

ユニタリ行列は一般に,

$$U = e^{i\Lambda/2} \begin{pmatrix} e^{i\Psi/2} & 0 \\ 0 & e^{-i\Psi/2} \end{pmatrix} \begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} e^{i\Phi/2} & 0 \\ 0 & e^{-i\Phi/2} \end{pmatrix} \quad (0.0.1)$$

と分解できる. 具体的に  $U$  を計算すると,

$$U = e^{i\Lambda/2} \begin{pmatrix} e^{i\Psi/2} & 0 \\ 0 & e^{-i\Psi/2} \end{pmatrix} \begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2) \\ -\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} e^{i\Phi/2} & 0 \\ 0 & e^{-i\Phi/2} \end{pmatrix} \quad (0.0.2)$$

$$= e^{i\Lambda/2} \begin{pmatrix} e^{i\Psi/2} \cos(\Theta/2) & e^{i\Psi/2} \sin(\Theta/2) \\ -e^{-i\Psi/2} \sin(\Theta/2) & e^{-i\Psi/2} \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} e^{i\Phi/2} & 0 \\ 0 & e^{-i\Phi/2} \end{pmatrix} \quad (0.0.3)$$

$$= e^{i\Lambda/2} \begin{pmatrix} e^{i(\Psi+\Phi)/2} \cos(\Theta/2) & e^{i(\Psi-\Phi)/2} \sin(\Theta/2) \\ -e^{-i(\Psi-\Phi)/2} \sin(\Theta/2) & e^{-i(\Psi+\Phi)/2} \cos(\Theta/2) \end{pmatrix} \quad (0.0.4)$$

であり,  $\alpha = \Psi + \Phi$ ,  $\beta = \Psi - \Phi$  とすると,

$$U = e^{i\Lambda/2} \begin{pmatrix} e^{i\alpha/2} \cos(\Theta/2) & e^{i\beta/2} \sin(\Theta/2) \\ -e^{-i\beta/2} \sin(\Theta/2) & e^{-i\alpha/2} \cos(\Theta/2) \end{pmatrix} \quad (0.0.5)$$

$$= \begin{pmatrix} e^{i(\Lambda+\alpha)/2} \cos(\Theta/2) & e^{i(\Lambda+\beta)/2} \sin(\Theta/2) \\ -e^{i(\Lambda-\beta)/2} \sin(\Theta/2) & e^{i(\Lambda-\alpha)/2} \cos(\Theta/2) \end{pmatrix} \quad (0.0.6)$$

と書ける.

*Proof.* 任意  $2 \times 2$  の行列は, 実数  $r_{ij}$  と  $\theta_{ij}$  を用いて,

$$M = \begin{pmatrix} r_{11}e^{i\theta_{11}} & r_{12}e^{i\theta_{12}} \\ r_{21}e^{i\theta_{21}} & r_{22}e^{i\theta_{22}} \end{pmatrix} \quad (0.0.7)$$

と書けて,

$$M^\dagger M = \begin{pmatrix} r_{11}e^{-i\theta_{11}} & r_{21}e^{-i\theta_{21}} \\ r_{12}e^{-i\theta_{12}} & r_{22}e^{-i\theta_{22}} \end{pmatrix} \begin{pmatrix} r_{11}e^{i\theta_{11}} & r_{12}e^{i\theta_{12}} \\ r_{21}e^{i\theta_{21}} & r_{22}e^{i\theta_{22}} \end{pmatrix} \quad (0.0.8)$$

$$= \begin{pmatrix} r_{11}^2 + r_{21}^2 & r_{11}r_{12}e^{-i(\theta_{11}-\theta_{12})} + r_{21}r_{22}e^{-i(\theta_{21}-\theta_{22})} \\ r_{11}r_{12}e^{i(\theta_{11}-\theta_{12})} + r_{21}r_{22}e^{i(\theta_{21}-\theta_{22})} & r_{12}^2 + r_{22}^2 \end{pmatrix} \quad (0.0.9)$$

$$MM^\dagger = \begin{pmatrix} r_{11}e^{i\theta_{11}} & r_{12}e^{i\theta_{12}} \\ r_{21}e^{i\theta_{21}} & r_{22}e^{i\theta_{22}} \end{pmatrix} \begin{pmatrix} r_{11}e^{-i\theta_{11}} & r_{21}e^{-i\theta_{21}} \\ r_{12}e^{-i\theta_{12}} & r_{22}e^{-i\theta_{22}} \end{pmatrix} \quad (0.0.10)$$

$$= \begin{pmatrix} r_{11}^2 + r_{12}^2 & r_{11}r_{21}e^{i(\theta_{11}-\theta_{21})} + r_{11}r_{22}e^{i(\theta_{12}-\theta_{22})} \\ r_{11}r_{21}e^{-i(\theta_{11}-\theta_{21})} + r_{12}r_{22}e^{-i(\theta_{12}-\theta_{22})} & r_{21}^2 + r_{22}^2 \end{pmatrix} \quad (0.0.11)$$

