AP Calculus BC Sequences and Series

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Contents

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1 Introduction

1.1 Motivation

This area of calculus is actually the oldest, dating back to the ancient Greeks. It predates derivatives, integrals, and limits. The Greeks were especially interested in the following problem.

1.1.1 Zeno's Paradox

Zeno's paradox, particularly the Paradox of the Arrow, challenges our understanding of motion and infinity. It states that in order for an arrow to reach its target, it must first cover half the distance to the target. Then it must cover half of the remaining distance, and then half of that, and so on. This process continues infinitely, as there will always be some distance left to cover, no matter how small.

Each step divides the remaining distance in half, creating an infinite sequence of smaller and smaller steps. Zeno argued that since completing an infinite number of steps seems impossible, the arrow would never reach its target.

This raises a fundamental question: Will the arrow ever reach the end? Intuitively, we know the arrow does reach the target, but the paradox highlights the difficulty in reconciling the concept of infinite division with the finite act of reaching a destination.

The resolution of this paradox lies in the concept of convergence from mathematics, where the sum of an infinite series of diminishing steps can still be finite. In this case, the total time and distance required for the arrow to reach its target can indeed add up to a finite value, allowing it to hit the target despite the infinite subdivisions.

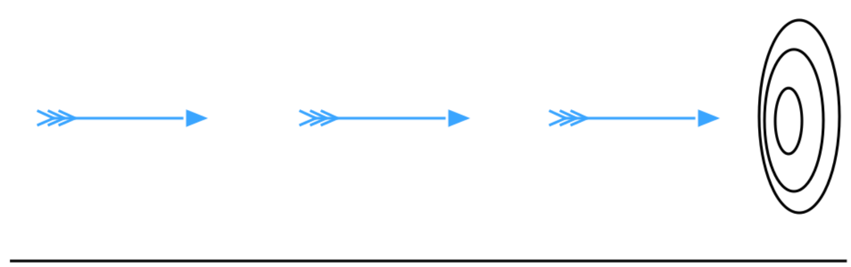

Figure 1: Zeno's Arrow Paradox

If we were to write this as a decreasing **sequence**, with total distance of 1 unit, it can be represented at each time step as

$$a_n = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$$

Can you identify the pattern? The sequence is defined by:

$$a_n = \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$$

So, to find the total distance is given by:

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$$

We will represent this as an **infinite series.**

Definition 1.1. *Infinite Series Notation*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

1.2 Question on the Paradox

There is a question that we have here, namely

Question 1.2. *What is the **limit** of a sequence? That is*

$$\lim_{n\to\infty} a_n = L$$

Question 1.3. *What is the **limit** of a series? That is given a partial sum, s_n (up to a number)*

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

What is the limit of the partial sums?

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$$

and if such a limit exists, does it converge (go to a finite number), or diverge (tends to infinity)?

We will answer these questions by the end of this document

2 Sequences

This section will go over sequences, which although important, will not be the main focus on the AP exam.

2.1 Important Sequences

Here are some important sequences:

Definition 2.1. *Alternating Sequence:*

$$\begin{aligned} a_n &= \{(-1)^n\} \\ &= \{-1, 1, -1, 1, \ldots\} \end{aligned}$$

Definition 2.2. *Definition of e in limit form*

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \\ &= e\end{aligned}$$

Note: This is a really important limit, and is one of the three definitions of e, the third we will cover soon!

Definition 2.3. *Properties of Limits of Sequences:*

Let

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} b_n = K$$

The following holds

1.
$$\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$$

2.
$$\lim_{n\to\infty} ca_n = cL$$

3.
$$\lim_{n\to\infty}(a_n b_n) = LK$$

4.
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{L}{K}$$

Definition 2.4. *Squeeze Theorem for Sequences*

Let

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L$$

You should also know how to identify patterns

Definition 2.5. *Common patterns start at $n = 0$:*

- *Even numbers:*
$$a_n = \{2n\} = \{0, 2, 4, 6, 8, \ldots\}$$
- *Odd numbers:*
$$a_n = \{2n + 1\} = \{1, 3, 5, 7, \ldots\}$$
- *Powers of base 2 (can be other bases):*
$$a_n = \{2^n\} = \{2^0, 2^1, 2^2, 2^3, \ldots\} = \{1, 2, 4, 8, \ldots\}$$
- *Factorials:*
$$a_n = \{n!\} = \{0!, 1!, 2!, 3!, 4!, 5!, \ldots\} = \{1, 1, 2, 6, 24, 120, \ldots\}$$

Tip 2.6.
$$0! = 1$$
(this is also true in computer science, but the $! =$ means something different here!!)

