Lab 03: Recursion and Tree Recursion

1. Instructions

> Please download lab materials `lab03.zip` from our QQ group if you don't have one.

In this lab, you have one task:

- Complete the required problems described in section 3 and submit your code to our OJ website as instructed in lab00. The starter code for these problems is provided in `lab03.py`, which is distributed as part of the lab materials in the `code` directory.

Submission: As instructed above, you just need to submit your answer for problems described in section 3 to our OJ website. You may submit more than once before the deadline; only the final submission will be scored. See lab00 for more instructions on submitting assignments.

Readings: You might find the following reference to the textbook useful:

- Section 1.7

2. Review

2.1 Recursion

A recursive function is a function that calls itself in its body, either directly or indirectly. Recursive functions have three important components:

1. Base case(s), the simplest possible form of the problem you're trying to solve.
2. Recursive case(s), where the function calls itself with a *simpler argument* as part of the computation.
3. Using the recursive calls to solve the full problem.

Let's look at the canonical example, `factorial`.

> Factorial, denoted with the ! operator, is defined as:
>
> ```
> n! = n * (n-1) * ... * 1
> ```
>
> For example, `5! = 5 * 4 * 3 * 2 * 1 = 120`.

The recursive implementation for factorial is as follows:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n - 1)
```

We know from its definition that 0! is 1. Since `n == 0` is the smallest number we can compute the factorial of, we use it as our base case. The recursive step also follows from the definition of factorial, i.e., `n! = n * (n-1)!`.

The next few questions in lab will have you writing recursive functions. Here are some general tips:

- Paradoxically, to write a recursive function, you must assume that the function is fully functional before you finish writing it; this is called the recursive leap of faith.
- Consider how you can solve the current problem using the solution to a simpler version of the problem. The amount of work done in a recursive function can be deceptively little: remember to take the leap of faith and trust the recursion to solve the slightly smaller problem without worrying about how.
- Think about what the answer would be in the simplest possible case(s). These will be your base cases - the stopping points for your recursive calls. Make sure to consider the possibility that you're missing base cases (this is a common way recursive solutions fail).
- It may help to write an iterative version first.

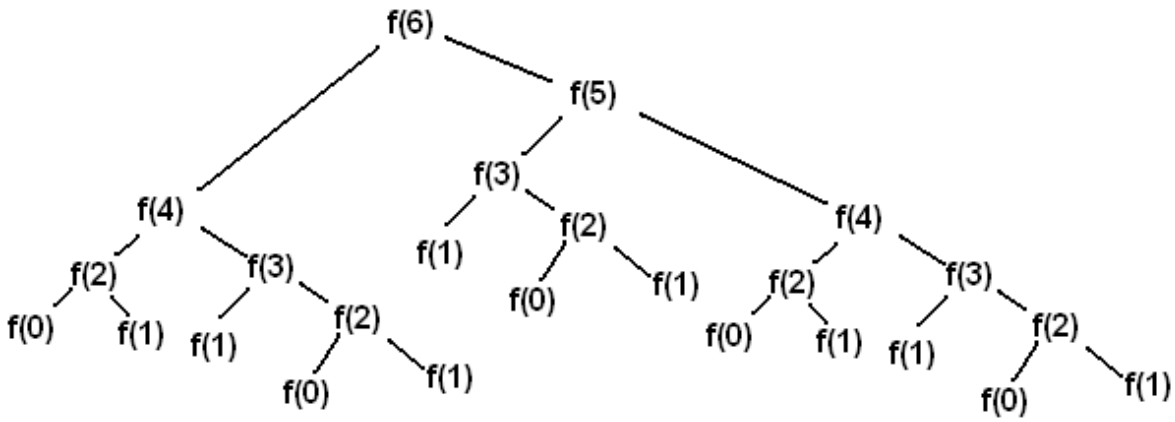
2.2 Tree Recursion

A tree recursive function is a recursive function that makes more than one call to itself, resulting in a tree-like series of calls.

A classic example of a tree recursion function is finding the nth Fibonacci number:

```
def fib(n):
    if n == 0 or n == 1:
        return n
    return fib(n - 1) + fib(n - 2)
```

Calling `fib(6)` results in the following call structure (where `f` is `fib`):



Each `f(i)` node represents a recursive call to `fib`. Each recursive call makes another two recursive calls. `f(0)` and `f(1)` do not make any recursive calls because they are the base cases of the function. Because of these base cases, we are able to terminate the recursion and beginning accumulating the values.

Generally, tree recursion is effective when you want to explore multiple possibilities or choices at a single step. In these types of problems, you make a recursive call for each choice or for a group of choices. Here are some examples:

- Given a list of paid tasks and a limited amount of time, which tasks should you choose to maximize your pay? This is actually a variation of the Knapsack problem, which focuses on finding some optimal combination of different items.
- Suppose you are lost in a maze and see several different paths. How do you find your way out? This is an example of path finding, and is tree recursive because at every step, you could have multiple directions to choose from that could lead out of the maze.
- Your dryer costs $2 per cycle and accepts all types of coins. How many different combinations of coins can you create to run the dryer? This is similar to the partitions problem from the textbook.

3. Required Problems

In this section, you are required to complete the problems below and submit your code to `Contest lab03` in our OJ website as instructed in lab00 to get your answer scored.

3.1 Recursion

Problem 1: Skip Add (100pts)

Write a recursive function `skip_add` that takes a single argument n and returns `n + n-2 + n-4 + n-6 + ... + 0`. Assume n is non-negative.

```
this_file = __file__

def skip_add(n):
    """ Takes a number n and returns n + n-2 + n-4 + n-6 + ... + 0.

