{A}} \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) \mathrm{e}^{-t\hat{A}} \tag{0.0.61}$$
$$= \mathrm{e}^{t\hat{A}} \left[\hat{A}, \hat{B}\right] \mathrm{e}^{-t\hat{A}} \tag{0.0.62}$$

である．よって，

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{t=0} = \left[\hat{A}, \hat{B}\right] \tag{0.0.63}$$

である．2 階以上の微分では，**式** (0.0.62) において，$\hat{B} \to \left[\hat{A}, \hat{B}\right]$ とすればよい．よって，**式** (0.0.58) に**式** (0.0.62) を代入すると，

$$f(t) = B + \left[\hat{A}, \hat{B}\right] t + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] t^2 + \cdots \tag{0.0.64}$$

である．$t = 1$ とすれば,

$$\mathrm{e}^{\hat{A}}\hat{B}\mathrm{e}^{\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \cdots \tag{0.0.65}$$

となる．特に,

$$\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] = 0 \tag{0.0.66}$$

のとき,

$$\mathrm{e}^{\hat{A}}\hat{B}\mathrm{e}^{\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] \tag{0.0.67}$$

である． □

0.0.5 Baker-Campbell-Hausdorff の公式 2

$$\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}} = \exp\left(\hat{A} + \hat{B} + \frac{1}{2}\left[\hat{A}, \hat{B}\right] + \frac{1}{12}\left[\left(\hat{A} - \hat{B}\right), \left[\hat{A}, \hat{B}\right]\right] + \cdots\right) \tag{0.0.68}$$

なる式を示す．

Proof. $f(t)$ と $g(t)$ を,

$$\begin{cases} f(t) \coloneqq \mathrm{e}^{t\hat{A}}\mathrm{e}^{t\hat{B}} = \mathrm{e}^{g(t)} \\ g(t) \coloneqq \log\left(\mathrm{e}^{t\hat{A}}\mathrm{e}^{t\hat{B}}\right) = \log\left(f(t)\right) \end{cases} \tag{0.0.69}$$

と定義する．$f(t)$ の Taylor 展開を考える．

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}f(t) &= \hat{A}\mathrm{e}^{t\hat{A}}\mathrm{e}^{t\hat{B}} + \mathrm{e}^{t\hat{A}}B\mathrm{e}^{t\hat{B}} && (0.0.70)\\
&= \mathrm{e}^{t\hat{A}}\left(\hat{A} + \hat{B}\right)\mathrm{e}^{t\hat{B}} && (0.0.71)\\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t) &= \mathrm{e}^{t\hat{A}}\left\{\hat{A}\left(\hat{A} + \hat{B}\right) + \left(\hat{A} + \hat{B}\right)\hat{B}\right\}\mathrm{e}^{t\hat{B}} && (0.0.72)\\
&= \mathrm{e}^{t\hat{A}}\left\{\hat{A}^2 + 2\hat{A}\hat{B} + \hat{B}^2\right\}\mathrm{e}^{t\hat{B}} && (0.0.73)\\
&= \mathrm{e}^{t\hat{A}}\left\{\hat{A}^2 + \hat{A}\hat{B} + \hat{B}\hat{A} + \hat{B}^2 + \hat{A}\hat{B} - \hat{B}\hat{A}\right\}\mathrm{e}^{t\hat{B}} && (0.0.74)\\
&= \mathrm{e}^{t\hat{A}}\left\{\left(\hat{A} + \hat{B}\right)^2 + \left[\hat{A}, \hat{B}\right]\right\}\mathrm{e}^{t\hat{B}} && (0.0.75)\\
\frac{\mathrm{d}^3}{\mathrm{d}t^3}f(t) &= \mathrm{e}^{t\hat{A}}\left\{\hat{A}\left(\hat{A} + 2\hat{A}\hat{B} + \hat{B}^2\right) + \left(\hat{A} + 2\hat{A}\hat{B} + \hat{B}^2\right)\hat{B}\right\}\mathrm{e}^{t\hat{B}} && (0.0.76)\\
&= \mathrm{e}^{t\hat{A}}\left\{\hat{A}^3 + 3\hat{A}^2\hat{B} + 3\hat{A}\hat{B}^2 + \hat{B}^3\right\}\mathrm{e}^{t\hat{B}} && (0.0.77)\\
&= \mathrm{e}^{t\hat{A}}\left\{\hat{A}^3 + \hat{A}^2\hat{B} + \hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}^2 + \hat{B}^2\hat{A} + \hat{B}\hat{A}\hat{B} + \hat{A}\hat{B}^2 + \hat{B}^3 + 2\hat{A}^2\hat{B} - \hat{A}\hat{B}\hat{A} - \hat{B}\hat{A}^2 - \hat{B}^2\hat{A} - \hat{B}\hat{A}\hat{B} + 2\hat{A}\hat{B}^2\right\}\mathrm{e}^{t\hat{B}} && (0.0.78)\\
&= \mathrm{e}^{t\hat{A}}\left\{\left(\hat{A} + \hat{B}\right)^3 + \hat{A}^2\hat{B} - \hat{A}\hat{B}\hat{A} + \hat{A}^2\hat{B} - \hat{B}\hat{A}^2 + \hat{A}\hat{B}^2 - \hat{B}^2\hat{A} + \hat{A}\hat{B}^2 - \hat{B}\hat{A}\hat{B}\right\}\mathrm{e}^{t\hat{B}} && (0.0.79)\\
&= \mathrm{e}^{t\hat{A}}\left\{\left(\hat{A} + \hat{B}\right)^3 + \hat{A}\left[\hat{A}, \hat{B}\right] + \left[\hat{A}^2, \hat{B}\right] + \left[\hat{A}, \hat{B}^2\right] + \left[\hat{A}, \hat{B}\right]\hat{B}\right\}\mathrm{e}^{t\hat{B}} && (0.0.80)
\end{aligned}$$

