```
clear

global k
global n
global nd
n = 8;   % number of points
k = 8;   % number of polynomial coefficients (deg+1)
m = 3;   % 3 jerk, 4 snap
nd = 64;

points = [
    3, 0, 0;
    0, 3, 0;
    -3, 0, 2;
    0, -4, 0;
    0, -4, 4;
    4, 0, 3;
    0, 5, 4;
    -5, 5, 4;
];
tangents = [
    0, 0, 1;
    1, 0, 0;
    1, 0, 0;
    0, 1, 0;
    0, 1, 0;
    0, 1, 0;
    1, 0, 0;
    0, 1, 0;
];
%points(:, 3) = points(:, 3) * 0.01;
%tangents(:, 3) = tangents(:, 3) * 0.01;
```

Minimum jerk/snap

```
% set cost
t = sym('t', 'real');
d0 = t .^ (0:k-1);
d1 = diff(d0, t);
d2 = diff(d1, t);
d3 = d0;
for i = 1:m
    d3 = diff(d3, t);
    disp(i)
    disp(int(d3'*d3, t, 0, 1))
end
```

        1

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{4}{3} & \frac{3}{2} & \frac{8}{5} & \frac{5}{3} & \frac{12}{7} & \frac{7}{4} \\ 0 & 1 & \frac{3}{2} & \frac{9}{5} & 2 & \frac{15}{7} & \frac{9}{4} & \frac{7}{3} \\ 0 & 1 & \frac{8}{5} & 2 & \frac{16}{7} & \frac{5}{2} & \frac{8}{3} & \frac{14}{5} \\ 0 & 1 & \frac{5}{3} & \frac{15}{7} & \frac{5}{2} & \frac{25}{9} & 3 & \frac{35}{11} \\ 0 & 1 & \frac{12}{7} & \frac{9}{4} & \frac{8}{3} & 3 & \frac{36}{11} & \frac{7}{2} \\ 0 & 1 & \frac{7}{4} & \frac{7}{3} & \frac{14}{5} & \frac{35}{11} & \frac{7}{2} & \frac{49}{13} \end{pmatrix}$$

2

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 0 & 6 & 12 & 18 & 24 & 30 & 36 \\ 0 & 0 & 8 & 18 & \frac{144}{5} & 40 & \frac{360}{7} & 63 \\ 0 & 0 & 10 & 24 & 40 & \frac{400}{7} & 75 & \frac{280}{3} \\ 0 & 0 & 12 & 30 & \frac{360}{7} & 75 & 100 & 126 \\ 0 & 0 & 14 & 36 & 63 & \frac{280}{3} & 126 & \frac{1764}{11} \end{pmatrix}$$

3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 72 & 120 & 180 & 252 \\ 0 & 0 & 0 & 72 & 192 & 360 & 576 & 840 \\ 0 & 0 & 0 & 120 & 360 & 720 & 1200 & 1800 \\ 0 & 0 & 0 & 180 & 576 & 1200 & \frac{14400}{7} & 3150 \\ 0 & 0 & 0 & 252 & 840 & 1800 & 3150 & 4900 \end{pmatrix}$$

```
Ti = int(d3'*d3, t, 0, 1);
T = zeros(k*(n-1), k*(n-1));
invTi = inv(Ti(m+1:end, m+1:end));
invTi = double(invTi / gcd(invTi));
Ti = double(Ti / gcd(Ti));
disp(Ti)
```

