

```

clear

global k
global n
global nd
n = 8; % number of points
k = 8; % number of polynomial coefficients (deg+1)
m = 3; % 3 jerk, 4 snap
nd = 64;

points = [
    3, 0, 0;
    0, 3, 0;
    -3, 0, 2;
    0, -4, 0;
    0, -4, 4;
    4, 0, 3;
    0, 5, 4;
    -5, 5, 4;
];
tangents = [
    0, 0, 1;
    1, 0, 0;
    1, 0, 0;
    0, 1, 0;
    0, 1, 0;
    0, 1, 0;
    0, 1, 0;
    1, 0, 0;
    0, 1, 0;
];
%points(:, 3) = points(:, 3) * 0.01;
%tangents(:, 3) = tangents(:, 3) * 0.01;

```

Minimum jerk/snap

```

% set cost
t = sym('t', 'real');
d0 = t .^ (0:k-1);
d1 = diff(d0, t);
d2 = diff(d1, t);
d3 = d0;
for i = 1:m
    d3 = diff(d3, t);
    disp(i)
    disp(int(d3'*d3, t, 0, 1))
end

```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{4}{3} & \frac{3}{2} & \frac{8}{5} & \frac{5}{3} & \frac{12}{7} & \frac{7}{4} \\ 0 & 1 & \frac{3}{2} & \frac{9}{5} & 2 & \frac{15}{7} & \frac{9}{4} & \frac{7}{3} \\ 0 & 1 & \frac{8}{5} & 2 & \frac{16}{7} & \frac{5}{2} & \frac{8}{3} & \frac{14}{5} \\ 0 & 1 & \frac{5}{3} & \frac{15}{7} & \frac{5}{2} & \frac{25}{9} & 3 & \frac{35}{11} \\ 0 & 1 & \frac{12}{7} & \frac{9}{4} & \frac{8}{3} & 3 & \frac{36}{11} & \frac{7}{2} \\ 0 & 1 & \frac{7}{4} & \frac{7}{3} & \frac{14}{5} & \frac{35}{11} & \frac{7}{2} & \frac{49}{13} \end{pmatrix}$$

2

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 0 & 6 & 12 & 18 & 24 & 30 & 36 \\ 0 & 0 & 8 & 18 & \frac{144}{5} & 40 & \frac{360}{7} & 63 \\ 0 & 0 & 10 & 24 & 40 & \frac{400}{7} & 75 & \frac{280}{3} \\ 0 & 0 & 12 & 30 & \frac{360}{7} & 75 & 100 & 126 \\ 0 & 0 & 14 & 36 & 63 & \frac{280}{3} & 126 & \frac{1764}{11} \end{pmatrix}$$

3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 72 & 120 & 180 & 252 \\ 0 & 0 & 0 & 72 & 192 & 360 & 576 & 840 \\ 0 & 0 & 0 & 120 & 360 & 720 & 1200 & 1800 \\ 0 & 0 & 0 & 180 & 576 & 1200 & \frac{14400}{7} & 3150 \\ 0 & 0 & 0 & 252 & 840 & 1800 & 3150 & 4900 \end{pmatrix}$$

```
Ti = int(d3'*d3, t, 0, 1);
T = zeros(k*(n-1), k*(n-1));
invTi = inv(Ti(m+1:end, m+1:end));
invTi = double(invTi / gcd(invTi));
Ti = double(Ti / gcd(Ti));
disp(Ti)
```

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	126	252	420	630	882
0	0	0	252	672	1260	2016	2940
0	0	0	420	1260	2520	4200	6300
0	0	0	630	2016	4200	7200	11025
0	0	0	882	2940	6300	11025	17150

```

invT = zeros((n-1)*(k-m), (n-1)*(k-m));
for i = 0:n-1-1
    i1 = (k-m)*i;
    invT(i1+1:i1+k-m, i1+1:i1+k-m) = invTi;
    i1 = k*i;
    T(i1+1:i1+k, i1+1:i1+k) = Ti;
end

% set point constraints
Dp0 = zeros(n-1, k*(n-1));
Dp1 = zeros(n-1, k*(n-1));
for i = 0:n-1-1
    i1 = k*i;
    Dp0(i+1, i1+1) = 1;
    Dp1(i+1, i1+1:i1+k) = ones(1, k);
end
pp0 = points(1:n-1, :);
pp1 = points(2:n, :);

% set derivative constraints
Dd = zeros((m-1)*(n-2), k*(n-1));
dd0 = zeros(m-1, k);
dd1 = zeros(m-1, k);
dd = d0;
for i = 1:m-1
    dd = diff(dd, t);
    dd0(i, :) = double(subs(dd, t, 1));
    dd1(i, :) = double(subs(dd, t, 0));
end
disp(dd0)