Tip 2.7. *You should also be familiar with factorial algebra.*
$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 1$$
and be comfortable in simplifying factorial expressions.

2.2 Bounded and Monotonic

Definition 2.8. *Definition of a monotonic sequence.*
*A sequence a_n is **monotonic** if its terms are non decreasing*

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots$$

or if terms are non increasing

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$$

Definition 2.9. *Bounded Sequence*

1. *A sequence a_n is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called the **upper bound** of the sequence.*
2. *A sequence a_n is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called the **lower bound** of the sequence.*
3. *A sequence a_n is **bounded** if it is bounded above and below.*

Theorem 2.10. *Bounded Monotonic Sequences:*
If a sequence a_n is bounded and monotonic, then it converges.

Example 2.11. *Bounded and Monotonic sequences*

1. *The sequence*
$$a_n = \left\{\frac{1}{n}\right\}$$
is both bounded and monotonic, thus it converges
2. *The divergent sequence*
$$b_n = \left\{\frac{n^2}{n+1}\right\}$$
is monotonic, but not bounded (as it is bounded below)
3. *The divergent sequence*
$$c_n = \{(-1)^n\}$$
is bounded, but not monotonic

3 Series and Convergence

This section will cover how to identify whether or not a series (the sum of a sequence) converges or diverges.

3.1 nth term test

The following theorem states that if a series converges, the limit of its nth term must be 0. The contrapositive is the test.

Theorem 3.1. *nth-Term Test for Divergence*

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Tip 3.2. *Do the nth term test first! It is useful in determining whether or not the series diverges.*

NOTE: *This test only tests for divergence. It is inconclusive in determining convergence, as more tests are required.*

Example 3.3. *Applying the nth-Term Test for divergence*

1. *For*
$$\sum_{n=1}^{\infty} 2^n$$
applying the nTT
$$\lim_{n\to\infty} 2^n = \infty$$
The nth term is not 0, and the series diverges.
2. *For*
$$\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$$
applying the nTT
$$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2}$$
The nth term is not 0, and the series diverges
3. *For*
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
applying the nTT
$$\lim_{n\to\infty} \frac{1}{n} = 0$$
Since the nth term is 0, the nth-Term test for divergence does no apply, and is inconclusive

Tip 3.4. *One way to remember this is that at the 'end' of the series, we are repeatedly adding a number, thus the sum tends to infinity and diverges.*

3.2 Geometric Series

Definition 3.5. *Geometric Series:*
They look like:

$$a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots$$

Index at $n = 0$

$$\sum_{n=0}^{\infty} ar^n$$

Index at $n = 1$

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Theorem 3.6. *A geometric series with ratio* r *diverges if* $|r| \geq 1$*. If* $0 < |r| < 1$*, then the series converges to the sum*

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Where a *is the first term and* r *is the geometric ratio*

Lets take a look back at Zeno's paradox.

Example 3.7. *We previously stated that*

$$\begin{aligned} S &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \end{aligned}$$

Applying our geometric series sum. Note $r = \frac{1}{2}$ *and* $0 < |\frac{1}{2}| < 1$

$$\begin{aligned} \sum_{n=1}^{\infty} ar^{n-1} &= \frac{a}{1-r} \\ \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n &= \frac{1/2}{1-(1/2)} = 1 \end{aligned}$$

So with calculus, we showed that yes indeed, an arrow in Zeno's paradox will hit its target and won't be stuck in a world of infinitesimals.

