AP Calculus BC Integration Techniques

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1 Introduction

Integration is considered a "fun-house" compared to differentiation. With differentiation, we have clear rules such as the product rule and chain rule. However, when it comes to integrals, it can be considered as a wacky fun house where basic rules don't apply.

Solving integrals can be considered as an "art." Knowing which integration technique to use comes with practice and time.

2 A Review of AB Techniques

2.1 Know Your Antiderivatives!

Definition 2.1. ***Basic Integration Rules*** *Below is a list of basic integration rules*

$$\frac{d}{dx}C = 0 \qquad \int 0\,dx = C$$

$$\frac{d}{dx}(kx) = k \qquad \int k\,dx = kx + C$$

$$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}f(x) \qquad \int kf(x)\,dx = k\int f(x)\,dx$$

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \qquad \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1)$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \int \cos x\,dx = \sin x + C$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \int \sin x\,dx = -\cos x + C$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \int \sec x \tan x\,dx = \sec x + C$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \int \csc x \cot x\,dx = -\csc x + C$$

2.2 Tips

1. **Always** put in the differential dx or dy when doing an integral. Without it, the integral becomes meaningless.

2. **Always** add the constant of integration, $+C$, after an indefinite integral (one without bounds)

3. Remember the Fundamental Theorem of Calculus part II for definite integrals:

Definition 2.2. ***The Fundamental Theorem of Calculus part II***

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

4. Rewrite in power form:

Example 2.3. ***Rewriting Before Integrating***

$$\int \frac{1}{x^3}dx \implies \int x^{-3}dx$$

$$\int \sqrt{x}dx \implies \int x^{1/2}dx$$

5. Break apart integrals when possible

Example 2.4.

$$\int \frac{x+1}{\sqrt{x}+1}dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$$

$$\int \frac{\sin x}{\cos^2 x}dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx$$

2.2.1 Practice

Exercises 2.5. ***Find the antiderivatives***

1. $\int \frac{2}{\sqrt{x}}dx$

2. $\int (t^2+1)^2 dt$

3. $\int \frac{x^3+3}{x^2}dx$

4. $\int \sqrt[3]{x}(x-4)dx$

5. $\int y^2\sqrt{y}dy$

6. $\int (\tan^2 y + 1)dy$

7. $\int \sec y(\tan y - \sec y)dy$

2.3 u-substitution

When we take derivatives, we apply the chain rule. u-substitution is the reverse of doing that for integrals.

Theorem 2.6. *Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then*

$$\int f(g(x)g'(x)\ dx = F(g(x)) + C$$

If we let $u = g(x)$, then $du = g'(x)$ and

$$\int f(u)du = F(u) + C$$

Although we can expand the polynomial in the integrand, performing a substitution makes it much easier.

It is also key to see the composed function is multiplied by its derivative in the integrand.

Example 2.7.

$$\int (x^2+1)^2(2x)\ dx$$

Let $u = x^2 + 1 \implies du = 2xdx \implies dx = \frac{1}{2x}du$

$$\int (u^2)2x \cdot \frac{1}{2x}du$$

$$\int (u^2)\cancel{2x} \cdot \frac{1}{\cancel{2x}}du$$

$$\int u^2\ du = \frac{u^3}{3}$$

$$= \frac{(x^2+1)^3}{3} + C$$

Example 2.8.

$$\int \sin^2 3x \cos 3x\ dx$$

Let $u = \sin 3x \implies du = \cos 2x \cdot 3\ dx \implies dx = \frac{du}{3\cos 3x}$

$$\int u^2 \cos 3x\ \frac{du}{3\cos 3x}$$

$$\int u^2 \cancel{\cos 3x}\ \frac{du}{3\cancel{\cos 3x}}$$

Pull out the $\frac{1}{3}$ *and bring back the* u

$$\frac{1}{3}\int u^2\ du = \frac{1}{3}\left(\frac{u^3}{3}\right) = \frac{1}{9}\sin^3 3x + C$$

2.3.1 Guidelines for u-substitution

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)dx$
3. Rewrite the integral in terms of the variable u
4. Find the resulting integral in terms of u
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating

2.3.2 Practice

Exercises 2.9. ***Solve the following integrals using u-substitution:***

1. $\int (x^2 + 5x)e^{x^3+\frac{5}{2}x^2}\, dx$
2. $\int \frac{\cos(x)}{\sin^2(x)}\, dx$
3. $\int \sqrt{x^2+4} \cdot x\, dx$
4. $\int \frac{(2x+3)}{\sqrt{x^2+3x+2}}\, dx$
5. $\int \frac{\ln(x)}{x}\, dx$
6. $\int (5x^4 + 3x^2) \cdot (x^5 + x^3)^4\, dx$
7. $\int_0^4 x^2\sqrt{x^3+1}\, dx$
8. $\int_1^3 e^{x^2}(2x)\, dx$
9. $\int_0^\pi \sin(x)\cos(x)\, dx$
10. $\int_0^1 \frac{4x}{(x^2+1)^2}\, dx$

2.4 More Integration Rules

Definition 2.10. ***Basic Integration Rules*** $a > 0$

1.
$$\int [f(u) \pm g(u)]\, du = \int f(u)\, du \pm \int g(u)\, du$$

2.
$$\int kf(u)\, du = k \int f(u)\, du, \quad k \in \mathbb{R}$$

3.
$$\int \frac{1}{u}\, du = \ln|u| + C, \quad u \neq 0$$

4.
$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

5.
$$\int e^u\, du = e^u + C$$

6.
$$\int \frac{1}{u}\, du = \ln|u| + C$$

7.
$$\int a^u\, du = \frac{a^u}{\ln a} + C, \quad a > 0,\ a \neq 1$$

8.
$$\int \sin u\, du = -\cos u + C$$

9.
$$\int \cos u\, du = \sin u + C$$

10.
$$\int \tan u\, du = -\ln|\cos u| + C$$

11.
$$\int \cot u\, du = \ln|\sin u| + C$$

12.
$$\int \sec u\, du = \ln|\sec u + \tan u| + C$$

13.
$$\int \csc u\, du = -\ln|\csc u + \cot u| + C$$

14.
$$\int \sec^2 u\, du = \tan u + C$$

15.
$$\int \csc^2 u\, du = -\cot u + C$$

16.

$$\int \sec u \tan u \, du = \sec u + C$$

17.

$$\int \csc u \cot u \, du = -\csc u + C$$

18.

$$\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C, \quad a > 0$$

19.

$$\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \arcsin\left(\frac{u}{a}\right) + C, \quad a > 0$$

20.

$$\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C, \quad |u| > a > 0$$

Perhaps the most important ones are 3, 4, and 18.

2.5 Integrating Functions Using Long Division and Completing the Square

2.5.1 Polynomial Long Division Before Integrating

When we have a rational polynomial function, we may need to simplify it down by performing polynomial long division. Consider the example.

Example 2.11. *Find*

$$\int \frac{x^2 + x + 1}{x^2 + 1} \, dx$$

We divide $x^2 + x + 1$ *by* $x^2 + 1$ *as follows:*

$$\begin{array}{r|l} & 1 \\ \hline x^2 + 1 & x^2 + x + 1 \\ & -\ (x^2 + 1) \\ \hline & x \end{array}$$

Thus the polynomial is 1 and its remainder:

$$1 + \frac{x}{x^2 + 1}$$

Resulting in the integral:

$$\int 1 + \frac{x}{x^2 + 1} \, dx$$

Which is an exercise left to the reader (break apart, and use u *- sub)*

2.5.2 Completing the Square

Recall the method for completing the square of a quadratic:

We have an equation like:

$$ax^2 + bx + c = 0$$

and want to turn it into

$$a(x + d)^2 + e = 0$$

With $d = \frac{b}{2a}$ and $e = c - \frac{b^2}{4a}$

Example 2.12. *Complete the square of the following quadratic:*

$$x^2 + 6x + 7$$

We will add and subtract a $(b/2)^2$ *term. Here* $b = 6$

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 + 7 - \left(\frac{6}{2}\right)^2$$

$$(x + 3)^2 + 7 - 9 = (x + 3)^2 - 2$$

Example 2.13. *Complete the square of the following quadratic:*

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4) - 4) + 5 \\ &= 2[(x - 2)^2 + 1] \end{aligned}$$

Now, we see an example with an integral:

Example 2.14. *Find*

$$\int \frac{dx}{x^2 - 4x + 7}$$

Completing the square:

$$\begin{aligned} x^2 - 4x + 7 &= (x^2 - 4x + 4) - 4 + 7 \\ &= (x - 2)^2 + 3 = u^2 + a^2 \end{aligned}$$

In this completed square form, let $u = x - 2$ *and* $a = \sqrt{3}$.

