AP Calculus BC Parametric, Polar, and Vector Valued Functions

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Contents

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1 Introduction

1.1 Motivation

These notes will serve as a brief introduction to a *new* type of functions, namely parametric functions, vector valued functions, and polar functions.

Definition 1.1. *Cartesian functions studied thus far:*

$$y = f(x), \quad x = g(y)$$

Although studying these functions in relation to calculus have been helpful thus far, we want other options to model motion, or have more convenient coordinate systems to describe their locations.

These *new* functions are just a rewritten form of what we have seen before. These types of functions are often useful for describing 2-dimensional space (and later 3-d space) in a new manner. As well as being useful in Physics, such as describing circular motion or electricity and magnetism.

1.2 Conic Sections

The ancient Greeks discovered that the curves we studied in school (parabolas, circles, ellipses, and hyperbolas) are projections of a cone.

In real life, a remarkably brilliant scientist named Johannes Kepler made a groundbreaking discovery about the motion of celestial objects in our solar system. He determined that planets, moons, comets, and asteroids follow specific paths known as conic sections—ellipses, parabolas, and hyperbolas—depending on their trajectories and the gravitational forces acting on them. These conic sections became the foundation for Kepler's Three Laws of Planetary Motion, which he published in 1609.

However, while Kepler's work was based on meticulous observations and empirical data, it lacked a formal mathematical foundation to explain why these laws were true. The proof would not come until nearly 50 years later, with the development of calculus by another extraordinary mind, Sir Isaac Newton, in the late 1660s.

Newton's invention of calculus—a mathematical framework for understanding continuous change—was crucial to proving Kepler's laws. Using his newly developed tools, Newton was able to derive the laws of motion and the law of universal gravitation. He demonstrated that the gravitational force between two objects decreases with the square of the distance between them and is proportional to the product of their masses. With this principle, he showed that the elliptical orbits described by Kepler's First Law naturally arise from the influence of gravity and inertia.

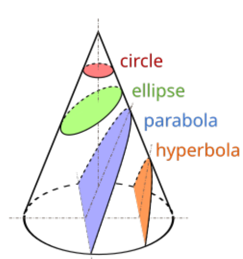


Figure 1: Conic Sections

1.2.1 Equations of Conic Sections

Definition 1.2. *Equations of the Conic Sections*

1. ***Parabola***

   A parabola is the set of points equidistant from a fixed point (focus) and a fixed line (directrix).

   Standard equations:

   - *Vertical axis of symmetry:*

   $$y = ax^2 + bx + c \quad or \quad (x-h)^2 = 4p(y-k)$$

   - *Horizontal axis of symmetry:*

   $$x = ay^2 + by + c \quad or \quad (y-k)^2 = 4p(x-h)$$

2. ***Circle***

   A circle is the set of all points equidistant from a fixed point (center).

   General equation:

   $$(x-h)^2 + (y-k)^2 = r^2$$

   where (h, k) is the center, and r is the radius.

3. ***Ellipse***

   An ellipse is the set of points where the sum of distances to two fixed points (foci) is constant.

   Standard equations:

   - *Horizontal major axis:*

   $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b$$

   - *Vertical major axis:*

   $$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \quad a > b$$

4. ***Hyperbola***

   A hyperbola is the set of points where the absolute difference of distances to two fixed points (foci) is constant.

   Standard equations:

   - *Horizontal transverse axis:*

   $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

   - *Vertical transverse axis:*

   $$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

2 Parametric Equations

2.1 Definition

Parametric equations are useful in describing motion with time, t, being the independent variable. Any cartesian or rectangular equation (in the form (x, y)) can be converted into a parametric equation by using the dummy variable t.

Definition 2.1. *Definition of a Plane Curve. If f and g are continuous functions of t on an interval Im then the equations*

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called ***parametric equations*** *and t is called the parameter.*

We will show how to convert from parametric to cartesian, and how to graph it. We should note that parametric equations always follows a time domain.

