Math Modeling Project

Jingyuan(Harry) Liu
NetID: jl10624

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Catalog

1 Abstract

This paper aims to perform a detailed analysis on the dynamics of a simple pendulum. It would first analyze the motion of the pendulum via free body diagram. Non-linear ordinary differential equations are examined to capture the relationship between important variables involved. Linear approximation is utilized to reduce the problem into linear ODEs for information. Numerical methods like Euler's Method are used to solve the original non-linear ODEs. Figures are included to analyze the motion under different conditions. Drag force is also considered in the case of simple and physical pendulum (rod).

2 Introduction

Motion of pendulums is a well-studied phenomenon in Physics. The Italian scientist Galileo first noted the constancy of a pendulum's period by comparing the movement of a swinging lamp in a Pisa cathedral with his pulse rate. The Dutch mathematician and scientist Christiaan Huygens invented a clock controlled by the motion of a pendulum in 1656.

Pendulums are in common usage: the mechanism, can be found in clocks, swings, etc. Under small displacements and reasonable approximation, it is a simple harmonic oscillator.

The problem we are trying to study at hand first is a simple pendulum consisting of a ball with mass m anchored to a fixed point O via a rod of length l. We firstly analyze the problem when the rod is mass-less and there's no drag force (air resistance). Drag force and physical pendulum will be considered in later stages.

3 Model Setup

We assume that the rod is mass-less and there's no air friction (described in Introduction section). We can get the free body diagram of the system as below.

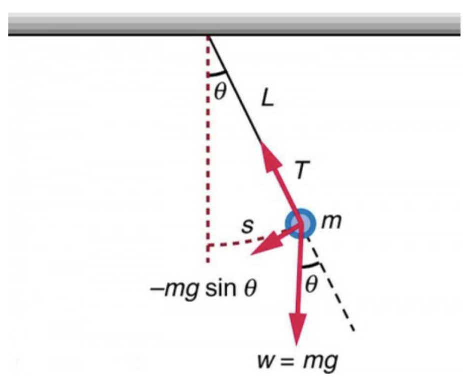

Figure 1: Free Body Diagram - Cited from Lumen Learning

We assume $l = 1m$ throughout the entire analysis. We let the angle between the vertical line and the rod be θ, length of the rod be L, the arc corresponding to θ be S, which is $L \cdot \theta$.

4 Model Analysis

4.1 Dimensional Analysis

We first perform a dimensional analysis on the system. Variables are T,θ,m,l,g. By Buckingham Theorem, we get one π term, which is

$$\frac{g \cdot T^2}{l}$$

Rearranging the equation, we get

$$T = \sqrt{\pi_1} \cdot \sqrt{\frac{l}{g}}$$

4.2 Linear Approximation of the Governing ODE

On the other hand, by Newton's second law, we get that

$$-mg \cdot sin\theta = ma \tag{1}$$

We can also get the acceleration a here, which is the arc length S divided by t^2. Thus, we can write as

$$a = l \cdot \frac{\partial^2 \theta}{\partial t^2}$$

After simplification, we get a second non-linear ODE :

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{g}{l} \cdot \sin\theta = 0 \tag{2}$$

For small deviation angles, we get

$$sin\theta = \theta - \frac{\theta^3}{3\,!} + \mathcal{O}(\theta^5)$$

by Taylor series, and we could approximate $\sin\theta$ by θ.

As in Figure 2 (see below), we can see when θ is between 0 to about 30 degrees, $sin\theta$ and θ have very little difference.

