MAT 135B Term Paper
Unraveling the Formula: A Comprehensive Exploration of the Derivation of the Black-Scholes Model in Financial Option Pricing

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1 Introduction

1.1 Introduction to the Black-Scholes Model

The Black-Scholes Model is a mathematical model that seeks to explain the behavior of financial derivatives. In particular, and most commonly options. The model was published by Fischer Black and Myron Scholes in 1973 under the article "The Pricing of Operations and Corporate Liabilities" In the *Journal of Political Economy.* From this model, we are able to calculate the price of an option based on different variables and factors. Today, there are numerous variations of the Black-Scholes model, each seeking to improve the original model based on certain criteria available. This paper will focus on the solution derivation of the Black-Scholes Formula in PDE form and the mathematics behind it.

The mathematics behind finance, and especially this famous equation for pricing financial derivatives has been sought after throughout history. Take for example, the great Isaac Newton, who dabbled in the stock market late in his life. He famously lost money in the market after incorrectly pricing the value of his investment. He states

Quote 1.1. *I can calculate the motions of the heavenly bodies, but not the madness of the people.*

Isaac Newton

If only Newton had used his own tool of calculus to understand the dynamic of financial markets, he might of fared better. Indeed, the mathematics behind finance, including derivative pricing stems from the principles of calculus and probability theory.

1.2 Terminology

Let us begin be defining some terminology that will be helpful

Probability Terms

Definition 1.1. *A probability space* $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ *for which*

- $\mathbf{\Omega}$ *is the sample space, set of all possible outcomes*
- $\mathcal{F}$ *is the event space, the set of outcomes*
- $\mathbf{P}$ *is the probability, which assigns the prior to the probability, which is a number between 0 and 1*

Definition 1.2. *Cumulative distribution function,* F*, for the random variable* X *defined for* $b \in \mathbb{R}$ *by*

$$F(b) = \mathbf{P}(X \leq b)$$

X admits a probability density function (PDF) or density f if

$$\mathbf{P}(X \leq b) = F(b) = \int_{\infty}^{b} f(x)\, dx$$

for a non negative function f

Definition 1.3. *X is a random normal variable with parameter μ and $\sigma^2 > 0$ if the density of X given by*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Definition 1.4. *Expected value for continuous functions*

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Definition 1.5. *X is log-normally distributed if for some normally distributed variable Y,*

$$X = e^{Y}$$

That is, $\ln X$ is normally distributed

Finance terms

Definition 1.6. *A call option is a contract between two parties in which the holder of the option has the right(not an obligation) to buy an asset at a certain time in the future for a specific price, called the strike price*

Definition 1.7. *A put option is a contract between two parties in which the holder of the option has the right(not an obligation) to sell an asset at a certain time in the future for a specific price, called the strike price*

2 The Black-Scholes Model

2.1 Black-Scholes Formula Part I-Assumptions

The Black-Scholes formula provides the price of plain vanilla European call and put options, under the assumption that the price of the underlying asset (stocks, futures, indices, etc) has log-normal distribution – that is its continuous probability distribution of a random variable whose logarithm is normally distributed

Let us now introduce the Black-Scholes formula. We will assume that for any t_1a and t_2 with $t_1 < t_2$. The random variable $\frac{S(t_2}{S(t_1)}$ is log-normal with parameters $\left(\mu - q - \frac{\sigma^2}{2}\right)(t_2 - t_1)$ and $\sigma^2(t_2 - t_1)$. That is,

Definition 2.1. *Black-Scholes Formula Assumption:*

$$\ln\left(\frac{S(t_2)}{S(t_1)}\right) = \left(\mu - q - \frac{\sigma^2}{2}\right)(t_2 - t_1) + \sigma\sqrt{t_2 - t_1}Z$$

Where

- Z is the standard normal variable
- μ, σ are the drift and volatility of the price $S(t)$ of the underlying asset $S(t)$ of the underlying asset and represent the expected value and the standard deviation of the asset returns
- q is the continuous rate at which the asset pays dividends

2.2 Black-Scholes Formula Part II-Formula

The Black-Scholes model depends on the following parameters:

- S is the spot price of the underlying asset at time t
- T is the maturity of the option
- K is the strike price of the option
- r is the risk-free interest rate which is assumed to be constant over the life of the option $(t \leq r \leq T)$
- σ is the volatility of the underlying asset, also known as the standard deviation of the returns of the asset
- q is the dividend rate of the underlying asset, which is assumed to pay dividends at a continuous rate

Given the parameters and assumptions mentioned above.

Let $C(S,t)$ be the value at time $t$of a call option with strike K and maturity T, and let $P(S,t)$ be the value at time t of a put option with strike K and maturity T, then

Definition 2.2. *Call Option Pricing:*

$$C(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Definition 2.3. *Put Option Pricing:*

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1)$$

Where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

We also have that $N(z)$ is the cumulative distribution of the standard normal variable:

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$$

2.3 Black-Scholes Formula for non-dividend paying asset ($q = 0$)

For a non-dividend paying asset, we have a simplified form of the formula:

Definition 2.4. *Call Option Pricing ($q = 0$):*

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2$$

Definition 2.5. *Put Option Pricing ($q = 0$):*

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

Where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

2.4 The Greeks

The Greeks are the quantities representing the sensitivity of the price of a derivative instrument, such as an option to changes in one or more underlying parameter on which the value of an instrument or portfolio of the financial instruments is dependent. These are also known as risk sensitivities, risk measures, or hedge parameters.
Let V be the price of a portfolio of derivative securities on one underlying asset. Below are the most widely used Greeks.

Definition 2.6. *Delta (Δ): The rate of change of the value of the portfolio with respect to the sport price S of the underlying asset*

$$\Delta(V) = \frac{\partial V}{\partial S}$$

Definition 2.7. *Gamma (Γ): The rate of change of the delta of the portfolio with respect to the spot price S of the underlying asset, i.e. the second partial derivative of the value of the portfolio with respect to S*

$$\Gamma(V) = \frac{\partial^2 V}{\partial S^2}$$

Definition 2.8. *Theta (Θ): The rate of change of the value of the portfolio with respect to* *time t (not maturity T)*

$$\Theta(V) = \frac{\partial V}{\partial t}$$

Definition 2.9. *Rho (ρ): The rate of change of the value of the portfolio with respect to the risk-free interest r*

$$\rho(V) = \frac{\partial V}{\partial r}$$

Definition 2.10. *Vega (ν): The rate of change of the value of the portfolio with respect to the volatility σ of the underlying asset*

$$\nu(V) = \frac{\partial V}{\partial \sigma}$$

We can take the calculus derivative of the Black-Scholes Formula to obtain the appropriate Greek, which will be left as an exercise to the reader.

2.5 Benefits and Limitations to the model

Below are the **benefits** of the Black-Scholes model

- **Simplicity**: Option pricing is made constant, this places emphasis on the effects of time and volatility in relation to asset price. This provides easy pricing of options
- **Hedging**: Due to the simple assumptions made in the model, the model can easily hedge between the option markets and other financial instruments involved
- **Widely used**: The model has been around for more than half a century and is the most prevalent options pricing framework as it allows analysts to make computations in excel
- **Speed**: The simplifications make it possible to calculate a vast number of options quickly.

