Maths for ML Specialisation Course 1

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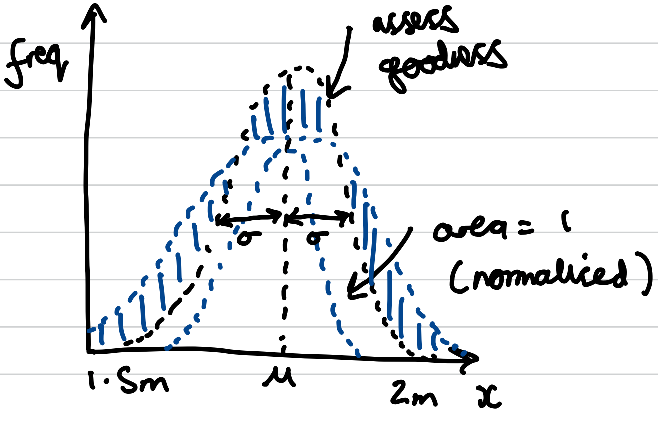
2022

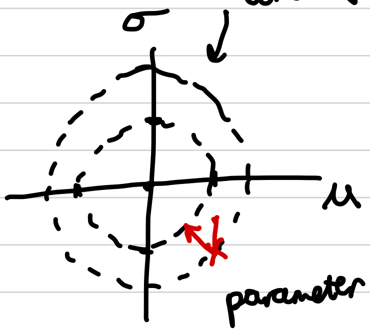
Week 1: ML, LA, Vectors and Matrices



(histogram)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} \quad \mathbf{p} = \begin{bmatrix} \mu \\ \sigma \end{bmatrix}$$

Gaussian distribution: (centre of μ and width of σ)



want to find gradients

vectors = lists

$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} \begin{bmatrix} \mu' \\ \sigma' \end{bmatrix}$

$\begin{bmatrix} £ \\ gCO_2 \\ Nox \\ \vdots \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$

Want to fit distribution with equation describing variation of heights in the population

(2 parameters: μ and σ)

Vector = list of numbers, pos. in 3D of space and 1D of time, something which moves in a space of fitting parameters

Each parameter vector $\mathbf{p}$ represents a different bell curve, each with its own value for the sum of squared residuals: $SSR(\mathbf{p}) = |\mathbf{f} - \mathbf{g_p}|^2$

Histogram as vector: (fit a model to data to represent underlying distribution)

$$\mathbf{f} = \begin{bmatrix} f_{150.0,152.5} \\ f_{152.5,155.0} \\ f_{155.0,157.5} \\ \vdots \end{bmatrix}$$

(record freq. of people in 2.5cm height intervals)

components = size of each bar in the histogram

Different sample = different frequencies

One task of ML is to fit a model to the data in order to represent the underlying distribution

May use a Normal (Gaussian) distribution to predict frequencies of heights of a population

A model allows us to predict data in a distribution

For a sufficiently large sample, the data will represent the population it is taken from

Can start with parameter vector $\mathbf{p}$ and convert it to a vector of expected frequencies $\mathbf{g_p}$

$$\mathbf{g_p} = \begin{bmatrix} g_{150.0,\,152.5} \\ g_{152.5,\,155.0} \\ g_{155.0,\,157.5} \\ \vdots \end{bmatrix}$$

A model is only considered good if it fits the measured data well (some parameter values will be better than others for the model)

Calculate "residuals" to quantify how good the fit is
(difference between measured data and modelled prediction for each histogram bin)

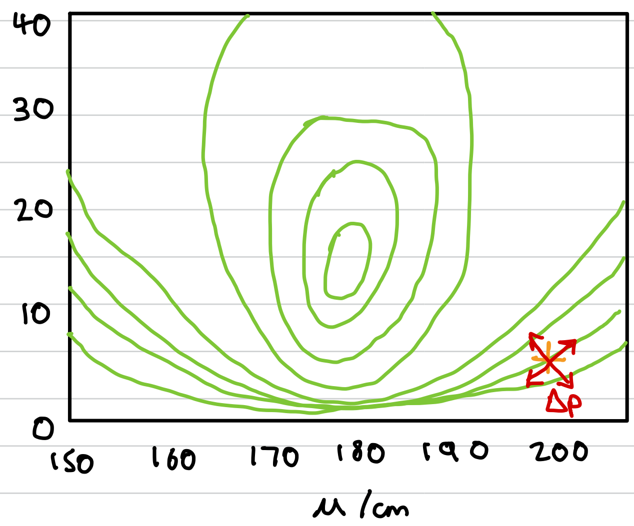
Good fit = lots of overlap (performance of model can be quantified in a single number, e.g. SSR)

Each $\mathbf{p}$ represents a different bell curve, each with its own value for the SSR

Can draw the SSR surface over the space spanned by $\mathbf{p}$

Goal in ML is to find the parameter set where the model fits the data as well as it possibly can (i.e. finding the global minimum in this space)

(top down view = contour map)



Every point on the surface represents the SSR of a choice of **p**, with some bell curves performing better at representing the data than others

Moving at right angles to contour lines in the parameter space will have the greatest effect on the fit than moving in other directions (moving along the lines has no effect)

Often, can't see whole parameter space, so instead of picking lowest point, need to make educated guesses where better points will be

Can define another vector, $\Delta \mathbf{p}$ in the same space as $\mathbf{p}$ which tells us what change can be made to $\mathbf{p}$ to get a better fit

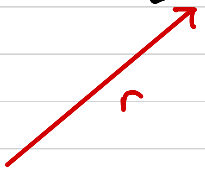
e.g. a model with parameters $\mathbf{p}' = \mathbf{p} + \Delta \mathbf{p}$ will produce a better fit to the data, if we can find a suitable $\Delta \mathbf{p}$

Solving simultaneous equations is the process of finding the values of the variables that satisfy the system of equations

(subtract ② from ①, substitute values)

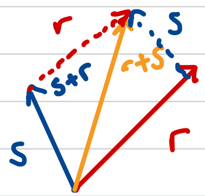
(elimination = multiply both sides of equation to make no. in front of x/y the same)

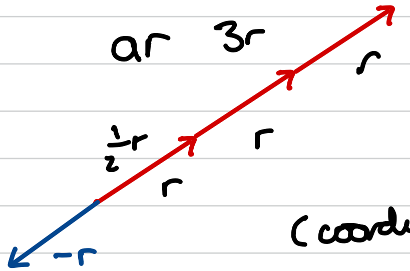
(substitution method = re-arrange an equation and substitute in) (can have more than 2 unknowns)

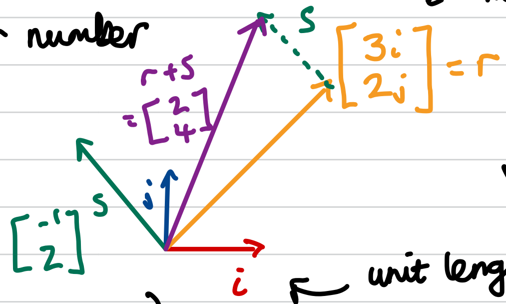


vector (move around space)

- In ML: features (e.g. $\begin{bmatrix} 120 \\ 2 \\ 1 \\ 150 \end{bmatrix}$) describes a house
- Obeys addition and multiplication by a scalar number







Scalar multiple of 3i's and 2j's as vector sum

(coordinate system)

unit length (length 1)

Vector addition: can add components component by component

Associative (doesn't matter which additions you do) → $(r+s)+t$, $r+(s+t)$

Multiplication by a scalar: e.g. $r = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$,

$$2r = \begin{bmatrix} 2*3 \\ 2*2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

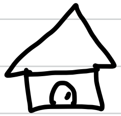
across 6i's, up 4j's

$$r + (-r) = \begin{bmatrix} 3 + -3 \\ 2 + -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Vector subtraction: vector addition of $-1 \times$ vector

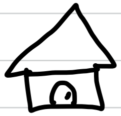
e.g. $r = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $s = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\therefore -s = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

so $r - s = r + (-s) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$



$$\begin{array}{cl} \begin{bmatrix} 120 \\ 2 \\ 1 \\ 150 \end{bmatrix} & \begin{array}{l} \text{sqm} \\ \text{beds} \\ \text{baths} \\ £,000 \end{array} \end{array}$$

+



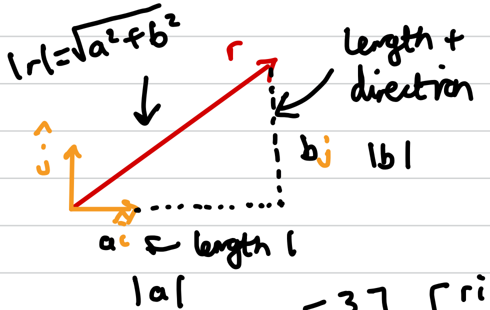
$$= 2\begin{bmatrix} 120 \\ 2 \\ 1 \\ 150 \end{bmatrix} = \begin{bmatrix} 240 \\ 4 \\ 2 \\ 300 \end{bmatrix}$$

Useful to define a coordinate System

Basis vectors define the coordinate system

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

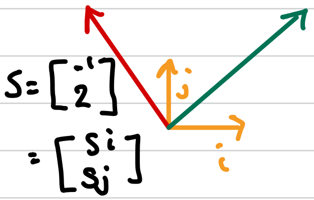
Week 2: Vectors



$r = a\hat{i} + b\hat{j} = \begin{bmatrix} a \\ b \end{bmatrix}$

$r = \left| \begin{matrix} a \\ b \end{matrix} \right|$ ← size

$|r| = \sqrt{a^2+b^2}$ ↙ size of vector = sum of squares of vectors components



$r.s = r_i s_i + r_j s_j$ (dot product)

$= 3 \times -1 + 2 \times 2$

$= 1$

↑ Scalar output

Dot product has 3 properties:

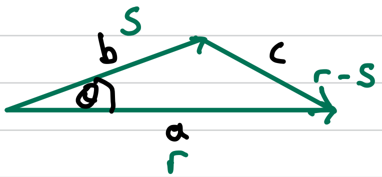
- commutative (order doesn't matter)
  $(r.s = s.r)$
- distributive over addition
  $(r.(s+t) = r.s + r.t)$

$$r.(s+t) = r_1(s_1+t_1) + r_2(s_2+t_2) + \dots + r_n(s_n+t_n)$$
$$= r_1 s_1 + r_1 t_1 + r_2 s_2 + r_2 t_2 + \dots + r_n s_n + r_n t_n$$
$$= r.s + r.t$$