2

```
         0            0            0            0            0            0            0            0
         0            0            0            0            0            0            0            0
         0            0            0            0            0            0            0            0
         0            0            0          126          252          420          630          882
         0            0            0          252          672         1260         2016         2940
         0            0            0          420         1260         2520         4200         6300
         0            0            0          630         2016         4200         7200        11025
         0            0            0          882         2940         6300        11025        17150
```

```matlab
invT = zeros((n-1)*(k-m), (n-1)*(k-m));
for i = 0:n-1-1
    i1 = (k-m)*i;
    invT(i1+1:i1+k-m, i1+1:i1+k-m) = invTi;
    i1 = k*i;
    T(i1+1:i1+k, i1+1:i1+k) = Ti;
end

% set point constraints
Dp0 = zeros(n-1, k*(n-1));
Dp1 = zeros(n-1, k*(n-1));
for i = 0:n-1-1
    i1 = k*i;
    Dp0(i+1, i1+1) = 1;
    Dp1(i+1, i1+1:i1+k) = ones(1, k);
end
pp0 = points(1:n-1, :);
pp1 = points(2:n, :);

% set derivative constraints
Dd = zeros((m-1)*(n-2), k*(n-1));
dd0 = zeros(m-1, k);
dd1 = zeros(m-1, k);
dd = d0;
for i = 1:m-1
    dd = diff(dd, t);
    dd0(i, :) = double(subs(dd, t, 1));
    dd1(i, :) = double(subs(dd, t, 0));
end
disp(dd0)
```

```
     0     1     2     3     4     5     6     7
     0     0     2     6    12    20    30    42
```

```matlab
disp(dd1)
```

```
     0     1     0     0     0     0     0     0
     0     0     2     0     0     0     0     0
```
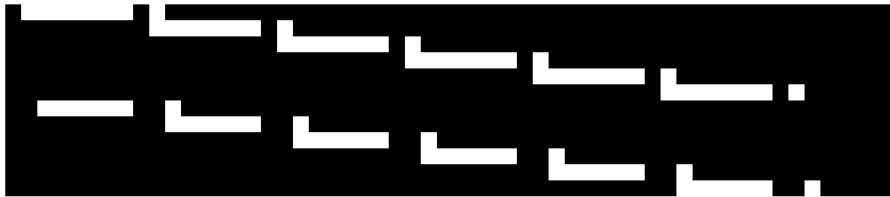
```matlab
for i = 0:n-2-1
    i1 = k*i;
    for j = 1:m-1
        j1 = (j-1)*(n-2);
        Dd(j1+i+1, i1+1:i1+k) = dd0(j, :);
```

```
            Dd(j1+i+1, i1+k+1:i1+k+k) = -dd1(j, :);
        end
    end
end
figure(); imshow(log10(abs(Dd))+6);
```



```
pd = zeros((m-1)*(n-2), 3);

% set tangent constraints
d10 = double(subs(diff(d0, t), t, 0));
d11 = double(subs(diff(d0, t), t, 1));
disp(d10)
```

```
     0     1     0     0     0     0     0     0
```

```
disp(d11)
```

```
     0     1     2     3     4     5     6     7
```

```
d00 = zeros(size(d0));
Dt = zeros(2*n, 3*k*(n-1));
for i = 0:n-1
    w = tangents(i+1, :)';
    u = cross(w, [1e-8, 2e-8, 1]');
    u = u / norm(u);
    v = cross(u, w);
    if i == n-1
        d = kron(eye(3), d11);
```

```matlab
        i1 = k*(i-1);
    else
        d = kron(eye(3), d10);
        i1 = k*i;
    end
    ud = u' * d;
    vd = v' * d;
    for j = 0:3-1
        j1 = i1 + j*(k*(n-1));
        Dt(2*i+1, j1+1:j1+k) = ud(k*j+1:k*j+k);
        Dt(2*i+2, j1+1:j1+k) = vd(k*j+1:k*j+k);
    end
end
disp(Dt)
```

```
     0   -0.8944        0        0        0        0        0        0        0        0
     0    0.4472        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
     0         0        0        0        0        0        0        0        0        0
```
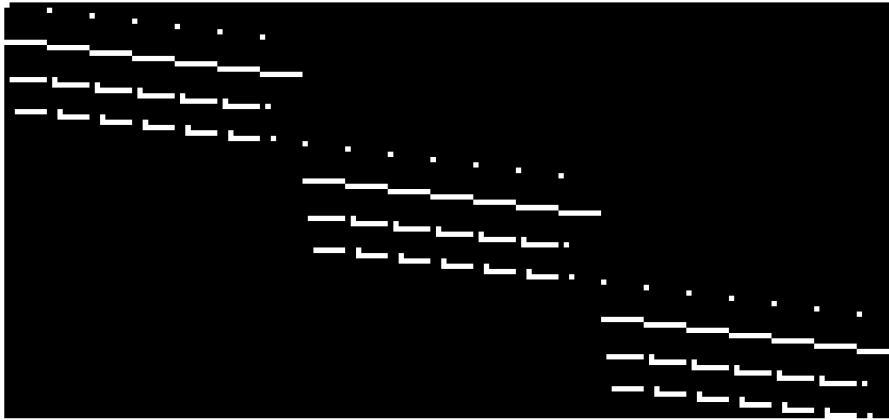
```matlab
pt = zeros(2*n, 1);

% constraints
D = [Dp0; Dp1; Dd];
D = kron(eye(3), D);
figure(); imshow(log10(abs(D))+6);
```

```
p = [pp0; pp1; pd];
ceq = [p(:, 1); p(:, 2); p(:, 3)];
if 1
    D = [D; Dt];
    ceq = [ceq; pt];
end
nc = size(D); nc = nc(1)
```

