UC Davis Math Club Integration Bee 2024

University of California, Davis

January 18, 2024

Contents I

Problem 1

$$\int \frac{2024e^x}{e^{2x} + \pi} dx$$

Problem 1 Solution

We have that

$$\int \frac{1}{u^2 + a^2} du = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + c$$

So,

$$\int \frac{2024e^x}{e^{2x} + \pi} dx = \int \frac{2024e^x}{(e^x)^2 + (\sqrt{\pi})^2} dx$$

$$\boxed{\frac{2024}{\sqrt{\pi}} \arctan\left(\frac{e^x}{\sqrt{\pi}}\right) + C}$$

Problem 2

$$\int \frac{\sqrt{x}}{1+x}dx$$

Problem 2 Solution

Let $u = \sqrt{x} \implies du = \frac{1}{2\sqrt{x}}dx$ Applying the substitution:

$$\int \frac{u}{1+x} \cdot 2\sqrt{x}du = 2\int \frac{u^2}{1+x}dx$$

Since $u = \sqrt{x} \implies u^2 + 1 = x + 1$
So,

$$2\int \frac{u^2}{u^2+1}du = 2\int 1 - \frac{1}{u^2+1}$$

$$2u - \arctan(u)$$

$$\boxed{2\sqrt{x} - \arctan(\sqrt{x}) + C}$$

Problem $e = 2.71828\ldots$

$$\int_1^{\int_1^{\int_1^{\vdots} 2x\,dx} 2x\,dx} 2x\,dx$$

Problem e Solution

We define the integral as and use a sequential definition

$$a_1 = I(t) = \int_1^t 2x dx = t^2 - 1$$

$$a_2 = I(I(t)) = \int_1^{\int_1^t 2x dx} 2x dx = \int_1^{t^2-1} 2x dx$$

$$\vdots$$

$$a_n(t) = I^n(t) = \int_1^{a_{n-1}(t)} 2x dx = (a_{n-1}(t))^2 - 1$$

We need to analyze the following:

$$\lim_{n\to\infty} a_n(t) = \lim_{n\to\infty} I^n(t) = \lim_{n\to\infty} ((a_{n-1}(t))^2 - 1)$$

Problem e Solution Continued

Ansatz of integral:

$$I = \int_1^I 2x dx = x^2\Big|_1^I$$

$$I = I^2 - 1 \implies I^2 - I - 1 = 0$$

$$I = \frac{1 \pm \sqrt{5}}{2} = \varphi, \bar{\varphi} \quad \textbf{Golden ratio (1.618...) and its conjugate}$$

But we have one integral with two solutions? Plugging this into the limit

$$a_1(\varphi) = I(\varphi) = \int_1^\varphi 2x dx = \varphi^2 - 1$$

Note

$$\varphi^2 - \varphi - 1 = 0 \implies \varphi^2 - 1 = \varphi$$

Thus $I(\varphi) = \varphi$

$$a_2(\varphi) = I(I(\varphi)) = I(\varphi) = \varphi$$

Thus

$$a_n(\varphi) = \varphi$$

$$\boxed{\lim_{n\to\infty} a_n(\varphi) = \varphi}$$

Problem e Solution Continued

We also have that $-\varphi$ would work as well. In addition, $\bar{\varphi}$ also satisfies $\bar{\varphi}^2 - 1 = \bar{\varphi}$, as well as $-\bar{\varphi}$.
We have convergence for the fixed points, φ and $\bar{\varphi}$, and "eventual points" $-\varphi$ and $-\bar{\varphi}$. This provides multiple solutions for which a fixed point exists:

$$t^2 - 1 = \varphi \implies t = \pm\sqrt{1+\varphi}$$

Now, for divergence:

$$|t| > \varphi$$

$$\lim_{n\to\infty} I^n(t) \to \infty$$

There are also points $t \in (-\varphi, \varphi)$ that also diverge, like $t = 0$ (providing an oscillatory sequence between -1 and 0).

Problem 3

$$\int \sqrt{e^x - 1}dx$$

Problem 3 Solution

Let $u = \sqrt{e^x - 1} \implies du = \frac{e^x}{2\sqrt{e^x-1}}dx \implies dx = \frac{2u}{e^x}du$ Applying the substitution:

$$2\int \frac{u^2}{e^x}du$$

$$u = \sqrt{e^x - 1} \implies u^2 = e^x - 1 \implies e^x = u^2 + 1$$

So,

$$2\int \frac{u^2}{u^2+1}du$$

$$2\int \frac{u^2+1-1}{u^2+1}du = 2\int 1 - \frac{1}{u^2+1}du$$

$$2u - 2\arctan(u)$$

$$\boxed{2\sqrt{e^x - 1} - 2\arctan(\sqrt{e^x - 1}) + C}$$

Problem 4

$$\int \sqrt{(x+3)(x+2)(x+4)(x+1)+\sin^2 x+\cos^2 x}dx$$

Problem 4 Solution

FOIL and use the Pythagorean identity:

$$\int \sqrt{(x^2+5x+6)(x^2+5x+4)+1}dx$$

Let $\alpha = x^2 + 5x$

$$\int \sqrt{(\alpha+6)(\alpha+4)+1}dx = \int \sqrt{(\alpha^2+10\alpha+24)+1}$$

$$= \int \sqrt{\alpha^2+10\alpha+25}$$

This is a perfect square

$$\int \sqrt{(\alpha+5)^2}dx = \int \alpha + 5dx = \int x^2+5x+5dx$$

$$\boxed{\frac{x^3}{3} + \frac{5x^2}{2} + 5x + C}$$

Problem 5

$$\int_0^\infty \cos(x^2)dx$$

Problem 5 Solution

By Euler's Formula: $e^{ix} = \cos x + i \sin x$, $e^{ix^2} = \cos x^2 + i \sin x^2$. With the real part, $\cos x^2 = e^{ix^2}$. This provides the following:

$$\int_0^\infty \cos(x^2) dx = \int_0^\infty e^{ix^2} dx = \int_0^\infty e^{(i\sqrt{x})^2} dx$$

We have that $\sqrt{i} = a + bi$, then $i = a^2 - b^2 + 2abi$. We see that $a = b$

$$(a, b) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \implies \sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

Thus,

$$\int_0^\infty e^{(i\sqrt{x})^2} dx = \int_0^\infty e^{((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)x)^2} dx$$

Problem 5 Solution Continued

We make the substitution

$$iu = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}ix\right) \implies u = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}ix\right) \implies \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) du = dx$$

$$\int_0^\infty e^{((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)x)^2} \to \lim_{n\to\infty} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \int_0^{\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)n} e^{(iu)^2} du$$

$$\lim_{n\to\infty} \frac{\sqrt{2}}{2}(1+i) \int_0^{\frac{\sqrt{2}}{2}(1-i)n} e^{-u^2} du$$

Since this integral is entirely real, we take the real part and reduce it down to the Gaussian Integral.

$$\text{Re}\left(\lim_{n\to\infty} \frac{\sqrt{2}}{2}(1+i) \int_0^{\frac{\sqrt{2}}{2}(1-i)n} e^{-u^2} du\right) = \frac{\sqrt{2}}{2} \int_0^\infty e^{-u^2} du = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{\pi}}{2}\right)$$

$$= \boxed{\frac{\sqrt{2\pi}}{4}}$$

Problem 6

$$\int x^{\frac{1}{\ln x}}\,dx$$

Problem 6 Solution

Note

$$e^{\ln x} = x$$

$$\int \left(e^{\ln x}\right)^{\frac{1}{\ln x}} dx$$

$$\int e dx = \boxed{ex + c}$$

Problem 7

$$\int x^x(\ln(x)+1)dx$$

Problem 7 Solution

Note the derivative of x^x is the following $(x^x)' = x^x(\ln(x) + 1)$

So the FTC yields the following:

$$\int x^x(\ln(x) + 1)dx = \boxed{x^x + C}$$

Problem 8

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}dx$$

Problem 8 Solution

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \cdot \sqrt{\frac{1+x}{1+x}} dx = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} dx$$

$$= \arcsin x - \sqrt{1-x^2}\Big|_{-1}^{1} = \boxed{\pi}$$

Problem 9

$$\int_0^1 \ln(x)\ln(1-x)dx$$

Problem 9 Solution

Using the Taylor series for $\ln(1-x)$, we obtain

$$-\int_0^1 \ln x \sum_{n=1}^{\infty} \frac{x^n}{n} dx = -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln x dx$$

Let

$$I_n = \int_0^1 x^n \ln x dx \xRightarrow{\text{IBP}} I_n = -\frac{1}{(n+1)^2}$$

Thus,

$$\int_0^1 \ln(x) \ln(1-x) dx = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=2}^{\infty} \frac{1}{(n-1)n^2} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) - \frac{1}{n^2}$$

$$\sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) - \sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{1}{2-1} - \left(\frac{\pi^2}{6} - 1\right) = \boxed{2 - \frac{\pi^2}{6} = 2 - \zeta(2)}$$

Problem 10 (from 1980 Putnam A3)

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^{\sqrt{2}} x}$$

Problem 10 Solution

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \left(\frac{\sin x}{\cos x}\right)^{\sqrt{2}}} = \int_0^{\frac{\pi}{2}} \frac{\cos^{\sqrt{2}} x \; dx}{\cos^{\sqrt{2}} x + \sin^{\sqrt{2}} x}$$

$$\xrightarrow[\textbf{phase shift}]{x \mapsto \frac{\pi}{2} - x} \int_0^{\frac{\pi}{2}} \frac{\sin^{\sqrt{2}} x \; dx}{\sin^{\sqrt{2}} x + \cos^{\sqrt{2}} x} = I$$