Below are the **limitations** of the Black-Scholes Model

- **Volatility not considered**: The model does not consider the volatility index, as it tends to vary over time and over strike price.
- **Risk-free rate not constant**: In our assumption, we have the risk-free rate as constant, in reality, it isn't.
- **Assumption of price changes being normally distributed**: In reality, stocks do not follow a standard distribution of returns.
- **Random Walk assumption**: The model works on the basis that the price changes are completely random. In reality, there are a multitude of factors that make it not really random, such as momentum and market performance.

2.6 Is this used in the real world?

Quote 2.1. *"Black-Scholes is an attempt to measure the market value of options, and it cranks in certain variables. But the most important variable it cranks in that might be subject well, might be a case where if you had differing views you could make some money but it's based upon the past volatility of the asset involved. And past volatilities are not the best judge of value."*

Warren Buffett

This is an important remark that the Black-Scholes model is only as good as its input parameters. We should not take this Partial differential equation as if it was a tool handed to us from above to predict market prices and to make profit. Much like Newton's law of universal gravitation when it comes to relativity, Black-Scholes also has its pitfalls at certain conditions in areas such as volatility and long-term behavior.

The case Warren Buffett presents it that the Black-Scholes model produces wildly inappropriate values when applied to long-dated options. That is the model can only forecast conditions locally, and has issues extrapolating with accuracy. As a result, many financial firms use tree methods and in-house models based on the model to more accurately predict derivative pricing values.

The key takeaway here is that **theory will take you only so far**, it takes a team of quantitative analysts and financial engineers with experience to accurately develop financial models based on the Black-Scholes model. This paper will not discuss such methods, but rather provide theoretical insight and derivation into the machinery of this equation. As if one were to master the theory behind it, they can easily apply that theory into practical applications.

3 Partial-Differential Equation form of Black-Scholes

3.1 Definition

The following is the statement for the Black Scholes PDE

Definition 3.1. *Black-Scholes PDE*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

With terminal condition

$$V(T = t, S) = (S - K)^+$$

We note this is a backward parabolic partial differential equation, which we could use the *Feynman-Kac* formula later to solve, but we will solve it in this section without any need of probability theory using Mr. Fourier's Heat equation.

3.2 The Greeks and The Black-Scholes PDE

Recall from section 2.4 that the Greeks represent the rates of change of the price of a derivative security with respect to some parameter. We can rewrite the Black-Scholes PDE as the following

Definition 3.2. *Greeks Version of Black-Scholes PDE*

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + (r - q)S\Delta - rV = 0$$
$$\iff \Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r(V - \Delta S) + q\Delta S$$

The RHS of the bottom line is small compared to the terms Θ and $\sigma^2 S^2 \Gamma$. This means, we can use the small-value approximation

$$1 + \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma}{\Theta} \approx 0$$

Therefore Θ, Γ have, in most instances, different signs. We also have that is Θ is large in absolute value then Γ is also large in absolute value, and vice versa.

3.3 PDE Solution using the Heat Equation

We will solve the Black-Scholes Partial Differential Equation. We note that solving this does not require any probability theory, with t and x as dummy variables, not random variables
Before we begin, let us make a couple assumptions:

- The option can only be exercised at the expiration date, by definition of a European Option
- Constant composition returns are normally distributed
- There are efficient markets
- Risk-free rate is known and constant
- No taxes or transaction costs
- There are no dividends during the life of the option ($q = 0$)

With these assumptions made, our task of solving the PDE becomes possible

So, the ultimately goal is the following:

$$\underset{\textbf{Black-Scholes PDE}}{\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0}} \xRightarrow{\textbf{(Heat Equation)}} \underset{\textbf{Formula}}{\boxed{V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)}}$$

The formula part was mentioned earlier in section 2, this is what financial engineers and quantitative analysts use. The "**(Heat Equation)**" part will be lengthy and will require some help from our friend Mr. Fourier.

We want the Black-Scholes equation to hold for all $x \geq 0$ and $t \in [0, T)$ which will hold regardless of of initial paths the stock price follows.

We begin with the statement of the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

Set $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$ and solving for τ:

$$\frac{\tau}{\frac{1}{2}\sigma^2} = T - t \implies \tau = \frac{1}{2}(T-t)\sigma^2$$

Next set $S = ke^x$

$$e^x = \frac{S}{K} \implies x = \ln\left(\frac{S}{K}\right)$$

We now set the following

$$V(S,t) = Kv(x,\tau) \tag{2}$$

The next steps is to obtain the Greeks, or calculus derivatives of V with respect to stock price, S, and the first derivative with respect to time, t.

$$\frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x}\left[\ln\left(\frac{S}{K}\right)\frac{\partial}{\partial S}\right] = K\frac{\partial v}{\partial x}\cdot\frac{1}{S}$$

$$\frac{\partial V}{\partial t} = k\frac{\partial v}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} = K\frac{\partial v}{\partial \tau}\left[(T-t)\frac{1}{2}\sigma^2\frac{\partial}{\partial t}\right] = K\frac{\partial v}{\partial \tau}\cdot\frac{-\sigma^2}{2}$$

Using $\frac{\partial x}{\partial S} = \frac{1}{\frac{S}{K}}\cdot\frac{1}{K} = \frac{1}{S}$

$$\begin{aligned}
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial x}\cdot\frac{1}{S}\right)\\
&= K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2} + K\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\frac{1}{S}\\
&= K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2} + K\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial S}\cdot\frac{1}{S}\\
&= K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2}
\end{aligned}$$

With these equations, we now set the terminal equation:

$$V(S,T) = \max(S-K,0) - \max(Ke^x - K, 0)$$

$$V(S,T) = Kv(x,0) \quad \text{and} \quad v(x,0) = \max(e^x - 1, 0)$$

We now take the derivatives and plug them into the the Black-Scholes PDE (1):

$$\left(K\frac{\partial v}{\partial \tau}\cdot\frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}S^2\left(K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2}\right) + rS\left(K\frac{\partial v}{\partial x}\cdot\frac{1}{S}\right) - rKv = 0$$

Cancellations provide the following

$$\left(\frac{\partial v}{\partial \tau}\cdot\frac{-\sigma^2}{2}\right) + \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\left(\frac{\partial v}{\partial x}\right) - rv = 0$$

Solving for $\frac{\partial v}{\partial \tau}$:

$$\frac{\partial v}{\partial \tau}\cdot\frac{\sigma^2}{2} = \frac{\sigma^2}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r\frac{\partial v}{\partial x} - rv$$

Factor out $\frac{\sigma^2}{2}$ and let $k = \frac{r}{\frac{\sigma^2}{2}}$, we obtain

$$\frac{\partial v}{\partial \tau} = \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + \frac{r}{\frac{\sigma^2}{2}} \cdot \frac{\partial v}{\partial x} - \frac{r}{\frac{\sigma^2}{2}} v$$

$$\frac{\partial v}{\partial \tau} \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + k\frac{\partial v}{\partial x} - kv \tag{3}$$