- associative over scalar multiplication (can pull a scalar out)
  $(r.(as) = a(r.s))$ $\quad r_1(as_1) + r_2(as_2) = a(r_1 s_1 + r_2 s_2) = a(r.s)$

$r.r = r_1 r_1 + r_2 r_2 = r_1^2 + r_2^2 = \left(\sqrt{r_1^2 + r_2^2}\right)^2 = |r|^2$ ← modulus (size of vector)

↑ dot product vector with itself and take square root



Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos\theta$

$|r-s|^2 = \cancel{|r|^2} + \cancel{|s|^2} - 2|r||s|\cos\theta$

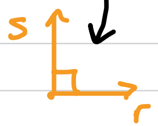
$(r-s).(r-s) = r.r - s.r - s.r - s.-s$

$= \cancel{|r|^2} - 2s.r + \cancel{|s|^2}$

$-2s.r = -2|r||s|\cos\theta$ $(\times -1)$ (2's cancel)

$\therefore \boxed{r.s = |r||s|\cos\theta}$ (tells us extent to which the 2 vectors go in the same direction)

orthogonal vectors



if $\theta = 0°$, $\cos\theta = 1$, $\therefore$ $r.s = |r||s|$

if $\theta = 90°$, $\cos\theta = 0$, $\therefore$ $r.s = 0$

if $\theta = 180°$, $\cos\theta = -1$, $\therefore$ $r.s = -|r||s|$

going in opposite directions

Projection:

$\frac{r}{|r|}$ unit length vector

$\cos\theta = \frac{adj}{hyp} = \frac{adj}{|s|}$

$r.s = |r|\underbrace{|s|\cos\theta}_{adj}$

projection of S onto r × |r|

cos(90°) = 0, so
Scalar projection = 0

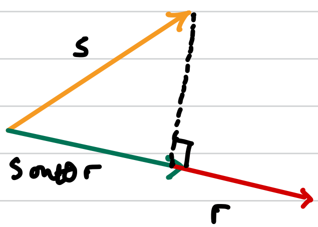
(think of shadow)

(dot product also known as projection product - takes projection of one vector onto another)

vector of unit length → $\frac{r.s}{|r|} = |s|\cos\theta$ (scalar projection)

how much s goes along r

Vector projection: $r\frac{r.s}{|r||r|} = \frac{r.s}{r.r}r$ ($\frac{r}{|r|}$ × scalar projection of s onto r)



Scalar projection = size of green vector
Vector projection = green vector
Projection will have a -ve sign if angle between s and r is > π/2

e.g. $r = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ and $s = \begin{bmatrix} 10 \\ 5 \\ -6 \end{bmatrix}$, ∴ Scalar projection of s onto r $= \frac{r.s}{|r|}$

$= \frac{(3\times10)+(-4\times5)}{\sqrt{3^2+4^2}}$

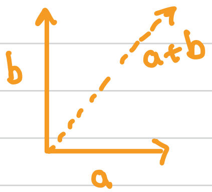
$= \frac{10}{5} = \underline{\underline{2}}$

vector projection of s onto r $= \frac{r \cdot s}{r \cdot r} r$

$$= \frac{(3 \times 10) + (-4 \times 5)}{3 \times 3 + (-4 \times -4)} \cdot \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \frac{10}{25} \cdot \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6/5 \\ -8/5 \\ 0 \end{bmatrix}$$

Triangle inequality: $|a+b| \leq |a| + |b|$ for every pair of vectors a and b



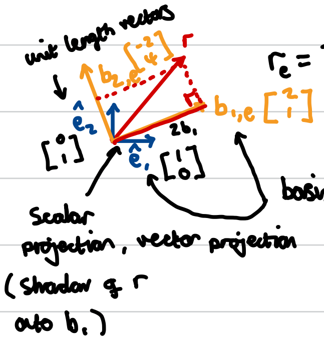
(for any triangle, sum of lengths of any two sides must be $\geq$ the length of the remaining side)

We can find the angle between two vectors using the dot product

Size of vector = Square root of dot product of vector with itself

Changing Basis

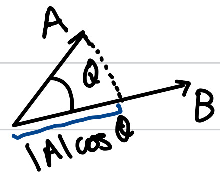
- Changing Basis (co-ordinate systems) example using projection product

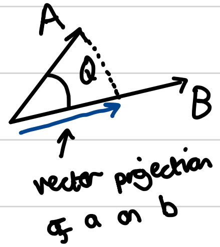


$$r_e = 3\hat{e}_1 + 4\hat{e}_2 = \begin{bmatrix}3\\4\end{bmatrix}$$

$$r_b = \begin{bmatrix}?\\?\end{bmatrix} = \begin{bmatrix}2\\1/2\end{bmatrix}$$

- basis vectors describe space of data
- can change basis by taking dot / projection product in case where new basis vectors are orthogonal to each other





Components of vector are scalar projections in directions of coordinate axis

- r has existence independent of coordinate system we use to describe numbers in r
- if b_1 and b_2 are at 90°: can use dot product to get r_b

$$\cos\theta = \frac{\overbrace{b_1 . b_2}^{0}}{|b_1||b_2|} = 90, \therefore \text{ can do projection}$$

$$b_1 . b_2 = 2\times -2 + 1\times 4 = 0$$

$$\frac{r_e . b_1}{|b_1|^2} = \frac{3\times 2 + 4\times 1}{2^2+1^2} = 2$$

$$\frac{r_e . b_1}{|b_1|^2} b_1 = 2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

$$\frac{r_e . b_2}{|b_2|^2} = \frac{3\times -2 + 4\times 4}{-2^2 + 4^2} = \frac{10}{20} = \frac{1}{2}$$

$$\frac{r_e . b_2}{|b_2|^2} b_2 = \frac{1}{2}\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix},$$

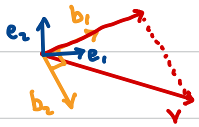
$$r_b = \frac{r_e . b_1}{|b_1|^2} b_1 + \frac{r_e . b_2}{|b_2|^2} b_2$$

$$= \begin{bmatrix}4\\2\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} = r_e$$

(converted from e basis vectors to b basis vectors)

Changing Basis Questions

1. $v_e = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (vector v projected onto new basis vectors)



$$\cos\theta = \frac{b_1 \cdot b_2}{|b_1||b_2|} = \frac{1\times 1 + 1\times -1}{|b_1||b_2|} = 90, \therefore \text{ orthogonal}$$

b_1 Scalar projection : $\dfrac{V_e \cdot b_1}{|b_1|^2} = \dfrac{5\times 1 + -1\times 1}{1^2+1^2} = \dfrac{4}{2} = 2$

b_1 Vector projection : $\dfrac{V_e \cdot b_1}{|b_1|^2} b_1 = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

b_2 Scalar projection : $\dfrac{V_e \cdot b_2}{|b_2|^2} = \dfrac{5\times 1 + -1\times -1}{1^2 + -1^2} = \dfrac{6}{2} = 3$

b_2 vector projection : $\dfrac{V_e \cdot b_2}{|b_2|^2} b_2 = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} = V_b$$

$\therefore V_b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (v in basis defined by b_1 and b_2)

2. $V_e = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

b_1 Scalar projection : $\dfrac{V_e \cdot b_1}{|b_1|^2} = \dfrac{10\times 3 + -5\times 4}{3^2+4^2} = \dfrac{30-20}{25} = \dfrac{2}{5}$

b_2 Scalar projection : $\dfrac{V_e \cdot b_2}{|b_2|^2} = \dfrac{10\times 4 + -5\times -3}{4^2 + -3^2} = \dfrac{55}{25} = \dfrac{11}{5}$, $\therefore V_b = \begin{bmatrix} 2/5 \\ 11/5 \end{bmatrix}$

3. $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $b_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

b_1 Scalar projection: $\frac{2\times -3 + 2\times 1}{-3^2 + 1^2} = \frac{-6+2}{10} = -\frac{2}{5}$

b_2 Scalar projection: $\frac{2\times 1 + 2\times 3}{1^2+3^2} = \frac{2+6}{10} = \frac{4}{5}$, $\therefore V_b = \begin{bmatrix} -2/5 \\ 4/5 \end{bmatrix}$

4. $V_e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $b_3 = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}$

b_1 Scalar projection: $\frac{1\times 2 + 1\times 1 + 1\times 0}{2^2+1^2+0^2} = \frac{3}{5}$, ...

5. $V_e = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, $b_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

b_1 Scalar projection: $\frac{1\times 1}{1} = 1$

b_2 Scalar projection: $\frac{2\times 1 + 2\times -1}{2^2 + -1^2} = 0$

b_3 Scalar projection: $\frac{1\times 1 + 2\times 2}{1^2+2^2} = 1$

b_4 Scalar projection: $\frac{3\times 3}{3^2} = 1$

Basis, Vector Space, Linear Independence

- Basis: set of n vectors that: are not linear combinations of each other (linearly independent), and span the space
- The space is then n-dimensional

may have vectors which are linear combinations of others and aren't needed

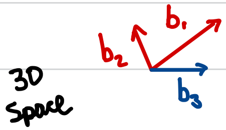
$\bar{v}_1, \bar{v}_2, \dots \bar{v}_n$ in vector space

vector of 0's

$a_1\bar{v}_1 + a_2\bar{v}_2 + a_3\bar{v}_3 = \bar{0}$

general matrix, get into row echelon form

if set of a's are not 0's and satisfy the equation, then set of v vectors are not linearly independent, otherwise will be (requires all scalars to be 0, so look for nonzero scalars)

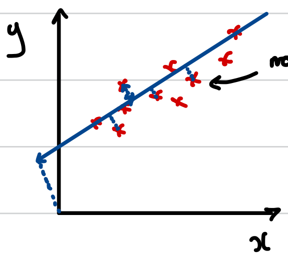


linearly independent

Has to be impossible for $b_3 \neq a_1 b_1 + a_2 b_2$, for any a_1 or a_2

- b_3 does not lie in the plane spanned by b_1 and b_2
- don't have to be unit vectors, or orthogonal (but would be easier if they were)
  - ↳ want to use orthonormal basis vector sets of unit length
- map from one basis to another: projection keeps grid evenly spaced, ∴ any mapping from one set of basis vectors to another keeps vector space being regularly spaced grid (vector rules still work)
- may be stretched, rotated or inverted, but everything remains evenly spaced and linear combinations still work
- need to use matrices if we want to change basis where new basis vectors are not orthogonal