nc = 94

```
disp([size(D) rank(D)]);
```

    94   168   94

```
A = zeros(3*k*(n-1)+nc, 3*k*(n-1)+nc);
A = [kron(eye(3), T), D'; D, zeros(nc, nc)];
disp([size(A) rank(A)])
```
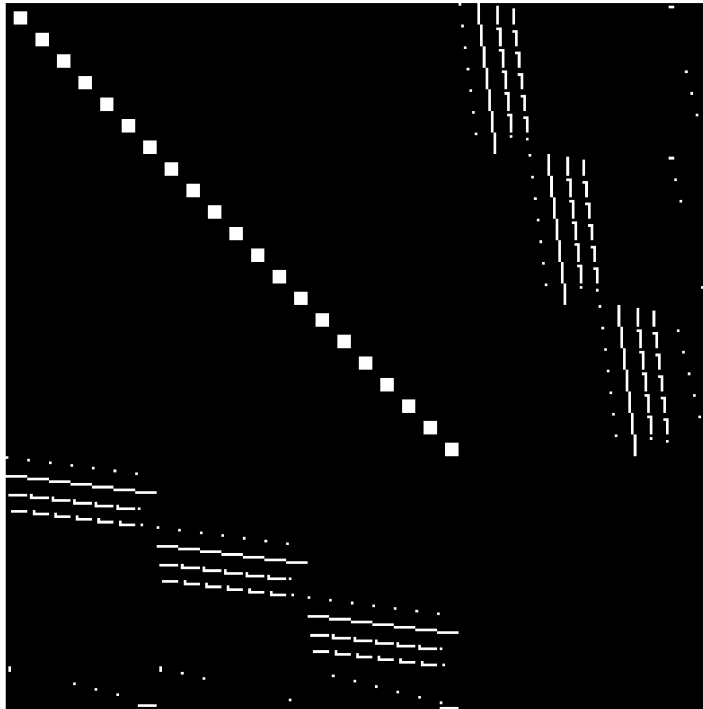
   262   262   262

```
disp(cond(A))
```

   2.0181e+08

```
figure(); imshow(log10(abs(A))+6)
```

```matlab
%figure(); imshow((abs(inv(A)).^0.4), [])
%figure(); boxplot(log10(abs(eig(A))))
b = [zeros(3*k*(n-1), 1); ceq];
c = A \ b;
c = [c(1:k*(n-1), :), c(k*(n-1)+1:2*k*(n-1)), c(2*k*(n-1)+1:3*k*(n-1))];

% plot
for i = 0:n-1-1
    t = (0:0.01:1)' .^ (0:k-1);
    ci = c(k*i+1:k*(i+1), :);
    disp(ci)
    p = t * ci;
    plot3(p(:, 1), p(:, 2), p(:, 3))
    hold on
end
```

```
    3.0000         0         0
   -0.0000    0.0000    1.6314
   -2.9487    7.1253   -2.1752
   -0.0000   -0.0000   -0.0000
   -0.5911   -6.3760    0.0000
    0.5398    2.2506    0.5438
    0.0000    0.0000    0.0000
   -0.0000   -0.0000   -0.0000

         0    3.0000         0
   -5.5628   -0.0000    0.0000
   -1.0974   -8.6240    3.2628
```

```
    3.0335    -2.9975     5.4380
    2.1079    16.8671   -10.6644
   -1.4812    -8.2456     3.9636
   -0.0000    -0.0000     0.0000
    0.0000     0.0000    -0.0000

   -3.0000         0     2.0000
    2.3686    -0.0000     0.0000
    5.8388     1.1304    -4.7736
   -3.3466   -17.9848     2.4164
   -5.2979    17.6770    -0.5121
    3.4370    -4.8226     0.8692
    0.0000    -0.0000     0.0000
   -0.0000     0.0000    -0.0000

         0    -4.0000          0
    0.0000    -5.0987     0.0000
   -1.6182     5.0118     8.0956
    9.8320     4.4970     9.0605
  -14.8093    -6.4361   -22.4079
    6.5955     2.0259     9.2518
    0.0000     0.0000     0.0000
   -0.0000    -0.0000    -0.0000

         0    -4.0000     4.0000
   -0.0000     2.8012     0.0000
    4.9773     0.1454    -6.6526
   16.5502    -0.9882    11.9467
  -28.0324     3.6935    -8.9356
   10.5049    -1.6519     2.6415
    0.0000    -0.0000     0.0000
   -0.0000     0.0000    -0.0000

    4.0000         0     3.0000
    0.0000     6.6417     0.0000
   -8.5178     2.8223     1.9890
    9.4693    -2.7337     2.6194
   -6.9457    -4.5662    -6.2057
    1.9942     2.8360     2.5973
    0.0000     0.0000     0.0000
   -0.0000    -0.0000    -0.0000

         0     5.0000     4.0000
   -6.4394    -0.0000    -0.0000
   -1.8418    -4.4166    -1.4137
    1.6288     7.3610     3.7699
    3.0255    -3.6805    -3.2987
   -1.3731     0.7361     0.9425
   -0.0000     0.0000     0.0000
    0.0000    -0.0000    -0.0000
```

```
axis equal
```
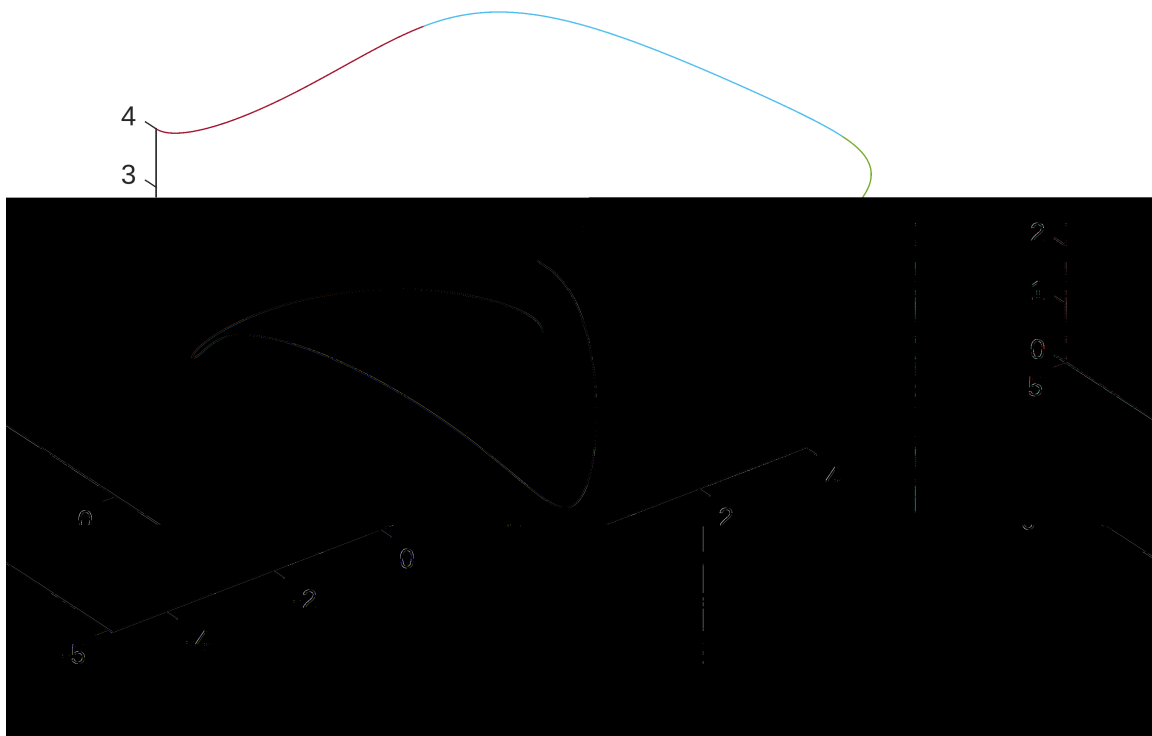
```matlab
%view([-90, 90])
```