This provides a single dimensionless parameter, k. We rescale the v equation such that

$$v = e^{\alpha x + \beta \tau} u(x, \tau) \tag{4}$$

Differentiate with respect to x and τ provides the following:

$$\frac{\partial v}{\partial x} = \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial \tau} = \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial \tau}$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x^2}$$

Plugging these derivatives into equation (3):

$$\beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial \tau} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau} \frac{\partial^2 u}{\partial x^2} + (k-1) \left(\alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x}\right) - k e^{\alpha x + \beta \tau} u$$

We now divide by $e^{\alpha x + \beta \tau}$ and combine the like terms

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku$$

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku$$

$$\frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku - \beta u$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(k - 1 + 2\alpha) + u(\alpha^2 + k\alpha - \alpha - k - \beta) \tag{5}$$

The coefficients are equal to zero, this implies that $u = 0$ and $\frac{\partial u}{\partial x} = 0$. Choose $\alpha = \frac{-(k-1)}{2}$ and $\beta = \alpha^2 + (k-1)\alpha - k = \frac{-(k+1)^2}{4}$ and plug this into equation 5. With some help from the Heat Equation, this provides the following:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left[k - 1 + 2\left(-\frac{k-1}{2}\right)\right] + u\left[\left(-\frac{k-1}{2}\right)^2 + k\left(-\frac{k-1}{2}\right) - \left(-\frac{k-1}{2}\right) - k - \left(-\frac{(k+1)^2}{4}\right)\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left[k - 1 - (k-1)\right] + u\left[\left(\frac{k^2 - 2k + 1}{4}\right) - \left(\frac{k^2 - k}{2}\right) + \left(\frac{k-1}{2}\right) - k + \left(\frac{k^2 + 2k + 1}{4}\right)\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[0] + u\left[\left(\frac{k^2-2k+1}{4}\right) - \left(\frac{2k^2-2k}{4}\right) + \left(\frac{2k-2}{4}\right) - \frac{4k}{4} + \left(\frac{k^2+2k+1}{4}\right)\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + u\left[\frac{k^2-2k+1-2k^2+2k+2k-2-4k+k^2+2k+1}{4}\right]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + u[0]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$$u_\tau = y_{xx}$$

The initial condition is then transformed into

$$u(x,0) = \max\left(\exp\left(\frac{k+1}{2}x\right) - \exp\left(\frac{k-1}{2}x\right), 0\right) \tag{6}$$

This leads to the Heat equation solution, providing the following transformation to use for the Black-Scholes equation:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) \exp\left(\frac{-(x-s)^2}{4\tau}\right) ds$$

We will make a change of variables such that $s = z\sqrt{2\tau} + x$. Our goal here is to get the exponent into the form of $\frac{-y^2}{2}$ which is why $z = \frac{x-s}{\sqrt{2\tau}}$, to get the equation of the standard normal deviation. This will then be used later in this deviation to find the final solution. Taking the derivative, we obtain $ds = dx$ and $dx = \sqrt{2\tau}dz$:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(z\sqrt{2\tau}+x) e^{\frac{-z^2}{2}} dz \tag{7}$$

From the transformation, there is a change in (6) for the x value

$$u_0 = \exp\left(\frac{k+1}{2}(x+z\sqrt{2\tau})\right) - \exp\left(\frac{k-1}{2}(x+z\sqrt{2\tau})\right) \tag{8}$$

Since the time value must be positive, it must be that $u_0 > 0$. So, $x > -\frac{x}{\sqrt{2\tau}}$ which transforms the base of the domain of (7):

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left(\frac{k+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2}\right) - \exp\left(\frac{k-1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2}\right) dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left(\frac{k+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2}\right) dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \exp\left(\frac{k-1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2}\right) dz \tag{9}$$

We now complete the square of the exponential argument portion of the integrand.

$$\begin{aligned}\frac{k+1}{2}(x+z\sqrt{2\tau})-\frac{z^2}{2} &= -\frac{1}{2}\left[z^2-z\sqrt{2\tau}(k+1)\right]+\frac{x(k+1)}{2}\\ &= -\frac{1}{2}\left[z^2-z\sqrt{2\tau}(k+1)+\frac{\tau}{2}(k+1)^2\right]+\frac{x(k+1)}{2}-\left[-\frac{\tau(k+1)^2}{4}\right]\\ &= -\frac{1}{2}\left[z-\sqrt{\frac{\tau}{2}}(k+1))\right]^2+\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}\end{aligned}$$

We now substitute this into (9). This the the exponential argument. In addition we can pull out constants.

$$\frac{\exp\left(\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}\right)}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}\exp\left(-\frac{1}{2}\left[z-\frac{\sqrt{\tau}}{2(k+1)}\right]^2\right)dz$$

We will perform a change of variables $y=z-\sqrt{\frac{\tau}{2}}(k+1), dy=dz$ and $z=\frac{-x}{\sqrt{2\tau}}$ to obtain the following

$$\frac{\exp\left(\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}\right)}{\sqrt{2\pi}}\int_{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty}e^{\frac{-y^2}{2}}dy$$

Equation (9) then yields the following:

$$u(x,\tau)=\frac{\exp\left(\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(k+1)}e^{\frac{-y^2}{2}}dy-\frac{\exp\left(\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}\right)}{\sqrt{2\pi}}\int_{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty}e^{\frac{-y^2}{2}}dy \tag{10}$$

We can thus define the following improper integral

$$N(d)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}e^{\frac{-y^2}{2}}dy$$

Where,

$$d=\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(k+1)$$

We have that $-\infty\to d$ is equivalent to $-d\to\infty$. Thus the value of d_2 is the same as d_1 with the exception that $(k+1)$ is $(k-1)$, so:

$$d_1=\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(k+1)$$

$$d_2=\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(k-1)$$

Substituting N into equation(10), we obtain:

$$u(x,\tau)=\exp\left(\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}\right)N(d_1)-\exp\left(\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}\right)N(d_2) \tag{11}$$

Now plugging α, β and equation (11) into equation (4):

$$
\begin{aligned}
v(x,\tau) &= \exp\left(\frac{-x(k-1)}{2} - \frac{\tau(k+1)^2}{4}\right)u(x,\tau) \\
&= \exp\left(\frac{-x(k-1)}{2} - \frac{\tau(k+1)^2}{4}\right)\left[\exp\left(\frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}\right)N(d_1) - \exp\left(\frac{x(k-1)}{2} + \frac{\tau(k-1)^2}{4}\right)N(d_2)\right]
\end{aligned}
$$