Applications of Changing Basis



noise: distance from line

noise dimension tells you how good the fit is

orthogonal lines: can use dot product to do projection to map data from x,y space to along the line and away from the line

want to make transformations of pixels into new basis that describes facial features, and discard actual pixel data

goal of learning process of nn: derive set of basis vectors that extract most information rich features of human faces

Linear dependency of set of vectors questions

Variable that can take any value : free variable

1. Linearly dependent : $a = q_1 b + q_2 c$, $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$c_1 + 2c_2 = 0$

$c_1 + 2c_2 = 0$, so $c_1 = 0$, $c_2 = 0$, ∴ vectors are linearly independent (not linearly dependent)

↳ nothing we can multiply the vector by to get the other

↳ can also use square matrix from set of vectors, ∴ can find determinant : if it's 0 → vectors are linearly dependent, otherwise linearly independent (non-zero determinant → linearly independent)

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $1 \times 2 - 2 \times 1 = 0$, ∴ linearly dependent

↳ $a = \frac{1}{2} b$ (scalar multiple)

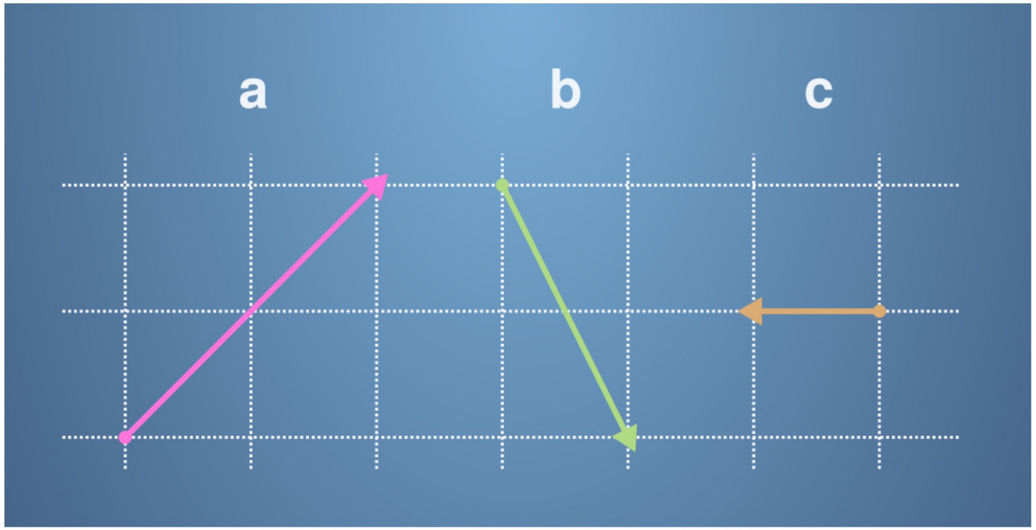
2.

- not linearly dependent : can't write one of the vectors as a linear combination of the others

$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, determinant $= 1 \times 1 - 2 \times 1 = -1$, non-zero, ∴ linearly independent

(one is not a scalar multiple of the other)

3. We also saw in the lectures that three vectors that lie in the same two dimensional plane must be linearly dependent. This tells us that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent in the following diagram:



What are the values of q_1 and q_2 that allow us to write $\mathbf{a} = q_1\mathbf{b} + q_2\mathbf{c}$? Put your answer in the following codeblock:

$a = q_1 b + q_2 c$

$$\begin{bmatrix}2\\2\end{bmatrix} = q_1\begin{bmatrix}1\\-2\end{bmatrix} + q_2\begin{bmatrix}-1\\0\end{bmatrix}$$

(2) $2 = -2q_1$, $\therefore q_1 = -1$

(1) $2 = q_1 - q_2$, $\therefore q_2 = -3$

4. An n-dimensional space can have n linearly independent vectors — one can't be written as a linear sum of the other two

$$a = \begin{bmatrix}1\\0\\0\end{bmatrix},\ b = \begin{bmatrix}1\\1\\0\end{bmatrix},\ c = \begin{bmatrix}1\\0\\1\end{bmatrix} \rightarrow \begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix},$$

determinant $|B|$ $= 1\times1\times1 - 1\times1\times0 = 1$, non-zero, $\therefore$ linearly independent

5. $$a = \begin{bmatrix}1\\0\\1\end{bmatrix},\ b = \begin{bmatrix}2\\-1\\1\end{bmatrix},\ c = \begin{bmatrix}-3\\1\\-2\end{bmatrix} \rightarrow \begin{bmatrix}1&2&-3\\0&-1&1\\1&1&-2\end{bmatrix},$$

determinant $|B|$ $= 1\times(-1\times-2 - 1\times1) - 2\times(0\times-2 - 1\times1) + -3\times(0\times1 - (-1\times1))$

$= 1\times(1) - 2\times(-1) + -3\times(1)$

$= 1 + 2 - 3 = 0$,

$\therefore$ linearly dependent

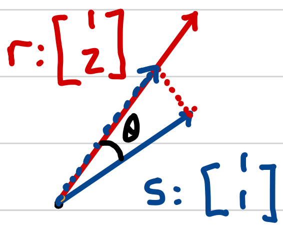
↳ 3x3 matrix determinant: $A = \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$, $|A| = a\times\begin{vmatrix}e&f\\h&i\end{vmatrix} - b\times\begin{vmatrix}d&f\\g&i\end{vmatrix} + c\times\begin{vmatrix}d&e\\g&h\end{vmatrix}$ $(-+-+\dots)$

6. $a = \begin{bmatrix}1\\2\\0\end{bmatrix}$, $b = \begin{bmatrix}-2\\1\\3\end{bmatrix}$, $c = \begin{bmatrix}4\\3\\-3\end{bmatrix}$, why can't vectors be used as basis for 3D Space?

$c = 2a - b$, $\therefore$ vectors are linearly dependent (can be written as linear combination)
vectors don't span 3D space

Vector Operations Assessment

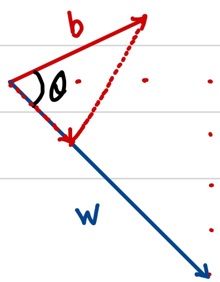
1. Ship travels with velocity $\begin{bmatrix}1\\2\end{bmatrix}$, current flowing in direction $\begin{bmatrix}1\\1\end{bmatrix}$ w.r.t some coordinate axis
Velocity of Ship in direction of current?



Vector projection of velocity of Ship (r) onto velocity of current (s): $\dfrac{s.r}{s.s} s$

$$= \frac{1\times1 + 2\times1}{1\times1 + 1\times1}\begin{bmatrix}1\\1\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3/2\\3/2\end{bmatrix}$$

2. Ball velocity $\begin{bmatrix}2\\1\end{bmatrix}$, wind direction $\begin{bmatrix}3\\-4\end{bmatrix}$, size of velocity of ball in direction of wind?



Scalar projection of b onto w: $\dfrac{w.b}{|w|} = \dfrac{3\times2 + 1\times-4}{\sqrt{3^2 + -4^2}} = \dfrac{2}{\sqrt{25}} = \dfrac{2}{5}$

3. $v = \begin{bmatrix} -4 \\ -3 \\ 8 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}$, v in basis b_1, b_2, b_3? ← all pairwise orthogonal

$$\frac{V_e \cdot b_1}{|b_1|^2} = \frac{-4\times1 + -3\times2 + 8\times3}{1^2+2^2+3^2} = \frac{-4-6+24}{1+4+9} = \frac{14}{14} = 1 \quad \text{(Scalar projection)}$$

$$\frac{V_e \cdot b_2}{|b_2|^2} = \frac{-4\times-2 + -3\times1 + 8\times0}{-2^2+1^2+0^2} = \frac{8-3}{5} = 1 \quad \text{(Scalar projection)}$$

$$\frac{V_e \cdot b_3}{|b_3|^2} = \frac{-4\times-3 + -3\times-6 + 8\times5}{-3^2+-6^2+5^2} = \frac{12+18+40}{9+36+25} = \frac{70}{70} = 1 \quad \text{(Scalar projection)}$$

4. Are $a = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ -8 \\ 7 \end{bmatrix}$ linearly independent?

$x_1 a + x_2 b + x_3 c = 0$

↑ if any x_i combos satisfy equation, then vectors are linearly dependent

$$X = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -4 & -8 \\ -1 & 5 & 7 \end{bmatrix}, \quad \det(X) = 1\times(-4\times7 - -8\times5) - 3\times(2\times7 - -8\times-1) + 1\times(2\times5 - -4\times-1)$$

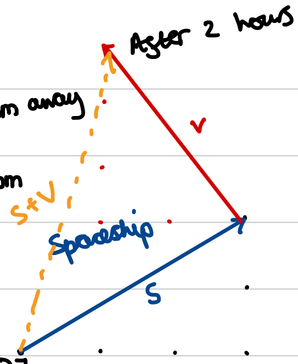
$$= 1\times(-28+40) - 3\times(14-8) + 1\times(10-4)$$

$$= 12 - 18 + 6$$

$= 0$, ∴ linearly dependent ($\det(X) = 0$) ← i.e. one can be written as linear combination of other two

5. $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ km away

12 pm



$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Location of ship after 2 hours?