となる．$t = 0$ 周りで $f(t)$ を Taylor 展開すると,

$$f(t) = f(0) + \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}t}f(t)\bigg|_{t=0} t + \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\bigg|_{t=0} t^2 + \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}t^3}f(t)\bigg|_{t=0} t^3 + \cdots \tag{0.0.81}$$

$$
= 1 + \frac{1}{1!}\Big(\hat{A}+\hat{B}\Big)t + \frac{1}{2!}\left\{\Big(\hat{A}+\hat{B}\Big)^2 + \Big[\hat{A},\hat{B}\Big]\right\}t^2 + \frac{1}{3!}\left\{\Big(\hat{A}+\hat{B}\Big)^3 + \hat{A}\Big[\hat{A},\hat{B}\Big] + \Big[\hat{A}^2,\hat{B}\Big] + \Big[\hat{A},\hat{B}^2\Big] + \Big[\hat{A},\hat{B}\Big]\hat{B}\right\}t^3 + \cdots \tag{0.0.82}
$$

$$
= 1 + \hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3 + \cdots \tag{0.0.83}
$$

$$
\tag{0.0.84}
$$

と書ける．ただし，

$$
\hat{P}_1 \coloneqq \frac{1}{1!}\Big(\hat{A}+\hat{B}\Big) \tag{0.0.85}
$$

$$
\hat{P}_2 \coloneqq \frac{1}{2!}\left\{\Big(\hat{A}+\hat{B}\Big)^2 + \Big[\hat{A},\hat{B}\Big]\right\} \tag{0.0.86}
$$

$$
\hat{P}_3 \coloneqq \frac{1}{3!}\left\{\Big(\hat{A}+\hat{B}\Big)^3 + \hat{A}\Big[\hat{A},\hat{B}\Big] + \Big[\hat{A}^2,\hat{B}\Big] + \Big[\hat{A},\hat{B}^2\Big] + \Big[\hat{A},\hat{B}\Big]\hat{B}\right\} \tag{0.0.87}
$$

と定義した．$F(t)$ を，

$$
F(t) \coloneqq f(t) - 1 \tag{0.0.88}
$$

$$
= \hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3 + \cdots \tag{0.0.89}
$$