Debug curvature gradient

```matlab
t = sym('t', 'real');
C = sym('c', [k, 3], 'real');
M = C * C';
kn = (d1*M*d1') .* (d2*M*d2') - (d1*M*d2').^2;
km = (d1*M*d1') .^ 2;
kc = kn ./ km;

gn = d2'*d1*M*d1'*d2 - 2*d2'*d1*M*d2'*d1 + d1'*d2*M*d2'*d1;
gm = 2*d1'*d1*M*d1'*d1;
gc = gn * (1/km) - gm * (kn/km^2);

tc = 0:0.01:1;
cc = randn([k, 3]);
%cc(:, 3) = 0.01*cc(:, 3);

D1 = sym('D1', [1, k], 'real');
D2 = sym('D2', [k, k], 'real');
D3 = sym('D3', [k, 1], 'real');
%M = sym('M', [k, k], 'real');
v = D1*M*D2*M*D3;
```

```
v = kc;
G0 = reshape(gradient(v, reshape(C, [3*k, 1])), [k, 3]);
G = D3*D1*M*D2 + D2*M*D3*D1;
G = gc;
G = (G+G') * C;
dG = G0-G;
dG = subs(subs(dG, C, randn(size(C))), t, rand());
disp(double(dG))
```

```
     0     0     0
     0     0     0
     0     0     0
     0     0     0
     0     0     0
     0     0     0
     0     0     0
     0     0     0
```
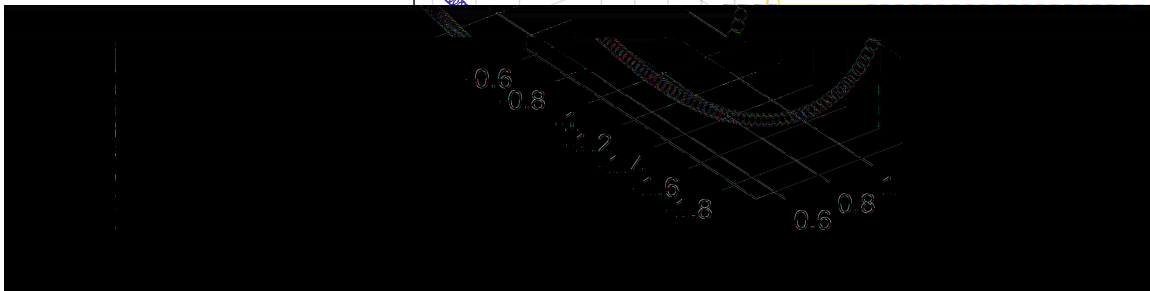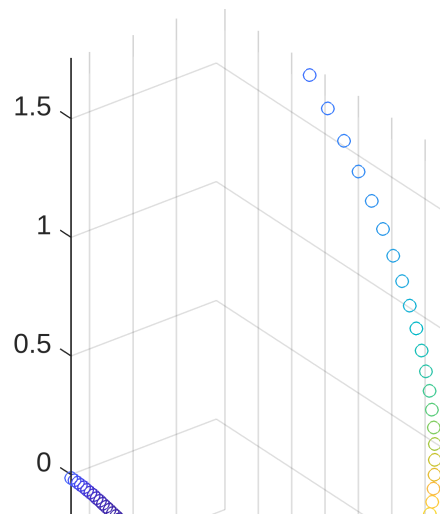
```
pc = tc' .^ (0:k-1) * cc;
kcp = double(subs(subs(kc, C, cc), t, tc));
figure
scatter3(pc(:, 1), pc(:, 2), pc(:, 3), 20, kcp);
%view([-90, 90]);
axis equal;
```