After 2 hours, ship should have travelled: $2\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}$ km

$$s + v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}$$

Week 3: Matrices, vectors, simultaneous equations

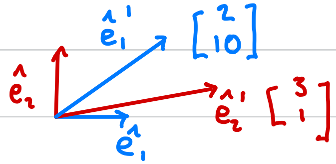
$2a + 3b = 8$ (apples and bananas problem)

$10a + 1b = 13$

↙ unit basis vector

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 8 \\ 13 \end{bmatrix}}_{2\times 1} \qquad \begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a + 3b \\ 10a + 1b \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$$



function that operates on input vectors which gives other output vectors

(set of simultaneous equations says what vector you need in order to get a transformed product at position $\begin{bmatrix} 8 \\ 13 \end{bmatrix}$)

Linear algebra (mathematical system for manipulating vectors and spaces described by vectors)

Can make any vector out of a vector sum of the scaled versions of $\hat{e}_1$ and $\hat{e}_2$

↳ result of transformation just a sum of transformed vectors

Rules:

$Ar = r'$

$A(nr) = nr'$

$A(r+s) = Ar + As$ ($\hat{e}_1$ and $\hat{e}_2$ are transformed and can add vectors with them)

$A(n\hat{e}_1 + m\hat{e}_2) = nA\hat{e}_1 + mA\hat{e}_2 = ne_1' + me_2'$

$$\begin{bmatrix}2&3\\10&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}12\\32\end{bmatrix}$$

$$\downarrow\ \begin{bmatrix}2&3\\10&1\end{bmatrix}\left(3\overset{\hat{e}_1}{\begin{bmatrix}1\\0\end{bmatrix}}+2\overset{\hat{e}_2}{\begin{bmatrix}0\\1\end{bmatrix}}\right)=3\left(\overbrace{\begin{bmatrix}2&3\\10&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}}^{\hat{e}_1'}\right)+2\left(\overbrace{\begin{bmatrix}2&3\\10&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}}^{\hat{e}_2'}\right)$$

$$=3\begin{bmatrix}2\\10\end{bmatrix}+2\begin{bmatrix}3\\1\end{bmatrix}$$

$$=\begin{bmatrix}12\\32\end{bmatrix}$$ (∴ rules work, matrix multiplication: multiply vector sum of transformed basis vectors)

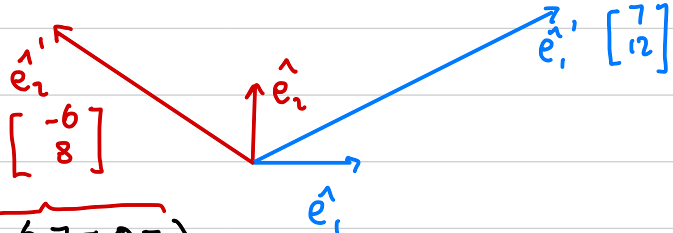
Example:

$$\begin{bmatrix}7&-6\\12&8\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}\qquad Ar=r'$$

$$\begin{bmatrix}7&-6\\12&8\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}7\times1+-6\times0\\12\times1+8\times0\end{bmatrix}=\overset{\hat{e}_1'}{\begin{bmatrix}7\\12\end{bmatrix}}$$

$$\begin{bmatrix}7&-6\\12&8\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}7\times0+-6\times1\\12\times0+8\times1\end{bmatrix}=\begin{bmatrix}-6\\8\end{bmatrix}\hat{e}_2'$$

$$\downarrow\ \begin{bmatrix}7&-6\\12&8\end{bmatrix}\left(5\overset{\hat{e}_1}{\begin{bmatrix}1\\0\end{bmatrix}}+6\overset{\hat{e}_2}{\begin{bmatrix}0\\1\end{bmatrix}}\right)$$



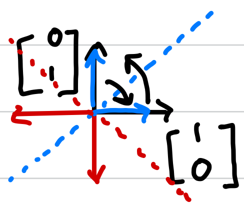
$$=5\left(\overbrace{\begin{bmatrix}7&-6\\12&8\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}}^{\hat{e}_1'}\right)+6\left(\overbrace{\begin{bmatrix}7&-6\\12&8\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}}^{\hat{e}_2'}\right)$$ (sum of scalar multiplications of transformed basis vectors)

$$=5\left(\begin{bmatrix}7\\12\end{bmatrix}\right)+6\left(\begin{bmatrix}-6\\8\end{bmatrix}\right)=\begin{bmatrix}35\\60\end{bmatrix}+\begin{bmatrix}-36\\48\end{bmatrix}=\begin{bmatrix}-1\\108\end{bmatrix}$$

Types of matrix transformation:

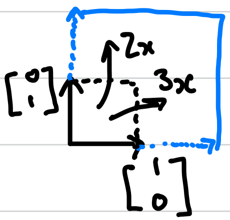
$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I \text{ (identity matrix)}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

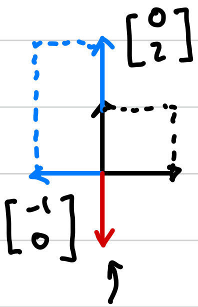
$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$



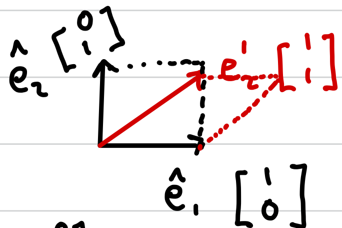
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$ (vertical mirror ⋮)

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ (horizontal mirror -------)

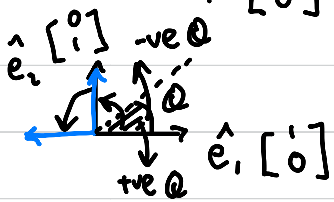
$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$ (inversion)



$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: Shears square to parallelogram



rotation in 2D: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\hat{e}_1$ 90° ↺ becomes $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\hat{e}_2$ 90° ↺ becomes $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

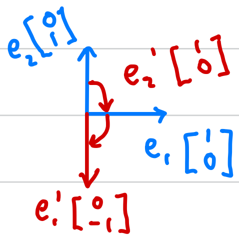
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

rotation by 90° anticlockwise

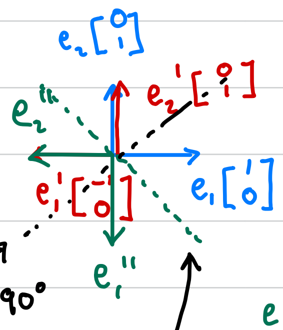
Composition or combination of matrix transformations

$A_2 (A_1 r)$ ← then

①



②



$$A_1 = \begin{array}{cc} e_1' & e_2' \\ \end{array} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(90° rotation)

$$A_2 = \begin{array}{cc} e_1'' & e_2'' \\ \end{array} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(vertical mirror)

Apply A_1 to A_2 (rotate 90°, then flip):

$$A_2 A_1 = \begin{array}{cc} e_1'' & e_2'' \\ \end{array} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \overset{②}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \underbrace{\overset{①}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}}_{\text{transformed basis vectors}}$$

(rotate then flip)

$$= \begin{bmatrix} -1\times 0 + 0\times -1 & -1\times 1 + 0\times 0 \\ 0\times 0 + 1\times -1 & 0\times 1 + 1\times 0 \end{bmatrix} \quad \text{(row × column for all combos)}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

(example)

$$A_1 \cdot A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\times 1 + 1\times 0 & 0\times 1 + 1\times 1 \\ -1\times 1 + 0\times 0 & -1\times 1 + 0\times 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A_1 \cdot A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\times -1 + 1\times 0 & 0\times 0 + 1\times 1 \\ -1\times -1 + 0\times 0 & -1\times 0 + 0\times 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

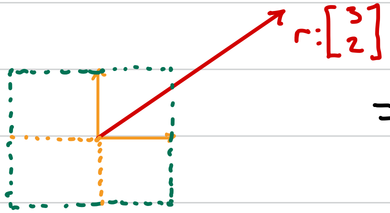
(flip then rotate)

$A_1 . A_2 \neq A_2 . A_1$ (not commutative)

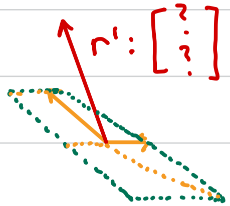
$A_3 (A_2 A_1) = (A_3 A_2) A_1$ (associative)

Matrix Transformations Assessment

1. Matrices make transformations on vectors, potentially changing their magnitude and direction



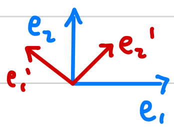
$\Rightarrow$ apply $A : \begin{bmatrix} 1/2 & -1 \\ 0 & 3/4 \end{bmatrix}$ to r $\Rightarrow$

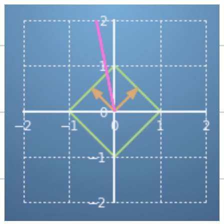


$$r' = Ar = \begin{bmatrix} 1/2 & -1 \\ 0 & 3/4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \times 3 + -1 \times 2 \\ 0 \times 3 + 3/4 \times 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$$

2. $$s = \begin{bmatrix} 1/2 & -1 \\ 0 & 3/4 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 \times -2 + -1 \times 4 \\ 0 \times -2 + 3/4 \times 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

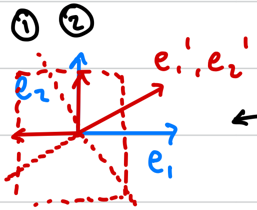
3. $$M = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$





$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

4. Image rotation, which matrix?



$\leftarrow$

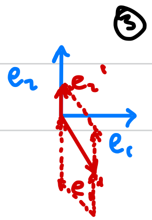
≈ 0.8660

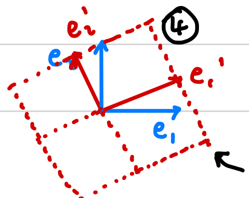
① $\begin{bmatrix} \sqrt{3}/2 & \sqrt{3}/2 \\ 1/2 & 1/2 \end{bmatrix}$ (Squashes onto line)

② $\begin{bmatrix} -1/2 & 0 \\ 0 & \sqrt{3}/2 \end{bmatrix}$ (inverts and scales in y)

③ $\begin{bmatrix} 1/2 & 0 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ (rotates and squashes in y)

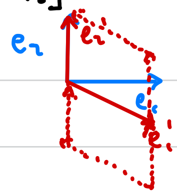
④ $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ (rotates left and squashes in y)





$\nwarrow$ correct

$$\begin{bmatrix}1&0\\-1/2&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\1&-1/2\end{bmatrix}$$



5. Undo skull transformation: $M=\begin{bmatrix}1&0\\0&8\end{bmatrix}\begin{bmatrix}1&0\\-1/2&1\end{bmatrix}$

(Shear in y direction, then scaling in y direction)

$$=\begin{bmatrix}1\times1+0\times-1/2 & 1\times0+0\times1\\ 0\times1+8\times-1/2 & 0\times0+8\times1\end{bmatrix}=\begin{bmatrix}1&0\\-4&8\end{bmatrix}$$