There will be cancellations, yielding the following

$$v(x,\tau) = e^x N(d_1) - e^{-kr}N(d_2)$$

We then substitute back in $x = \ln\left(\frac{S}{K}\right)$ and $\tau = \frac{1}{2}\sigma^2(T-t)$ to obtain the following:

$$v(x,\tau) = e^{\ln(S/K)}N(d_1) - e^{-\frac{k}{2}\sigma^2(t-t)}N(d_2)$$

$$v(x,\tau) = \frac{S}{K}N(d_1) - e^{-\frac{k}{2}\sigma^2(t-t)}N(d_2) \tag{12}$$

We will now solve for the d-values

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S}{K}\right)}{\sqrt{2\left(\frac{1}{2}\sigma^2(T-t)\right)}} + \sqrt{\frac{\frac{1}{2}\sigma^2(T-t)}{2}}(k+1) \\
&= \frac{\ln\left(\frac{S}{K}\right)\left(\frac{\sigma^2}{2}k + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

We have that the risk-free interest rate, $r = \frac{k}{2}\sigma^2$. So then equation (12) and the d-value becomes:

$$v(x,\tau) = \frac{S}{K}N(d_1) - e^{-r(T-t)}N(d_2) \tag{13}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right)\left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

We now plug-in equation (13) into equation (2):

$$
\begin{aligned}
V(S,t) &= K\frac{S}{K}N(d_1) - Ke^{-r(T-t)}N(d_2) \\
&= SN(d_1) - Ke^{-r(T-t)}N(d_2)
\end{aligned}
$$

And with that, we have finally solved the Black-Scholes Partial-Differential Equation

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

with,

$$d_1 = \frac{\ln\left(\frac{S}{K}\right)\left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right)\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

□

4 Brownian Motion

The term *classical Brownian motion* describes the random movement of particles suspended within a liquid or gas. Originally discovered by Robert Brown to analyze the pollen movement of plants, it has found multiple uses in the areas of physics and later mathematical finance.

4.1 Motivation

We are motivated to understand the idea of Brownian motion due to its intrinsic connection to the unpredictable nature of financial markets. The motivation behind using Brownian motion lies in its ability to model seemingly random asset prices over time. This makes it a valuable tool to understand the financial markets and the Black-Scholes. We need to have an understanding of Brownian motion in order to make the assumption hold and later derive the Black-Scholes formula as they are related in the use of modeling asset prices. Brownian motion is a basic mathematical assumption needed to understand pricing options and risk associated with financial instruments by considering the probabilistic nature of price movements over time.

Originally introduced by Louis Bachelier in 1900 to describe stock price fluctuations, Brownian motion has since been embraced for its elegance to capture the continuous and stochastic nature of financial markets. In the area of mathematical finance, Brownian motion is assumed to be true and hold, where assets are changing continually over small intervals of time and position. The use of Brownian motion is one of the most fundamental tools on which all financial asset pricing and derivative pricing models are based.

In this section we will outline the basic definitions and properties

4.2 Stochastic Process

The main topic of this class, a stochastic process is defined as the following

Definition 4.1. *A* ***Stochastic Process*** *is a parameterized collection of random variables*

$$\{X_t\}_{t\in T}$$

defined on a probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ *and assuming values in* $\mathbb{R}^n$

Or a looser defintion would be, just a collection of random variables.

4.3 Brownian motion is a Wiener Process

We have the Brownian motion is a Wiener stochastic process, $W(t)$ with values in $\mathbb{R}$ defined for $t \in [0, \infty)$ Such that

Definition 4.2. *Conditions for a* ***Wiener process***

- $W(0) = 0$
- *If* $0 < s < t$ *then* $W(t) - W(s)$ *has a normal distribution* $\sim N(0, t-s)$ *with mean 0 and variance* $(t - s)$

- *If* $0 \le s \le t \le u \le v$ *(the two intervals* $[s,t]$ *and* $[u,v]$ *do not overlap). Then* $W(t) - W(s)$ *and* $W(v) - W(u)$ *are independent random variables. We have the the Wiener process is the only time-homogeneous stochastic process with independent increments that has continuous trajectories.*

- *The sample paths* $t \mapsto W(t)$ *are almost surely continuous*

Fact 4.3. *The probability density function of a Wiener processes* $W(t)$ *is the following:*

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

4.4 Properties

Below are some basic properties

- B_t is a Gaussian process (For all $0 \le t_1 \le t_2 \le \ldots \le t_k$, the random vector $\mathbb{Z} = (B_{t_1}, \ldots, B_{t_k}) \in \mathbb{R}$ has a multi normal distribution)

- B_t has stationary increments, that is the process $(B_{t+h} - B_t)_{h \ge 0}$ has the same distribution for all t. We have that $\mathbb{E}(B_{t+h} - B_t) = 0$ and $\mathrm{Var}(B_{t+h} - B_t) = h$

- B_t has continuous paths but is not differentiable anywhere at all.

- Brownian motion is a martingale:

  Definition 4.4. *The process* M_t *is a* ***martingale*** *is*

  1. $|\mathbb{E}(M_t)| < \infty$ *for all* t
  2. M_t *is* F_t *measurable for all* t
  3. *The conditional expectation* $\mathbb{E}(M_t | \mathcal{F}_s) = M_s$ *if* $s < t$

- $\mathrm{Cov}(B_s, B_t) = \min(s,t)$

4.5 Brownian Motion definition

We define Brownian motion as the limit of random walks $W^{(n)}(t)$ as $n \to \infty$. We have that the random walk $W^{(n)}(t)$ is defined to be $W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$. For $nt \in \mathbb{Z}$ and with M_{nt} representing a standard normal variable. This is known as a scaled symmetric random walk. This is described to have a symmetric distribution, meaning that the probability density function of the distribution is symmetric about its mean. We also have that it is scaled, ensuring that its limit converges. Additionally this is a random walk, which is a mathematical model for a paht that consists of random steps. In this case, the steps are determined by the values of the standard normal random variable M_{nt} and the scaling factor controls how these steps accumulate over time.
With that out of the way, let us define Brownian motion

Definition 4.5. ***Brownian motion***
Let $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space. For each* $\omega \in \Omega$ *suppose there is a continuous function*

$W(t)$ of $t \geq 0$ such that $W(0) = 0$ and that depends on ω. Then $W(t), t \geq 0$ is a Brownian motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0$$

$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i$$

4.6 Distributional properties of Brownian motion

- Spatial Homogeneity: $B_t + x$ for any $x \in \mathbb{R}$ is a Brownian motion started at x
- Symmetry: B_t is also a Brownian Motion
- Scaling:$\sqrt{c}B_{t/c}$ for any $c > 0$ is a Brownian motion
- Time inversion:

  $$Z_t = \begin{cases} 0 & t = 0 \\ tB_{1/t} & t > 0 \end{cases}$$

  is a Brownian motion
- Time reversibility: For any given $t > 0$:

  $$\{B_s | 0 \leq s \leq t\} \sim \{B_{t-s} - B_t | 0 \leq s \leq t\}$$

4.7 Special Facts about Brownian Motion

- Brownian motion is continuous everywhere yet nowhere differentiable.
- Fractal-like behavior. A small piece of Brownian motion trajectory looks like the entire trajectory.
- Brownian motion will hit any and every real value. It can jump around to hit every real values
- Once Brownian motion hits 0 or any particular values, it will hit it infinitely again and infinitely often

4.8 Brownian Motion options pricing

The Black-Scholes model has the underlying assumption of Brownian motion. We have the following condition for something to follow a *Geometric Brownian Motion*

Definition 4.6. *A stochastic process S_t is said to follow a Geometric Brownian Motion if it satisfies the stochastic differential equation:*

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

With μ representing percentage drift, and σ the percentage volatility

This stochastic differential equation has the following solution

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma dB_t\right)$$

for an initial value S_0.