と定義する．$h(x) \coloneqq \log(1+x)$ は $x=0$ のまわりで，

$$
h(x) = h(0) + \frac{1}{1!}\left(\frac{\mathrm{d}}{\mathrm{d}x}h(x)\bigg|_{x=0}\right)x + \frac{1}{2!}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}h(x)\bigg|_{x=0}\right)x^2 + \frac{1}{3!}\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}h(x)\bigg|_{x=0}\right)x^3 + \cdots \tag{0.0.90}
$$

$$
= 0 + \frac{1}{1!}\frac{1}{1+0}x + \frac{1}{2!}\left(-\frac{1}{(1+0)^2}\right)x^2 + \frac{1}{3!}\left(\frac{2}{(1+0)^3}\right)x^3 + \cdots \qquad = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \tag{0.0.91}
$$

であることと，$t \to 0$ で $F(t) \to 0$ であることを用いて $g(t)$ を $t=0$ の周りで Taylor 展開すると，

$$
g(t) = \log(f(t)) \tag{0.0.92}
$$

$$
= \log(1+F(t)) \tag{0.0.93}
$$

$$
= F(t) - \frac{F(t)^2}{2} + \frac{F(t)^3}{3} - \cdots \tag{0.0.94}
$$

$$
= \Big(\hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3\Big) - \frac{1}{2}\Big(\hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3\Big)^2 + \frac{1}{3}\Big(\hat{P}_1 t + \hat{P}_2 t^2 + \hat{P}_3 t^3\Big)^3 - \cdots \tag{0.0.95}
$$

$$
= \hat{P}_1 t + \left(\hat{P}_2 - \frac{1}{2}\hat{P}_1^2\right)t^2 + \left\{\hat{P}_3 - \frac{1}{2}\Big(\hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1\Big) + \frac{1}{3}\hat{P}_1^3\right\}t^3 - \cdots \tag{0.0.96}
$$

となる．$\hat{P}_1$，$\hat{P}_2$，$\hat{P}_3$ の定義を思い出せば，

$$
(t^1 \text{の係数}) = \hat{P}_1 = \Big(\hat{A}+\hat{B}\Big) \tag{0.0.97}
$$

$$
(t^2 \text{の係数}) = \hat{P}_2 - \frac{1}{2}\hat{P}_1^2 \tag{0.0.98}
$$

$$
= \frac{1}{2}\left\{\Big(\hat{A}+\hat{B}\Big)^2 + \Big[\hat{A},\hat{B}\Big]\right\} - \frac{1}{2}\Big(\hat{A}+\hat{B}\Big)^2 \tag{0.0.99}
$$

$$
= \frac{1}{2}\Big[\hat{A},\hat{B}\Big] \tag{0.0.100}
$$

$$
(t^3 \text{の係数}) = \hat{P}_3 - \frac{1}{2}\Big(\hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1\Big) + \frac{1}{3}\hat{P}_1^3 \tag{0.0.101}
$$

$$
= \frac{1}{3!}\left\{\Big(\hat{A}+\hat{B}\Big)^3 + \hat{A}\Big[\hat{A},\hat{B}\Big] + \Big[\hat{A}^2,\hat{B}\Big] + \Big[\hat{A},\hat{B}^2\Big] + \Big[\hat{A},\hat{B}\Big]\hat{B}\right\}
$$

$$
- \frac{1}{2}\left[\Big(\hat{A}+\hat{B}\Big)\frac{1}{2!}\left\{\Big(\hat{A}+\hat{B}\Big)^2 + \Big[\hat{A},\hat{B}\Big]\right\} + \frac{1}{2!}\left\{\Big(\hat{A}+\hat{B}\Big)^2 + \Big[\hat{A},\hat{B}\Big]\right\}\Big(\hat{A}+\hat{B}\Big)\right]
$$

$$+\frac{1}{3}\left(\hat{A}+\hat{B}\right)^3 \tag{0.0.102}$$

$$=\frac{1}{6}\left(\hat{A}+\hat{B}\right)^3+\frac{1}{6}\hat{A}\left[\hat{A},\hat{B}\right]+\frac{1}{6}\left\{\hat{A}\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{B}\right]\hat{A}\right\}+\frac{1}{6}\left\{\hat{B}\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{B}\right]\hat{B}\right\}+\frac{1}{6}\left[\hat{A},\hat{B}\right]\hat{B}$$
$$-\frac{1}{2}\left(\hat{A}+\hat{B}\right)^3-\frac{1}{4}\hat{A}\left[\hat{A},\hat{B}\right]-\frac{1}{4}\hat{B}\left[\hat{A},\hat{B}\right]-\frac{1}{4}\left[\hat{A},\hat{B}\right]\hat{A}-\frac{1}{4}\left[\hat{A},\hat{B}\right]\hat{B}$$
$$+\frac{1}{3}\left(\hat{A}+\hat{B}\right)^3 \tag{0.0.103}$$