$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x - 2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan\left(\frac{x - 2}{\sqrt{3}}\right) + C$$

Example 2.15. *Find*

$$\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}} dx$$

Completing the square of the denominator,

$$\begin{aligned} 3x - x^2 &= -(x^2 - 3x) \\ &= -\left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] \\ &= \left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2 \end{aligned}$$

Thus,

$$\begin{aligned} \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}} dx &= \int_{3/2}^{9/4} \frac{1}{\sqrt{(3/2)^2 - [x - (3/2)]^2}} dx \\ &= \arcsin\left(\frac{x - (3/2)}{3/2}\right)\Bigg|_{3/2}^{9/4} \\ &= \arcsin\left(\frac{1}{2}\right) - \arcsin(0) \\ &= \frac{\pi}{6} \end{aligned}$$

2.6 Procedures for Fitting Integrands to Basic Rules

Definition 2.16. *These are some basic procedures for making an integral more "friendly"*

- *Expand (numerator)*

$$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$$

- *Separate numerator*

$$\frac{1 + x}{x^2 + 1} = \frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$$

- *Complete the square*

$$\frac{1}{\sqrt{2x - x^2}} = \frac{1}{\sqrt{1 - (x - 1)^2}}$$

- *Divide improper rational function*

$$\frac{x^2}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}$$

- *Add and subtract terms in the numerator*

$$\frac{1}{1 + e^x} = \frac{1 + e^x - e^x}{1 + e^x} = 1 - \frac{1}{1 + e^x}$$

- *Use trigonometric identities*

$$\cot^2 x = \csc^2 x - 1$$

- *Multiply and divide by the Pythagorean conjugate*

$$\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$$

3 Integration by Parts (BC)

With all the techniques we know in BC, there are still some integral we cannot do until now.

Example 3.1. *Integration by Parts Integrals*

$$\int \ln x dx, \quad \int x^2 e^x dx, \quad \int e^x \sin x dx$$

Integration by Parts is the "reverse product rule." The product rule is:

$$\begin{aligned} \frac{d}{dx}[uv] &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= uv' + vu' \end{aligned}$$

If we were to integrate both sides:

$$\begin{aligned} uv &= \int uv' dx + \int vu' dx \\ &= \int u\ dv + \int v\ du \end{aligned}$$

Rewriting, we obtain

Theorem 3.2. *If u and v are functions of x and have continuous derivatives, then*

$$\int u\ dv = uv - \int v\ du$$

3.1 Guidelines for Integration by parts

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand
2. Try letting u be the portion of the integrand whose derivative is simpler than u. Then dv will be the remaining factor(s) of the integrand

3.1.1 LIATE for selecting u

We have the term *LIATE* for selecting the u when we integration by parts.

- L - Logarithms, i.e. $\ln x$
- I - Inverse functions, i.e $\arctan x, \arcsin x$
- A - Algebraic (polynomials), i.e. x^2, x^3

- T - Trigonometric, i.e. $\sin x, \cos x$
- E - Exponential, i.e. e^x

Alternatively, if you like you can select the dv with *DETAIL*, the same but in reverse order, with "D" standing for dv.