We note that the Black-Scholes model and geometric Brownian motion has the same $S(t)$ expression. This assumption is necessary and fundamental to the derivation of options pricing formulas such as Black-Scholes. Additionally, we interpret Brownian motion to ahve a volatility parameter (σ) that captures the stochastic nature.

5 The Itô Calculus

We now will extend our favorite calculus operation, the chain rule, into stochastic calculus. This is the general idea behind the Itô calculus. Itô's lemma tries to calculate the dynamics of $f(B_t)$ for B a Brownian motion. At its core, Itô calculus seeks to generalize our understanding of differentiation to the realm of stochastic processes. Itô's lemma serves to unveil the dynamics of a Brownian motion.

As we embark on our brief exploration of Itô calculus, its relevance to the Black-Scholes model becomes apparent. One key assumption behind Black-Shcoles is the asset price following a geometric Brownian motion, a stochastic process central to Itô calculus. The extension of the chain rule in stochastic calculus, embodied by Itô's lemmas becomes essential in analyzing the dynamics of option pricing within Black-Scholes. By studying Itô calculus, we can make more effectively model the complex and unpredictable nature of financial markets, providing a nuanced perspective on the dynamics of financial instruments and providing insight into

5.1 Itô's Lemma Integral Form

Itô's Lemma bridges our traditional calculus and the stochastic processes. It extends our notion of differentiation to functions involving stochastic variables. Itô's Lemma provides a method for calculating the differential of a function that depends on a stochastic process, typically with Brownian motion. We now will state Itô's Lemma.

Theorem 5.1. ***(Itô's Lemma I)*** *Suppose $f \in C^2$ function and B_t is a standard Brownian motion. Then for all t*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$$

Equivalently in differential form:

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

Proof (using Taylor Series): Without Loss of Generality, we will assume $t = 1$ such that

$$f(B_1) = f(B_0) + \int_0^1 f'(B_s)dB_s + \frac{1}{2}\int_0^1 f''(B_s)ds$$

$$f(B_1) = f(B_0) - f(B_0) + f(B_{1/n}) - f(B_{1/n}) + \ldots + f(B_{(n-1)/n}) - f(B_{(n-1)/n}) + f(B_1)$$
$$= f(B_0) + \sum_{j=1}^{n} [f(B_{j/n} - f(B_{(j-1)/n}]$$

Thus,

$$f(B_1) - f(B_0) = \sum_{j=1}^{n} [f(B_{j/n} - f(B_{(j-1)/n}] \qquad (*)$$

Performing a second degree Taylor approximation, we obtain:

$$f(B_{j/n}) = f(B_{(j-1)/n}) + f'(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})$$
$$+ \frac{1}{2} f''(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})^2 + o((B_{j/n} - B_{(j-1)/n})^2)$$

Rearranging yields:

$$f(B_{j/n}) - f(B_{(j-1)/n}) = f'(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})$$
$$+ \frac{1}{2} f''(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})^2 + o((B_{j/n} - B_{(j-1)/n})^2) \qquad (**)$$

Combining (*) and (**) together yields:

$$f(B_1) - f(B_0) = \sum_{j=1}^{n} [f'(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})]$$
$$+ \frac{1}{2} f''(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})^2 + o((B_{j/n} - B_{(j-1)/n})^2)$$

As we take $n \to \infty$ on the RHS and LHS, we have the following limits of sums for $f(B_1) - f(B_0)$

$$\lim_{n\to\infty} \sum_{j=1}^{n} [f'(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})]$$

$$\lim_{n\to\infty} \sum_{j=1}^{n} [\frac{1}{2} f''(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})^2]$$

$$\lim_{n\to\infty} \sum_{j=1}^{n} [o((B_{j/n} - B_{(j-1)/n})^2)]$$

We note for the first sum, it is the definition of a **Simple Process** (which we will return to later), thus

$$\lim_{n\to\infty} \sum_{j=1}^{n} f' B_{(j-1)/n}(B_{j/n} - B_{(j-1)/n}) = \int_0^1 f'(B_t) dB_t.$$

For the second sum, let $f''(B_t) = h(t)$. We have that $f \in C^2$, thus $h(t)$ is continuous. Thus for all $\epsilon > 0$, there exists a step function $h_\epsilon(t)$ such that for every $, |h(t) - h_\epsilon(t)| < \epsilon$. Consider the following:

$$\lim_{n\to\infty} \sum_{j=1}^{n} h_\epsilon(t)[(B_{j/n}) - B_{(j-1)/n}]^2 = \int_0^1 h_\epsilon(t) dt$$

Breaking apart the absolute value, we see

$$\left|\sum_{j=1}^{n}[h(t) - h_\epsilon(t)][(B_{j/n}) - B_{(j-1)/n}]^2\right| \leq \epsilon \sum_{j=1}^{n}[(B_{j/n}) - B_{(j-1)/n}]^2 \to \epsilon$$

We now have the following

$$\int_0^1 h_\epsilon(t)dt = \int_0^1 h(t)dt = \int_0^1 f''(B_t)dt$$

Combing results, we obtain

$$\lim_{n\to\infty} \frac{1}{2}\sum_{j=1}^{n} f''(B_{(j-1)/n})[B_{j/n} - B_{(j-1)/n}]^2 = \frac{1}{2}\int_0^1 f''(B_t)dt$$

For our last limit sum, since B_t is a standard Brownian motion,

$$[B_{j/n} - B_{(j-1)/n}]^2 \approx 1/n$$

and they tend to smaller and smaller terms until it disappears to 0 as $n \to \infty$ Thus we have shown that

$$f(B_1) - f(B_0) = \int_0^1 f'(B_t)dB_t + \frac{1}{2}\int_0^1 f''(B_t)dt + 0$$

In this proof, we assumed $t = 1$, however in a general case we can use the interval $[0, t]$ as well. Thus,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$$

□

5.2 Itô's Lemma Differential Form

Before we move on to the second part of Itô's Lemma, let us introdduce the definition of the Simple and Itô processes.

Definition 5.2. ***Simple process*** *A process A_t is a simple process if there exists $0 = t_0 < t_1 < \ldots < t_n < \infty$ and random variables Y_j for $j \in \mathbb{Z}^+$, that are F_{tj}-measurable such that $A_t = Y_j, t_j \leq t \leq t_{j+1}$*

Remark: If we set $t_{n+1} = \infty$ and assume $\mathbb{E}[Y_j^2] < \infty$ for each j. For a simple process A_t, we define

$$Z_t = \int_0^t A_s dB_s$$

by the following "Riemann Sum"

$$Z_{tj} = \sum_{i=0}^{j-1} Y_i [B_{ti+1} - B_{ti}]$$

With $Z_t = Z_{tj} + Y_j [B_t - B_{tj}]$.
This property was used earlier to prove the second limit-sum for the second degree term in the proof above.