$$=\frac{1}{12}\hat{A}\left[\hat{A},\hat{B}\right]-\frac{1}{12}\hat{B}\left[\hat{A},\hat{B}\right]+\frac{1}{12}\left[\hat{A},\hat{B}\right]\hat{A}-\frac{1}{12}\left[\hat{A},\hat{B}\right]\hat{B} \tag{0.0.104}$$

$$=\frac{1}{12}\left(\hat{A}-\hat{B}\right)\left[\hat{A},\hat{B}\right]-\frac{1}{12}\left[\hat{A},\hat{B}\right]\left(\hat{A}-\hat{B}\right) \tag{0.0.105}$$

$$=\frac{1}{12}\left[\left(\hat{A}-\hat{B}\right),\left[\hat{A},\hat{B}\right]\right] \tag{0.0.106}$$

よって,

$$\mathrm{e}^{t\hat{A}}\mathrm{e}^{t\hat{B}}=f(t)=\mathrm{e}^{g(t)} \tag{0.0.107}$$

$$=\exp\left\{\left(\hat{A}+\hat{B}\right)t+\frac{1}{2}\left[\hat{A},\hat{B}\right]t^2+\frac{1}{12}\left[\left(\hat{A}-\hat{B}\right),\left[\hat{A},\hat{B}\right]\right]t^3+\cdots\right\} \tag{0.0.108}$$

$$\tag{0.0.109}$$

となるから，$t=0$ とすれば,

$$\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}=\exp\left\{\left(\hat{A}+\hat{B}\right)+\frac{1}{2}\left[\hat{A},\hat{B}\right]+\frac{1}{12}\left[\left(\hat{A}-\hat{B}\right),\left[\hat{A},\hat{B}\right]\right]+\cdots\right\} \tag{0.0.110}$$

となる．特に,

$$\left[\left(\hat{A}-\hat{B}\right),\left[\hat{A},\hat{B}\right]\right]=0 \tag{0.0.111}$$

$$\iff \left[\hat{A},\left[\hat{A},\hat{B}\right]\right]=\left[\hat{B},\left[\hat{A},\hat{B}\right]\right]=0 \tag{0.0.112}$$

のときは,

$$\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}=\exp\left\{\left(\hat{A}+\hat{B}\right)+\frac{1}{2}\left[\hat{A},\hat{B}\right]\right\} \tag{0.0.113}$$

となる． □

0.0.6 ビームスプリッタハミルトニアン

ビームスプリッタ行列を再び考えよう．今度は入力電場と出力電場の位相差が存在することにして，$\Lambda=0$ のみ課しておく．するとビームスプリッタ行列は,

$$B=\begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2)\end{pmatrix} \tag{0.0.114}$$

と書ける．ビームスプリッタ行列を用いて,

$$\begin{pmatrix}\hat{a}_1'\\ \hat{a}_2'\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2) & \mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2)\\ -\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2) & \mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} \tag{0.0.115}$$

$$=\begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2)\hat{a}_1+\mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2)\hat{a}_2\\ -\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sin(\Theta/2)\hat{a}_1+\mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\cos(\Theta/2)\hat{a}_2\end{pmatrix} \tag{0.0.116}$$

$$=\begin{pmatrix}\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\sqrt{T}\hat{a}_1-\mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sqrt{R}\hat{a}_2\\ \mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sqrt{R}\hat{a}_1+\mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\sqrt{T}\hat{a}_2\end{pmatrix} \tag{0.0.117}$$