Minimize curvature

```
if length(c) == k*(n-1)
    c = reshape(c, [3*k*(n-1), 1])';
end
disp(sum(curvature(c)))
```

```
    1.6465e+03
```

```
c = lsqnonlin(@curvature, c, -Inf(size(c)), Inf(size(c)), [], [], D, ceq);
```

Solver stopped prematurely.

fmincon stopped because it exceeded the function evaluation limit,
options.MaxFunctionEvaluations = 3.000000e+03.

```
disp(sum(curvature(c)))
```

```
  614.5970
```

```
c = reshape(c', [k*(n-1), 3]);
disp(c)
```

```
    3.0000         0         0
         0         0    0.0007
   -0.2740    1.1923    4.2799
   -1.5776    2.6584   11.2606
    8.1394   13.8961  -32.0636
   -5.8205  -13.9825   -0.1082
  -13.3304   -9.3182   29.9609
    9.8630    8.5540  -13.3304
         0    3.0000         0
  -12.7667         0         0
   -7.2077   -7.4210   14.0745
   51.1518  -18.0769  -17.9063
  -29.1043   42.2017   -0.8410
  -23.0708   -4.5987    8.5733
   17.1925  -28.9951    0.6290
    0.8053   13.8900   -2.5295
   -3.0000         0    2.0000
    3.2941         0         0
   15.7131    2.3349   -2.6423
  -28.8463  -27.5777   -7.0811
    5.3732   27.7083    9.3464
   11.2043   -1.6069    2.7867
   -0.4731   -4.1470   -6.0766
   -3.2653   -0.7116    1.6670
         0   -4.0000         0
         0   -5.1277         0
   -2.2112   -7.3658    3.9162
    9.3859   18.8382   17.3270
  -10.6226    0.0021  -23.3506
    5.2154   -3.0281    5.7695
   -5.0508   -7.9493   -2.3761
    3.2833    4.6304    2.7140
         0   -4.0000    4.0000
         0    6.2411         0
    7.5514   -3.1190   -5.1584
    9.3762   -7.2330   10.2314
  -23.9964    7.6489  -10.6134
   13.0501    3.7046    4.0649
```

```
-1.3726    -2.1983     1.5767
-0.6086    -1.0443    -1.1013
 4.0000          0     3.0000
      0     6.9227          0
-11.1681     3.2160     3.0291
18.4296    -5.0396     0.7441
-11.6907    -5.8172    -6.1467
-2.0943     5.8267     3.4243
-0.3255     3.3405     0.4696
 2.8489    -3.4491    -0.5204
      0     5.0000     4.0000
-6.2920          0          0
 7.9783   -10.8624    -1.2603
-10.8021     8.2695     3.7014
-3.8864     4.3484    -2.7976
 4.3781     2.2483     0.4306
 8.9719    -3.6780    -0.9725
-5.3478    -0.3259     0.8984
```

```matlab
ps = zeros(0, 3);
for i = 0:n-1-1
    t = ((1:nd)'-0.5)/nd;
    t = t .^ (0:k-1);
    ci = c(k*i+1:k*(i+1), :);
```
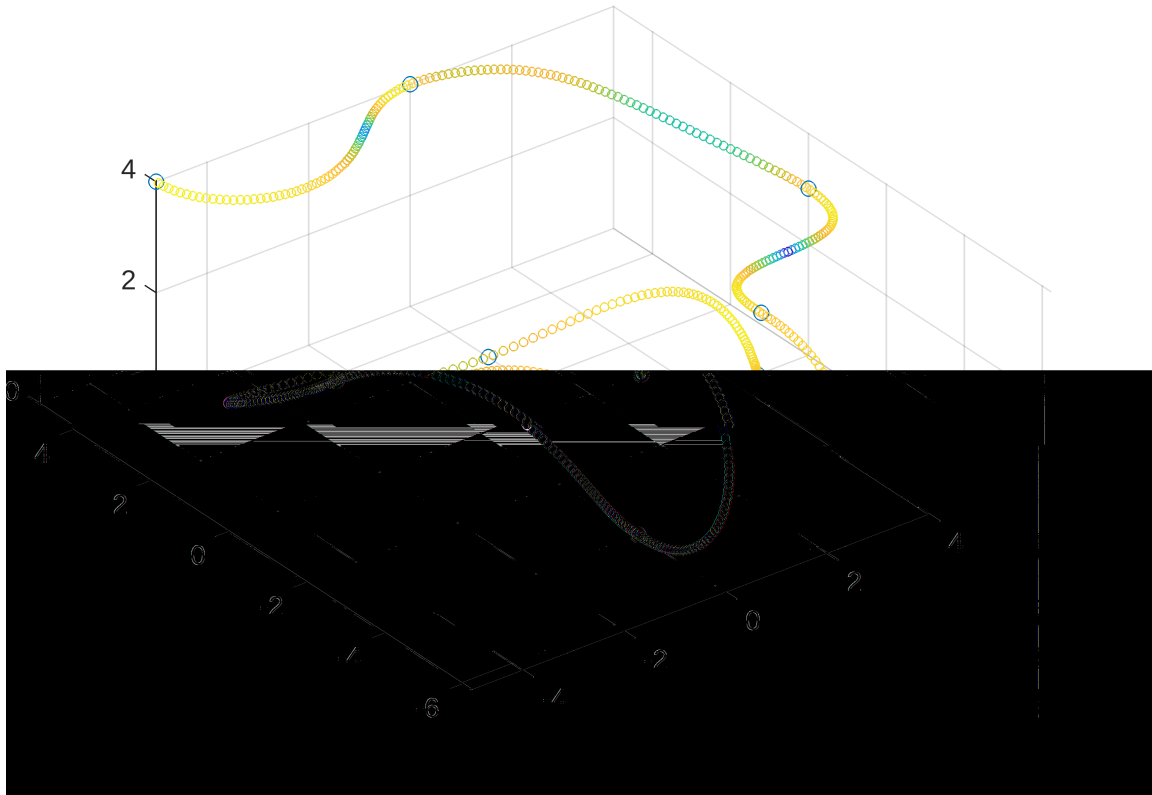
```
%view([-90 90])
```