Therefore, we simplify the equation into

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{g}{l} \cdot \theta = 0 \tag{3}$$

4.3 Analytical Solution to the Approximated Linear ODE

Analytically solving the equation, we get a general solution of

$$\theta = C_1 \cdot \cos\sqrt{\frac{g}{l}}t + C_2 \cdot \sin\sqrt{\frac{g}{l}}t \tag{4}$$

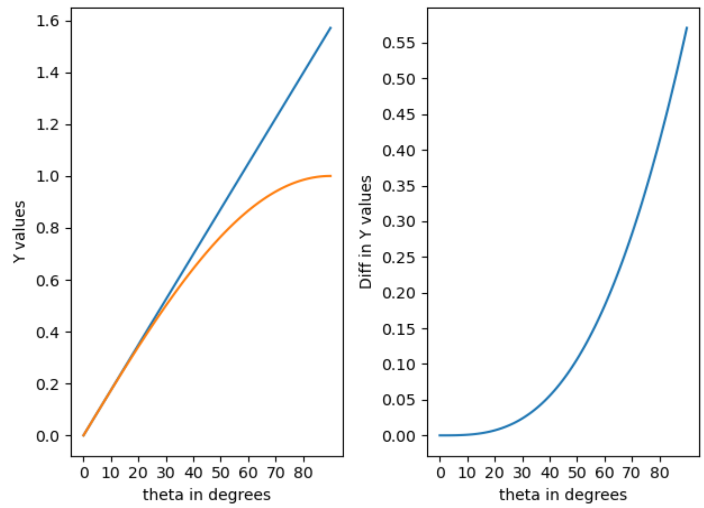

Figure 2: Small Angle Approximation

By initial condition that

$$\theta = \theta_0, \frac{\partial \theta}{\partial t} = 0$$

at t=0, we get that

$$C_1 = \theta_0, C_2 = 0$$

Hence, the function we want to find is

$$\theta = \theta_0 \cdot \cos \sqrt{\frac{g}{l}} t \tag{5}$$

When $t = 0$, $\theta = \theta_0$. The next t that makes the $\theta(t)$ function to be 0 is $t = 2\pi\sqrt{\frac{l}{g}}$. Thus, the period of oscillation is

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{6}$$

By comparing the two T equations, we can reach that $\sqrt{\pi_1} = 2\pi$, which verifies that the π term is dimensionless.

4.4 Numerical Solution to the Second-Order ODE

Once we get the τ in the previous section, we can use it to plot how the angle changes with respect to time (dimensionless).

We return back to the original ODE, which is

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{g}{l} \cdot \sin \theta = 0 \tag{7}$$

We let $u = \frac{\partial \theta}{\partial t}$, so $\frac{\partial \theta}{\partial t} = \frac{\partial^2 u}{\partial t^2}$, and the second-order ODE is in turn

$$\frac{\partial u}{\partial t} + \frac{g}{l} \cdot \sin \theta = 0$$

By Euler's Method, we get

$$\theta(t + \Delta t) = \theta(t) + \Delta t \cdot u(t) \tag{8}$$

$$u(t + \Delta t) = u(t) - \frac{g}{l} \sin\theta(t) \cdot \Delta t \tag{9}$$

Since angular velocity at first is 0, we let $u_0 = 0$. We choose θ_0 to be 5° (see 4.2 on small angle approximation). We iteratively solve the equations and the numerical results are plotted in Figure 3. Here, I non-dimensionalize the time by our calculated τ, the oscillation period. I also normalize the angle θ by θ_0 for generalization.

Qualitatively speaking, when Δt increases, it doesn't do a good job simulating the change of the angle. Since we've already said that $\theta(t + \Delta t) = \theta(t) + \Delta t \cdot u(t)$, a larger Δt would generate larger oscillations, hence a larger θ. Under a frictionless environment where the law of conservation of mechanical energy holds, we would expect θ to be consistent throughout the oscillation. It doesn't make any sense for the amplitude to keep increasing like the orange line in Figure 3. Whereas for the blue line with relatively small Δt, it gives us an ideal pattern.