Example 3.8. *Find the sum*

$$\sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n$$

Since $r = \frac{e}{\pi}$ *and* $0 < |\frac{e}{\pi}| < 1$*, we can find its geometric sum as:*

$$\sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n = \frac{\left(\frac{e}{\pi}\right)}{1-\left(\frac{e}{\pi}\right)}$$

Question 3.9. *What about*

$$\sum_{n=0}^{\infty}\left(\frac{\pi}{e}\right)^n$$

This diverges since $r = \frac{\pi}{e} \approx 1.1557 > 1$ *by the GST.*

Remark 3.10. *We really need to pay attention to the geometric ratio* r *which converges within* $0 < |r| < 1$ *or* $-1 < r < 1$*. Consider the following:*

$$\sum_{n=0}^{\infty}(-1)^n$$

If we were to blindly use the geometric sum formula, we get:

$$\frac{1}{1-(-1)} = \frac{1}{2}$$

So, $1-1+1-1+1-1+\ldots = \frac{1}{2}$ *This is nonsense! Adding 1 and subtracting 1 infinitely gives us a half?*

Tip 3.11. *You are always welcome to simplify or rearrange the expression to make it look like a geometric series*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{3}{2^n} &= \sum_{n=0}^{\infty}3\cdot\frac{1}{2^n}\\ &= \frac{3}{1-(1/2)}\\ &= 6\end{aligned}$$

Also you may have to convert from power form ($a^{b+c} = a^b a^c$)*, consider this*

$$\sum_{n=2}^{\infty}\frac{5^{n+1}}{7^{n-2}}$$

Tip 3.12. *On the FRQ portion of the AP exam,* ***do not*** *simplify the fraction. It is OK to leave it as is. You don't want to get the right answer but simplified wrong and get 0 points.*

3.3 Integral Test

An integral represents the area under a continuous curve, while a series is a summation of discrete values, typically defined only at integer points. The connection between the two lies in their behavior: we can use improper integrals to test whether a series converges. By comparing a series to the integral of a related function, we can determine if the series adds up to a finite value (converges) or grows without bound (diverges).

Theorem 3.13. *The Integral Test:*

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

Put it simply, take whats inside the sum and make it into an integral (swap the index n to the continuous x), if that integral converges then the series converges. If that integral diverges then the series diverges.

Example 3.14. *Converge or diverge?*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

We will use the function $f(x) = \frac{1}{x^2+1}$. $f(x)$ is continuous, decreasing, and positive for $x \geq 1$, satisfying the conditions of the integral test (check for yourself).

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1}dx &= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}dx \\
&= \lim_{b\to\infty} [\arctan x]_1^b \\
&= \lim_{b\to\infty} (\arctan b - \arctan 1) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$

So, since the integral converges, the series converges by the integral test

Ok, lets do one small change.

Example 3.15. *Converge or diverge?*

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

We will use the function $f(x) = \frac{x}{x^2+1}$. *To determine if* f *is decreasing, find the derivative.*

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

We have that $f'(x) < 0$ *for* $x > 1$ *and it follows that* f *satisfies the conditions of the integral test.*

$$\begin{aligned}
\int_1^{\infty} \frac{x}{x^2+1} dx &= \frac{1}{2} \int_1^{\infty} \frac{2x}{x^2+1} dx \\
&= \frac{1}{2} \lim_{b\to\infty} \left[\ln(x^2+1)\right]_1^b \\
&= \frac{1}{2} \lim_{b\to\infty} (\ln(b^2+1) - \ln 2] \\
&= \infty
\end{aligned}$$

So, since the integral diverges, the series diverges by the integral test

Tip 3.16. *You **MUST** list the conditions of the integral test on the AP exam. Write "f is positive, continuous, and decreasing for* $x \geq 1$*"*

Question 3.17. *Does the sum* $\sum_{n=1}^{\infty} a_n$ *converge to the value of* $\int_1^{\infty} f(x)dx$*?*

ANS: *No, the integral test only tests for convergence. Let's see an example*

Example 3.18. *Series* $\neq$ *Integral: Consider the following series*

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

We will use $f(x) = \frac{1}{x^2}$. f *is positive, continuous, and decreasing for* $x \geq 1$, *and satisfies the* *conditions for the integral test.*

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx\\ &= \lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b\\ &= \lim_{b\to\infty}\left(\frac{-1}{b}-\frac{-1}{1}\right)\\ &= 1\end{aligned}$$

