AP Calculus BC Taylor and Maclaurin Series

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Mr. Harry Yang*

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*harry.yang@compscihigh.org

1 Introduction

1.1 Motivation

After studying all the series convergence tests, we will finally see how it all comes together.

The Taylor series is a powerful mathematical tool that allows us to approximate complex functions as infinite sums of polynomial terms. The motivation for the Taylor series comes from the desire to simplify non-linear functions into forms that are easier to compute and analyze. By expressing a function as a sum of its derivatives evaluated at a specific point, the Taylor series provides a locally accurate approximation of the function. This method is particularly useful when the exact evaluation of a function is computationally intensive or impossible. For functions that are infinitely differentiable, the Taylor series offers a way to represent them exactly if the entire infinite series is used.

In physics, Taylor series are used extensively to simplify problems and find approximate solutions. Many physical phenomena involve functions that are difficult or impossible to solve analytically, such as the motion of particles, oscillatory systems, or the behavior of fields in quantum mechanics and electromagnetism. For example, in classical mechanics, the potential energy near an equilibrium point is often approximated using a second-order Taylor expansion, leading to the widely-used harmonic oscillator model. Similarly, in thermodynamics and statistical mechanics, Taylor expansions are used to approximate state equations near critical points.

Computers use Taylor series (or closely related expansions to polynomial techniques) to compute transcendental functions such as $\sin(x), \cos(x), \ln(x)$, and e^x . These functions do not have closed-form algebraic solutions and must be approximated numerically. The Taylor series provides a framework to compute these values efficiently by breaking them down into polynomial terms, which are easy for computers to handle.

1.2 Extension of Linear Approximations

Taylor polynomials can be thought of as natural extensions of linear tangent line approximations, designed to provide increasingly accurate representations of a function near a specific point. A linear tangent line approximation, $f(x) \approx f(a) + f'(a)(x - a)$, uses only the function's value and its first derivative at $x = a$, effectively describing the slope and intercept of the function near that point. However, this linear model lacks information about the function's curvature and higher-order behavior. By incorporating higher-order derivatives, Taylor polynomials account for the function's concavity (second derivative), rate of change of concavity (third derivative), and so on.

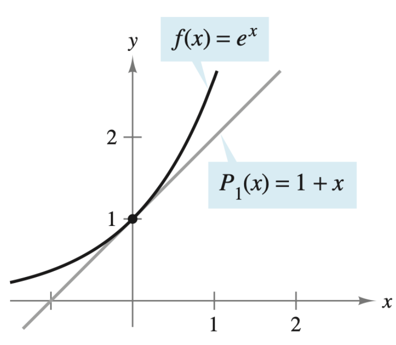

Figure 1: First order tangent line approximation of e^x at $x = 0$

2 Power Series

Let's begin with the definition of a power series

Definition 2.1. *Power Series*

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

is called a power series. More generally, they are in the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n + \ldots$$

is called a power series centered at c, where c is a constant.

Remark 2.2. *A power series is just a fancy way of saying geometric series, this time with a variable x.*

Just as with geometric series, a power series converges for $|x| < 1$

And just as a geometric series, it converges to

$$\frac{a_0}{1-x}$$

3 Taylor Series

This section may seem confusing at first, so be sure to read it slowly and many times until you get it. If needed, make a cup of tea and come back to it.

3.1 Taylor Series Definition

Definition 3.1. *Definition of nth Taylor Polynomial and nth Maclaurin Polynomial If a function f has derivatives at a point c, then the polynomial*

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor polynomial for f at c. If $c = 0$, then the polynomial

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the nth Maclaurin polynomial for f.

Remark 3.2. *A Taylor series is the more general form. The Maclaurin series is just centered at the origin.*
The reason why we have these two terms is that Taylor is from England, and Maclaurin is from Scotland, and as always the English have beef with the Scottish.

Corollary 3.3. *The general Taylor series for a function $f(x)$ centered at $x = c$ is given by:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

In this series, the 0-th derivative of f is simply the original function, i.e., $f^{(0)}(x) = f(x)$.