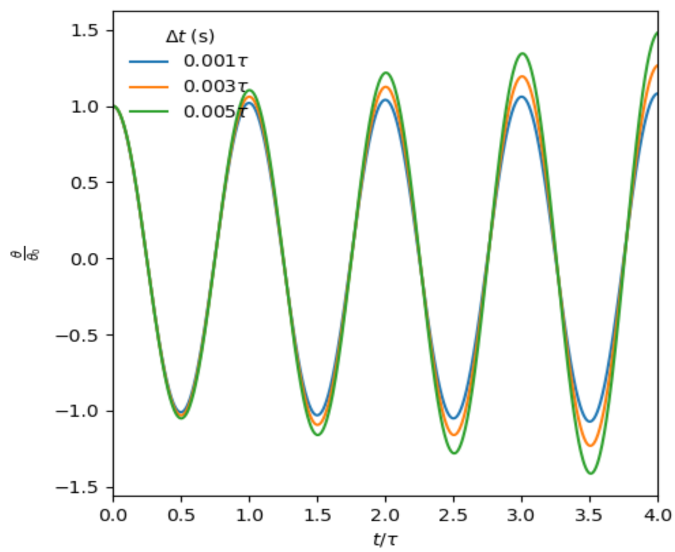

Figure 3: θ vs. time : Different Δt

Next, we plot the graph of θ value with respect to time (see Figure 4) using Euler's Method again, varying initial θ degrees. We also plot the analytical solution in the same figure for better comparison. We use solid lines to plot Euler's Method, and dotted lines to plot the analytical solution. Observing Figure 4, we can see that the curves becomes smoother and flatter as θ_0 decreases. The numerical solution tells us that the oscillation tends to stay the same for small θ_0; the amplitude of oscillation will increase, however, for larger θ_0. This doesn't make too much sense, since the only forces we are accounting for are the weight of the sphere and the tension on the string. The amplitude shouldn't change due to conservation of mechanical energy. Thus, numerical solution is ideal for smaller angles.

Moreover, for small θ_0 (like 5°), the analytical solution under linear approximation does a very good job simulating the numerical solution, as we can see the solid and dotted red lines overlapping each other. However, it does not work for larger θ_0 values. Therefore, we should only use linear approximation when the initial angle is small, verifying our conclusion in section 4.2.

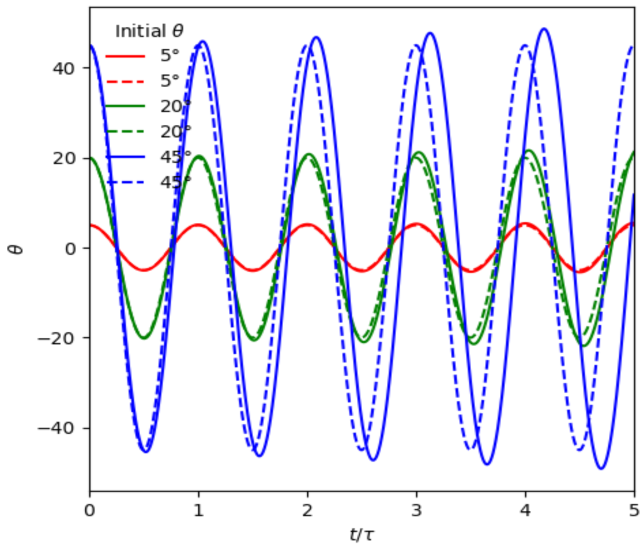

Figure 4: θ vs. t/τ : Numerical vs. Analytical Solution under different θ_0

5 Considering Drag Force

5.1 New Model Analysis

We now revisit the problem, taking drag force, F_D, into account this time. We redo the free body diagram, where the drag force would be in the opposite direction of the motion (see Figure 5).

5.2 Applying Stokes' Law

We assume that the pendulum is placed in a room under dry air condition with 15 °C. Under such conditions, the air density ρ is about 1.225 kg/m^3 and air viscosity μ is about $1.802 \times 10^{-5} kg/m \cdot s$. Let initial $\theta_0 = 5°$. We further assume the blob to be a spherical object with homogeneous material and a smooth surface.

With these premises, we are able to apply Stokes' Law to the problem. By the Stokes' law, the Reynolds number, denoted as Re, should be less than 1. So we try to find D and V that satisfies the condition.