2.2 Examples

Example 2.2. *Given the parametric equations:*

$$x = t^2 - 4 \quad \text{and } y = \frac{t}{2}, \quad -2 \leq t \leq 3$$

We convert to cartesian by solving for it. One way would be:
Since

$$y = \frac{t}{2} \implies 2y = t$$

and substituting it in to x

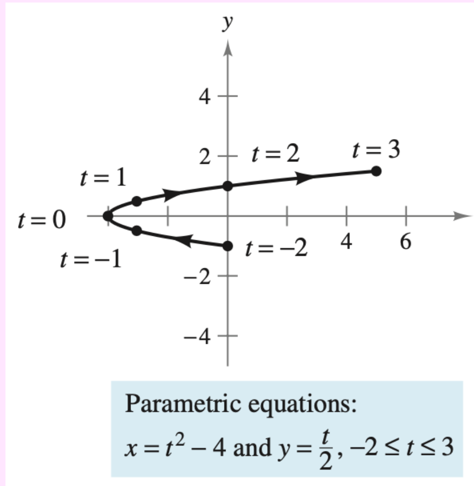
$$x = (2y)^2 - 4$$

Following the equations of a right facing parabola.

To graph, we should make a table of (x, y) values.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

Resulting in the following graph:



NOTE: From the Vertical Line Test, you can see that the graph shown in graph above does not define y as a function of x. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions.

We can also use trigonometric functions to express parametric equations,

Example 2.3. *Sketch the curve defined by*

$$x = 3\cos\theta \quad \text{and } y = 4\sin\theta, \quad 0 \le \theta \le 2\pi$$

We will first eliminate the parameter θ and find the corresponding cartesian equation.

Solve for $\cos\theta$ and $\sin\theta$ in the given equation

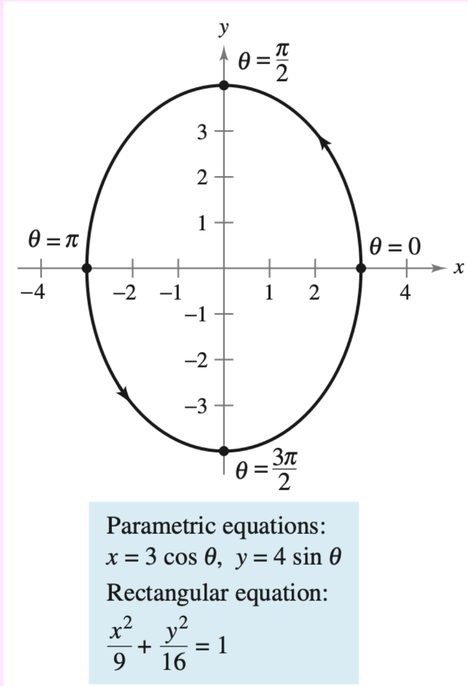
$$\cos\theta = \frac{x}{3} \quad \text{and} \quad \sin\theta = \frac{y}{4}$$

We will make use of the Pythagorean Identity.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$
$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

We can make a table and plug in nice values on the unit circle to get the (x, y) coordinate pairs.

When we do so, we get the graph:



2.3 Important Note on Parametrization

Here are some quick tips on parametrization based on what conic sections to use.

Tip 2.4. *Given a cartesian ellipse:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Its parametric form would be:

$$x = a\cos t \quad y = b\sin t$$

This graphs the ellipse starting at $(a, 0)$ *moving in the counter-clockwise direction along* $0 \le t \le 2\pi$

We can also parameterize using:

$$\begin{aligned} x = a\cos(\omega t) &\quad y = b\sin(\omega t) \\ x = a\sin(\omega t) &\quad y = b\cos(\omega t) \\ x = a\cos(\omega t) &\quad y = -b\sin(\omega t) \end{aligned}$$

Tip 2.5. *Given a cartesian circle*

$$x^2 + y^2 = r^2$$

We parameterize a circle with

$$x = r\cos t \quad y = r\sin t$$

Tip 2.6. *A general cartesian function in the form*

$$y = f(x) \quad x = h(y)$$

We parameterize this as

$$\begin{aligned} x = t \quad & y = h(t) \\ y = f(t) \quad & y = t \end{aligned}$$

2.4 Calculus with Parametric Equations

With our introduction out of the way, we can now do some fun stuff with calculus!