```

0	1	2	3	4	5	6	7
0	0	2	6	12	20	30	42

```
disp(dd1)
```

0	1	0	0	0	0	0	0
0	0	2	0	0	0	0	0

```

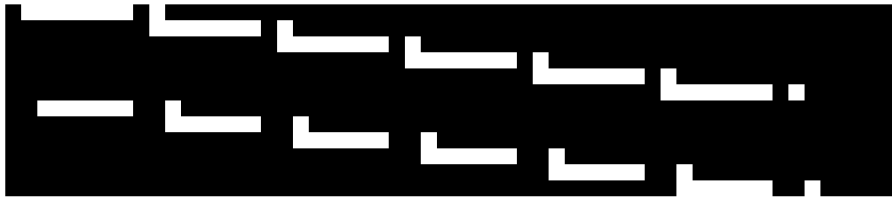
for i = 0:n-2-1
    i1 = k*i;
    for j = 1:m-1
        j1 = (j-1)*(n-2);
        Dd(j1+i+1, i1+1:i1+k) = dd0(j, :);
    end
end

```

```

        Dd(jl+i+1, il+k+1:il+k+k) = -dd1(j, :);
    end
end
figure(); imshow(log10(abs(Dd))+6);

```



```

pd = zeros((m-1)*(n-2), 3);

% set tangent constraints
d10 = double(subs(diff(d0, t), t, 0));
d11 = double(subs(diff(d0, t), t, 1));
disp(d10)

```

```

0    1    0    0    0    0    0    0

```

```
disp(d11)
```

```

0    1    2    3    4    5    6    7

```

```

d00 = zeros(size(d0));
Dt = zeros(2*n, 3*k*(n-1));
for i = 0:n-1
    w = tangents(i+1, :)' ;
    u = cross(w, [1e-8, 2e-8, 1]');
    u = u / norm(u);
    v = cross(u, w);
    if i == n-1
        d = kron(eye(3), d11);
    end
end

```

```

        i1 = k*(i-1);
    else
        d = kron(eye(3), d10);
        i1 = k*i;
    end
    ud = u' * d;
    vd = v' * d;
    for j = 0:3-1
        j1 = i1 + j*(k*(n-1));
        Dt(2*i+1, j1+1:j1+k) = ud(k*j+1:k*j+k);
        Dt(2*i+2, j1+1:j1+k) = vd(k*j+1:k*j+k);
    end
end
disp(Dt)

```

```

0    -0.8944         0         0         0         0         0         0         0         0
0     0.4472         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0

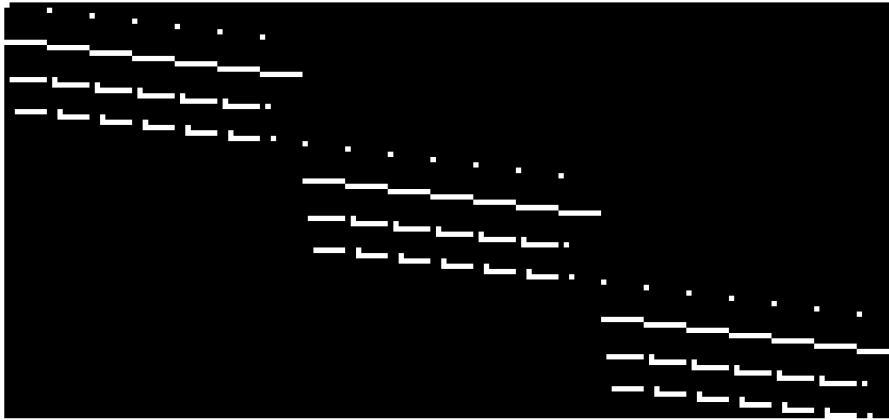
```

```

pt = zeros(2*n, 1);

% constraints
D = [Dp0; Dp1; Dd];
D = kron(eye(3), D);
figure(); imshow(log10(abs(D))+6);