と書ける．出力それぞれでの光の強度は，$\hat{a}_1$ と $\hat{a}_2^\dagger$ や $\hat{a}_1^\dagger$ と $\hat{a}_2$ が交換することを思い出せば，

$$\hat{a}_1'^\dagger\hat{a}_1' = \left(\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\sqrt{T}\hat{a}_1 - \mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sqrt{R}\hat{a}_2\right)^\dagger\left(\mathrm{e}^{\mathrm{i}(\Psi+\Phi)/2}\sqrt{T}\hat{a}_1 - \mathrm{e}^{\mathrm{i}(\Psi-\Phi)/2}\sqrt{R}\hat{a}_2\right) \tag{0.0.118}$$

$$= T\hat{a}_1^\dagger\hat{a}_1 + R\hat{a}_2^\dagger\hat{a}_2 - \sqrt{T}\sqrt{R}\left(\mathrm{e}^{\mathrm{i}\Phi}\hat{a}_1\hat{a}_2^\dagger + \mathrm{e}^{-\mathrm{i}\Phi}\hat{a}_1^\dagger\hat{a}_2\right) \tag{0.0.119}$$

$$\hat{a}_2'^\dagger\hat{a}_2' = \left(\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sqrt{R}\hat{a}_1 + \mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\sqrt{T}\hat{a}_2\right)^\dagger\left(\mathrm{e}^{-\mathrm{i}(\Psi-\Phi)/2}\sqrt{R}\hat{a}_1 + \mathrm{e}^{-\mathrm{i}(\Psi+\Phi)/2}\sqrt{T}\hat{a}_2\right) \tag{0.0.120}$$

$$= R\hat{a}_1^\dagger\hat{a}_1 + T\hat{a}_2^\dagger\hat{a}_2 + \sqrt{T}\sqrt{R}\left(\mathrm{e}^{\mathrm{i}\Phi}\hat{a}_1\hat{a}_2^\dagger + \mathrm{e}^{-\mathrm{i}\Phi}\hat{a}_1^\dagger\hat{a}_2\right) \tag{0.0.121}$$

となる．**式** (0.0.119) と**式** (0.0.121) について，第 1 項と第 2 項はそれぞれモード 1 の入力光子数，モード 2 の入力光子数に対応する．これらの重ね合わせに依って位相が変化して，そのパラメータは T である．相互作用を表す項は第 3 項であるから，ビームスプリッタによる相互作用ハミルトニアン $\hat{H}_{\mathrm{int}}$ を，

$$\hat{H}_{\mathrm{int}} \coloneqq \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\Phi}\hat{a}_1\hat{a}_2^\dagger + \mathrm{e}^{-\mathrm{i}\Phi}\hat{a}_1^\dagger\hat{a}_2\right) \tag{0.0.122}$$

と定義する．

また，以下の演算子を定義する．

$$\hat{L}_0 \coloneqq \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2\right) \tag{0.0.123}$$

$$\hat{L}_1 \coloneqq \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger\right) \tag{0.0.124}$$

$$\hat{L}_2 \coloneqq \frac{1}{2\mathrm{i}}\left(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1\hat{a}_2^\dagger\right) \tag{0.0.125}$$

$$\hat{L}_3 \coloneqq \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right) \tag{0.0.126}$$

$\hat{L}_2$ と $\hat{H}_{\mathrm{int}}$ の関係を調べよう．唐突だが，

$$\mathrm{e}^{-\mathrm{i}\Theta\hat{L}_2}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}\mathrm{e}^{\mathrm{i}\Theta\hat{L}_2} \tag{0.0.127}$$

考える．**式** (0.0.127) の第 1 成分について，Baker-Campbell-Hausdorff の公式より，

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\Theta\hat{L}_2}\hat{a}_1\mathrm{e}^{\mathrm{i}\Theta\hat{L}_2} &= \hat{a}_1 + \left[-\mathrm{i}\Theta\hat{L}_2, \hat{a}_1\right] + \frac{1}{2!}\left[-\mathrm{i}\Theta\hat{L}_2, \left[-\mathrm{i}\Theta\hat{L}_2, \hat{a}_1\right]\right] \\ &+ \frac{1}{3!}\left[-\mathrm{i}\Theta\hat{L}_2, \left[-\mathrm{i}\Theta\hat{L}_2, \left[-\mathrm{i}\Theta\hat{L}_2, \hat{a}_1\right]\right]\right] + \frac{1}{4!}\left[-\mathrm{i}\Theta\hat{L}_2, \left[-\mathrm{i}\Theta\hat{L}_2, \left[-\mathrm{i}\Theta\hat{L}_2, \left[-\mathrm{i}\Theta\hat{L}_2, \hat{a}_1\right]\right]\right]\right] + \cdots\end{aligned} \tag{0.0.128}$$