Thus the series converges. If we were to take partial sums, or just add terms on a computer, we observe that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\approx 1.6449$$

clearly not the same result of the integral $\int_1^\infty \frac{1}{x^2}dx = 1$

Remark 3.19. *We know the series above converges, but to what value? In 1735, Euler figured out that*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Quite shocking right? Why is π *here? This result is too difficult to prove for now.*

3.3.1 p - Series

We can quickly determine the convergence of a series with the series p - test

Theorem 3.20. *Convergence of* p *-series:*

The p *-series*

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots$$

1. *converges if* $p>1$*, and*
2. *diverges if* $0<p\leq 1$

Remark 3.21. *This result is based on the integral* p*-test.*

Example 3.22. *The harmonic series:*

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\dots\qquad p=1$$

diverges.

Example 3.23. *The series (aka Basel Problem, named after Euler's hometown):*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \qquad p = 2$$

converges. (We have also shown this by the integral test)

Example 3.24. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots \qquad p = 3/2$$

converges.

3.4 Alternating Series

So far we have only dealt with positive terms, but we can also have terms that contain both positive and negative terms. Take the following

Example 3.25. *Alternating Geometric Series:*

$$\begin{aligned}\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} \\ &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots\end{aligned}$$

Remark 3.26. *Alternating series can occur in two ways: either the odd terms are negative or the even terms are negative.*

Theorem 3.27. *Alternating Series Test*

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following ***two*** *conditions are met*

1.
$$\lim_{n \to \infty} a_n = 0$$

2.
$$a_{n+1} \leq a_n, \quad \text{for all } n$$

   (it is a decreasing sequence)

Example 3.28. *The alternating harmonic series*

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

First verify the two conditions of the alternating series test (AST)

1. $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

   This condition is met.

2. $$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

   This condition is met.

Thus we conclude by AST, the series converges

Remark 3.29. *Harmonic Series*

1. $$\sum_{n=1}^{\infty}\frac{1}{n} \quad \textit{diverges}$$

2. $$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} \quad \textit{converges}$$

The harmonic series diverges, but the alternating harmonic series converges

Tip 3.30. *On the AP Exam be sure to write the following statement when dealing with an alternating series FRQ problem.*

"The series is alternating, limit is 0, and terms decreasing are in magnitude. Thus we can apply the alternating series test."

Example 3.31. *Determine if the following series in convergent or divergent*

$$\sum_{n=2} \frac{\cos(n\pi)}{\sqrt{n}}$$

We can make note that

$$\cos(n\pi) = (-1)^n$$

So,

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Verify the conditions of the AST

1.
$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

2.
$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$$

The two conditions of the test are met, and so by the Alternating Series Test the series converges.

3.5 Comparison Tests

Sometimes we may have a scary looking series that is not as nice to directly perform a test on. When this happens, we can always compare with a simpler known series (b_n). They may look like:

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1}, \quad \sum_{n=0}^{\infty} \frac{n}{2^n}, \quad \sum_{n=0}^{\infty} \frac{1}{5^n+n}$$

3.5.1 Direct Comparison Test

Theorem 3.32. *Direct Comparison Test*

Let $0 < a_n \leq b_n$ *for all* n

1.
$$\text{If } \sum_{n=1}^{\infty} b_n \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges}$$

2.
$$\text{If } \sum_{n=1}^{\infty} a_n \text{ diverges, then } \sum_{n=1}^{\infty} b_n \text{ diverges}$$

Example 3.33. *Converge or diverge?*

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

This series looks like

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

which is a convergent geometric series. Comparing term-by term:

$$a_n = \frac{1}{2+3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1$$

So by the direct comparison test (DCT), the series converges.

Example 3.34. *Converge or diverge?*

$$\sum_{n=1}^{\infty} \frac{1}{3+\sqrt{n}}$$

This series looks like

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

which is a divergent p- series. Comparing term-by term:

$$\frac{1}{3+\sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad n \geq 1$$

So by the direct comparison test (DCT), the series diverges.