Remark 3.4. *A Taylor polynomial terminates, it goes up to a power or degree. Whereas a Taylor Series is an infinite series.*

Example 3.5. *Find the 3rd-order Taylor polynomial of $\tan(x)$ centered at $x = \frac{\pi}{4}$.*
The general form of the Taylor polynomial is:

$$P_3(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3$$

Now, we compute the necessary derivatives of $f(x) = \tan(x)$.
1st derivative:

$$f'(x) = \sec^2(x).$$

Evaluating at $x = \frac{\pi}{4}$:

$$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2.$$

2nd derivative:

$$f''(x) = 2\sec^2(x)\tan(x).$$

Evaluating at $x = \frac{\pi}{4}$:

$$f''\left(\frac{\pi}{4}\right) = 2 \cdot 2 \cdot 1 = 4.$$

3rd derivative:

$$f^{(3)}(x) = 2\sec^4(x) + 4\sec^2(x)\tan^2(x)$$

Evaluating at $x = \frac{\pi}{4}$:

$$f^{(3)}\left(\frac{\pi}{4}\right) = 8 + 8 = 16$$

Now, we can construct the 4th-order Taylor polynomial:

$$P_3(x) = \tan\left(\frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

Simplifying each term:

$$P_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

Tip 3.6. *See for yourself by graphing* $\tan(x)$ *on desmos and inputting the Taylor polynomial and look at what happens at* $\pi/4$.

Tip 3.7. *The AP Exam may ask what the coefficient is for a certain power. Know which derivative to take, and what factorial to divide by.*

3.2 Interval of Convergence and Radius of Convergence

This is just an application of the ratio test. Here we want to know for what values of x is the series convergent

Theorem 3.8. *Convergence of a Power Series*

For a power series centered at c*, precisely one of the following is true:*

1. *The series converges only at* c.
2. *There exists a real number* $R > 0$ *such that the series converges absolutely for* $|x-c| < R$*, and diverges for* $|x-c| > R$.
3. *The series converges absolutely for all* x.

The number R *is called the* ***radius of convergence*** *of the power series. If the series converges only at* c*, the radius of convergence is* $R = 0$*. If the series converges for all* x*, the radius of convergence is* $R = \infty$.
The set of all values of x *for which the power series converges is called the* ***interval of convergence*** *of the power series.*

Example 3.9. *Find the radius of convergence of*

$$\sum_{n=0}^{\infty} 3(x-2)^n$$

For $x \neq 2$*, let* $a_n = 3(x-2)^n$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| \\ &= \lim_{n\to\infty}|x-2| \\ &= |x-2|\end{aligned}$$

By the ratio test, the series converges for $|x-2| < 1$ *and diverges if* $|x-2| > 1$*. Thus the radius of convergence is* $R = 1$

Example 3.10. *Find the radius of convergence of*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let $a_n = \frac{x^n}{n!}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}n!}{x^n(n+1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\ &= 0,\end{aligned}$$

since $n+1 \to \infty$ *as* $n \to \infty$.
By the ratio test, the series converges for all x. *Thus, the radius of convergence is* $R = \infty$.

Example 3.11. *Determine the Interval of Convergence for:*

$$\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n$$

We can do this one of two ways: ratio test, or using the fact that this is a power series. We are going to use the latter.

Recall that a geometric series converges for $|r| < 1$. *The* r *here is whats in the sum. So its interval of convergence is determined by*

$$\left|\frac{x-3}{2}\right| < 1$$

$$-1 < \frac{x-3}{2} < 1$$

$$-2 < x-3 < 2$$

$$1 < x < 5$$

But we are not done just yet. ***CHECK THE ENDPOINTS***

At $x = 1$*, we plug in*

$$\sum_{n=0}^{\infty}\left(\frac{1-3}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n$$

By nTT *or GST, this diverges*

At $x = 5$*, we plug in*

$$\sum_{n=0}^{\infty}\left(\frac{5-3}{2}\right)^n = \sum_{n=0}^{\infty}(1)^n$$

By nTT *or GST, this diverges*

Thus the Interval of Convergence for the series is $1 < x < 5$

Question 3.12. *What function does the series above converge to within its interval of convergence?*

Example 3.13. *Determine the Interval of Convergence for:*

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Using the ratio test, with $a_n = x^n/n$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)} \frac{n}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{nx}{n+1} \right| \\
&= |x|
\end{aligned}$$

So, by the ratio test, the radius of convergence is $R = 1$. *And we have to check the interval* $-1 < x < 1$

At $x = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is the convergent alternating harmonic series

At $x = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is the divergent harmonic series.