$$\begin{aligned}&= \hat{a}_1 + (-\mathrm{i}\Theta)\left[\hat{L}_2, \hat{a}_1\right] + \frac{(-\mathrm{i}\Theta)^2}{2!}\left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right] \\ &+ \frac{(-\mathrm{i}\Theta)^3}{3!}\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right]\right] + \frac{(-\mathrm{i}\Theta)^4}{4!}\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right]\right]\right] + \cdots\end{aligned} \tag{0.0.129}$$

となる．$\hat{L}_2$ と $\hat{a}_1$，$\hat{L}_2$ と $\hat{a}_2$ との交換関係についてそれぞれ，

$$\left[\hat{L}_2, \hat{a}_1\right] = \left[\frac{1}{2\mathrm{i}}\left(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1\hat{a}_2^\dagger\right), \hat{a}_1\right] \tag{0.0.130}$$

$$= \frac{1}{2\mathrm{i}}\left(\hat{a}_2\left[\hat{a}_1^\dagger, \hat{a}_1\right] - \hat{a}_2^\dagger[\hat{a}_1, \hat{a}_1]\right) \tag{0.0.131}$$

$$= -\frac{1}{2\mathrm{i}}\hat{a}_2 \tag{0.0.132}$$

$$\left[\hat{L}_2, \hat{a}_2\right] = \left[\frac{1}{2\mathrm{i}}\left(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1\hat{a}_2^\dagger\right), \hat{a}_2\right] \tag{0.0.133}$$

$$= \frac{1}{2\mathrm{i}}\left(\hat{a}_1^\dagger[\hat{a}_2, \hat{a}_2] - \hat{a}_1\left[\hat{a}_2^\dagger, \hat{a}_2\right]\right) \tag{0.0.134}$$

$$= \frac{1}{2\mathrm{i}}\hat{a}_1 \tag{0.0.135}$$

となる．ただし，$\hat{a}_1$ と $\hat{a}_2$ が交換することを用いた．よって，

$$\left[\hat{L}_2, \hat{a}_1\right] = -\left(\frac{1}{2\mathrm{i}}\right)^1 \hat{a}_2 \tag{0.0.136}$$

$$\left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right] = -\frac{1}{2\mathrm{i}}\left[\hat{L}_2, \hat{a}_2\right] = -\left(\frac{1}{2\mathrm{i}}\right)^2 \hat{a}_1 \tag{0.0.137}$$

$$\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right]\right] = -\left(\frac{1}{2\mathrm{i}}\right)^2 \left[\hat{L}_2, \hat{a}_1\right] = \left(\frac{1}{2\mathrm{i}}\right)^3 \hat{a}_2 \tag{0.0.138}$$

$$\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right]\right]\right] = \left(\frac{1}{2\mathrm{i}}\right)^3 \left[\hat{L}_2, \hat{a}_2\right] = \left(\frac{1}{2\mathrm{i}}\right)^4 \hat{a}_1 \tag{0.0.139}$$