3.5.2 Limit Comparison Test

Often when the given series looks like a geometric or p-series, but are not able to establish the term-by-term comparison to apply the DCT, we need to apply the Limit Comparison Test (LCT)

Theorem 3.35. *Limit Comparison Test*

Suppose that $a_n > 0, b_n > 0$, and

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = L$$

where L is finite and positive ($L > 0$), Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 3.36. *Show that the general harmonic series diverges*

$$\sum_{n=1}^{\infty} \frac{1}{an+bn}, \quad a > 0, b > 0$$

We will compare with

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

a known divergent series. Applying the LCT:

$$\lim_{n\to\infty} \frac{1/(an+bn)}{1/n} = \lim_{n\to\infty} \frac{n}{an+b} = \frac{1}{a}$$

Since $L = \frac{1}{a} > 0$ and we compared it with a known divergent, $\sum \frac{1}{n}$, we conclude by LCT that the series diverges.

Example 3.37. *Converge or diverge?*

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

We only care about the highest powers of n, so we compare it with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

a convergent p-series. Applying the LCT:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2+1}\right)\left(\frac{n^{3/2}}{1}\right) \\ &= \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1\end{aligned}$$

Since $L = 1 > 0$ and we compared it with a known convergent $\sum \frac{1}{n^{3/2}}$, we conclude by the LCT that the series converges

3.6 Absolute and Conditional Convergence

Lets look back at the following series from remark 3.29

Example 3.38. *Harmonic Series and alternating harmonic series*

1. $$\sum_{n=1}^{\infty} \frac{1}{n} \quad \textit{diverges}$$

2. $$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \quad \textit{converges}$$

*The harmonic series diverges, but the alternating harmonic series converges. We say that this series in **conditionally convergent***

Theorem 3.39. *Absolute Convergence:*

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges

Example 3.40. *We will show that the following series is absolutely convergent*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

This series converges by the alternating series test. When we put in the absolute value bars:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The series also converges by p-series. Thus this series is absolutely convergent.

Tip 3.41. *Absolute convergent means "absolute value"*

Tip 3.42. *A series is absolutely convergent even when regardless of alternating sign, is still convergent.*

Definition 3.43. *Definitions of Absolute and Conditional Convergence*

1. $\sum a_n$ *is **absolutely convergent** if* $\sum |a_n|$ *converges*
2. $\sum a_n$ *is **conditionally convergent** if* $\sum a_n$ *converges but* $\sum |a_n|$ *diverges*

3.7 Ratio Test

The last test we cover is the most important test. If all other tests fail, it is most likely to use the ratio test.

Theorem 3.44. *Ratio Test*

Let $\sum a_n$ *be a series with nonzero terms*

1. $$\sum a_n \text{ converges absolutely if } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$
2. $$\sum a_n \text{ diverges if } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \text{ or } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$$
3. $$\text{The Ratio Test is inconclusive if } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

Tip 3.45. *To simplify calculations multiply a_{n+1} by the **reciprocal** of a_n.*

Tip 3.46. a_{n+1} *means taking all the n's from a_n and replacing it by $(n+1)$*

Tip 3.47.

$$(n+k)! = (n+k)(n+k-1)(n+k-2)\dots(n+1)n!$$

Example 3.48. *Converge or diverge?*

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

Applying the Ratio test:

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!}, \quad a_n = \frac{2^n}{n!}$$

Taking the limit:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| \\
&= \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{2^n \cdot 2}{(n+1)n!} \cdot \frac{n!}{2^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\cancel{2^n} \cdot 2}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}}\right| \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0
\end{aligned}$$

Since $0 < 1$, *the series converges by the ratio test*

Tip 3.49. ***DON'T*** *be lazy, always write* $\lim_{n\to\infty}$ *and remember the absolute value bars.*

Tip 3.50. *This test works especially well for factorials* $n!$ *and powers* b^n.

Example 3.51. *Converge or diverge?*

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

By the ratio test

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n!}{n^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^n (n+1)}{(n+1)n!} \frac{n!}{n^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^n \cancel{(n+1)}}{\cancel{(n+1)}\cancel{n!}} \frac{\cancel{n!}}{n^n} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \\
&= e > 1 \quad \textit{definition of } e
\end{aligned}$$

Since $e > 1$, the series diverges by the ratio test.