```



```
p = [pp0; pp1; pd];
ceq = [p(:, 1); p(:, 2); p(:, 3)];
if 1
    D = [D; Dt];
    ceq = [ceq; pt];
end
nc = size(D); nc = nc(1)
```

```
nc = 94
```

```
disp([size(D) rank(D)]);
```

```
94    168    94
```

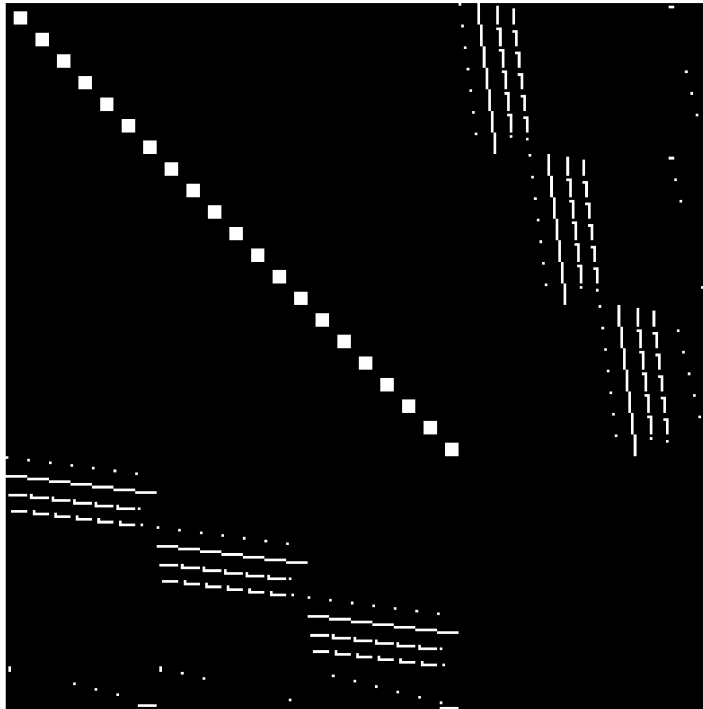
```
A = zeros(3*k*(n-1)+nc, 3*k*(n-1)+nc);
A = [kron(eye(3), T), D'; D, zeros(nc, nc)];
disp([size(A) rank(A)])
```

```
262    262    262
```

```
disp(cond(A))
```

```
2.0181e+08
```

```
figure(); imshow(log10(abs(A))+6)
```



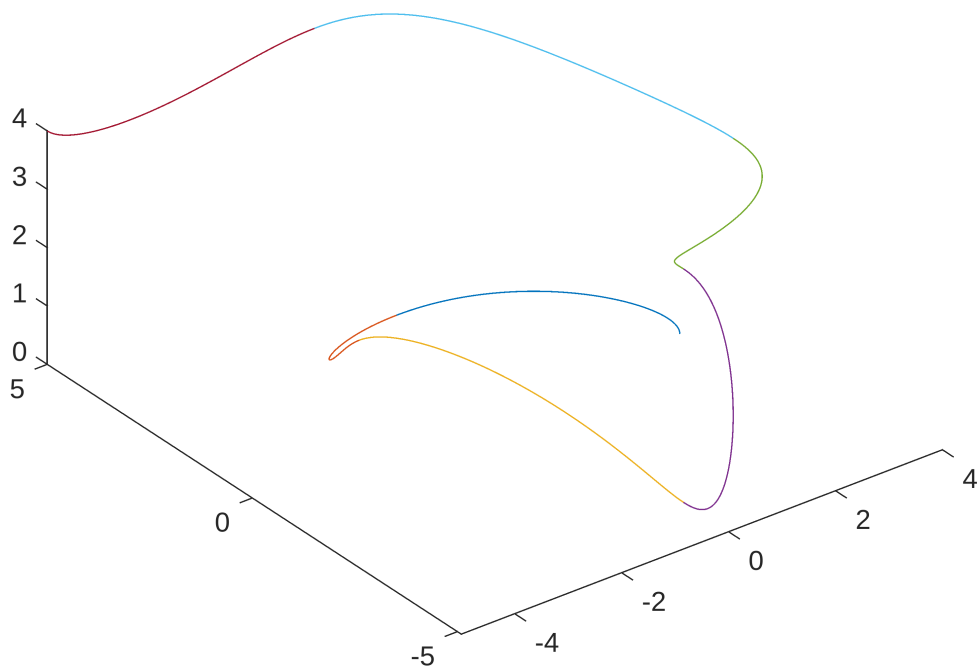
```
%figure(); imshow((abs(inv(A)).^0.4), [])
%figure(); boxplot(log10(abs(eig(A))))
b = [zeros(3*k*(n-1), 1); ceq];
c = A \ b;
c = [c(1:k*(n-1), :), c(k*(n-1)+1:2*k*(n-1)), c(2*k*(n-1)+1:3*k*(n-1))];

% plot
for i = 0:n-1-1
    t = (0:0.01:1)' .^ (0:k-1);
    ci = c(k*i+1:k*(i+1), :);
    disp(ci)
    p = t * ci;
    plot3(p(:, 1), p(:, 2), p(:, 3))
    hold on
end
```