Gaussian Elimination (row reduction)

$$\begin{pmatrix}2&3\\10&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix}$$ (apples and bananas problem)

$$A \quad r \quad = \quad s$$

inverse → $A^{-1}A=I$

$$\times A^{-1} \qquad Ar=s \Rightarrow \underbrace{A^{-1}A}_{I}r=A^{-1}s \Rightarrow \boxed{r=A^{-1}s}$$

e.g. $$\begin{array}{l}1.\\2.\\3.\end{array}\begin{pmatrix}1&1&3\\1&2&4\\1&1&2\end{pmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}15\\21\\13\end{bmatrix}$$

elimination ↓

$$\begin{array}{l}1.\\2.-1.\\3.-1.\end{array}\underbrace{\begin{pmatrix}1&1&3\\0&1&1\\0&0&+1\end{pmatrix}}_{\text{triangular matrix (reduced to echelon form)}}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}15\\6\\+2\end{bmatrix}$$

all values below diagonal are 0

$c=2$ (now do back substitution: put c back into first 2 rows)

$$\begin{array}{l}1.-3\times3.\\2.-3.\\3.\end{array}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{bmatrix}a\\b\\c(2)\end{bmatrix}=\begin{bmatrix}15-3\times c\\6-2\times c\\c\end{bmatrix}=\begin{bmatrix}9\\4\\2\end{bmatrix}$$ (Subtract c off)

$$\begin{array}{l}1.-2.\\2.\\3.\end{array}\overbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}^{I}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}9-4\\4\\2\end{bmatrix}=\begin{bmatrix}5\\4\\2\end{bmatrix}$$

$b=4$ (subtract b off)

example: $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{matrix}(b=1) \\ (c=1)\end{matrix} \Rightarrow \begin{bmatrix} 1 & \cdot & -2\,\cdot \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0-2\times c \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Finding the Inverse Matrix

$AA^{-1} = I$ (verify)

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A^{-1}A = I$ $\quad B = A^{-1}$

$AB = I$

transformation of x axis

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (A)\begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;\ldots\ldots$$

(can do elimination and back substitution all at once)

example:

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}, \quad A' = \begin{pmatrix} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 - I_1 \\ I_3 - I_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$ ($\times R_3$ by -1)

$$= \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & +1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ +1 & 0 & -1 \end{pmatrix}$$ (now can put R_3 into R_2 and R_1)

$\downarrow$ row elimination $\quad \uparrow$ back substitution

(Subtract 3 lots of R_3 off R_1 to make it 0)

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 3 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

(Subtract R_1 by R_2 both sides)

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{transformed } A \text{ to identity matrix}}\begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = A^{-1} \; (B)$$

(can solve sets of linear equations in general case by easy procedure to implement)

Solving linear equations using inverse matrix

1.

$$1a + 1b + 1c = 15$$
$$3a + 2b + 1c = 28$$
$$2a + 1b + 2c = 23$$

(apples, bananas, carrots)

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 28 \\ 23 \end{bmatrix}$$

2. System $Br = t$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} 4 & 6 & 2 \\ 3 & 4 & 1 \\ 2 & 8 & 13 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 2 \end{bmatrix}$$

(convert to echelon form using elimination)

$$\div 4 \quad \begin{array}{c} 1' \\ 2' \\ 3' \end{array}\begin{bmatrix} 1 & 3/2 & 1/2 \\ 3 & 4 & 1 \\ 2 & 8 & 13 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ 7 \\ 2 \end{bmatrix} \qquad \begin{array}{c} 1'' \\ 2'' \\ 3'' \end{array}\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 2 & 8 & 13 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 2 \end{bmatrix}$$

$(2'' = [2' - 3\,1'] \times (-2))$

(made new second row linear combination of previous rows)

3.

$$\begin{array}{c} 1'' \\ 2'' \\ 3'' \end{array}\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 2 & 8 & 13 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 2 \end{bmatrix}$$

make it 0

(fix row 3 to be linear combination of other two)

$$3''' = -2 \times 1'' + 3'' = \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 5 & 12 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ -5/2 \end{bmatrix},$$

make it 0

$$3''' = -5 \times 2'' + 3'' = \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$$

make it 1

$$3''' = 3''/7 = \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$$

(echelon form)

4. Back substitution to solve the system (use linear combinations of the rows to turn the matrix into the identity matrix)

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$$

$$(R_1 - \tfrac{1}{2}R_3): \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 - \frac{1}{2} \times 0 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$$

← make it 0

$$(R_2 - R_3): \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$$

$$(R_1 - \tfrac{3}{2}R_2): \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 - 3/2 \times -1/2 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1/2 \\ 0 \end{bmatrix}$$

5. $$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 28 \\ 23 \end{bmatrix}$$

(convert to echelon form)

$$(R_2 - 3R_1) \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ -17 \\ 23 \end{bmatrix}$$

$$(R_3 - 2R_1) \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ -17 \\ -7 \end{bmatrix}$$

$$(R_2 \times -1) \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ -7 \end{bmatrix}$$

$$\frac{(R_3 + R_2)}{2} \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 5 \end{bmatrix}$$

Back substitution to solve the system (use linear combinations of the rows to turn the matrix into the identity matrix)

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 5 \end{bmatrix} \rightarrow \begin{array}{l} (R_2 - 2R_3) \\ c = 5, \therefore 2c = 10, \; 17 - 10 = 7 \end{array}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \\ 5 \end{bmatrix} \rightarrow (R_1 - R_2 - R_3)$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \quad \therefore a = 3, \; b = 7, \; c = 5$$

(3×3) (1×3)

A = [[...],[...],[...]], s = [...]

r = np.linalg.solve(A,s) => [...]
(1×3)

more computationally efficient to solve L.A. system

(don't is not absolutely necessary)

7. Solve the system in general: inverse matrix

$$AB = I, \quad B = A^{-1}, \quad A^{-1}A = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate on I

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad A' = \begin{bmatrix} R_1 \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{bmatrix}, \quad A' \begin{bmatrix} R_1 \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$(R_2 \times -1) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \left(\frac{R_3 + R_2}{2}\right) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad (R_1 - R_2) \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

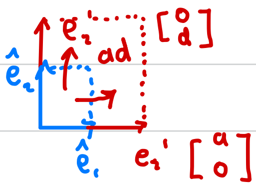
$$(R_1 + R_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$(R_2 - 2R_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Determinants and Inverses

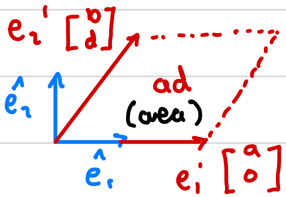
↙ amount it scales space

$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$



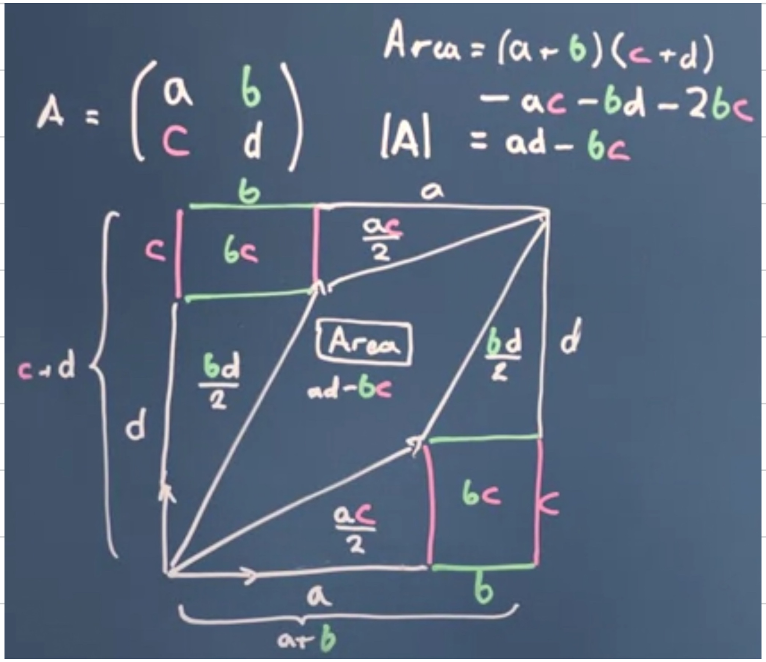
(Scaled the space by 'ad': determinant)

$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$



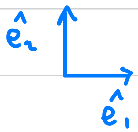
(area of parallelogram: base × ⊥er height)

↑ determinant still ad



|A|: determinant of A

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$



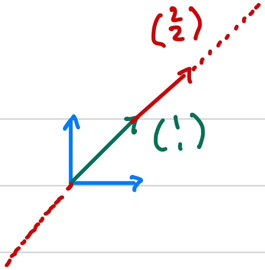
Inv$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: flip terms on leading diagonal, and take minus of off diagonal terms

$$\frac{1}{ad-bc}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{A^{-1}} = \underbrace{\begin{pmatrix} \frac{1}{ad-bc}\,(ad-bc) & 0 \\ 0 & \frac{1}{ad-bc}\,(ad-bc) \end{pmatrix}}_{\text{determinant}} = I$$

↑ undoes the scaling back to scale of 1

find determinant in general case: QR decomposition

$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$



(transforms all points onto a line)

(multiple of each other : not linearly independent)

$|A| = 0$

e.g. 3x3 matrix, with 1 basis vector being a linear multiple of the other two (not linearly independent)

↳ new space: a plane, or if only 1 independent basis vector: a line (volume = 0, det = 0)

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \\ 29 \end{pmatrix}$$

(transformation matrix doesn't describe 3 independent basis vectors, because one is linearly dependent on other two)

↳ ∴ collapses into a 2D space

row 3 = row 1 + row 2

col 3 = 2 × col 1 + col 2

Reduce to row echelon form:

$R_2 - R_1$,
$R_3 - R_2 - R_1$:

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \\ 0 \end{pmatrix}$$

$0c = 0$, true but not useful (infinite no. of solutions for c)

↳ can't solve equations, not enough info (i.e. third order of a, b, c was a copy of first two orders)

↳ when basis vectors describing the matrix aren't linearly independent, then det = 0, ∴ can't solve system of S.A. and can't invert (matrix has no inverse)

Inverse matrix: lets you undo your transformation (get from new vectors to original vectors)

extra dimension

If you dump a dimension to go from 2D space to a line: can't undo it, as you have lost some info during the transformation

You want the new basis vectors to be linearly independent so you can undo the transformation

