Mathematical Modelling

Week 5/6

Homework 6.4:

Show that the Jacobi iterative method can be written as

$$\overrightarrow{V}^{n+1} = -D^{-1}(Q - D)\overrightarrow{V}^{n} + D^{-1}\overrightarrow{F}$$

where D is the diagonal part of the matrix Q. Show that error $\overrightarrow{E}^{n} := \overrightarrow{V}^{n} - \overrightarrow{V}$ satisfies

$$\overrightarrow{E}^{n+1} := -D^{-1}R\overrightarrow{E}^{n}.$$

Solution:

As in first year linear algebra, we have a system of n equations with n variables, this may be represented in matrix form as $A\mathbf{x} = \mathbf{b}$ where:

A is the coefficient matrix, of the form: $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$

$\mathbf{b}$ is the solution matrix, of the form: $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$\mathbf{x}$ is the variable matrix, of the form: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

This provides the system of linear equations in matrix form.
We may represent the square coefficient matrix as the sum of a diagonal, upper triangular and lower triangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ -a_{21} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ -a_{n1} & \dots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ 0 & \dots & \dots & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & \dots & 0 & 0 \end{bmatrix} = D - L - U$$

thus we may represent the system of linear equations as $A\mathbf{x} = \mathbf{b}$ as $(D - L - U)\mathbf{x} = \mathbf{b}$ this is also

$\implies D\mathbf{x} - (L + U)\mathbf{x} = \mathbf{b}$

$\implies D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b}$

$\implies \mathbf{x} = D(L + U)\mathbf{x} + D\mathbf{b}$

Applying this to a current approximation for instance $\mathbf{x}^{(k)} = x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}$ we can find new values by solving for $\mathbf{x}^{(k+1)} = x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}, \ldots, x_n^{(k+1)}$ in

$$\begin{aligned}
a_{11}x_1^{(k+1)} + a_{12}x_2^{(k)} + \cdots + a_{1n}x_n^{(k)} &= \mathbf{b}_1 \\
a_{11}x_1^{(k)} + a_{12}x_2^{(k+1)} + \cdots + a_{1n}x_n^{(k)} &= \mathbf{b}_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad &\ \ \vdots \\
a_{11}x_1^{(k)} + a_{12}x_2^{(k)} + \cdots + a_{1n}x_n^{(k+1)} &= \mathbf{b}_n
\end{aligned}$$

The system can then be written as

$$\begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \ldots & a_{1n} \\ a_{21} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & \ldots & a_{n,n-1} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\implies Dx^{(k+1)} + (L + U)x^{(k)} = \mathbf{b}$
$\implies Dx^{(k+1)} = -(L + U)x^{(k)} + \mathbf{b}$
$\implies x^{(k+1)} = -D^{-1}(L + U)x^{(k)} + D^{-1}\mathbf{b}$

Thus the matrix form of the Jacobi iterative method can be written as:
$x^{(k+1)} = -D^{-1}(L + U)x^{(k)} + D^{-1}\mathbf{b}$

By replaying variable names, we see:

$$\vec{V}^{n+1} = -D^{-1}(Q - D)\vec{V}^n + D^{-1}\vec{F}$$

We define the error as $\vec{E}^n := \vec{V}^n - \vec{V}$

Thus we have $\vec{V}^n := \vec{E}^n + \vec{V}$

Using this in the method defined above we have:

$$\vec{E}^{n+1} + \vec{V} = -D^{-1}(Q - D)(\vec{E}^n + \vec{V}) + D^{-1}\vec{F}$$

or

$$\vec{E}^{n+1} + \vec{V} = -D^{-1}R(\vec{E}^n + \vec{V}) + D^{-1}\vec{F}$$

$$\implies \vec{E}^{n+1} + \vec{V} = -D^{-1}R\vec{E}^n - D^{-1}R\vec{V} + D^{-1}\vec{F}$$

$$\implies \vec{E}^{n+1} + \vec{V} + D^{-1}R\vec{V} - D^{-1}\vec{F} = -D^{-1}R\vec{E}^n$$

$$\implies \vec{E}^{n+1} + \vec{V} - (D^{-1}R\vec{V} + D^{-1}\vec{F}) = -D^{-1}R\vec{E}^n$$

$$\implies \vec{E}^{n+1} + \vec{V} - \vec{V} = -D^{-1}R\vec{E}^n$$

$$\implies \vec{E}^{n+1} = -D^{-1}R\vec{E}^n$$