Definition 5.3. ***Itô Process*** *A n-dimensional Itô process satisfies the following stochastic differential equation*

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

for the following:

- *W is a m-dimensional standard Brownian motion*
- *m, a, b are are n-dimensional and $n \times m$-dimensional adapted process respectively*

Note the following is a solution to the stochastic differential equation above:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s$$

For the constant X_0. This integral can be done through methods in calculus.

Theorem 5.4. *(**Itô's Lemma II**) Let $f(t, X_t)$ be an Itô's process which satisfies the stochastic differential equation:*

$$dX_t = Z_t dt + y_t dB_t$$

If B_t is a standard Brownian motion and $f \in C^2$, then $f(t, X_t)$ is also an Itô process given by the following differential equation:

This version of Itô's lemma, which is equivalent to the first, just in differential form, will be helpful in deriving the Black-Scholes equation later in this paper.
Proof: Consider a stochastic process $f(t, X_t)$. We note that since X_t is a standard Brownian motion, $X_0 = 0$. Applying a Taylor approximation, we yield

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 + \frac{\partial^2 f}{\partial f \partial Xt}dtdX_t + \ldots$$

We note that since the quadratic variation of W_t is t, the term $(dW_t)^2$ contributes another dt term. However this is small and negligible, so it can be treated like a zero. Simplifying this down using an Itô multiplication table, we obtain

$$d(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t}[Z_t dt + y_t dB_t] + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(Z_t dt + y_t dB_t)^2)$$

We have the following equality

$$(Z_t dt + y_t dB_t)^2 = Z_t^2 (dt)^2 + 2Z_t y_t dB_t + y_t^2 (dB_t)^2 = y_t^2 dt$$

Performing the substituion into the previous line, we obtain

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t}[Z_t dt + y_t dB_t] + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2} y_t^2 dt$$

$$= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} Z_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2} y_t^2\right] dt + \frac{\partial f}{\partial X_t} y_t dB_t$$

This concludes the proof of the differential form of Itô's lemma. □

From this we now have all the necessary tools to develop the Black-Scholes model.But before we derive the models, let us take a quick look at the Feynman-Kac theorem, which allows us to efficiently calculate solutions to the Black-Scholes PDE

6 Feynman-Kac Theorem

In finance, the Feynman-Kac formula is used to efficiently calculate solutions to the Black-Scholes equation.

This theorem is named after Nobel Laureate Richard Feynman and Mathematician Mark Kac, while they were working at Cornell University. Originally developed to deal with Quantum Mechanics, it later found its application to finance. This work linked the problem of parabolic partial differential equation and Stochastic Processes. As mentioned earlier, the Black-Scholes PDE is a parabolic partial differential equation.

We have previously solved the Black-Scholes PDE using the heat equation, which is also a parabolic partial differential equation. Additionally the PDEs for mean curvature flow and Ricci Flow are also of this type.

In this section we will define the Feynman-Kac theorem followed by finding a solution to the PDE using Feynman-Kac

6.1 Feynman-Kac Theorem

The Feynman-Kac Theorem allows us to go from a parabolic partial differential equation to a conditional expectation under a probability measure Q.

Theorem 6.1. ***Feynman-Kac Theorem*** *For a particular partial differnetial equation in the form:*

$$\frac{\partial u}{\partial t}(x,t)+\mu(x,t)\frac{\partial u}{\partial x}(x,t)+\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t)-V(x,t)u(x,t)+f(x,t)=0$$

defined for all $x \in \mathbb{R}$ and $t \in [0,T]$ subject to the terminal condition: $u(x,T)=\psi(x)$
For μ,σ,ψ,V,f known functions, T is a parameter, and $u:\mathbb{R}\times[0,T]\to\mathbb{R}$ is unknown. The solution to the PDE is of the following conditional expectation:

$$u(x,t)=\mathbb{E}^Q\left[\int_t^T \exp\left(-\int_t^\tau V(X_s,s)ds\right)f(X_\tau,\tau)d\tau+\exp\left(-\int_t^T V(X_\tau,\tau)d\tau\right)\psi(X_T)\,\middle|\,X_t=x\right]$$

Under probability measure Q such that X is an Itô process by the following equation

$$dX_t=\mu(X,t)dt+\sigma(X,t)dW_t^Q$$

With $W^Q(t)$ as a Wiener process (Brownian motion) under measure Q, and initial condition for $X(t)$ is $X(t)=x$

6.2 Black-Scholes PDE Solution via Feynman Kac

We shall once again solve the Black-Shcoles PDE, this time using Feynman-Kac. For notation sake, we will state the conditional expectation as

$$\mathbb{E}^Q_{t,x}[\dots]=\mathbb{E}^Q[\dots|X_t=x]$$

and make the following substitutions,

- $f(x,t)=0$ Black-Scholes omits this term in the generalized parabolic PDE
- $u=F=\pi$ Where π represents the option price
- $\Phi(x)=\max(X_t-K,0)$ This variable is defined as the max between the terminal X and strike K
- $\exp\left(-\int_t^T V(X_\tau,\tau d\tau\right)\xrightarrow{\text{B-S}}\exp\left(-\int_t^T r d\tau\right)=e^{-r(T-t)}$. This is the "integrating factor" that we see in Black-Scholes

From this we obtain the following

$$\pi(t,x)=\mathbb{E}^Q_{t,s}\left[e^{-r(T-t)}\Phi(S_T)\right]$$

$$\pi(t,x)=\mathbb{E}^Q_{t,s}\left[e^{-r(T-t)}\max(X_t-K,0)\right]$$

$$\iff \quad \pi=\underbrace{\mathbb{E}^Q_{t,s}\left[e^{-r(T-t)}S_T\mathbb{1}_{(S_T\geq K)}\right]}_{\pi_1}-\underbrace{\mathbb{E}^Q_{t,s}\left[e^{-r(T-t)}K\mathbb{1}_{(S_T\geq K)}\right]}_{\pi_2}$$

With $\mathbb{1}$ representing the indicator function. Let us first analyze and solve π_2. Since K is a fixed amount and a binary option, with unconditionally it can be pulled out of the expectation.

$$\begin{aligned}\pi_2 &= -Ke^{-r(T-t)}\mathbb{E}^{Q}_{t,s}\left[\mathbb{1}_{(S_t\geq K)}\right]\\ &= -Ke^{-r(T-t)}Q[S_T \geq K]\end{aligned}$$

With the Q representing under probability measure Q. This now provides the following

$$\pi_2 = -Ke^{-r(T-t)}Q\left[S_T \exp\left((r-\frac{1}{2}\sigma^2)(T-t)+\sigma W^Q_{T-t}\geq K\right)\right]$$

We now divide by S_t, take the natural log, and move the argument to the other side to obtain:

$$\pi_2 = -Ke^{-r(T-t)}Q\left[\sigma W^Q_{T-t}\geq \ln\left(\frac{K}{S}\right)-(r-\frac{1}{2}\sigma^2)(T-t)\right]$$