We denote the velocity of the pendulum as $V = l \cdot \omega$, where we can approximate ω by $\frac{\theta_0}{\tau}$. We have

$$V = l \cdot \frac{\frac{\pi}{36}}{2\pi\sqrt{\frac{l}{g}}} = \frac{\sqrt{gl}}{72}$$

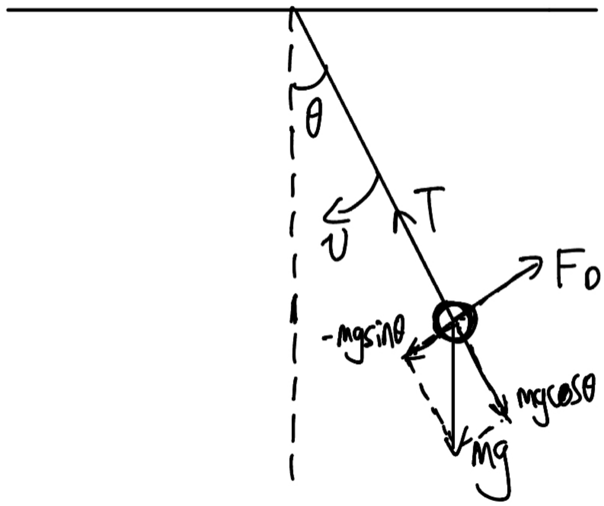

Figure 5: Free Body Diagram with Drag Force

We assume $D = 0.01m$ for the diameter of the blob and $l = 1m$ for the string length. By calculation, we have that

$$Re = \frac{\rho}{\mu} \cdot \frac{\sqrt{g \cdot l}}{72} \cdot D \approx 2.94\text{e}{-9}$$

which is a value that is way less than 1. Thus we are able to use the Stokes' Law, as Re is very small for certain conditions.

The drag coefficient $C_D = \frac{24}{Re}$ (given in HW1). Since we have that

$$F_D = \frac{1}{2} C_D \rho A V^2$$

$$Re = \frac{\rho V D}{\mu}$$

we are able to get the magnitude of the drag force:

$$|F_D| = 3\pi\mu V D$$

Since the drag force is in opposite direction of its velocity, we write it as

$$F_D = -3\pi\mu V D$$

We list the new governing ODE below as

$$-mg \cdot \sin\theta - 3\pi\mu V D \ = ml \cdot \frac{\partial^2 \theta}{\partial t^2} \tag{10}$$

We non-dimensionalize the equation. Since θ is dimensionless, we only need to non-dimensionalize the time variable. We denote $\tilde{t} = \frac{t}{\tau}$ and substitute it into equation (10). We simplify it and get

$$\frac{d^2\theta}{d\tilde{t}^2} + \frac{3\pi\mu D\tau}{m} \cdot \frac{d\theta}{d\tilde{t}} + \frac{g\tau^2}{l} \cdot \sin\theta = 0 \tag{11}$$

Since the coefficient of $\dot{\theta}$ is dimensionless, we let $\epsilon = \frac{3\pi\mu D}{m}$ to get

$$\frac{d^2\theta}{d\tilde{t}^2} + \epsilon \cdot \frac{d\theta}{d\tilde{t}} + \frac{g\tau^2}{l} \sin\theta = 0$$

5.3 Numerical Solution and Further Analysis

We can again use numerical methods like Euler's method to iteratively solve for relationship between θ and t/τ, while varying the value of ϵ. This is equivalent to varying the ratio between sphere diameter and its mass, since air viscosity, π, and τ are constant under our assumptions.

We plot the graph of θ versus t/τ under the damped scenario in Figure 6. Different from the previous plots under no drag force, the damped pendulum has a decreasing oscillation amplitude, where it stops at the neutral position eventually (see $\epsilon = 1$ in particular).