Special Matrices Lab

example

(to make diagonal 1)

$$\begin{bmatrix}1&2&3\\3&4&5\\5&6&7\end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix}1&2&3\\0&-2&-4\\5&6&7\end{bmatrix} \xrightarrow{R_2/-2} \begin{bmatrix}1&2&3\\0&1&2\\5&6&7\end{bmatrix} \xrightarrow{R_3-5R_1} \begin{bmatrix}1&2&3\\0&1&2\\0&-4&-8\end{bmatrix} \xrightarrow{R_3 - -4\times R_2} \begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix}$$

Singular, ∴ not invertible

Test if a 4x4 matrix is singular (i.e. determine if an inverse exists)

↳ convert matrix to echelon form, if zeros remain on leading diagonal that can't be removed: fail)

NumPy matrix: A[n,m], n'th row m'th column starting at 0

2 - 2 x 1

A[1,0]

fix row One: A[1] = A[1] - A[1,0] x A[0] $(R_2 - 2R_1)$

fix row Two: A[2] = A[2] - A[2,0] x A[0,0] $(R_3 - 3R_1)$ (3: value in R_3)

A[2] = A[2] - A[2] x A[1] $(R_3 - -10.5R_2)$

applied across whole row

fix row Three: (see example)

(see notebook)

Week 4: matrices make linear mappings

Einstein Summation Convention

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \\ \vdots & \vdots & & \\ a_{n1} & & & a_{nn} \end{pmatrix}} \underset{B}{\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \\ \vdots & \vdots & & \\ b_{n1} & & & b_{nn} \end{pmatrix}} = \underset{AB}{\begin{pmatrix} & 0 & \end{pmatrix}}$$

row, col   row, col

can write as $A_{ij}V_j$

$$\sum_{j=1}^{3} A_{ij}V_j = A_{i1}V_1 + A_{i2}V_2 + A_{i3}V_3$$

$$(ab)_{23} = a_{21}b_{13} + a_{22}b_{23} + \dots + a_{2n}b_{n3}$$

quick way of coding operations

$ab_{ik} = \sum_j a_{ij}b_{jk} = a_{ij}b_{jk}$ (Einstein convention: write sum without sigma, compact: sum over j as it appears twice)

gives you whole matrix

$AB = C$ (can multiply non-square matrices, as long as j's are equal)

$C_{ik} = a_{ij}b_{jk}$

$$2\overset{3}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}} \overset{4}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}3 = 2\overset{4}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}$$

(rows in A)   4 (cols in B)

(2×3)   (4×3)

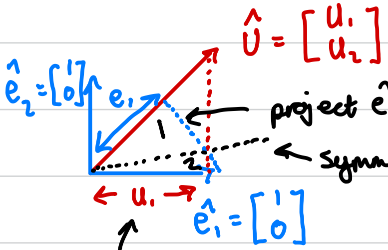
$$U \qquad V$$

$$\begin{pmatrix} u_i \\ \\ \end{pmatrix} \cdot \begin{pmatrix} v_i \\ \\ \end{pmatrix}$$

(matrix multiplication same as dot product)

$$\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$u_i v_i$ (over all i's)



project $\hat{e}_1$ onto $\hat{U}$

symmetry : triangles 1 and 2 are the same, $\hat{e}_1 . \hat{U} = \hat{U} . \hat{e}_1$, $\hat{e}_2 \; \hat{U} = \hat{U} . \hat{e}_2$

(projection is symmetric, dot product is symmetric, projection is dot product)

↓ connection between matrix multiplication and geometric projection

↓ matrix multiplication with vector : projection of vector onto vectors (columns) composing the matrix

project $\hat{U}$ onto $\hat{e}_1$

Non-Square Matrix Multiplication Quiz

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $C = AB$

$C_{mn} = A_{mj} B_{jn}$ $\qquad$ $C_{21} = A_{2j} B_{j1} = A_{21} B_{11} + A_{22} B_{21} + A_{23} B_{31}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 4 \times 1 + 0 \times 0 + 1 \times 1 = 5$

↑ multiply A's rows with B's cols

2. If j is same in $C_{mn} = A_{mj} B_{jn}$, can multiply, C will have A rows and B columns

$C = AB$

$C_{12} = A_{1j} B_{j2}$

$$\begin{array}{ccc} A & B & C \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = & \begin{bmatrix} 4 & 3 & 5 \\ 5 & 4 & 1 \end{bmatrix} \\ 2\times 3 & 3\times 3 & 2\times 3 \end{array}$$

Same, so can multiply

3. $\underset{1\times 4}{\begin{bmatrix} 2 & 4 & 5 & 6 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}} = 2\times 1 + 4\times 3 + 5\times 2 + 6\times 1 = 2 + 12 + 10 + 6 = 30$

(1x1 matrix)

4. $\underset{(4\times 1)}{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}} \underset{(1\times 4)}{\begin{bmatrix} 2 & 4 & 5 & 6 \end{bmatrix}} = \begin{bmatrix} 2 & 4 & 5 & 6 \\ 6 & 12 & 15 & 18 \\ 4 & 8 & 10 & 12 \\ 2 & 4 & 5 & 6 \end{bmatrix}$

(4x4 matrix)

5. $\underset{3\times 2}{\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} 0 & 1 & 4 & -1 \\ -2 & 0 & 0 & 2 \end{bmatrix}} = \underset{3\times 4}{\begin{bmatrix} 2 & 2 & 8 & -4 \\ -6 & 0 & 0 & 6 \\ 0 & 1 & 4 & -1 \end{bmatrix}}$

6. $D = ABC$ where $\overbrace{A = 5\times 3, B = 3\times 7}^{5\times 7}$, $C = 7\times 4$, $\therefore D = 5\times 4$

7. $\underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$

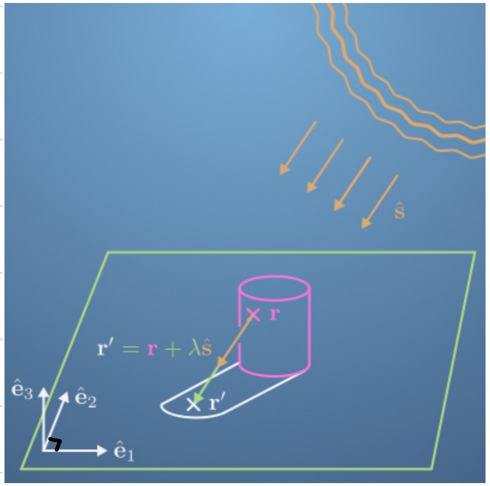
make sure indices are same in E. convention

8. U and V are vectors with n elements, dot product: $U_i V_i$, $\begin{bmatrix} U_1 & U_2 \dots U_n \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$, $U.V$, $\sum_{i=1}^{n} U_i V_i$

can't be other way round for dot product!

Non-Square Matrix Projection Quiz

Shadows : transformation that reduces no. of dimensions
↳ 3D objects cast shadows on 2D surfaces



e_3 and r' are mutually perpendicular

$r' = r + \lambda\hat{s}$ $\qquad s_3 = \hat{s}.\hat{e}_3$

$r.\hat{e}_3 + \lambda s_3 = 0$

↓

$\lambda s_3 = -r.\hat{e}_3 \ , \quad \lambda = \dfrac{-r.\hat{e}_3}{s_3}$

$\therefore\ r' = r + \left(\dfrac{-r.\hat{e}_3}{s_3}\right)\hat{s} = r - \dfrac{\hat{s}(r.\hat{e}_3)}{s_3}$

The sun is sufficiently far away that effectively all of its rays come in parallel to each other. We can describe their direction with the unit vector $\hat{\mathbf{s}}$.

We can describe the 3D coordinates of points on objects in our space with the vector $\mathbf{r}$. Objects will cast a shadow on the ground at the point $\mathbf{r}'$ along the path that light would have taken if it hadn't been blocked at $\mathbf{r}$, that is, $\mathbf{r}' = \mathbf{r} + \lambda\hat{\mathbf{s}}$.

The ground is at $\mathbf{r}'_3 = 0$; by using $\mathbf{r}'.\hat{\mathbf{e}}_3 = 0$, we can derive the expression, $\mathbf{r}.\hat{\mathbf{e}}_3 + \lambda s_3 = 0$, (where $s_3 = \hat{\mathbf{s}}.\hat{\mathbf{e}}_3$).

Rearrange this expression for λ and substitute it back into the expression for $\mathbf{r}'$ in order to get $\mathbf{r}'$ in terms of $\mathbf{r}$.

$\times A^{-1} \quad \times A^{-1}$

$A^{-1}A = I \ , \quad \cancel{I}A^{-1} = A^{-1}\cancel{I} \ , \quad A^{-1} = A^{-1} \ , \quad AI = A$

2. From your answer above, you should see that $\mathbf{r}'$ can be written as a linear transformation of $\mathbf{r}$. This means we should be able to write $\mathbf{r}' = A\mathbf{r}$ for some matrix A.

To help us find an expression for A, we can re-write the expression above with Einstein summation convention.

Which of the answers below correspond to the answer to Question 1? (Select all that apply)

- [x] $r'_i = (I_{ij} - s_i[\hat{e}_3]_j/s_3)r_j$
- [ ] None of the other options.
- [x] $r'_i = r_i - s_i r_3/s_3$
- [x] $r'_i = (I_{ij} - s_i I_{3j}/s_3)r_j$
- [x] $r'_i = r_i - s_i[\hat{e}_3]_j r_j/s_3$

$[Ar]_i = A_{ij}r_j$

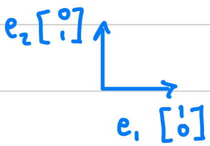
$r' = Ar \ , \quad A^{-1}r' = (A^{-1}A)r \ , \quad A^{-1}r' = Ir \ , \quad r = A^{-1}r'$

correct + concise (difficult to see mat. mul on r)

$[\hat{e}_a]_j = I_{aj}$ $\qquad [\hat{e}_1]_j = I_{1j}$

shows naturally

$r' = r - \dfrac{\hat{s}(r.\hat{e}_3)}{s_3} \ , \quad r' = Ar \ , \quad Ar = r - \dfrac{\hat{s}(r.\hat{e}_3)}{s_3}$



$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C_{in} = a_{ij} . b_{ij}$ (convention) , $r.I = r$, $r.\hat{e}_3 = r_3$

3. Based on your answer to the previous question, or otherwise, you should now be able to give an expression for A in its component form by evaluating the components A_{ij} for each row i and column j.

Since A will take a 3D vector, $\mathbf{r}$, and transform it into a 2D vector, $\mathbf{r}'$, we only need to write the first two rows of A. That is, A will be a 2×3 matrix. Remember, the columns of a matrix are the vectors in the new space that the unit vectors of the old space transform to - and in our new space, our vectors will be 2D.