We begin with derivatives and tangent lines.

Theorem 2.7. *Parametric form of the Derivative:*

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \quad \frac{dx}{dt} \neq 0$$

Corollary 2.8. *The second derivative of a parametric equation is given by:*

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

Example 2.9. *For the curve given by*

$$x = \sqrt{t} \quad \text{and } y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point $(2, 3)$

Slope (first derivative):

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} = t^{3/2}$$

Concavity (second derivative):

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\frac{d}{dt}[t^{3/2}]}{dx/dt} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}} = 3t$$

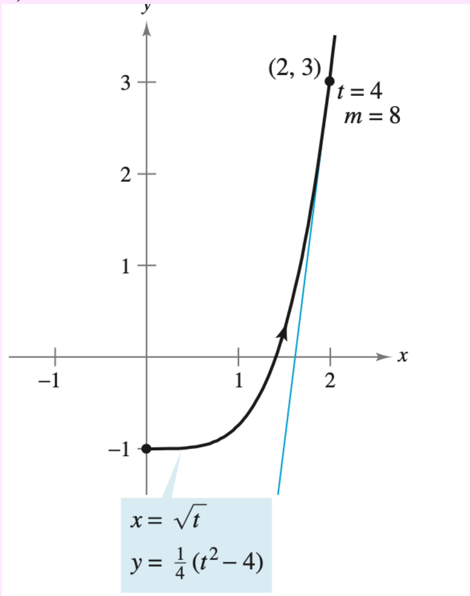
If we plug in $(x, y) = (2, 3)$ in the parametric equations, we get $t = 4$

$$\left.\frac{dy}{dx}\right|_{t=4} = (4)^{3/2} = 8$$

and

$$\left.\frac{d^2y}{dx^2}\right|_{t=4} = 3(4) = 12 > 0$$

Thus concave upward at $(2, 3)$



The graph is concave upward at (2, 3), when $t = 4$.

Next up we want to find how "long" a parametric curve is by finding its arc length.

Theorem 2.10. *If a smooth curve* C *is given by* $x = f(t)$ *and* $y = g(t)$ *such that* C *does not intersect itself on the interval* $a \leq t \leq b$*, the the arc length of* C *over the interval is given by*

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt \end{aligned}$$

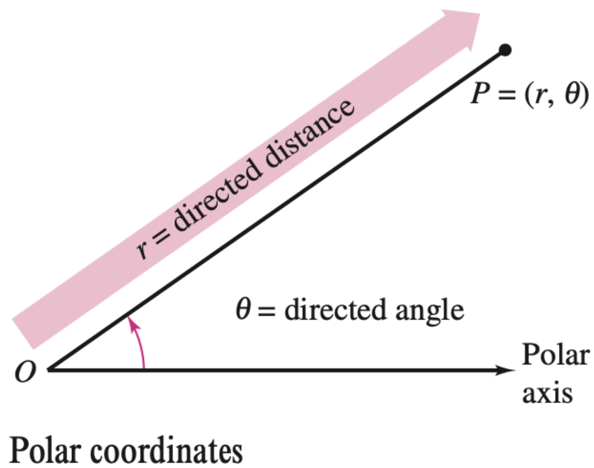
Tip 2.11. *Often times with* x *and* y *parameterized by trigonometric functions, make use of the Pythagorean identity or other trigonometric identities.*

Tip 2.12. *Know how to do this on your graphing calculator on the AP exam. They often will not make you integrate by hand, but you are expected on how to set it up.*

3 Polar Coordinates

We can think of the polar coordinate system as a coordinate system based on the unit circle.