3.0000	0	0
-0.0000	0.0000	1.6314
-2.9487	7.1253	-2.1752
-0.0000	-0.0000	-0.0000
-0.5911	-6.3760	0.0000
0.5398	2.2506	0.5438
0.0000	0.0000	0.0000
-0.0000	-0.0000	-0.0000
0	3.0000	0
-5.5628	-0.0000	0.0000
-1.0974	-8.6240	3.2628

3.0335	-2.9975	5.4380
2.1079	16.8671	-10.6644
-1.4812	-8.2456	3.9636
-0.0000	-0.0000	0.0000
0.0000	0.0000	-0.0000
-3.0000	0	2.0000
2.3686	-0.0000	0.0000
5.8388	1.1304	-4.7736
-3.3466	-17.9848	2.4164
-5.2979	17.6770	-0.5121
3.4370	-4.8226	0.8692
0.0000	-0.0000	0.0000
-0.0000	0.0000	-0.0000
0	-4.0000	0
0.0000	-5.0987	0.0000
-1.6182	5.0118	8.0956
9.8320	4.4970	9.0605
-14.8093	-6.4361	-22.4079
6.5955	2.0259	9.2518
0.0000	0.0000	0.0000
-0.0000	-0.0000	-0.0000
0	-4.0000	4.0000
-0.0000	2.8012	0.0000
4.9773	0.1454	-6.6526
16.5502	-0.9882	11.9467
-28.0324	3.6935	-8.9356
10.5049	-1.6519	2.6415
0.0000	-0.0000	0.0000
-0.0000	0.0000	-0.0000
4.0000	0	3.0000
0.0000	6.6417	0.0000
-8.5178	2.8223	1.9890
9.4693	-2.7337	2.6194
-6.9457	-4.5662	-6.2057
1.9942	2.8360	2.5973
0.0000	0.0000	0.0000
-0.0000	-0.0000	-0.0000
0	5.0000	4.0000
-6.4394	-0.0000	-0.0000
-1.8418	-4.4166	-1.4137
1.6288	7.3610	3.7699
3.0255	-3.6805	-3.2987
-1.3731	0.7361	0.9425
-0.0000	0.0000	0.0000
0.0000	-0.0000	-0.0000

axis equal



```
%view([-90, 90])
```

Debug curvature gradient

```
t = sym('t', 'real');
C = sym('c', [k, 3], 'real');
M = C * C';
kn = (d1*M*d1') .* (d2*M*d2') - (d1*M*d2').^2;
km = (d1*M*d1') .^ 2;
kc = kn ./ km;

gn = d2'*d1*M*d1'*d2 - 2*d2'*d1*M*d2'*d1 + d1'*d2*M*d2'*d1;
gm = 2*d1'*d1*M*d1'*d1;
gc = gn * (1/km) - gm * (kn/km^2);

tc = 0:0.01:1;
cc = randn([k, 3]);
%cc(:, 3) = 0.01*cc(:, 3);

D1 = sym('D1', [1, k], 'real');
D2 = sym('D2', [k, k], 'real');
D3 = sym('D3', [k, 1], 'real');
%M = sym('M', [k, k], 'real');
v = D1*M*D2*M*D3;
```

```

v = kc;
G0 = reshape(gradient(v, reshape(C, [3*k, 1])), [k, 3]);
G = D3*D1*M*D2 + D2*M*D3*D1;
G = gc;
G = (G+G') * C;
dG = G0-G;
dG = subs(subs(dG, C, randn(size(C))), t, rand());
disp(double(dG))