Thus, the interval of convergence is $-1 \leq x < 1$

Tip 3.14.

$$|x| < k \iff -k < x < k$$

Tip 3.15. *We only care about the variable* n *when taking the limit. Leave* x *alone.*

Tip 3.16. ***ALWAYS*** *check the endpoints of the interval of convergence.*

3.3 Functions Represented by Taylor Series

3.3.1 Important Functions

Here are some important Taylor Series, that I'd highly recommend you memorize.

Definition 3.17. *Definition of Power Series*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$
$$= \sum_{n=0}^{\infty} x^n$$

Interval of Convergence: $-1 < x < 1$

Definition 3.18. *Definition of* e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Interval of Convergence: $-\infty < x < \infty$

Definition 3.19. *Definition of* $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Interval of Convergence: $-\infty < x < \infty$

Definition 3.20. *Definition of* $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Interval of Convergence: $-\infty < x < \infty$

Tip 3.21. *Here's how to remember the trig Taylor Series*

- $\cos x$ *is an **even** function, and only has **even** powers.*
- $\sin x$ *is an **odd** function, and only has **odd** powers.*

Definition 3.22. *Definition of* $\ln(1+x)$

$$\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}\end{aligned}$$

Interval of Convergence: $-1 < x \leq 1$

Definition 3.23. *Definition of* $\arctan x$

$$\begin{aligned}\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n+1}}{2n+1}\end{aligned}$$

Interval of Convergence: $-1 \leq x \leq 1$

Question 3.24. *Verify for yourself that by Taylor Series:*

$$\frac{d}{dx}(\sin x) = \cos x$$

and

$$\int \cos x \; dx = \sin x + C$$

Question 3.25. *This was an extra credit problem on a college calculus exam.*

Given $i^2 = -1$. *Show through Taylor Series that:*

$$e^{ix} = \cos x + i \sin x$$

Also show that $e^{i\pi} + 1 = 0$

4 Manipulating Series

Given all the series definitions from the previous section, we want to know how to manipulate these series into other series of functions.

4.1 Examples

Example 4.1. *Write the Taylor Series for:*

$$f(x) = e^{-x^2}$$

centered at $c = 0$

We can do this in one of two ways:

1. *Take derivatives and use the Taylor series formula, such as in example 3.5.*
2. *Manipulate an already known series.*

We will do the second option.

Given this already known Taylor Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

We just swap out the x *to* $(-x^2)$*:*

$$\begin{aligned} e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^{2n})}{n!} \\ &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \end{aligned}$$

Example 4.2. *Find the Power Series representation for*

$$f(x) = \frac{1}{(1-x)^2}$$

We observe that:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

Thus, in series form

$$\begin{aligned}\frac{d}{dx}\sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty}\frac{d}{dx}(x^n)\\ &= \sum_{n=0}^{\infty} nx^{n-1}\\ &= 1 + 2x + 3x^2 + \dots\\ &= \frac{1}{(1-x)^2}\end{aligned}$$

Remark 4.3. *Differentiating and integrating a Power/Taylor series maintains the same radius of convergence. But it may change end point behavior in the interval of convergence.*

Example 4.4. *Find the Maclaurin Series for:*

$$f(x) = \arctan x$$

Once again, we can do this by using Taylor's formula (long and tedious) or manipulate already known series (easy). We will do the easy option

We know that:

$$\int \frac{1}{1+x^2}\, dx = \arctan x + C$$

and we can rewrite the integrand as

$$\frac{1}{1-(-x^2)}$$

which looks similar to the sum of the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$*. Here, we switch the* x *to a* $(-x^2)$*. Thus,*

$$\begin{aligned}\frac{1}{1-(-x^2)} &= \sum_{n=0}^{\infty}(-x^2)^n\\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n}\\ &= \frac{1}{1+x^2}\end{aligned}$$