We can now make this argument that σW^Q_{T-t} is approximately normally distributed to 0 mean and $\sigma^2(T-t)$ variance and define it to be a new normalized standard random variable γ

$$\sigma W^Q_{T-t}\sim N(0,\sigma^2(T-t)) \xrightarrow{\text{normalize}} \frac{x-\mu}{\sigma}\rightarrow \frac{\sigma W^Q_{T-t}-0}{\sigma\sqrt{T-t}} := \gamma$$

This then provides the following

$$\pi_2 = -ke^{-r(T-t)}Q\left[\gamma \leq \frac{\ln\left(\frac{S}{K}\right)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right]$$

This is starting to look a lot like the first part of Black-Scholes, in fact we get that the argument is d_2. so

$$\pi_2 = -ke^{-r(T-t)}N(d_2)$$

Let us now solve for π_1

$$\pi_1 = \mathbb{E}^{Q}_{t,s}\left[\boxed{e^{-r(T-t)}}\ S_T\mathbb{1}_{(S_T\geq K)}\right]$$

We first observe

$$\boxed{e^{-r(T-t)}} = \frac{1}{e^{r(T-t)}} = \frac{1}{B(T)}$$

This represents the reciprocal of the bond price at time T, $B(T)$ We now change $Q\mapsto Q^s$, this yields the following

$$\begin{aligned}\frac{\pi_1}{S(t)} &= \mathbb{E}^{Q^s}_{t,s}\left[\frac{\cancel{S_T}\mathbb{1}_{(S_t\geq K)}}{\cancel{S_T}}\right]\\ \pi_1 &= S(t)\mathbb{E}^{Q^s}_{t,s}\left[\mathbb{1}_{(S_T\geq K)}\right]\\ \pi_1 &= S(t)Q^s[S_t\geq K]\end{aligned}$$

By a similar argument to π_2 and with Girsanov's theorem (not mentioned here, but interesting), we have

$$\pi_1 = S(t)Q^s\left[S(t)\exp\left((r+\frac{1}{2}\sigma^2)(T-t)+\sigma^2 W^{Q^s}_{T-t}\right)\geq K\right]$$

Once again we divide by $S(t)$, take the natural logarithm, and rearrange to obtain the following:

$$\pi_1 = S(t)Q^s \left[\sigma W_{T-t}^{Q^s} \geq \ln\left(\frac{K}{S}\right) - \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right]$$

By a similar normalization argument, we obtain

$$\pi_1 = S(t)Q^s \left[\gamma \leq \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]$$

Under measure Q^s, providing us the same thing as d_1 Thus

$$\pi = \pi_1 + \pi_2 = \boxed{SN(d_1) - Ke^{-r(T-t)}N(d_2)}$$

Remarkably, using Feynman-Kac, we have once again solved the Black-Scholes PDE and yielded the same result (with much less work) from section 3.3!!

7 Deriving the PDE form of Black-Scholes

In this section we are going to go over the four different ways the Black-Scholes PDE is derived. Recall the Black-Scholes PDE is of the following:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

To derive the PDE, we need to make the following assumptions

- A risk less bond B that evolves with the process $dB = rBdt$
- An underlying security which evolves with accordance with the Itô process $dS = \mu Sdt + \sigma SdW$
- A option V written on the underlying security, by Itô's lemma evolves with the following process:
$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\sigma S\frac{\partial V}{\partial S}\right)dW$$

For notation simplicity $S = S(t), B = B(t), V = V(t)$ and $dW = dW(t)$. We also make the assumptions that the portfolios are self-financing, which means that the value of the portfolio is only dependent on the assumptions listed above.

7.1 Hedging

Using a hedging argument is the original derivation by Black and Scholes.

Consider a portfolio Π that is composed of one option and an amount Δ of underlying stock. Suppose the portfolio is risk less (insensitive to changes in the price of the security). We have the price of the portfolio at time t is of the following

$$\Pi(t) = V(t) + \Delta S(t)$$

Rewriting this in differential form provides the following:

$$\begin{aligned} d\Pi &= dV + \Delta ds \\ &= \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \Delta\mu S\right)dt + \left(\sigma S\frac{\partial V}{\partial S} + \Delta\sigma S\right)dW \end{aligned}$$

This portfolio has the feature that is is risk less, implying that the second term containing the Brownian motion dW is zero such that $\Delta = -\frac{\partial V}{\partial S}$. Substituting this in provides the following:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

Additionally, there is a feature that the portfolio must earn the risk free rate. implying that the diffusion of the risk less portfolio is $d\Pi = r\Pi dt$. Which we write

$$\begin{aligned} d\Pi &= r\Pi dt \\ \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt &= r\left(V - \frac{\partial V}{\partial S}S\right)dt \end{aligned}$$

Cancelling out the dt-term from both sides yields the Black-Scholes PDE. This is a method known as delta hedging. Where Δ is known as the hedge ratio. This method requires that in order to hedge a single option, we need to hold Δ shares of the stock

We will now show how Black and Scholes derived their famous PDE. As opposed to the Δ hedging technique, this is comprised of one share and $1/\Delta$ shares of the option. They define their portfolio as $\Pi(t) = \theta V(t) + S(t)$

$$\begin{aligned} d\Pi &= \theta dV + dS \\ &= \left(\theta\frac{\partial V}{\partial t} + \theta\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\theta\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \mu S\right)dt + \left(\theta\sigma S\frac{\partial V}{\partial S} + \sigma S\right)dW \end{aligned}$$

In order for the portfolio to be risk less, they set $\theta = -\left(\frac{\partial V}{\partial S}\right)^{-1}$. Substituting this in, and equate $d\Pi = r\Pi dt = r[\theta V + S]dt$ and drop the term involving μS to obtain

$$\left(\theta\frac{\partial V}{\partial t} + \frac{1}{2}\theta\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt = r[\theta V + S]dt$$

Dropping the dt form both sides, and dividing by θ, we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Moving everything to the LHS to make it homogeneous, yields the Black-Scholes equation.

7.2 Arbitrage

Pricing by arbitrage means in a complete market, all derivatives can be expressed in terms of a self-financing replicating strategy, and that this replicating strategy is unique. We can set up a replicating portfolio and use a risk neutral measure to calculate the value of the derivative.

We begin with a couple definitions

Definition 7.1. ***Trading Strategy***. *Given N assets with values $Z_1, Z_2, \ldots, Z_N$ at a time t, a trading strategy is a N-dimensional stochastic process $a_1, a_2, \ldots, a_N$ that represents the allocations into the assets at time t. The time value of the portfolio is $\Pi(t) = \sum_{i=1}^{N} a_i(t) Z_i(t)$.*

Definition 7.2. ***Self-Financing Trading Strategy***. *The trading strategy is self-financing if the change in the value of the portfolio is due only to changes in the value of the assets and not to inflows or outflows of funds. This implies that*

$$d\Pi(t) = d\left(\sum_{i=1}^{N} a_i(t) Z_i(t)\right) = \sum_{i=1}^{N} a_i(t) dZ_i(t)$$

In integral form:

$$\Pi(t) = \Pi(0) = \sum_{i=1}^{N} \int_0^t a_i(u) dX_i(u)$$

Definition 7.3. ***Arbitrage Opportunity*** *is a self-financing trading strategy that produces the following properties on the portfolio value:*

$$\Pi(t) \leq 0 \tag{14}$$

$$\mathbb{P}[\Pi(T) > 0] = 1 \tag{15}$$

Implying that the initial value of the portfolio is zero or negative, and the value of the portfolio at time T will be greater than zero with absolute certainty. This means that we start with a portfolio with zero value, or with debt (negative value). At some future time we have positive wealth, and since the strategy is self-financing, no funds are required to produce this wealth wealth.