```

```

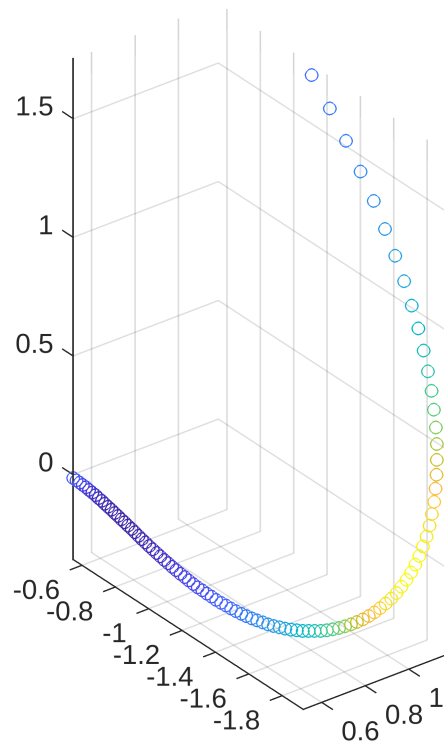
0      0      0
0      0      0
0      0      0
0      0      0
0      0      0
0      0      0
0      0      0
0      0      0

```

```

pc = tc' .^ (0:k-1) * cc;
kcp = double(subs(subs(kc, C, cc), t, tc));
figure
scatter3(pc(:, 1), pc(:, 2), pc(:, 3), 20, kcp);
%view([-90, 90]);
axis equal;

```



Minimize curvature

```

if length(c) == k*(n-1)
    c = reshape(c, [3*k*(n-1), 1])';
end
disp(sum(curvature(c)))

```

1.6465e+03

```

c = lsqnonlin(@curvature, c, -Inf(size(c)), Inf(size(c)), [], [], D, ceq);

```

Solver stopped prematurely.

fmincon stopped because it exceeded the function evaluation limit,
options.MaxFunctionEvaluations = 3.000000e+03.

```

disp(sum(curvature(c)))

```

614.5970

```

c = reshape(c', [k*(n-1), 3]);
disp(c)

```

3.0000	0	0
0	0	0.0007
-0.2740	1.1923	4.2799
-1.5776	2.6584	11.2606
8.1394	13.8961	-32.0636
-5.8205	-13.9825	-0.1082
-13.3304	-9.3182	29.9609
9.8630	8.5540	-13.3304
0	3.0000	0
-12.7667	0	0
-7.2077	-7.4210	14.0745
51.1518	-18.0769	-17.9063
-29.1043	42.2017	-0.8410
-23.0708	-4.5987	8.5733
17.1925	-28.9951	0.6290
0.8053	13.8900	-2.5295
-3.0000	0	2.0000
3.2941	0	0
15.7131	2.3349	-2.6423
-28.8463	-27.5777	-7.0811
5.3732	27.7083	9.3464
11.2043	-1.6069	2.7867
-0.4731	-4.1470	-6.0766
-3.2653	-0.7116	1.6670
0	-4.0000	0
0	-5.1277	0
-2.2112	-7.3658	3.9162
9.3859	18.8382	17.3270
-10.6226	0.0021	-23.3506
5.2154	-3.0281	5.7695
-5.0508	-7.9493	-2.3761
3.2833	4.6304	2.7140
0	-4.0000	4.0000
0	6.2411	0
7.5514	-3.1190	-5.1584
9.3762	-7.2330	10.2314
-23.9964	7.6489	-10.6134
13.0501	3.7046	4.0649