Adding the two,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\sqrt{2}} x + \cos^{\sqrt{2}} x \; dx}{\sin^{\sqrt{2}} x + \cos^{\sqrt{2}} x} = \int_0^{\frac{\pi}{2}} dx$$

$$2I = \frac{\pi}{2} \implies \boxed{I = \frac{\pi}{4}}$$

Problem 🤓

$$\int_0^\infty e^{-x^2} \cos\left(\begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}\right) dx$$

Problem 🤓Solution

We note that for any matrix $A = \begin{bmatrix} 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}$

$$\cos A = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{(2n)!} \quad A^0 = I_3, \quad A^2 = A \cdot A = \begin{bmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A^{2n} = A^4 = O_3 \ \textbf{(0 matrix), for } n \geq 2$$

$$\cos A = I_3 - \frac{1}{2} A^2 \implies \cos A = \begin{bmatrix} 1 & 0 & -\frac{1}{2}x^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So,

$$\int_0^{\infty} e^{-x^2} \cos A dx = \int_0^{\infty} \begin{bmatrix} e^{-x^2} & 0 & -\frac{1}{2}x^2 e^{-x^2} \\ 0 & e^{-x^2} & 0 \\ 0 & 0 & e^{-x^2} \end{bmatrix} dx$$

Problem 🤓Solution Continued

The integral of a matrix is a matrix of integrals.

$$\begin{bmatrix} \int_0^\infty e^{-x^2}dx & \int_0^\infty 0dx & \int_0^\infty -\frac{1}{2}x^2e^{-x^2}dx \\ \int_0^\infty 0dx & \int_0^\infty e^{-x^2}dx & \int_0^\infty 0dx \\ \int_0^\infty 0dx & \int_0^\infty 0dx & \int_0^\infty e^{-x^2}dx \end{bmatrix}$$

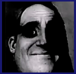
The diagonals of the matrix are the Gaussian integrals: $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

$$\int_0^\infty x^2e^{-x^2}dx \xRightarrow[u=x^2 \quad x=\sqrt{u}]{du=\frac{1}{2\sqrt{u}}du} \frac{1}{2}\int_0^\infty ue^{-u}u^{-1/2}du = \int_0^\infty u^{1/2}e^{-u}du = \Gamma\left(\frac{1}{2}+1\right)$$

$$*\Gamma(z+1), = z\Gamma(z) \quad \frac{1}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}$$

$$\boxed{\frac{\sqrt{\pi}}{2}\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Problem



$$\int_0^1 \frac{x^2 \cosh(\arctan(x))}{\sqrt{1+x^2}}\,dx$$

Problem  Solution

Symmetry: $\cosh x, x^2, 1+x^2$ are even

$$\frac{1}{2}\int_{-1}^{1} \frac{x^2 \cosh(\arctan(x))}{\sqrt{1+x^2}}\, dx$$

$* \cosh(x) = \frac{e^x + e^{-x}}{2}$

$$\Longrightarrow \frac{1}{4}\left(\underbrace{\int_{-1}^{1} \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}}\, dx}_{I_1} + \underbrace{\int_{-1}^{1} \frac{x^2 e^{-\arctan x}}{\sqrt{1+x^2}}\, dx}_{I_2} \right)$$

For I_2, make the substitution $x \mapsto -x$

$$I_2 = \int_{+1}^{-1} \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}} (-dx) \implies I_2 = I_1$$

Problem  Solution Continued

We now have that

$$\stackrel{\text{☝}}{\Longrightarrow} \frac{1}{4}\cdot 2\int_{-1}^{1}\frac{x^2e^{\arctan x}}{\sqrt{1+x^2}}dx = \frac{1}{2}\int_{-1}^{1}\frac{x^2e^{\arctan x}}{\sqrt{1+x^2}}dx$$

Applying the substitution: $x = \tan u \quad dx = \sec^2 u du$

$$I = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\frac{\tan^2 u e^u}{\cancelto{\sec u}{\sqrt{1+\tan^2 u}}}\quad \sec^{\cancelto{1}{2}} u du$$

$$I = \frac{1}{2}\int_{-\pi/4}^{\pi/4} e^u \tan^2 u \sec u du$$

$$\xRightarrow{\textbf{IBP}} \frac{1}{2}\int_{-\pi/4}^{\pi/4} e^u \tan u d(\sec u)$$

$$\frac{1}{2}\left((e^u \tan u \sec u)\Big|_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4} \sec u(e^u \tan u + e^u \sec^2 u)du \right)$$