3.1.2 Examples of IBP

Example 3.3. *Find*

$$\int \ln x \ dx$$

Although this may look like a tricky one, we can make a quick cosmetic change:

$$\int \ln x \ dx = \int 1 \cdot \ln x \ dx$$

Thus, we first select the u *and* dv*, before filling out with* du *and* v

$$u = \ln x \quad v = x$$

$$du = \frac{1}{x} \ dx \quad dv = 1dx$$

Integration by parts formula: $uv - \int v \ du$

$$x \ln x - \int x \cdot \frac{1}{x} dx$$

$$x \ln x - x + C$$

Example 3.4. *Find*

$$\int x^2 \sin x \ dx$$

With LIATE, we set

$$u = x^2 \quad v = \int \sin x \ dx = -\cos x$$

$$du = 2x \ dx \quad dv = \sin x$$

Now, integration by parts produces:

$$\int x^2 \sin x \ dx = -x^2 \cos x + \int 2x \cos x dx$$

The second term requires integration by parts again:

$$u = 2x \quad v = \int \cos x \ dx = \sin x$$

$$du = 2dx \ dx \quad dv = \cos x dx$$

The second term,

$$\begin{aligned}\int 2x\cos x\;dx &= 2x\sin x - \int 2\sin x dx\\ &= 2x\sin x + 2\cos x + c\end{aligned}$$

Combining the two results,

$$\int x^2 \sin x\;dx = -x^2\cos x + 2x\sin x + 2\cos x + C$$

3.1.3 IBP - Tabular Integration

Watch this awesome video to learn a better way of performing integration by parts.

Example 3.5. *Find*

$$\int x^2 e^{2x}\;dx$$

D		I
x^2	$+$	e^{2x}
$2x$	$-$	$\frac{1}{2}e^{2x}$
2	$+$	$\frac{1}{4}e^{2x}$
0		$\frac{1}{8}e^{2x}$

Remark: This works really well when the u is algebraic.

4 Partial Fraction Decomposition

If integrating a ration expression involving polynomials, in the form of

$$\int \frac{P(x)}{Q(x)}\;dx$$

where the degree of $P(x)$ is smaller than the degrees of $Q(x)$. Factor the denominator as completely as possible, and find the partial fraction decomposition of the ration expression. Integrate the partial fraction decomposition. For each factor in the denominator we get term(s) in the decompostion as follows:

Factor of $Q(x)$	Terms in PFD	Factor is $Q(x)$	Term in PFD
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+b}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\ldots+\frac{A_kx+B_k}{(ax^2+bx+c)^2}$

4.1 PFD Method

Example 4.1. *We will perform the PFD for the following*

$$\frac{1}{x^2 - 5x + 6}$$

Factoring the denominator:

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

Where A, B *are to be determined. Multiplying the equation by the least common denominator* $(x-3)(x-2)$ *yields*

$$1 = A(x-2) + B(x-3)$$

We can substitute any convenient values for x *to obtain* A *and* B*. The most convenient values are the ones that make particular factors equal to 0.*

To solve for A*, let* $x = 3$*, then*

$$\begin{aligned} 1 &= A(3-2) + B(3-3) \\ 1 &= A \cdot 1 + B \cdot 0 \\ A &= 1 \end{aligned}$$

To solve for B*, let* $x = 2$*, then*

$$\begin{aligned} 1 &= A(2-2) + B(2-3) \\ 1 &= A \cdot 0 + B \cdot (-1) \\ B &= -1 \end{aligned}$$

So the decomposition is:

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x-3} + \frac{-1}{x-2}$$

Example 4.2. *Find*

$$\int \frac{5x^2+20x+6}{x^2+2x^2+x}\ dx$$

Looking at the integrand:

$$\frac{5x^2+20x+6}{x^2+2x^2+x} = \frac{5x^2+20x+6}{x(x+1)^2}$$

Setting up the PFD:

$$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Note: we need to "build up" the power for the last two terms.

Multiplying by the least common denominator $x(x+1)^2$, *yields the basic equation*

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

To get A,*let* $x=0$. *Which makes the* B, C *terms disappear*

$$\begin{aligned} 6 &= A \cdot 1 + 0 + 0 \\ A &= 6 \end{aligned}$$

To get C, *let* $x=-1$, *which makes the* A, B *terms disappear and*

$$\begin{aligned} 5-20 &= 0+0-c \\ C &= 9 \end{aligned}$$

To find the value of B, *we use any other value of* x *along with what we got for* A *and* C. *Using* $x=1, A=6, C=9$

$$\begin{aligned} 5+20+6 &= A\cdot 4 + B \cdot 2 + C \\ 31 &= 6(4) + 2B + 9 \\ -2 &= 2B \\ B &= -1 \end{aligned}$$