    >>> skip_add(5)  # 5 + 3 + 1 + 0
    9
    >>> skip_add(10) # 10 + 8 + 6 + 4 + 2 + 0
    30
    >>> # Do not use while/for loops!
    >>> from construct_check import check
    >>> # ban iteration
    >>> check(this_file, 'skip_add',
    ...       ['While', 'For'])
    True
    """
    "*** YOUR CODE HERE ***"
```

You can use doctest to test your code:

```
$ python -m doctest lab03.py
```

Problem 2: Summation (100pts)

Now, write a recursive implementation of `summation`, which takes a positive integer `n` and a function `term`. It applies `term` to every number from `1` to `n` including `n` and returns the sum of the results.

```
def summation(n, term):

    """Return the sum of the first n terms in the sequence defined by term.
    Implement using recursion!

    >>> summation(5, lambda x: x * x * x) # 1^3 + 2^3 + 3^3 + 4^3 + 5^3
    225
    >>> summation(9, lambda x: x + 1) # 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10
    54
    >>> summation(5, lambda x: 2**x) # 2^1 + 2^2 + 2^3 + 2^4 + 2^5
    62
    >>> # Do not use while/for loops!
    >>> from construct_check import check
    >>> # ban iteration
```

```
    >>> check(this_file, 'summation',
    ...       ['While', 'For'])
    True
    """
    assert n >= 1
    "*** YOUR CODE HERE ***"
```

Problem 3: GCD (100pts)

The greatest common divisor of two positive integers `a` and `b` is the largest integer which evenly divides both numbers (with no remainder). Euclid, a Greek mathematician in 300 B.C., realized that the greatest common divisor of `a` and `b` is one of the following:

- the smaller value if it evenly divides the larger value, or
- the greatest common divisor of the smaller value and the remainder of the larger value divided by the smaller value.

In other words, if `a` is greater than `b` and `a` is not divisible by `b`, then

```
gcd(a, b) = gcd(b, a % b)
```

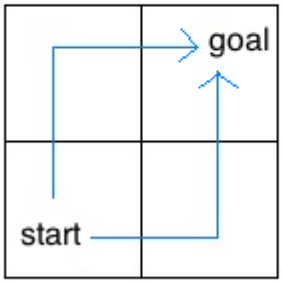
Write the `gcd` function recursively using Euclid's algorithm.

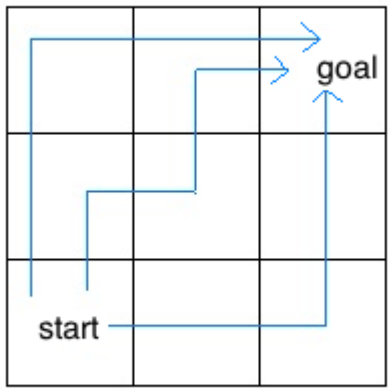
```
def gcd(a, b):
    """Returns the greatest common divisor of a and b.
    Should be implemented using recursion.

    >>> gcd(34, 19)
    1
    >>> gcd(39, 91)
    13
    >>> gcd(20, 30)
    10
    >>> gcd(40, 40)
    40
    """
    "*** YOUR CODE HERE ***"
```

3.2 Tree Recursion

Problem 4: Insect Combinatorics (200pts)

Consider an insect in an `M` by `N` grid. The insect starts at the bottom left corner, `(0, 0)`, and wants to end up at the top right corner, `(M-1, N-1)`. The insect is only capable of moving right or up. Write a function paths that takes a grid length and width and returns the number of different paths the insect can take from the start to the goal. (There is a closed-form solution to this problem, but try to answer it procedurally using recursion.)





For example, the 2 by 2 grid has a total of two ways for the insect to move from the start to the goal. For the 3 by 3 grid, the insect has 6 different paths (only 3 are shown above).

```
def paths(m, n):
    """Return the number of paths from one corner of an
    M by N grid to the opposite corner.

    >>> paths(2, 2)
    2
    >>> paths(5, 7)
    210
    >>> paths(117, 1)
    1
    >>> paths(1, 157)
    1
    """
    "*** YOUR CODE HERE ***"
```

Problem 5: Maximum Subsequence (200pts)

A subsequence of a number is a series of (not necessarily contiguous) digits of the number. For example, 12345 has subsequences that include 123, 234, 124, 245, etc. Your task is to get the maximum subsequence below a certain length.

```
def max_subseq(n, l):
    """
    Return the maximum subsequence of length at most l that can be found in the
given number n.
    For example, for n = 20125 and l = 3, we have that the subsequences are
        2
        0
        1
        2
        5
        20
        21
        22
        25
        01
        02
        05
        12
```

```
        15
        25
        201
        202
        205
        212
        215
        225
        012
        015
        025
        125
    and of these, the maximum number is 225, so our answer is 225.

    >>> max_subseq(20125, 3)
    225
    >>> max_subseq(20125, 5)
    20125
    >>> max_subseq(20125, 6) # note that 20125 == 020125
    20125
    >>> max_subseq(12345, 3)
    345
    >>> max_subseq(12345, 0) # 0 is of length 0
    0
    >>> max_subseq(12345, 1)
    5
    """
    "*** YOUR CODE HERE ***"
```

There are two key insights for this problem:

- You need to split into two cases, the one where the last digit is used and the one where it is not. In the case where it is, we want to reduce `l` since we used the last digit, and in the case where it isn't, we do not.
- In the case where we are using the last digit, you need to put the digit back onto the end, and the way to attach a digit `d` to the end of a number `n` is `10 * n + d`.