Integrating term-by-term:

$$\begin{aligned}
\int \frac{1}{1+x^2}\,dx &= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx \\
&= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \qquad \text{'reverse' power rule.} \\
&= \arctan x + C
\end{aligned}$$

Which is the result of definition 3.23

Tip 4.5. *If you are not comfortable doing this with series sigma summation notation, try writing out the terms of the series, and integrating/differentiating them one-by-one.*

5 Error Estimation

We want to know how far off we are from out Taylor estimation compared to the actual function. This is done through the following:

5.1 Lagrange Error Bound

Before we begin this section on Lagrange Error bound, I want to note that this is something a lot of students struggle with. Also make note, the AP rarely has this error bound question on the exam.

Lets begin with the remainder estimation theorem

Theorem 5.1. *The Remainder Estimation Theorem (Lagrange Error Bound)*

If there is a positive constant M such that $|f^{n+1}(t)| \leq M$ for all t between x and c inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-c|^{n+1}}{(n+1)!}$$

If this inequality holds for every n and other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$

Let's see an example.

Example 5.2. *Use the Lagrange error bound to estimate the error in using a 4th degree Maclaurin polynomial to approximate* $\cos(\frac{\pi}{4})$

Begin by writing the degree 4 Maclaurin polynomial of $\cos x$

$$T(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

For the error bound we need to find the $(n+1)$*-th derivative, here the 5th derivative of* $\cos x$*, we note that* $f^{(5)}(x) = -\sin x$

We need to find the upper bound M*, the largest that* $|-\sin x|$ *could be is 1. So* $M = 1$*, plugging into the error bound formula with* $n = 4$

$$\begin{aligned} Error &\leq \frac{M}{(n+1)!}(x-c)^{n+1} && (1)\\ &\leq \frac{1}{5!}(\frac{\pi}{4} - 0)^5 && (2)\\ &= \frac{(\pi/4)^5}{120} && (3)\\ &\approx 0.00249 && (4) \end{aligned}$$

Tip 5.3. *On the AP exam, this is usually on the no calculator portion, so line 3 will do.*

Tip 5.4. *Finding* M *is the hardest part, we may need to the first derivative test to find the max on the interval*

5.2 Alternating Series Error and Remainder

This is an easier and specific form of error estimation, only for alternating series, and is often asked on the AP exam.

Theorem 5.5. *Alternating Series Remainder*

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *be an alternating series where:*

1. $a_n > 0$ *for all* n*,*
2. $a_{n+1} \leq a_n$ *for all* n *(i.e., the terms* a_n *are non-increasing),*
3. $\lim_{n\to\infty} a_n = 0$.

Then the series converges, and the absolute error (remainder) R_N *after* N *terms is bounded by the next term in the series:*

$$|R_N| = |S - S_N| \leq a_{N+1},$$

where S *is the sum of the series and* S_N *is the partial sum after* N *terms.*

Tip 5.6. *Just look at the next term, and plug in to get the error for an alternating series.*

Example 5.7. *Use the first 3 terms of the alternating series expansion of e^{-x} at $x = 1$ to approximate e^{-1}. Estimate the error using the Alternating Series Remainder theorem.*

Begin with writing the first three terms

$$e^{-x} = 1 - x + \frac{x^2}{2!}$$

Substitute $x = 1$ to approximate the first 3 terms

$$S_3 = 1 - 1 + \frac{1^2}{2!} = \frac{1}{2}$$

Now applying the Alternating Series Error, write the next term

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

The error R_n is bounded by the magnitude of the next term in the series

$$R_3 \leq \left|\frac{1^3}{3!}\right| = \frac{1}{6}$$

The error in this approximation is at most

$$R_3 \leq \frac{1}{6}$$

Thus e^{-1} is within $\frac{1}{6}$ of $\frac{1}{2}$.

Tip 5.8. *Always write one extra term, but box in the terms/degree it must go up to.*