So it follows that

$$\begin{aligned} \int \frac{5x^2+20x+6}{x(x+1)^2}\ dx &= \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right)\ dx \\ &= 6\ln|x| - \ln|x+1| + 9\frac{(x+1)^{-1}}{-1} + C \\ &= \ln|\frac{x^6}{x+1}| - \frac{9}{x+1} + C \end{aligned}$$

4.2 PFD Tips

Watch this video on the cover up method for PFD to learn how to do this quickly and less error prone.
On the AP exam, we typically only have linear terms in the denominator, it is very unlikely to see one with a $Ax + B$ term.

5 Improper Integrals

5.1 Types of Improper Integrals

We have two types of improper integrals: **Infinite Limit** and **Discontinuous Integrand**

Definition 5.1. *Type I: Infinite Limit*

1. $$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$
2. $$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$
3. $$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

   Provided both integrals are convergent

Definition 5.2. *Type II: Discontinuous Integrand*

1. *Discontinuity at a:*
$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$
2. *Discontinuity at b:*
$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

   item Discontinuity at $a < c < b$:
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

   *provided **both** are convergent*

Definition 5.3. *Comparison Test for Improper Integrals: If $f(x) \geq g(x) \geq 0$ on $[a, \infty)$ then,*

1. *If $\int_a^\infty f(x)dx$ is convergent then $\int_a^\infty g(x)dx$ is convergent (if larger converges so does the smaller)*
2. *If $\int_a^\infty g(x)dx$ is divergent then $\int_a^\infty f(x)dx$ is divergent (if smaller diverges so does the larger)*

Fact: If $a > 0$ then $\int_a^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges for $p \leq 1$.

5.2 Improper Integrals Examples

Example 5.4. *Evaluate the following improper integral*

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx \\ &= \lim_{b\to\infty} \frac{-1}{x}\bigg|_1^b \\ &= \lim_{b\to\infty}\left(\frac{-1}{b} - \frac{-1}{1}\right) \\ &= 1\end{aligned}$$

Example 5.5. *Evaluate*

$$\int_0^1 \frac{1}{\sqrt[3]{x}}dx$$

There is a discontinuity at $x = 0$*, so this is of type II. Rewriting the integral with a limit:*

$$\begin{aligned}\int_0^1 x^{-1/3}dx &= \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_b^1 \\ &= \lim_{b\to 0^+}\frac{3}{2}(1 - b^{2/3}) \\ &= \frac{3}{2}\end{aligned}$$

Heres an example where it diverges (limit goes off to infinity)

Example 5.6. *Evaluate*

$$\int_1^\infty \frac{1}{x}dx$$

Rewriting

$$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \lim_{k\to\infty}\int_1^k \frac{1}{x}dx \\ &= \lim_{k\to\infty}(\ln|x|)\bigg|_1^k \\ &= \lim_{k\to\infty}(\ln|k| - \ln(1)) \\ &= \infty\end{aligned}$$

We could also apply the integral p*-test and note that* $p = 1$ *and if* $p \leq 1$ *it diverges*

Example 5.7. *Evaluate*

$$\int_{-1}^{2} \frac{1}{x^3} dx$$

We note that this is of type II, and the discontinuity occurs at $x = 0$. *So, rewriting the integral as:*

$$\int_{-1}^{2} \frac{1}{x^3} dx = \lim_{k \to 0^-} \int_{-1}^{k} \frac{1}{x^3} dx + \lim_{b \to 0^+} \int_{b}^{2} \frac{1}{x^3} dx$$

Now if we look at the second integral:

$$\begin{aligned} \lim_{b \to 0^+} \int_{b}^{2} \frac{1}{x^3} dx &= \lim_{b \to 0^+} \left[-\frac{1}{2x^2} \right]_{b}^{2} \\ &= \lim_{b \to 0^+} \left(\frac{-1}{8} + \frac{1}{2b^2} \right) \\ &= \infty \end{aligned}$$

We note that if one term diverges it all diverges, thus:

$$\int_{-1}^{2} \frac{1}{x^3} dx \quad \textit{diverges}$$