Minimize curvature variation

```
if length(c) == k*(n-1)
    c = reshape(c, [3*k*(n-1), 1])';
end
disp(sum(curvatureVariation(c)))
```

```
   3.2772e+06
```

```
c = lsqnonlin(@curvatureVariation, c, -Inf(size(c)), Inf(size(c)), [], [],
D, ceq);
```

Solver stopped prematurely.

fmincon stopped because it exceeded the function evaluation limit,
options.MaxFunctionEvaluations = 3.000000e+03.

```
disp(sum(curvatureVariation(c)))
```

```
   1.1269e+03
```

```
c = reshape(c', [k*(n-1), 3]);
```

```matlab
ps = zeros(0, 3);
for i = 0:n-1-1
    t = ((1:nd)'-0.5)/nd;
    t = t .^ (0:k-1);
    ci = c(k*i+1:k*(i+1), :);
    disp(ci)
    p = t * ci;
    ps = vertcat(ps, p);
end
```

```
     3.0000         0         0
     0.0000   -0.0000   -0.0000
    -1.7246    1.5257    4.7291
    -4.2983    3.8582   11.9653
    15.6368    8.6094  -34.2754
    -6.2120   -6.2825   -3.0109
   -17.6988  -15.3243   37.3415
    11.2969   10.6136  -16.7495


          0    3.0000         0
   -11.9714   -0.0000   -0.0000
   -11.1660   -5.0484   13.2454
    54.1314  -21.4845  -17.1296
   -27.8991   40.3895    0.7521
   -23.0193   -2.5086    7.0723
    16.0926  -25.9716   -0.1093
     0.8317   11.6235   -1.8308


    -3.0000         0    2.0000
     3.7756    0.0000   -0.0000
    12.4952    2.2694   -2.9953
   -24.5198  -27.4067   -8.2335
     5.7080   28.0617   13.1288
     9.3564   -2.1205    1.6512
    -1.8869   -4.5994   -8.7785
    -1.9284   -0.2045    3.2273