であるから**式** (0.0.129) は，

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}\Theta\hat{L}_2}\hat{a}_1\mathrm{e}^{\mathrm{i}\Theta\hat{L}_2} &= \hat{a}_1 + (-\mathrm{i}\Theta)\left[\hat{L}_2, \hat{a}_1\right] + \frac{(-\mathrm{i}\Theta)^2}{2!}\left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right] \\
&\quad + \frac{(-\mathrm{i}\Theta)^3}{3!}\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right]\right] + \frac{(-\mathrm{i}\Theta)^4}{4!}\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_1\right]\right]\right]\right] + \cdots && (0.0.140) \\
&= \hat{a}_1 + (-\mathrm{i}\Theta)(-1)\left(\frac{1}{2\mathrm{i}}\right)^1 \hat{a}_2 + \frac{(-\mathrm{i}\Theta)^2}{2!}(-1)\left(\frac{1}{2\mathrm{i}}\right)^2 \hat{a}_1 + \frac{(-\mathrm{i}\Theta)^3}{3!}\left(\frac{1}{2\mathrm{i}}\right)^3 \hat{a}_2 + \frac{(-\mathrm{i}\Theta)^4}{4!}\left(\frac{1}{2\mathrm{i}}\right)^4 \hat{a}_1 + \cdots && (0.0.141) \\
&= \hat{a}_1 + \left(\frac{\Theta}{2}\right)^1 \hat{a}_2 - \frac{1}{2!}\left(\frac{\Theta}{2}\right)^2 \hat{a}_1 - \frac{1}{3!}\left(\frac{\Theta}{2}\right)^3 \hat{a}_2 + \frac{1}{4!}\left(\frac{\Theta}{2}\right)^4 \hat{a}_1 + \cdots && (0.0.142) \\
&= \left[1 - \frac{1}{2!}\left(\frac{\Theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\Theta}{2}\right)^4 - \cdots\right]\hat{a}_1 + \left[\left(\frac{\Theta}{2}\right)^1 - \frac{1}{3!}\left(\frac{\Theta}{2}\right)^3 + \cdots\right]\hat{a}_2 && (0.0.143) \\
&= \cos(\Theta/2)\hat{a}_1 + \sin(\Theta/2)\hat{a}_2 && (0.0.144)
\end{aligned}$$

となる．同様に，**式** (0.0.127) の第 2 成分について，

$$\left[\hat{L}_2, \hat{a}_2\right] = \left(\frac{1}{2\mathrm{i}}\right)^1 \hat{a}_1 \tag{0.0.145}$$

$$\left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_2\right]\right] = \frac{1}{2\mathrm{i}}\left[\hat{L}_2, \hat{a}_1\right] = -\left(\frac{1}{2\mathrm{i}}\right)^2 \hat{a}_2 \tag{0.0.146}$$

$$\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_2\right]\right]\right] = -\left(\frac{1}{2\mathrm{i}}\right)^2 \left[\hat{L}_2, \hat{a}_2\right] = -\left(\frac{1}{2\mathrm{i}}\right)^3 \hat{a}_1 \tag{0.0.147}$$

$$\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_2\right]\right]\right]\right] = -\left(\frac{1}{2\mathrm{i}}\right)^3 \left[\hat{L}_2, \hat{a}_1\right] = \left(\frac{1}{2\mathrm{i}}\right)^4 \hat{a}_2 \tag{0.0.148}$$

なる関係を用いると，

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}\Theta\hat{L}_2}\hat{a}_2\mathrm{e}^{\mathrm{i}\Theta\hat{L}_2} &= \hat{a}_2 + (-\mathrm{i}\Theta)\left[\hat{L}_2, \hat{a}_2\right] + \frac{(-\mathrm{i}\Theta)^2}{2!}\left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_2\right]\right] \\
&\quad + \frac{(-\mathrm{i}\Theta)^3}{3!}\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_2\right]\right]\right] + \frac{(-\mathrm{i}\Theta)^4}{4!}\left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \left[\hat{L}_2, \hat{a}_2\right]\right]\right]\right] + \cdots && (0.0.149) \\
&= \hat{a}_2 + (-\mathrm{i}\Theta)\left(\frac{1}{2\mathrm{i}}\right)^1 \hat{a}_1 + \frac{(-\mathrm{i}\Theta)^2}{2!}(-1)\left(\frac{1}{2\mathrm{i}}\right)^2 \hat{a}_2 + \frac{(-\mathrm{i}\Theta)^3}{3!}(-1)\left(\frac{1}{2\mathrm{i}}\right)^3 \hat{a}_1 + \frac{(-\mathrm{i}\Theta)^4}{4!}\left(\frac{1}{2\mathrm{i}}\right)^4 \hat{a}_2 + \cdots && (0.0.150) \\
&= \hat{a}_2 - \left(\frac{\Theta}{2}\right)^1 \hat{a}_1 - \frac{1}{2!}\left(\frac{\Theta}{2}\right)^2 \hat{a}_2 + \frac{1}{3!}\left(\frac{\Theta}{2}\right)^3 \hat{a}_1 + \frac{1}{4!}\left(\frac{\Theta}{2}\right)^4 \hat{a}_2 + \cdots && (0.0.151)
\end{aligned}$$