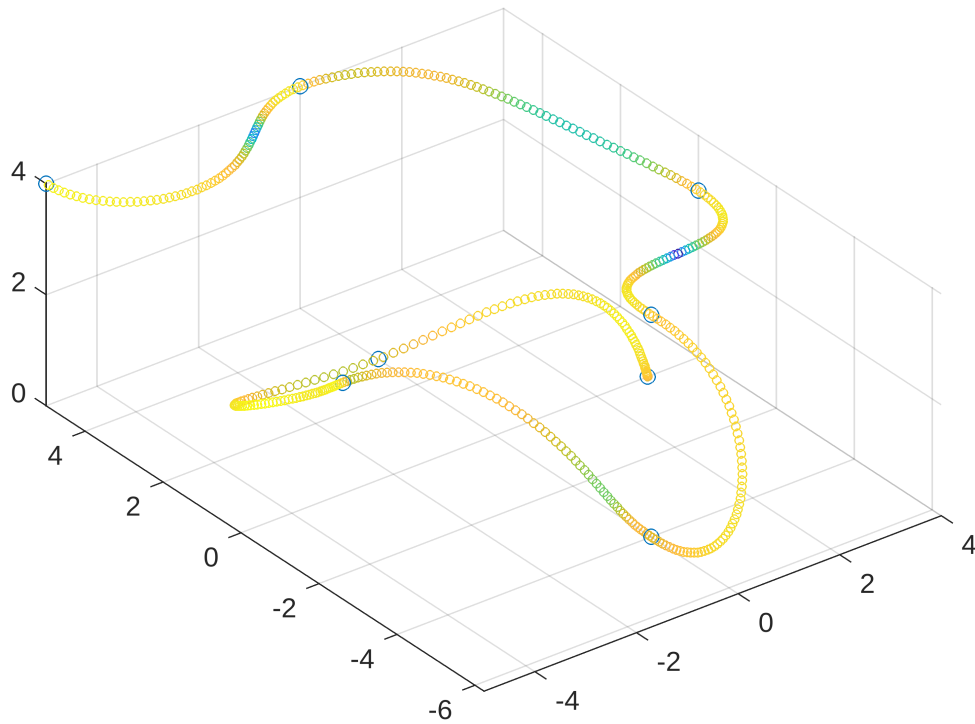
-1.3726	-2.1983	1.5767
-0.6086	-1.0443	-1.1013
4.0000	0	3.0000
0	6.9227	0
-11.1681	3.2160	3.0291
18.4296	-5.0396	0.7441
-11.6907	-5.8172	-6.1467
-2.0943	5.8267	3.4243
-0.3255	3.3405	0.4696
2.8489	-3.4491	-0.5204
0	5.0000	4.0000
-6.2920	0	0
7.9783	-10.8624	-1.2603
-10.8021	8.2695	3.7014
-3.8864	4.3484	-2.7976
4.3781	2.2483	0.4306
8.9719	-3.6780	-0.9725
-5.3478	-0.3259	0.8984

```

ps = zeros(0, 3);
for i = 0:n-1-1
    t = ((1:nd)'+0.5)/nd;
    t = t .^ (0:k-1);
    ci = c(k*i+1:k*(i+1), :);
    p = t * ci;
    ps = vertcat(ps, p);
end

figure
scatter3(points(:, 1), points(:, 2), points(:, 3), 30)
hold on
scatter3(ps(:, 1), ps(:, 2), ps(:, 3), 10, log10(curvature(c')))
axis equal

```



```
%view([-90 90])
```

Minimize curvature variation

```
if length(c) == k*(n-1)
    c = reshape(c, [3*k*(n-1), 1])';
end
disp(sum(curvatureVariation(c)))
```

```
3.2772e+06
```

```
c = lsqnonlin(@curvatureVariation, c, -Inf(size(c)), Inf(size(c)), [], [],
D, ceq);
```

Solver stopped prematurely.

fmincon stopped because it exceeded the function evaluation limit,
options.MaxFunctionEvaluations = 3.000000e+03.

```
disp(sum(curvatureVariation(c)))
```

```
1.1269e+03
```

```
c = reshape(c', [k*(n-1), 3]);
```

```

ps = zeros(0, 3);
for i = 0:n-1-1
    t = ((1:nd)')-0.5)/nd;
    t = t .^ (0:k-1);
    ci = c(k*i+1:k*(i+1), :);
    disp(ci)
    p = t * ci;
    ps = vertcat(ps, p);
end

```

```

3.0000      0      0
0.0000 -0.0000 -0.0000
-1.7246  1.5257  4.7291
-4.2983  3.8582 11.9653
15.6368  8.6094 -34.2754
-6.2120 -6.2825 -3.0109
-17.6988 -15.3243 37.3415
11.2969 10.6136 -16.7495

```

```

      0  3.0000      0
-11.9714 -0.0000 -0.0000
-11.1660 -5.0484 13.2454
54.1314 -21.4845 -17.1296
-27.8991 40.3895  0.7521
-23.0193 -2.5086  7.0723
16.0926 -25.9716 -0.1093
0.8317 11.6235 -1.8308

```

```

-3.0000      0  2.0000
 3.7756  0.0000 -0.0000
12.4952  2.2694 -2.9953
-24.5198 -27.4067 -8.2335
 5.7080 28.0617 13.1288
 9.3564 -2.1205  1.6512
-1.8869 -4.5994 -8.7785
-1.9284 -0.2045  3.2273