          0   -4.0000         0
     0.0000   -5.0646    0.0000
    -2.0530   -6.0699    3.6842
     9.2378   17.8688   17.5002
   -11.0325   -0.2578  -23.5001
     5.4268   -3.4918    6.1311
    -4.4424   -8.4070   -2.1836
     2.8633    5.4223    2.3682


          0   -4.0000    4.0000
     0.0000    5.4262   -0.0000
     7.2264   -1.1641   -6.5263
     8.5251   -9.4918   12.4247
   -21.2038    7.9980  -10.2718
    13.0187    5.2064    2.9289
    -4.6586   -2.3760    0.8902
     1.0922   -1.5987   -0.4457


     4.0000         0    3.0000
    -0.0000    7.1999   -0.0000
   -11.1777    1.2004    2.3988
    18.0412   -2.4424    1.6511
   -10.5928   -4.7592   -6.3309
    -2.3403    4.2329    3.4400
    -1.1435    1.3802    0.5143
```
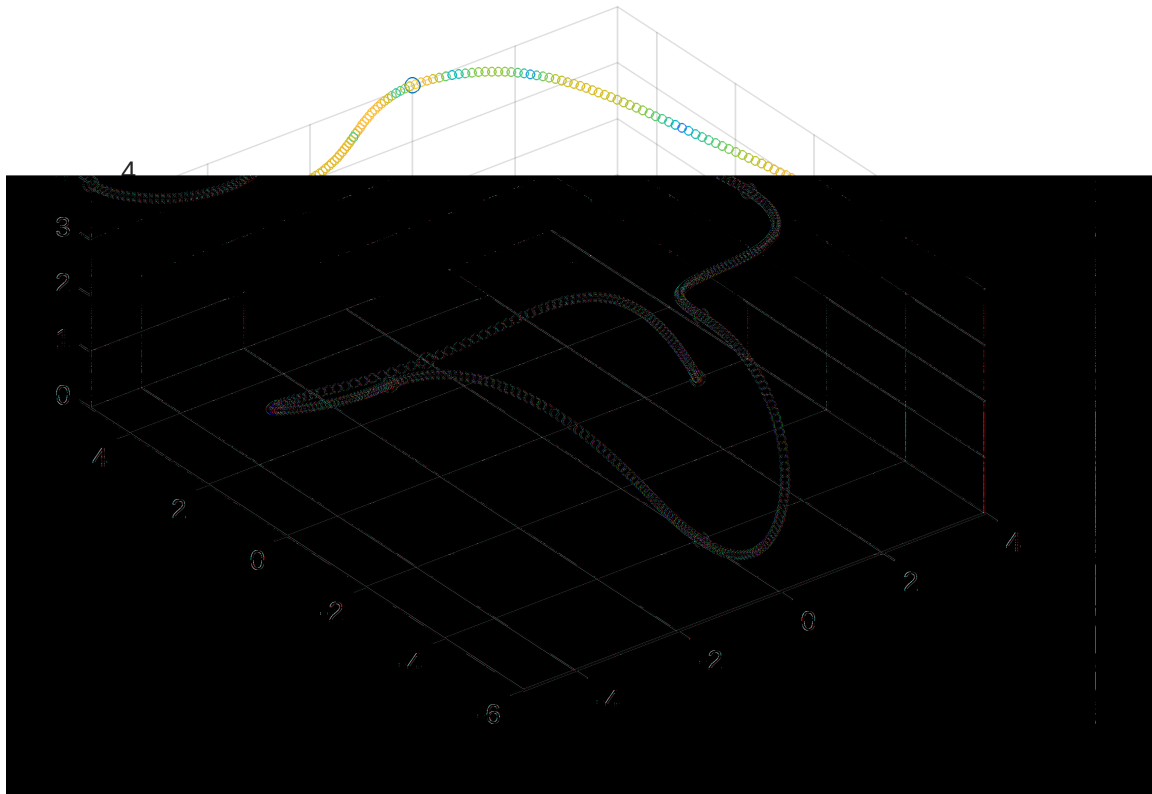
```
   3.2130    -1.8118    -0.6733

        0     5.0000     4.0000
  -6.6740    -0.0000     0.0000
   6.3078    -9.6972    -2.6580
  -7.9506     8.3143     6.3524
  -3.1792     3.6287    -2.7874
   3.4422     1.3620    -1.0129
   7.9605    -4.0860    -1.7317
  -4.9068     0.4782     1.8375
```
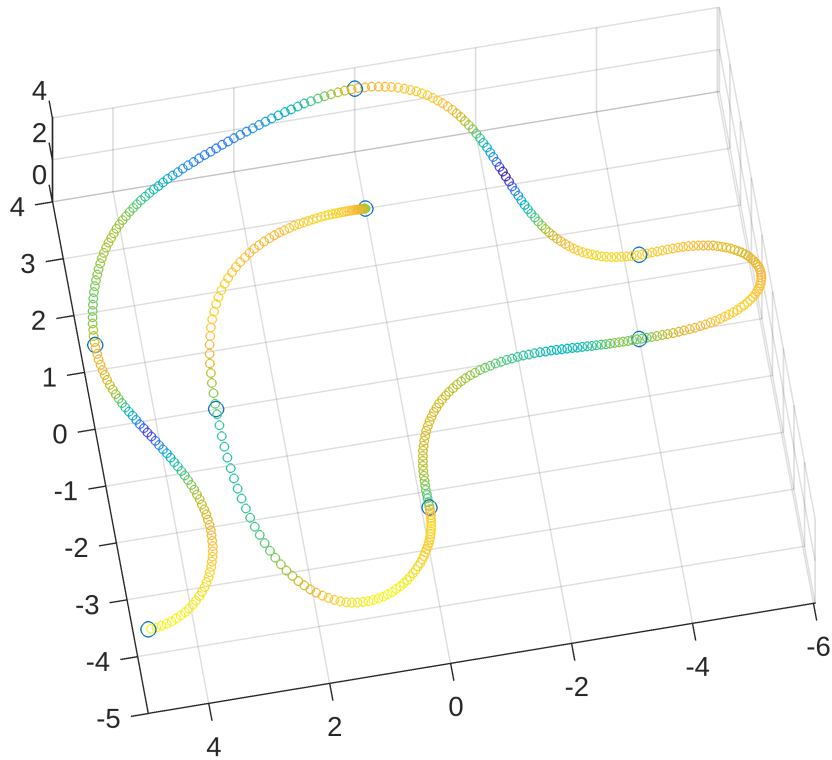
```matlab
figure
scatter3(points(:, 1), points(:, 2), points(:, 3), 30)
hold on
scatter3(ps(:, 1), ps(:, 2), ps(:, 3), 10, log10(curvatureVariation(c')))
axis equal
```



```matlab
figure
scatter3(points(:, 1), points(:, 2), points(:, 3), 30)
hold on
scatter3(ps(:, 1), ps(:, 2), ps(:, 3), 10, log10(curvature(c')))
axis equal

view([-100 70])
```