The unit circle has radius 1, in polar coordinates they have many radii and intersected by radial lines intersected through the pole.



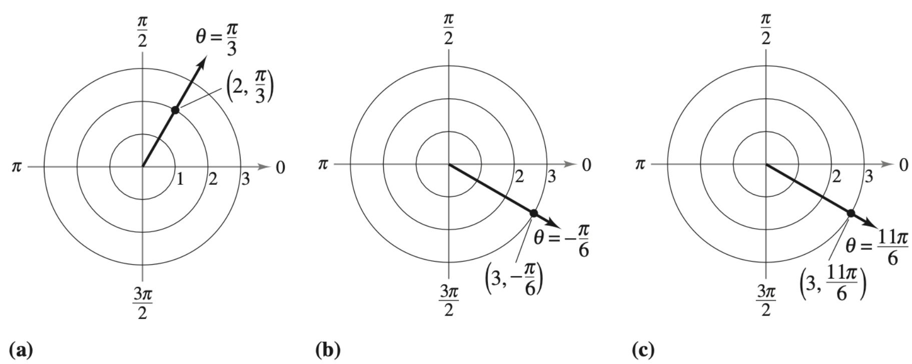


Figure 2: Polar Coordinate Examples

Theorem 3.1. *Coordinate Conversion*

The polar coordinates (r, θ) *of a point are related to the coordinates* (x, y) *of the point as follows*

1.

$$x = r\cos\theta, \quad y = r\sin\theta$$

2.

$$\tan\theta = \frac{y}{x}, \quad x^2 + y^2 = r^2$$

Example 3.2. *Convert* $(x, y) = (1, 1)$ *in cartesian to polar.*

With theorem 3.1, we have

$$\begin{aligned} x^2 + y^2 &= r^2 \\ 1^2 + 1^2 &= r^2 \implies r = \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ \tan\theta &= 1 \implies \theta = \frac{\pi}{4} \end{aligned}$$

So $(1, 1)$ *in cartesian is* $(\sqrt{2}, \frac{\pi}{4})$ *in polar*

Example 3.3. *Convert* $(r, \theta) = (2, \pi)$ *in polar to cartesian.*

With theorem 3.1, we have

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\pi = -2 \end{aligned}$$

and

$$\begin{aligned} y &= r\sin\theta \\ &= 2\sin\pi = 0 \end{aligned}$$

So $(2, \pi)$ *in polar is* $(-2, 0)$ *in cartesian*

Now, we are going to sketch polar graphs. This video is a good resource on how to do so. It can be a bit tricky at first to think in polar coordinates, so be sure to practice!

Tip 3.4. *Be sure you know how to graph polar equations on your graphing calculator. This can be done by going into* `mode` *and selecting polar, then* `Y=` *will become polar coordinates.*

3.1 Differential Calculus with Polar Equations

Theorem 3.5. *If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is*

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ)

Alternatively,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Tip 3.6. *It is easier to just memorize the one for parametric equations, and use*

$$x = r\cos\theta, \quad y = r\sin\theta$$

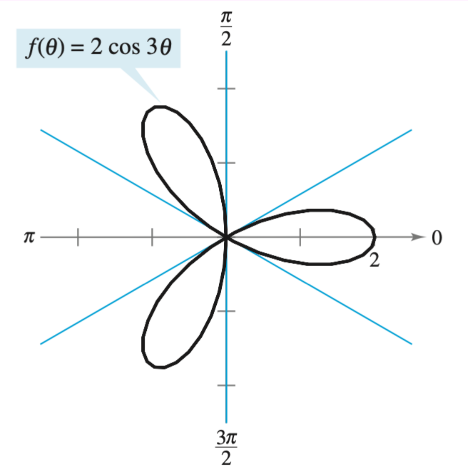
to derive it on the spot (use the product rule!!)