$$
\begin{aligned}
&= -\left[\left(\frac{\Theta}{2}\right)^1 - \frac{1}{3!}\left(\frac{\Theta}{2}\right)^3 + \cdots\right]\hat{a}_1 + \left[1 - \frac{1}{2!}\left(\frac{\Theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\Theta}{2}\right)^4 - \cdots\right]\hat{a}_2 && (0.0.152)\\
&= -\sin(\Theta/2)\hat{a}_1 + \cos(\Theta/2)\hat{a}_2 && (0.0.153)
\end{aligned}
$$

である．よって，**式** (0.0.127) は,

$$
\mathrm{e}^{-\mathrm{i}\Theta\hat{L}_2}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}\mathrm{e}^{\mathrm{i}\Theta\hat{L}_2} = \begin{pmatrix}\cos(\Theta/2) & \sin(\Theta/2)\\ -\sin(\Theta/2) & \cos(\Theta/2)\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} \tag{0.0.154}
$$

と書ける．**式** (0.0.154) の解釈を考えよう．相互作用ハミルトニアン $\hat{H}_{\mathrm{int}}$ の定義は,

$$
\hat{H}_{\mathrm{int}} \coloneqq \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\Phi}\hat{a}_1\hat{a}_2^\dagger + \mathrm{e}^{-\mathrm{i}\Phi}\hat{a}_1^\dagger\hat{a}_2\right) \tag{0.0.155}
$$

であった．$\Phi = \pi/2$ とすると,

$$
\begin{aligned}
\hat{H}_{\mathrm{int}} &= \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\pi/2}\hat{a}_1\hat{a}_2^\dagger + \mathrm{e}^{-\mathrm{i}\pi/2}\hat{a}_1^\dagger\hat{a}_2\right) && (0.0.156)\\
&= \frac{1}{2}\left(\mathrm{i}\hat{a}_1\hat{a}_2^\dagger - \mathrm{i}\hat{a}_1^\dagger\hat{a}_2\right) && (0.0.157)\\
&= \frac{1}{2\mathrm{i}}\left(\hat{a}_1^\dagger - \hat{a}_1\hat{a}_2^\dagger\right) && (0.0.158)\\
&= \hat{L}_2 && (0.0.159)
\end{aligned}
$$

と書ける．さらに，**式** (0.0.154) において，$\Theta = -t/\hbar$ とすれば,

$$
\exp\left(-\mathrm{i}\frac{\hat{H}_{\mathrm{int}}}{\hbar}t\right)\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}\exp\left(\mathrm{i}\frac{\hat{H}_{\mathrm{int}}}{\hbar}t\right) = \begin{pmatrix}\cos(-t/2\hbar) & \sin(-t/2\hbar)\\ -\sin(-t/2\hbar) & \cos(-t/2\hbar)\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} \tag{0.0.160}
$$

となる．左辺は,

$$
\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} \tag{0.0.161}
$$

なる消滅演算子のペアを時間発展演算子で挟んでいる格好である．となれば，右辺は Heisenberg 描像で表した消滅演算子であろう[2]．

[2]右辺に出てくる行列はビームスプリッタ行列でないことに注意する．確かに 2 つの入力電場間の位相ずれや，2 つの出力電場間の位相ずれがないと仮定したとき，ビームスプリッタ演算子は**式** (0.0.54) と書ける．しかし，$\Phi = \pi/2$ なる仮定のもと議論している．このような入力電場の位相ずれ Φ に対して，出力電場の位相ずれ Ψ をうまく定めれば**式** (0.0.54) の形を実現することができると思うかもしれないが，その試みははかなく終わる．そのような Ψ は，$\pi/2 + \Psi = 2n\pi$ かつ $\pi/2 - \Psi = 2m\pi$，$n, m \in \mathbb{Z}$ としなければいけないが，2 式を足して，$\pi = 2(n+m)\pi$ となり，そのような n，m は存在しない．要するに，**式** (0.0.154) の右辺の行列はビームスプリッタ行列ではないのだ．なお，テキストでの (1.106) は何を言いたいのかわからない．