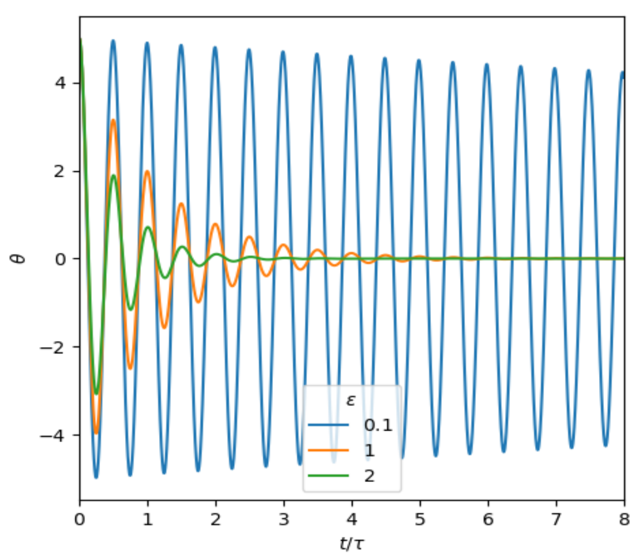

Figure 6: θ vs t/τ : Different θ_0 with Drag Force

We also plot the phase portrait of $\dot{\theta}$ versus θ separately with drag force and without it (see Figure 7 and 8).

We can see that the origin in Figure 7 without drag force is a center, which is stable. This means the pendulum resembles a simple harmonic motion (specifically for small angles): angular speed is the same at symmetric points on both sides of the pendulum. For small angles like 5°, it doesn't diverge or shrink to zero for θ. This indicates that the magnitude of the oscillation doesn't change and the pendulum can technically swing forever.

By comparison, the origin in Figure 8 with drag force is a stable spiral. From the plot, we can tell that whatever the initial angle is, they will stop at the neutral position eventually ($\theta = 0°$). If we draw a horizontal line on the plot, we can tell that at the same angular speed,

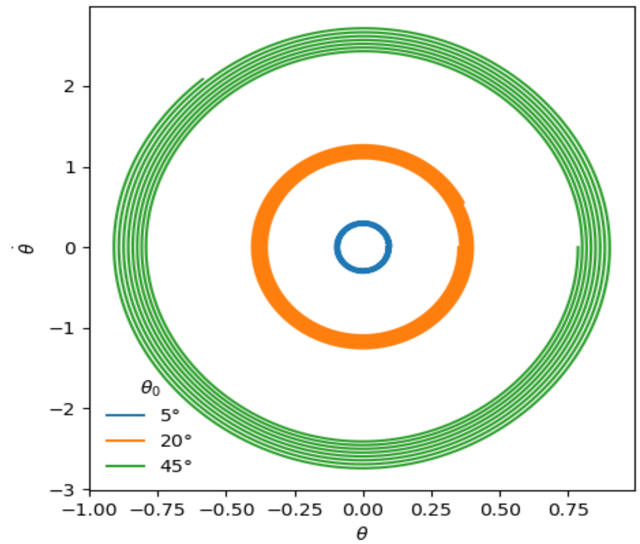

Figure 7: $\dot{\theta}$ vs θ without Drag

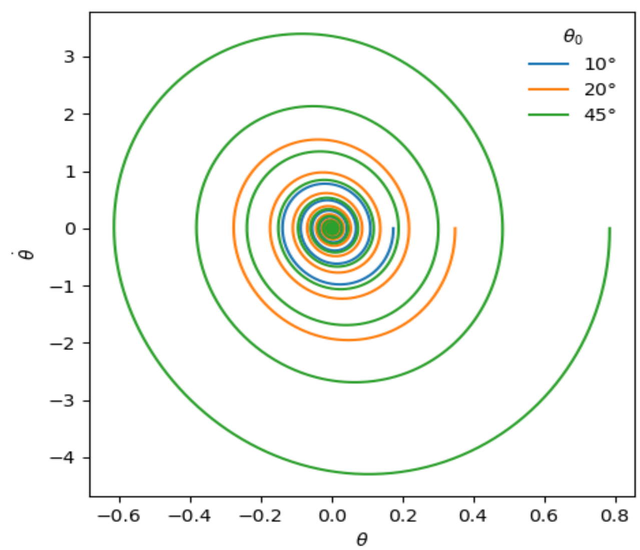

Figure 8: $\dot{\theta}$ vs θ with Drag

the magnitude of oscillation decreases and goes to zero eventually. This indeed corresponds to our analysis on the damped pendulum above.