Theorem 3.7. *If $f(\alpha)$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ os a tangent at the pole to the graph of $r = f(\theta)$*

Example 3.8. *Given a rose curve*

$$f(\theta) = 2\cos 3\theta$$

we have that $f(\theta) = 0$ when θ is $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$. Moreover the derivative $f'(\theta) = -6\sin 3\theta \neq 0$ for these values of θ



This rose curve has three tangent lines ($\theta = \pi/6$, $\theta = \pi/2$, and $\theta = 5\pi/6$) at the pole.

3.2 Important Polar Graphs

There are four "shape" types of polar functions:

- Limaçon
- Rose Curve
- Circle
- Lemniscate

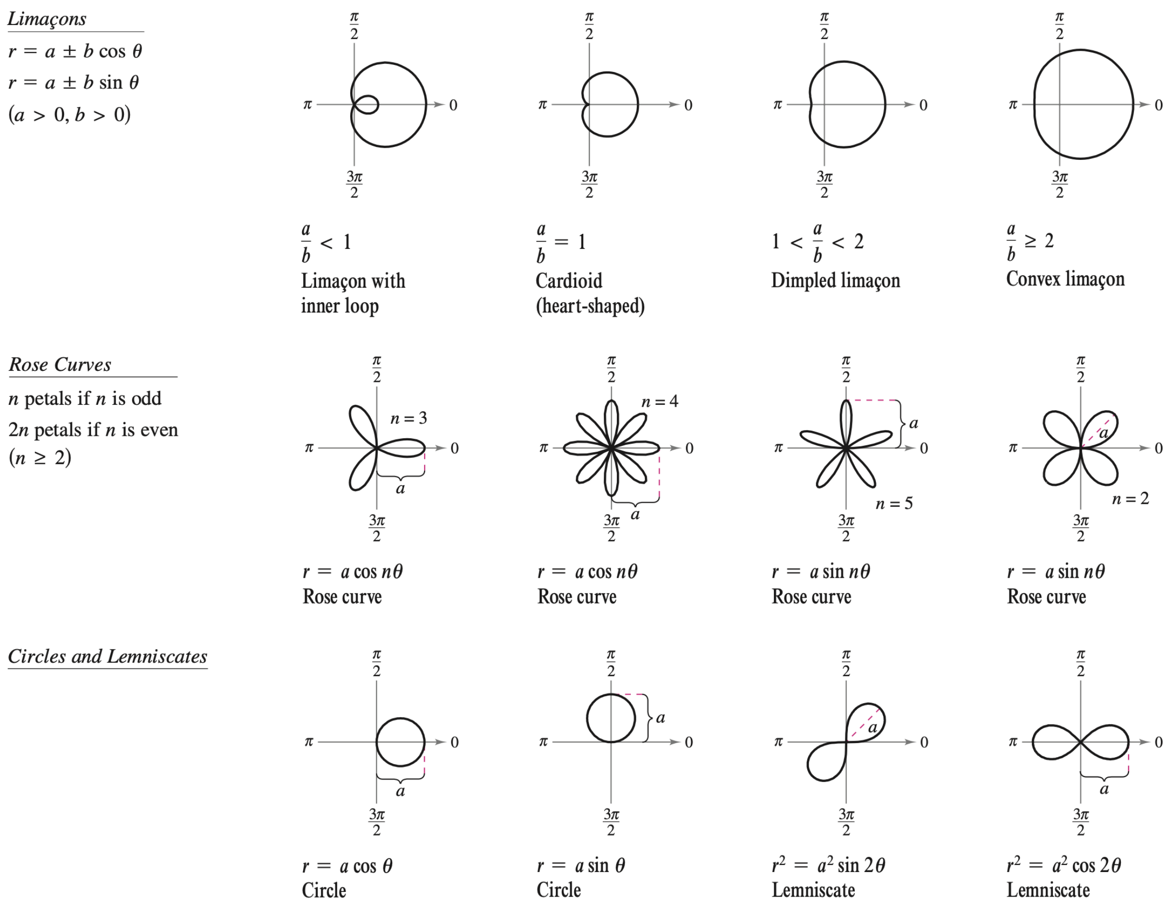


Figure 3: Special Polar Graphs

3.3 Integral Calculus with Polar Equations

This is probably one of the hardest concepts in BC, finding the area of a polar equation. The trickiest part is finding the limits of integration.