What is the value of A?

$$r' = Ar, \quad A = \left(I_{ij} - \frac{s_i I_{3j}}{s_3}\right), \quad r_i' = \left(I_{ij} - \frac{s_i I_{3j}}{s_3}\right) r_j$$

$$A_{ij} = \left(I_{ij} - \frac{s_i I_{3j}}{s_3}\right) \qquad j \to 1 \ldots$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$i=1, j=1:$ $$A_{11} = \left(I_{11} - \frac{s_1 I_{31}}{s_3}\right) = \left(1 - \frac{s_1 0}{s_3}\right) = 1$$

$i=1, j=2:$ $$A_{12} = \left(I_{12} - \frac{s_2 I_{32}}{s_3}\right) = \left(0 - \frac{s_1 0}{s_3}\right) = 0$$

$i=1, j=3:$ $$A_{13} = \left(I_{13} - \frac{s_2 I_{33}}{s_3}\right) = \left(0 - \frac{s_1 1}{s_3}\right) = -\frac{s_1}{s_3}$$

$i=2, j=1:$ $$A_{21} = \left(I_{21} - \frac{s_2 I_{31}}{s_3}\right) = \left(0 - \frac{s_2 0}{s_3}\right) = 0$$

$i=2, j=2:$ $$A_{22} = \left(I_{22} - \frac{s_2 I_{32}}{s_3}\right) = \left(1 - \frac{s_2 0}{s_3}\right) = 1$$

$i=2, j=3:$ $$A_{23} = \left(I_{23} - \frac{s_2 I_{33}}{s_3}\right) = \left(0 - \frac{s_2 0}{s_3}\right) = -\frac{s_2}{s_3}, \quad \therefore A = \begin{bmatrix} 1 & 0 & -s_1/s_3 \\ 0 & 1 & -s_2/s_3 \end{bmatrix}$$

4. A is 2x3 matrix, but what would components of third row be?

$i=3, j=1: \quad A_{31} = \left(I_{31} - \frac{s_3 I_{31}}{s_3}\right) = (0 - 1\times 0) = 0$

$r_3' = 0$ (doesn't exist for 2D objects : r' never has a value in the third direction)

$i=3, j=2: \quad A_{32} = \left(I_{32} - \frac{s_3 I_{32}}{s_3}\right) = (0 - 1\times 0) = 0$

$i=3, j=3: \quad A_{33} = \left(I_{33} - \frac{s_3 I_{33}}{s_3}\right) = (1 - 1) = 0$

5. Assume the Sun's rays come in at a direction $\hat{\mathbf{s}} = \begin{bmatrix} 4/13 \\ -3/13 \\ -12/13 \end{bmatrix}$.

Construct the matrix, A, and apply it to a point, $\mathbf{r} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$, on an object in our space to find the coordinates of that point's shadow.

Give the coordinates of $\mathbf{r}'$.

$r' = Ar, \quad \hat{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 4/13 \\ -3/13 \\ -12/13 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & -s_1/s_3 \\ 0 & 1 & -s_2/s_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/4 \end{bmatrix}$

$\therefore Ar = \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/4 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 7 \\ 1.25 \end{bmatrix}}$

6. Another use of non-square matrices is applying a matrix to a list of vectors.

Given our transformation $\mathbf{r}' = A\mathbf{r}$, this can be generalized to a matrix equation, $R' = AR$, where R' and R are matrices where each column are corresponding r' and r vectors, i.e.,

$$\begin{bmatrix} r'_1 & s'_1 & t'_1 & u'_1 & \dots \\ r'_2 & s'_2 & t'_2 & u'_2 & \dots \end{bmatrix} = A \begin{bmatrix} r_1 & s_1 & t_1 & u_1 & \dots \\ r_2 & s_2 & t_2 & u_2 & \dots \\ r_3 & s_3 & t_3 & u_3 & \dots \end{bmatrix}.$$

In Einstein notation, $r'_i = A_{ij} r_j$ becomes $R'_{ia} = A_{ij} R_{ja}$.

For the same $\hat{\mathbf{s}}$ as in the previous question, apply A to the matrix

$$R = \begin{bmatrix} 5 & -1 & -3 & 7 \\ 4 & -4 & 1 & -2 \\ 9 & 3 & 0 & 12 \end{bmatrix}.$$

Observe that it's the same result as treating the columns as separate vectors and calculating them individually.

$$\hat{S} = \begin{bmatrix} 4/13 \\ -3/13 \\ -12/13 \end{bmatrix}$$ populate A's third col

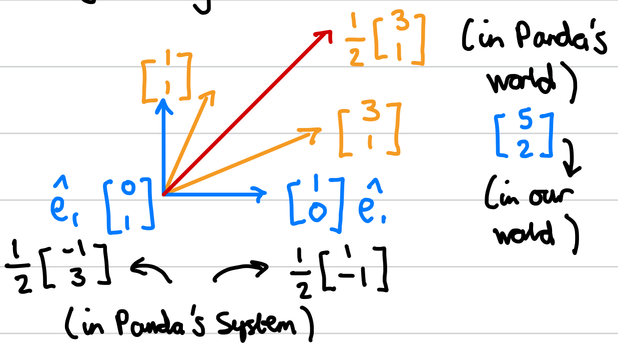
$$R' = AR, \quad A = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/4 \end{bmatrix}$$

$$R' = \underbrace{\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/4 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 5 & -1 & -3 & 7 \\ 4 & -4 & 1 & -2 \\ 9 & 3 & 0 & 12 \end{bmatrix}}_{3\times 4}$$

$$= \underbrace{\begin{bmatrix} 8 & 0 & -3 & 11 \\ 1.75 & -4.75 & 1 & -5 \end{bmatrix}}_{2\times 4}$$

Matrices Changing Basis

Columns of transformation matrix are axis of new basis vectors of the mapping



• : panda basis vectors

• : our basis vectors

Panda's basis vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in my frame

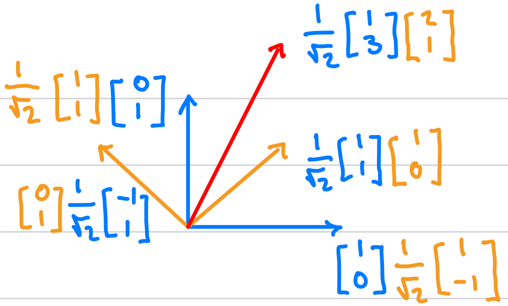
↳ transformation matrix $\underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{B Panda's basis in our coordinates}} \underbrace{\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}}_{\text{Panda's vector}} = \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{\text{Our vector}}$

(my basis in Panda's world)

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$ Panda's vector

if you want to take Panda's vector into my world you need Panda's basis in my coordinate frame



Panda's world will be orthonormal basis vector set

- : our world
- : Panda's world

B (Panda's transformation matrix: transforms vector of Panda's)

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$ (in our world)

Reverse transformation:

B^{-1}

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

If new basis vectors are orthogonal: can use projections

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}4 = 2$$ (first component of vector)

(my version of vector) (Panda's first axis in my world) (vector in Panda's world)

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2}2 = 1$$ (second component of vector)

↳ $\therefore \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (Panda's vector)

(used projections to translate my vector to Panda's vector using dot product)

lengths of Panda's vectors are 1, would have to normalise otherwise (∴ don't need matrix math, just use dot product only if Panda's vectors are orthogonal)

↳ if not orthogonal, use matrix math instead

Transformation in changed basis

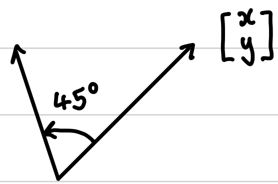
Panda's basis: $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ ↗ 1st axis, ↑ 2nd axis

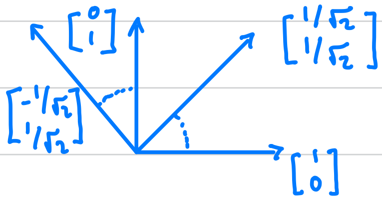
vector in Panda's basis,

want to do 45° rotation

(but don't know how to do 45° rotation in Panda's coordinate system,

only know in my normal $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ system)





$$\underbrace{\underset{B^{-1}}{\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}} \underbrace{\underset{\substack{R^{\circlearrowleft 45} \\ \text{(in my world)}}}{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vector in my frame}}}_{\text{vector in Panda's frame}}$$

$\frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$

$\frac{1}{2\sqrt{2}}\begin{bmatrix} -2 & -2 \\ 10 & 6 \end{bmatrix}$ (45° ↺ in Panda's coordinate system)

Panda wants rotation in his basis: transform into Panda's basis using B^{-1}

$B^{-1} R B = R_B$ (rotation in B's coordinate system)

↖ translation from my world to world of new basis system

Want to transform to non-orthonormal coordinate systems: transformation matrices also change

$$\frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2/\sqrt{2} & 0 \\ 4/\sqrt{2} & 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} - 2/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} + 2/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Orthogonal Matrices

$$A^T_{ij} = A_{ji} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$A_{n \times n}$: square matrix which defines transformation with new basis

$$\begin{bmatrix} \begin{pmatrix} a_1 \end{pmatrix} & \begin{pmatrix} a_2 \end{pmatrix} & \cdots & \begin{pmatrix} a_n \end{pmatrix} \end{bmatrix}$$

basis vectors in new space

$$\left.\begin{array}{lll} a_i \cdot a_j = 0 & i \neq j & \text{(orthogonal)} \\ = 1 & i = j & \text{(unit length)} \end{array}\right\}$$ orthonormal basis set (matrix composed of them = orthogonal matrix)

↳ scales space by factor of 1, ∴ determinant is + or − 1

in data science, want transformation matrix to be orthogonal, so inverse is easy to compute, transformation is reversible (doesn't collapse space), projection is just dot product

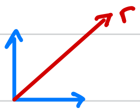
$$\underbrace{\begin{bmatrix} (& a_1 &) \\ (& a_2 &) \\ & \vdots & \\ (& a_n &) \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} \begin{pmatrix} a_1 \end{pmatrix} & \begin{pmatrix} a_2 \end{pmatrix} & \cdots & \begin{pmatrix} a_n \end{pmatrix} \end{bmatrix}}_{A_{n \times n}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\underbrace{A^T A}_{\text{can do it either way around}} = I$, ∴ A^T is valid inverse A