となる.  $M$  がユニタリ行列であることの必要十分条件は,

$$r_{11}^2 + r_{21}^2 = 1 \quad (0.0.12)$$

$$r_{12}^2 + r_{22}^2 = 1 \quad (0.0.13)$$

$$r_{11}^2 + r_{12}^2 = 1 \quad (0.0.14)$$

$$r_{21}^2 + r_{22}^2 = 1 \quad (0.0.15)$$

$$r_{11}r_{12}e^{i(\theta_{11}-\theta_{12})} + r_{21}r_{22}e^{i(\theta_{21}-\theta_{22})} = 0 \quad (0.0.16)$$

$$r_{11}r_{21}e^{i(\theta_{11}-\theta_{21})} + r_{11}r_{22}e^{i(\theta_{12}-\theta_{22})} = 0 \quad (0.0.17)$$

である.  $M^\dagger M$  や  $MM^\dagger$  の非対角成分は複素共役になっていることに注意する. 式 (0.0.12) から式 (0.0.15) を満たすような  $r_{ij}$  の組は, 実数  $\Theta$  を用いて,

$$r_{11} = r_{22} = \cos(\Theta/2) \quad (0.0.18)$$

$$r_{12} = -r_{21} = \sin(\Theta/2) \quad (0.0.19)$$

なるものである。また、これらの  $r_{ij}$  の値を式 (0.0.16) と式 (0.0.17) に代入すると、

$$e^{i(\theta_{11}-\theta_{12})} - e^{i(\theta_{21}-\theta_{22})} = 0 \quad (0.0.20)$$

$$-e^{i(\theta_{11}-\theta_{21})} + e^{i(\theta_{12}-\theta_{22})} = 0 \quad (0.0.21)$$

が成立する。

$$\Phi = \theta_{11} - \theta_{12} = \theta_{21} - \theta_{22} \quad (0.0.22)$$

$$\Psi = \theta_{11} - \theta_{21} = \theta_{12} - \theta_{22} \quad (0.0.23)$$

$$(0.0.24)$$

とすると、

$$\theta_{11} = \frac{\Lambda + \Psi + \Phi}{2} \quad (0.0.25)$$

$$\theta_{12} = \frac{\Lambda + \Psi - \Phi}{2} \quad (0.0.26)$$

$$\theta_{21} = \frac{\Lambda - \Psi + \Phi}{2} \quad (0.0.27)$$

$$\theta_{22} = \frac{\Lambda - \Psi - \Phi}{2} \quad (0.0.28)$$

となり、式 (0.0.4) を得る。つまり、任意のユニタリ行列は式 (0.0.4) で書けることが示された。  $\square$

実際に式 (0.0.4) がユニタリ行列であることを確かめる。

$$U^\dagger U = e^{-i\Lambda/2} \begin{pmatrix} e^{-i\alpha/2} \cos(\Theta/2) & -e^{i\beta/2} \sin(\Theta/2) \\ e^{-i\beta/2} \sin(\Theta/2) & e^{i\alpha/2} \cos(\Theta/2) \end{pmatrix} e^{i\Lambda/2} \begin{pmatrix} e^{i\alpha/2} \cos(\Theta/2) & e^{i\beta/2} \sin(\Theta/2) \\ -e^{-i\beta/2} \sin(\Theta/2) & e^{-i\alpha/2} \cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (0.0.29)$$

$$UU^\dagger = e^{i\Lambda/2} \begin{pmatrix} e^{i\alpha/2} \cos(\Theta/2) & e^{i\beta/2} \sin(\Theta/2) \\ -e^{-i\beta/2} \sin(\Theta/2) & e^{-i\alpha/2} \cos(\Theta/2) \end{pmatrix} e^{-i\Lambda/2} \begin{pmatrix} e^{-i\alpha/2} \cos(\Theta/2) & -e^{i\beta/2} \sin(\Theta/2) \\ e^{-i\beta/2} \sin(\Theta/2) & e^{i\alpha/2} \cos(\Theta/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (0.0.30)$$

となり、 $U$  はユニタリ行列であることが分かる。