Theorem 3.9. *If f is continuous and non negative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by*

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \end{aligned}$$

This video is good at explaining how to set this up.

Example 3.10. *Find the area of one petal of the rose curve given by*

$$r = 3\cos 3\theta$$

Due to the 3θ, we take $\frac{\pi}{2}$ and divide it by 3, giving us the angles $\alpha = -\pi/6$ and $\beta = \pi/6$

So,

$$\begin{aligned} A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta &= \int_{-\pi/6}^{\pi/6} (3\cos\theta)^2 d\theta \\ &= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2} d\theta \\ &= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6} \\ &= \frac{9}{4}\left[\frac{\pi}{6} + \frac{\pi}{6}\right] \\ &= \frac{3\pi}{4} \end{aligned}$$

Tip 3.11. *Most likely, a polar area question would be the **second** FRQ question, which is a calculator question. Make sure you know how to set it up and punch it in your calculator.*

Tip 3.12. *There may also be a question asking for the area of a region between two curves. Make sure you know which one is the outer and inner.*

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_O^2 - r_I^2\right) d\theta$$

Be very careful, sometimes the area of the outer and inner are opposite. Always graph to make sure!!

4 Vector Valued Functions

Vector valued functions and parametric functions are very similar. However, this time the components are part of a vector.

Remember that a vector has both magnitude and direction, whereas a parametric equation describes a path or motion in space by expressing coordinates as functions of a parameter.

Definition 4.1. *Vector-Valued Function:*

A function in the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$$

is a vector-valued function.

Where $\mathbf{i}, \mathbf{j}$ *are the standard basis unit vectors (a vector of length 1 pointing in the* x *direction and one in the* y *direction). We write vectors with a boldface in textbooks, and we draw a little arrow on them when hand writing to denote it is a vector.*

The component functions f, g *are real-valued functions of the parameter* t

Often (and on the AP), we denote them as

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle$$

Note, here we are working in 2-dimensions, when you study multi-variable calculus, there are 3 components representing 3-dimensional space.

4.1 Calculus With Vector Valued Functions

We are going to use our calculus trio - limits, derivatives and integrals.

Definition 4.2. *Limit of a Vector-Valued Function:*

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \left[\lim_{t \to a} f(t)\right], \left[\lim_{t \to a} g(t)\right] \right\rangle$$

Definition 4.3. *Derivative of a Vector-Valued Function*

$$\begin{aligned} \frac{d}{dt}\mathbf{r}(t) &= \left\langle \frac{d}{dt} f(t), \frac{d}{dt} g(t) \right\rangle \\ &= \langle f'(t), g'(t) \rangle \end{aligned}$$

Tip 4.4. *Remember!*
- $\mathbf{r}(t)$ *is the position or displacement vector*
- $\mathbf{v}(t) = \mathbf{r}'(t)$ *is the velocity vector*
- $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ *is the acceleration vector*

Definition 4.5. *Integral of a Vector-Valued Function*

$$\begin{aligned}\mathbf{r}(t) = \int \mathbf{v}(t)dt &= \left\langle \int f'(t)dt, \int g'(t)dt \right\rangle \\ &= \langle f(t), g(t) \rangle + \langle c_1, c_2 \rangle\end{aligned}$$

Where the $\langle c_1, c_2 \rangle$ *is given by the initial conditions*