Functions

```
function r = curvature(c)
    global k
    global n
    global nd
    t = ((1:nd)'-0.5)/nd;
    d10 = (0:k-1) .* t.^((0:k-1)-1);
    d20 = (0:k-1).*((0:k-1)-1) .* t.^((0:k-1)-2);
    c = reshape(c', [k*(n-1), 3]);
    r = zeros(nd*(n-1), 1);

    for pi = 0:n-1-1
        ci = c(k*pi+1:k*pi+k, :);
        M = ci * ci';
        for i = 1:nd
            d1 = d10(i, :);
            d2 = d20(i, :);
            kn = (d1*M*d1').*(d2*M*d2') - (d1*M*d2').^2;
            km = (d1*M*d1').^2;
            kc = kn / km^1.25;  % \kappa^2 ds
            %kc = kn^0.5 / km^0.5;   % \kappa ds
            %kc = kn / km^1.5;   % \kappa^2 dt
```

```matlab
            %kc = kn^0.5 / km^0.75;  % \kappa dt
            %kc = kn / km;  % \kappa^2 ds/dt ds
            r(pi*nd+i) = kc;
        end
    end
end

function r = curvatureVariation(c)
    global k
    global n
    global nd
    t = ((1:nd)'-0.5)/nd;
    d10 = (0:k-1) .* t.^((0:k-1)-1);
    d20 = (0:k-1).*((0:k-1)-1) .* t.^((0:k-1)-2);
    d30 = (0:k-1).*((0:k-1)-1).*((0:k-1)-2) .* t.^((0:k-1)-3);
    c = reshape(c', [k*(n-1), 3]);
    r = zeros(nd*(n-1), 1);
    for pi = 0:n-1-1
        ci = c(k*pi+1:k*pi+k, :);
        M = ci * ci';
        for i = 1:nd
            d1 = d10(i, :);
            d2 = d20(i, :);
            d3 = d30(i, :);
            v11 = d1*M*d1';
            g11 = d2*M*d1' + d1*M*d2';
            v22 = d2*M*d2';
            g22 = d3*M*d2' + d2*M*d3';
            v12 = d1*M*d2';
            g12 = d2*M*d2' + d1*M*d3';
            kn = v11*v22 - v12*v12;
            gn = g11*v22+v11*g22 - 2*v12*g12;
            km = v11^3;
            gm = 3*v11^2*g11;
            kc0 = kn / km;
            gc0 = (gn*km-kn*gm)/(km*km);
            kc = kc0^0.5;
            gc = gc0/(2.0*kc);
            ri = pi*nd+i;
            r(pi*nd+i) = gc.^2;
        end
    end
end
```