Definition 7.4. ***Derivatives and replication*** *The trading strategy is a replicating strategy and the portfolio is a replicating portfolio if a replicating strategy exists the derivative is attainable. If all derivatives are attainable, the economy is complete.*

We now will derive the PDE via Replication. In order to replicate the derivative V, we form a self-financing portfolio with stock S and bond B in the right proportion. We use the replicating strategy $(a(t), b(t)$ to form the replicating portfolio $V(t) = a(t)S(T) + b(t)B(t)$ and determine the value of $(a(t), b(t)$. Following the self-financing assumption, we obtain:

$$dV = adS + bdB$$

Substituting this into Itô's lemma for dB, dS produces

$$\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\sigma S \frac{\partial V}{\partial S}\right) dW = (a\mu S + brB)dt + a\sigma S dW$$

Equating coefficients for $dW \implies a = \frac{\partial V}{\partial S}$. Subbing this into the equation above produces

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} &= brB \\ &= br\left(\frac{V - aS}{b}\right) \\ &= rV - rS\frac{\partial V}{\partial S}\end{aligned}$$

We note that the far LHS and far RHS yields the Black-Scholes equation.

7.3 Capital Asset Pricing Model (CAPM)

We begin with the definition of the Capital Asset Pricing Model

Definition 7.5. ***Capital Asset Pricing Model (CAPM)*** *stipulates that the expected return of a security, i, in excess of the risk free rate is*

$$\mathbb{E}[r_i] - r = \beta_i(\mathbb{E}[r_M] - r)$$

For r_i is the return on the asset, r is the risk free rate, and r_m is the reutrn on the market, and

$$\beta_i = \frac{\text{Cov}[r_i, r_M]}{\text{Var}[r_M]}$$

is the security's beta.

Definition 7.6. ***Asset CAPM****, for a time increment dt, the expected stock price return $\mathbb{E}[r_S dt] = \mathbb{E}\left[\frac{dS_t}{S_t}\right]$ by which S_t follows the diffusion, $dS_t = rS_t dt + \sigma S_t dW_t$. Thus, the expected return is*

$$\mathbb{E}\left[\frac{dS_t}{S_t}\right] = rdt + \beta_S(\mathbb{E}[r_M] - r)dt$$

Similarly, for the turn on the derivative, $\mathbb{E}[r_V dt] = \mathbb{E}\left[\frac{dV_t}{V_t}\right]$

$$\mathbb{E}\left[\frac{dV_t}{V_t}\right] = rdt + \beta_V(\mathbb{E}[r_M] - r)dt$$

We will now derive the Black-Scholes PDE from the CAPM. We first note that the derivative follows the diffusion

$$dV_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial t}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$$

Divide everything by V_t yields

$$\frac{dV_t}{V_t} = \frac{1}{V_t}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \frac{\partial V}{\partial S}\frac{dS_t}{S_t}\frac{S_t}{V_t}$$

Which we note is

$$r_V dt = \frac{1}{V_t}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \frac{\partial V}{\partial S}\frac{S_t}{V_t}r_S dt$$

Cancelling out the dt form both sides and taking the covariance of r_V, r_M and take note that

$$\text{Cov}[r_V, r_M] = \frac{\partial V}{\partial S}\frac{S_t}{V_t}\text{Cov}[r_S, r_M]$$

This implies a relationship between the beta of the derivative ,β_V, and the beta of the stock, β_S

$$\beta_V = \left(\frac{\partial V}{\partial S}\frac{S_t}{V_t}\right)\beta_S$$

Multiplying the asset CAPM by V_t to obtain

$$\begin{aligned}\mathbb{E}[dV_t] &= rV_t dt + V_t\beta_V(\mathbb{E}[r_M] - r)dt \\ &= rV_t dt + \frac{\partial V}{\partial S}S_t\beta_S(\mathbb{E}[r_M] - r)dt\end{aligned}$$

Taking the expectation of Itô's lemma's process of V yields

$$\mathbb{E}[dV_t] = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}[rS_t dt + S_t\beta_S(\mathbb{E}[r_M - r])dt] + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt$$

Equating the previous two equations above yields cancellations and derives the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2} - rV_t = 0$$

7.4 Limit of the Binomial Model

We have that the stock price at time t is S_t. Let us define the following

$$u = e^{\sigma\sqrt{dt}}$$

$$d = e^{-\sigma\sqrt{dt}}$$

At time $t + dt$, the stock price moves up to $S^u_{t+dt} = uS_t$ with the probability

$$p = \frac{e^{rdt} - d}{u - d}$$

Or its complement, moving down to $S^d_{t+dt} = dS_t$ having probability $1 - p$.
Risk-neutral valuation of the derivative provides the following relationship:

$$Ve^{rdt} = pV_u + (1 - p)V_d = p(V_u - V_d) + V_d$$

Notation wise, $V = V(S_t), V_u = V(S^u_{t+dt}), V_d = V(S^d_{t+dt})$. Taking the Taylor expansion of the respective variable yields the following:

$$\begin{aligned}V_u &\approx V + \frac{\partial V}{\partial S}(S^u_{t+dt} - S_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(S^u_{t+dt} - S_t)^2 + \frac{\partial V}{\partial t}dt \\ &= V + \frac{\partial V}{\partial S}S_t(u - 1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S_t^2(u - 1)^2 + \frac{\partial V}{\partial t}dt\end{aligned}$$

For V_d,

$$V_d \approx V + \frac{\partial V}{\partial S}S_t(d - 1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S_t^2(d - 1)^2 + \frac{\partial V}{\partial t}dt$$

And the other expansions

$$
\begin{aligned}
e^{rdt} &\approx 1 + rdt \\
u &\approx 1 + \sigma\sqrt{dt} + \frac{1}{2}\sigma^2 dt \\
d &\approx 1 - \sigma\sqrt{dt} + \frac{1}{2}\sigma^2 dt
\end{aligned}
$$

We take note that $(u-1)^2 = (d-1)^2 = \sigma^2 dt$, implying

$$
p(V_u - V_d) = p(u-d)\frac{\partial V}{\partial S}S_t = \left(rdt + \sigma\sqrt{dt} - \frac{1}{2}\sigma^2 dt\right)\frac{\partial V}{\partial S}S_t
$$

Substituting the expansion for V_d into the equation above, we obtain

$$
V(1 + rdt) = rS_t\frac{\partial V}{\partial S}dt + V + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial t}dt
$$

Cancelling the V from both sides and dividing by dt yields the Black-Scholes equation

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