6 Physical Pendulum

Now let's revisit the problem, but this time the rod itself has mass m, with no mass connected to it (see Figure 9). We assume the rod is of homogeneous material across the whole length.

The rod is circular with radius a (thickness $2a$) and length l. We also account for drag force in this scenario, where F_D is linearly proportional to the magnitude of the normal velocity and the projected area. Let the linear coefficient be k, and let $\lambda = \frac{m}{l}$ be the the mass per length of the rod. We use torque and angular momentum to analyze the motion of the rod. Let the direction pointing inwards be the positive z direction.

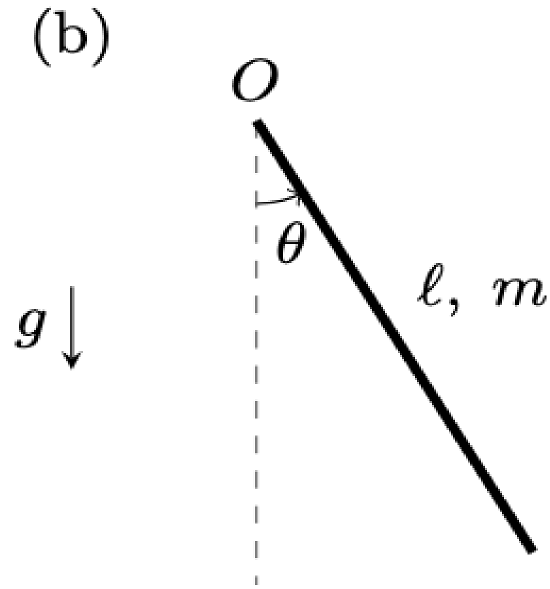

Figure 9: Cite from IMMProject.pdf Figure 1.(b)

6.1 Forces on a Small Segment on the Rod

Different segments of the rod have different velocity, as $V = l \cdot \omega$. Therefore, we try to analyze a tiny segment (can be seen as a point) on the rod that is x distance away from the anchor point O.

The drag force for the tiny segment can be written as:

$$dF_D = -k \cdot V \cdot (2a \cdot dx)$$

The weight of that tiny segment can be written as:

$$dG = dm \cdot g = dx\lambda \cdot g = \frac{m}{l} dx \cdot g$$

6.2 Calculating Torque

Next, we want to find the torque of that tiny segment, which can be represent as

$$d\vec{\tau} = d\vec{G} \times d\vec{x} + d\vec{F_D} \times d\vec{x} \tag{12}$$

$$d|\vec{\tau}| = \frac{m}{l} dx \cdot gx \sin\theta + (-k) \cdot V \cdot (2a \cdot dx) \cdot x \cdot (-1) = (\frac{mg\sin\theta}{l} \cdot x + 2kVa \cdot x) \cdot dx \tag{13}$$

To find the value of total torque, we integrate $d\vec{\tau}$ over the whole rod:

$$|\vec{\tau}| = \int_0^l (\frac{mg\sin\theta}{l} \cdot x + 2kVa \cdot x)\, dx = \frac{mg\sin\theta}{l} \cdot \frac{l^2}{2} + 2kVa \cdot \frac{l^2}{2} = \frac{mg\sin\theta}{2} \cdot l + kVa \cdot l^2 \quad (14)$$

6.3 Reaching the Governing ODE

From our lecture, we know that the inertia of an uniform thin stick around a perpendicular axis at its end is $\frac{1}{3}ml^2$, where m is the mass of the stick and l is the length of the stick. Though it's a circular rod, it has the same inertia as $\frac{1}{3}ml^2$ due to its homogeneous material.