```

```

      0 -4.0000      0
0.0000 -5.0646  0.0000
-2.0530 -6.0699  3.6842
 9.2378 17.8688 17.5002
-11.0325 -0.2578 -23.5001
 5.4268 -3.4918  6.1311
-4.4424 -8.4070 -2.1836
 2.8633  5.4223  2.3682

```

```

      0 -4.0000  4.0000
0.0000  5.4262 -0.0000
 7.2264 -1.1641 -6.5263
 8.5251 -9.4918 12.4247
-21.2038  7.9980 -10.2718
13.0187  5.2064  2.9289
-4.6586 -2.3760  0.8902
 1.0922 -1.5987 -0.4457

```

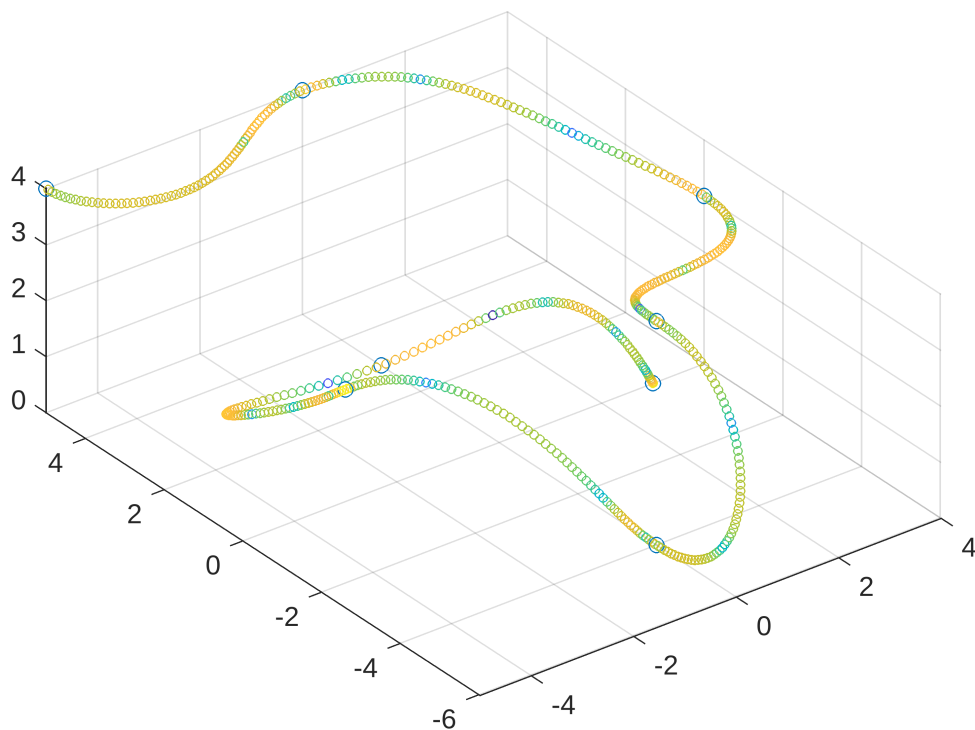
```

 4.0000      0  3.0000
-0.0000  7.1999 -0.0000
-11.1777  1.2004  2.3988
18.0412 -2.4424  1.6511
-10.5928 -4.7592 -6.3309
-2.3403  4.2329  3.4400
-1.1435  1.3802  0.5143