Problem  Solution Continued

$$\frac{1}{2}\left((e^{\pi/4}+e^{-\pi/4})\sqrt{2}-\int_{-\pi/4}^{\pi/4}e^{u}\,d(\sec u)-\underbrace{\boxed{\int_{-\pi/4}^{\pi/4}e^{u}\sec^{3}u\,du}}_{I_3}\right)$$

$$I_3=\boxed{\int_{-\pi/4}^{\pi/4}e^{u}\sec^{3}u\,du}_{I_3}=\int_{-\pi/4}^{\pi/4}e^{u}(1+\tan^{2}u)\sec u\,du$$

$$=\int_{-\pi/4}^{\pi/4}e^{u}\sec u\,du+\underbrace{\int_{-\pi/4}^{\pi/4}e^{u}\tan^{2}u\sec u\,du}_{=2I}$$

$$I_3=e^{u}\sec u\Big|_{-\pi/4}^{\pi/4}\cancel{-\int_{-\pi/4}^{\pi/4}e^{u}d(\sec u)}+2I$$

$$\frac{1}{2}\left\{(e^{\pi/4}+e^{-\pi/4})\sqrt{2}\cancel{-\int_{-\pi/4}^{\pi/4}e^{u}d(\sec u)}-\int_{\pi/4}^{\pi/4}e^{u}\sec^{3}u\,du\right\}$$

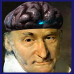
Problem  Solution Continued

$$I = \frac{1}{2}\left\{(e^{\pi/4} + e^{-\pi/4})\sqrt{2} - (e^{\pi/4} - e^{-\pi/4})\sqrt{2} - (e^{\pi/4} - e^{-\pi/4})\sqrt{2} - 2I\right\}$$

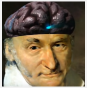
$$\implies 2I = \frac{1}{2}\{\cancel{(e^{\pi/4} + e^{-\pi/4})\sqrt{2}} - \cancel{(e^{\pi/4} - e^{-\pi/4})\sqrt{2}}\}$$

$$\implies I = \frac{1}{4} \cdot 2\sqrt{2}e^{-\pi/4}$$

$$\implies I = \boxed{\frac{\sqrt{2}}{2}e^{-\pi/4}}$$

Problem 

$$\int \sqrt{\tan x} dx$$

Problem  Solution

Let $u = \sqrt{\tan x}$, $\quad u^2 = \tan x \quad 2udu = \sec^2 x dx \implies \frac{2udu}{\sec^2 x} = dx$

$$\int u \frac{2u}{\sec^2 x} du$$

We also have that $(u^2)^2 = u^4 = \tan^2 x = \sec^2 x - 1 \implies u^4 + 1$

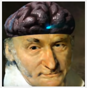
$$\int \frac{2u^2}{u^4+1} du = \int \frac{2u^2}{u^4+1} \cdot \frac{\frac{1}{u^2}}{\frac{1}{u^2}} du. = \int \frac{2}{u^2 + \frac{1}{u^2}} du$$

Note by CTS

$$u^2 + \frac{1}{u^2} \boxed{+\frac{2u}{u} - \frac{2u}{u}}$$

$$\implies \left(u^2 + \frac{2u}{u} + \frac{1}{u^2}\right) - \frac{2u}{u} = \boxed{\left(u + \frac{1}{u}\right)^2 - 2}_a$$

$$\implies \left(u^2 - \frac{2u}{u} + \frac{1}{u^2}\right) + \frac{2u}{u} = \boxed{\left(u - \frac{1}{u}\right)^2 + 2}$$

Problem  Solution Continued

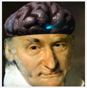
$$\int \frac{1+\frac{1}{u^2}-\frac{1}{u^2}}{u^2+\frac{1}{u^2}}du = \int \frac{1-\frac{1}{u^2}+1+\frac{1}{u^2}}{u^2+\frac{1}{u^2}}du$$

$$= \int \frac{1-\frac{1}{u^2}}{u^2+\frac{1}{u^2}}du + \int \frac{1+\frac{1}{u^2}}{u^2+\frac{1}{u^2}}du$$

$$= \int \frac{1+\frac{1}{u^2}}{\boxed{\left(u+\frac{1}{u}\right)^2-2}_a}du + \int \frac{1+\frac{1}{u^2}}{\boxed{\left(u-\frac{1}{u}\right)^2-2}_b}du$$

Apply the following substitution: $z = u + \frac{1}{u} \implies dz = 1 - \frac{1}{u^2}du$ and $w = u - \frac{1}{u} \implies dw = 1 + \frac{1}{u^2}du$

$$\implies \int \frac{1}{z^2-2}dz + \int \frac{1}{w^2+2}dw$$

Problem  Solution Continued

We have the following

$$\int \frac{1}{x^2 - a^2} dx = -\frac{1}{a} \operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Thus,

$$\boxed{-\frac{1}{\sqrt{2}} \operatorname{arctanh}\left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \arctan\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C}$$