Example 3.52. *Converge or diverge*

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$

Applying the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| (-1)^n \left(\frac{\sqrt{n+1}}{n+1} \right) \left(\frac{n+1}{\sqrt{n}} \right) \right| \\
&= \lim_{n\to\infty} \left[\sqrt{\frac{n+1}{n}} \left(\frac{n+1}{n+2} \right) \right] \\
&= \sqrt{1}(1) \\
&= 1
\end{aligned}$$

We see here that the ratio test yields 1, which in inconclusive. We must apply another test, in particular the alternating series test (which you should verify).

4 Which Test Do I Use?!

Identifying which series to use is a bit of an art. There are some key giveaways for what test is best to use. These are listed below. However, it is really important that you practice and see many series to a point where identifying the test becomes second nature.

If you have time (6 hours) watch this video on 100 series convergence tests in one take.

The main series convergence tests are:

1. nth term test
2. Geometric Series Test
3. Integral test
4. p-series test
5. Alternating series test
6. Direct comparison test
7. Limit comparison test
8. Ratio test

> **Tip 4.1.** *Always do the nth term test first*

> **Tip 4.2.** *Be very familiar with evaluating limits to infinity. You may have to use L'Hôpital's in some cases.*

> **Tip 4.3.** *Although the Limit Comparison Test and Ratio Test both involve limits, LCT has $L > 0$ and Ratio Test has $L < 1$. The key thing to remember is that the ratio test is a lot like the geometric series test, we do the $|a_{n+1}/a_n|$ to make sure $|r| < 1$. or $L < 1$*

4.1 Important Facts:

The following are important things to remember.

The fact:

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{bn} = e^{ab}$$

The list:

$$\text{As } n \to \infty, \quad \ln n << n^p << b^n << n! << n^n, \quad \text{where } p > 0 \text{ and } b > 1$$

The limit:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

note: do not say by L'Hôpital's rule!!

Geometric sum:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{where } |r| < 1$$

4.2 Important Series Convergence Table

Test	Series	Converges	Diverges	Remarks
For Divergence (TFD)	$\sum_{n=1}^{\infty} a_n$	CANNOT show convergence	$\lim_{n\to\infty} a_n \neq 0$	always check first!
Geometric	$\sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\text{sum} = \dfrac{\text{first term}}{1-r}$
Telescoping	$\sum_{n=1}^{\infty} (b_n - b_{n+k})$	$\lim_{n\to\infty} b_{n+k} = L$ L has to be finite	$\lim_{n\to\infty} b_{n+k}$ D.N.E. or inf	write out serval terms then cancel stuff to find paritial sum
***P*-Series**	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	famous sum $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
Integral	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ Converges	$\int_1^{\infty} f(x)\,dx$ Diverges	$f(x)$ has to be positive, continuous & decreasing for $x \geq 1$
Direct Comparison (DCT)	$\sum_{n=1}^{\infty} a_n$ $a_n > 0$	$\sum_{n=1}^{\infty} a_n \leq$ a known convergent	$\sum_{n=1}^{\infty} a_n \geq$ a known divergent	try to use p-series or geometric series to compare
Limit Comparison (LCT)	$\sum_{n=1}^{\infty} a_n$ $a_n > 0$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ & $\sum_{n=1}^{\infty} b_n$ is known to be convergent	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ & $\sum_{n=1}^{\infty} b_n$ is known to be divergent	this version of LCT is inconclusive if $L = 0$ or $L = \infty$
Alternating (AST)	$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $b_n \geq 0$	(1.) $\lim_{n\to\infty} b_n = 0$ (2.) $b_{n+1} \leq b_n$	use TFD $\lim_{n\to\infty} (-1)^{n-1} b_n \neq 0$	$(-1)^{n-1}$ $= \cos\big((n-1)\pi\big)$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L > 1$	inconclusive if $L = 1$ great for ! and $(\ )^n$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L > 1$	inconclusive if $L = 1$ great for $(\ )^n$

If $\sum_{n=1}^{\infty} \lvert a_n\rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ is **absolute convergent** (which implies $\sum_{n=1}^{\infty} a_n$ also converges)

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} \lvert a_n\rvert$ diverges, then $\sum_{n=1}^{\infty} a_n$ is **conditional convergent**

*table courtesy of blackpenredpen
**the telescoping and root tests will NOT be on the AP exam