```

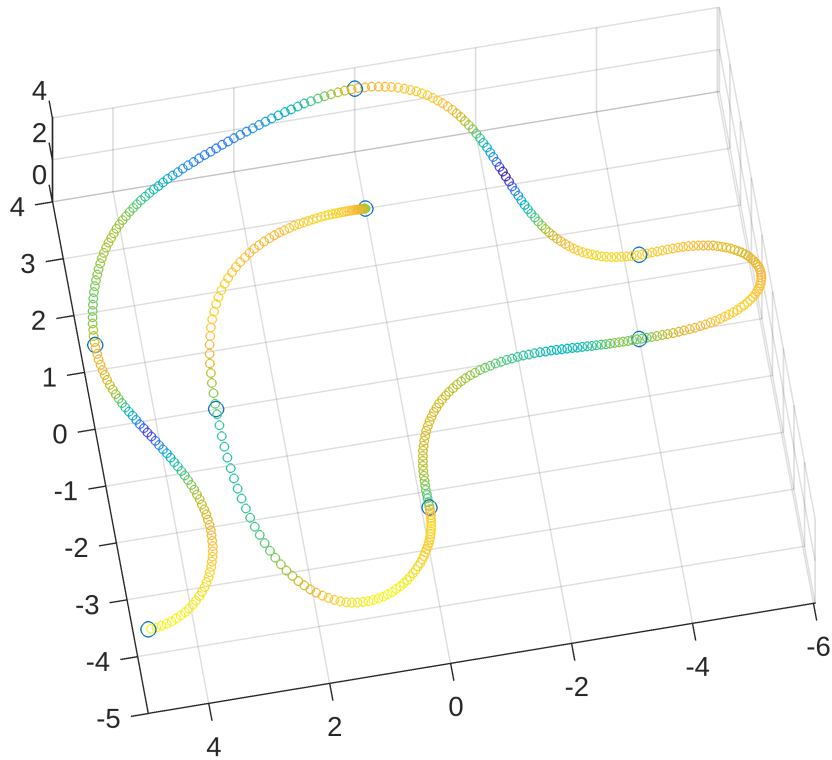
3.2130	-1.8118	-0.6733
0	5.0000	4.0000
-6.6740	-0.0000	0.0000
6.3078	-9.6972	-2.6580
-7.9506	8.3143	6.3524
-3.1792	3.6287	-2.7874
3.4422	1.3620	-1.0129
7.9605	-4.0860	-1.7317
-4.9068	0.4782	1.8375

```
figure
scatter3(points(:, 1), points(:, 2), points(:, 3), 30)
hold on
scatter3(ps(:, 1), ps(:, 2), ps(:, 3), 10, log10(curvatureVariation(c'))))
axis equal
```



```
figure
scatter3(points(:, 1), points(:, 2), points(:, 3), 30)
hold on
scatter3(ps(:, 1), ps(:, 2), ps(:, 3), 10, log10(curvature(c'))))
axis equal

view([-100 70])
```



Functions

```
function r = curvature(c)
    global k
    global n
    global nd
    t = ((1:nd)'-0.5)/nd;
    d10 = (0:k-1) .* t.^((0:k-1)-1);
    d20 = (0:k-1).*((0:k-1)-1) .* t.^((0:k-1)-2);
    c = reshape(c', [k*(n-1), 3]);
    r = zeros(nd*(n-1), 1);

    for pi = 0:n-1-1
        ci = c(k*pi+1:k*pi+k, :);
        M = ci * ci';
        for i = 1:nd
            d1 = d10(i, :);
            d2 = d20(i, :);
            kn = (d1*M*d1').*(d2*M*d2') - (d1*M*d2').^2;
            km = (d1*M*d1').^2;
            kc = kn / km^1.25; % \kappa^2 ds
            %kc = kn^0.5 / km^0.5; % \kappa ds
            %kc = kn / km^1.5; % \kappa^2 dt
        end
    end
end
```



```

        %kc = kn^0.5 / km^0.75; % \kappa dt
        %kc = kn / km; % \kappa^2 ds/dt ds
        r(pi*nd+i) = kc;
    end
end
end

function r = curvatureVariation(c)
    global k
    global n
    global nd
    t = ((1:nd)'.5)/nd;
    d10 = (0:k-1) .* t.^((0:k-1)-1);
    d20 = (0:k-1).*((0:k-1)-1) .* t.^((0:k-1)-2);
    d30 = (0:k-1).*((0:k-1)-1).*((0:k-1)-2) .* t.^((0:k-1)-3);
    c = reshape(c', [k*(n-1), 3]);
    r = zeros(nd*(n-1), 1);
    for pi = 0:n-1-1
        ci = c(k*pi+1:k*pi+k, :);
        M = ci * ci';
        for i = 1:nd
            d1 = d10(i, :);
            d2 = d20(i, :);
            d3 = d30(i, :);
            v11 = d1*M*d1';
            g11 = d2*M*d1' + d1*M*d2';
            v22 = d2*M*d2';
            g22 = d3*M*d2' + d2*M*d3';
            v12 = d1*M*d2';
            g12 = d2*M*d2' + d1*M*d3';
            kn = v11*v22 - v12*v12;
            gn = g11*v22+v11*g22 - 2*v12*g12;
            km = v11^3;
            gm = 3*v11^2*g11;
            kc0 = kn / km;
            gc0 = (gn*km-kn*gm)/(km*km);
            kc = kc0^0.5;
            gc = gc0/(2.0*kc);
            ri = pi*nd+i;
            r(pi*nd+i) = gc.^2;
        end
    end
end
end
end

```