Since $\frac{d\vec{L}}{dt} = \vec{\tau}$, we can apply it in terms of magnitude and get

$$\frac{d|\vec{L}|}{dt} = I \cdot \frac{dw}{dt} = |\vec{\tau}|$$

Plugging in the expression to get

$$\frac{1}{3}ml^2 \cdot \frac{d^2\theta}{dt^2} = \frac{mg\sin\theta}{2} \cdot l + kVa \cdot l^2 \quad (15)$$

Simplify equation (16) and we get

$$\frac{1}{3}ml^2 \cdot \frac{d^2\theta}{dt^2} = \frac{mg\sin\theta}{2} \cdot l + ka \cdot l^2 \cdot l \cdot \frac{d\theta}{dt} = \frac{mg\sin\theta}{2} \cdot l + kal^3 \cdot \frac{d\theta}{dt}$$

Finally, we reach the expression

$$\frac{d^2\theta}{dt^2} - \frac{3kal}{m} \cdot \frac{d\theta}{dt} - \frac{3g\sin\theta}{2l} = 0 \quad (16)$$

6.4 Numerical Solution and Further Analysis

We assume that $\theta_0 = 5°$. Similar to what we did in Section 5.3, we non-dimensionalize the equation and denote $\tilde{t} = \frac{t}{\tau}$. We substitute it into equation (16) and simplify to get

$$\frac{d^2\theta}{d\tilde{t}^2} - \frac{3kal\tau}{m} \cdot \frac{d\theta}{d\tilde{t}} - \frac{3g\sin\theta\tau^2}{2l} = 0 \quad (17)$$

We denote $\epsilon = \frac{3kal\tau}{m}$, and equation (17) becomes

$$\frac{d^2\theta}{d\tilde{t^2}} - \epsilon \cdot \frac{d\theta}{d\tilde{t}} - \frac{3g\sin\theta\tau^2}{2l} = 0 \quad (18)$$

We plot θ vs $\frac{t}{\tau}$ in Figure 10, varying ϵ again. However, $\epsilon = 0.1$ gives an undesirable result, indicating that we should choose larger ϵ for better simulation. Overall, from the plot, we can observe similar patterns like those in the damped simple pendulum: the amplitude of the rod decreases with time until it reaches the neutral position and stops eventually.

6.5 Conclusion

In this paper, we analyze the dynamics of simple pendulum with and without drag force, and also physical pendulum (with drag force). We applied dimensional analysis, linear approximation, linear and non-linear ordinary differential equations, Euler's Method, Stokes' Law, and basic kinematics to understand the problem.

We conclude that for small angles below or around 20°, linear approximation does a great job in simulating the actual motion of a simple pendulum. In terms of simple pendulum with relatively small angles, the amplitude tends to stay the same if we ignore drag force. And at same angular speed, the θ angles on both sides of the neutral position tends to stay the same. For the ones with drag force, the amplitude will decrease to zero and stops oscillating eventually; this also applies to the physical pendulum case.

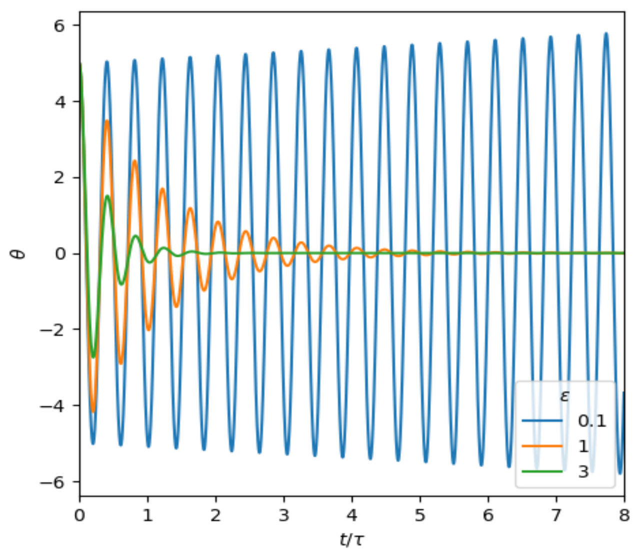

Figure 10: θ vs t/τ: Rod