$A^T = A^{-1}$

↳ ∴ rows of orthogonal matrix are orthonormal as well as columns

Transpose of orthonormal basis vector set is another orthogonal basis vector set



can do dot product of r with basis vectors, as long as basis vectors are orthogonal to each other

The Gram-Schmidt Process

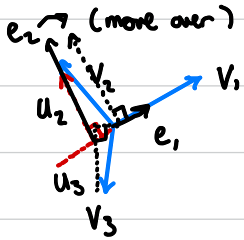
How to construct an orthonormal basis

↙ to check: write down cols in matrix and check if determinant isn't 0 (1 = linearly dependent)

$v = \{v_1, v_2, \ldots v_n\}$ (assume linear independence)

↖ not orthogonal to each other or of unit length

v_1 $\qquad e_1 = \dfrac{v_1}{|v_1|}$



↙ not in e_1, e_2 plane either

v_3: not linear combo of v_1 and v_2 (v_3 not in plane)

$v_2 = (v_2 . e_1)\dfrac{e_1}{\cancel{|e_1|}} + u_2$

$u_2 = v_2 - (v_2 \cdot e_1) e_1 \qquad \dfrac{u_2}{|u_2|} = e_2$ (unit vector that's normal to v_1)

Can project v_3 onto plane of e_1 and e_2 (projection will be some vector in the plane composed of e_1's and e_2's)

↙ normalise u_3

$u_3 = v_3 - \underbrace{(v_3 . e_1)}_{\text{component of } v_3 \text{ made up of } e_1\text{'s}} e_1 - \underbrace{(v_3 . e_2)}_{\text{component of } v_3 \text{ made up of } e_2\text{'s}} e_2 \qquad \dfrac{u_3}{|u_3|} = e_3$, now have unit vector which is normal to the plane and other two

Can carry on for all v's: gone from awkward non-orthogonal non-unit vectors to nice orthogonal unit vectors (orthonormal basis set)

↘ makes transformation matrices nice, can now do dot product projections, transpose as inverse

Gram-Schmidt Process Lab (see notebook)

Take a list of vectors and forms an orthonormal basis

↳ procedure allows us to determine dimension of space spanned by basis vectors (= or < space which the vectors sit)

A[n, m] : element at n'th row, m'th col (0-based)

u @ v : dot product

Take basis vectors as list of vectors as columns of matrix A

Set each vector to be orthogonal (one at a time) to all vectors that came before it

Then normalise

0^{th} column just needs normalising, has no other vectors to make it normal to ($\therefore$ divide by its modulus / norm)

1^{st} column: subtract any overlap with new 0^{th} vector

↳ anything left after subtraction: 1^{st} vector is linearly independent of 0^{th} vector ($\therefore$ can normalise, otherwise set vector to 0)

2^{nd} column: subtract overlap with 0^{th} vector, then subtract overlap with 1^{st} vector, then normalise new vector

3^{rd} column: subtract overlap with first 3 vectors, then normalise if possible

Can use Gram-Schmidt process to calculate dimension spanned by list of vectors

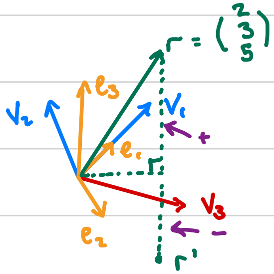
↳ each vector normalised to 1, or is 0, $\therefore$ sum of all norms will be dimension

1 vector which is linear combo of others: will be set to 0 and isn't an included dimension

Reflection in a Plane

$$\underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}\ \begin{pmatrix}2\\0\\1\end{pmatrix}}_{\text{in plane of mirror}}\ \begin{pmatrix}3\\1\\-1\end{pmatrix}$$

$v_1 \quad v_2 \quad v_3$



e_1: normalised version of v_1

e_2: $\perp$er part of v_2 to e_1, normalised to unit length

e_3: normal to e_1, e_2 plane, normalised to unit length

r: composed of vector that's in the plane (e_1's and e_2's) and some normal vector (some amount of e_3's)

Use Gram - Schmidt process to find orthonormal vectors describing plane

$$e_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$u_2 = v_2 - (v_2 . e_1)e_1 = \begin{pmatrix}2\\0\\1\end{pmatrix} - \left[\begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{1}{3}3\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$e_2 = \frac{u_2}{|u_2|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$u_3 = v_3 - (v_3 . e_1)e_1 - (v_3 . e_2)e_2 = \begin{pmatrix}3\\1\\-1\end{pmatrix} - \left[\begin{pmatrix}3\\1\\-1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} - \left[\begin{pmatrix}3\\1\\-1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}3\\1\\-1\end{pmatrix} - \frac{1}{3}3\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{2}2\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$e_3 = \frac{u_3}{|u_3|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$\therefore$ new transformation matrix $E = \big((e_1)(e_2)(e_3)\big) = \left(\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}\right)$

reflection in the plane's basis vector set

$$T_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$e_1 \; e_2 \; -e_3$

$$\boxed{r' = E T_E E^{-1} r}$$

basis vectors are orthonormal

$$(E^T = E^{-1})$$

$$\begin{tikzcd} r \arrow[r, "\text{hard!}"] \arrow[d, "E^{-1}"'] & r' \\ r_E \arrow[r, "T_E"'] & r'_E \arrow[u, "E"'] \end{tikzcd}$$

(my vector to Bear's basis)

(Bear's vector to my basis)

(transformed - reflection basis of plane)

$$T_E E^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3}\,(1 \; 1 \; 1) \\ 1/\sqrt{2}\,(1 \; -1 \; 0) \\ 1/\sqrt{6}\,(1 \; 1 \; -2) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3}\,(1 \; 1 \; 1) \\ 1/\sqrt{2}\,(1 \; -1 \; 0) \\ 1/\sqrt{6}\,(-1 \; -1 \; 2) \end{pmatrix}$$

$$E T_E E^T = \begin{pmatrix} 1/3 + 1/2 - 1/6 & 1/3 - 1/2 - 1/6 & 1/3 + 0 + 2/6 \\ 1/3 - 1/2 - 1/6 & 1/3 + 1/2 - 1/6 & 1/3 + 0 + 2/6 \\ 1/3 + 0 + 2/6 & 1/3 + 0 + 2/6 & 1/3 + 0 - 4/6 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix} = T$$

$$r' = Tr = T\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 14 \\ 5 \end{pmatrix}$$

(reflection in mirror)

Reflecting Bear Lab

Produce transformation matrix for reflecting vectors in an arbitrarily angled mirror

s.t , At , AB (vector dot product, matrix op. on vector, matrix multiplication) = @

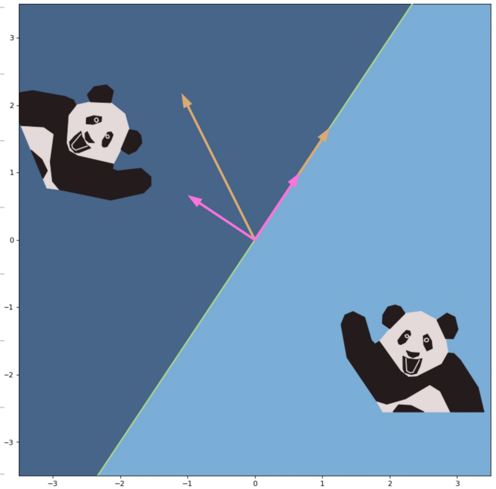
* : element-wise multiplication or multiplication by a scalar

Bear's basis

↳ Get mirror's orthonormal basis using Gram-Schmidt on Bear's basis

↳ Create matrix to perform mirror's reflection in mirror's basis (mirror : negate last vector component)

↳ $E^T . T_E \; E$



• : Bear's basis

• : Mirror's orthonormal basis

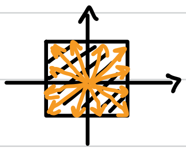
Shows Bear's reflection in his mirror

Week 5: Eigenvectors and Eigenvalues

Eigen-problem: finding characteristic properties of something

Express linear transformations using matrices (scalings, rotations, shears)

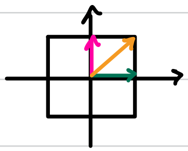
Can think about how matrix transforms all vectors in a space



Scaling of 2: becomes rectangle

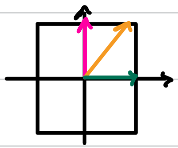
horizontal shear: parallelogram

Some vectors end up on same line they started on, some don't



vertical scaling

=>



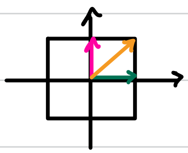
(pink is doubled, green is the same, orange was 45° but now increased as well as length)

Horizontal and vertical vectors are special: characteristic of particular transform => Eigenvectors

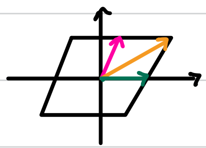
Green vector's length unchanged: has corresponding Eigenvalue of 1

Pink vector's length doubled: has Eigenvalue of 2

Take transform and look for vectors which lie on same span as before, and measure how much their length has changed



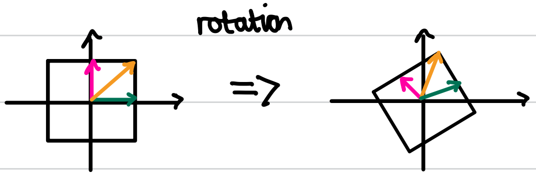
=> "pure" shear



1 Eigenvector

(only green vector still laying on original span, all other vectors are shifted)

(no scaling, rotation, ∴ area is unchanged)

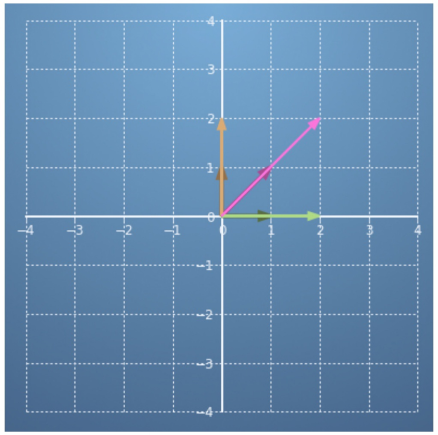


(no Eigenvectors as all vectors have been rotated from original span)

Selecting eigenvectors by inspection quiz

Eigenvector: vector lies in same span after applying linear transformation

1.

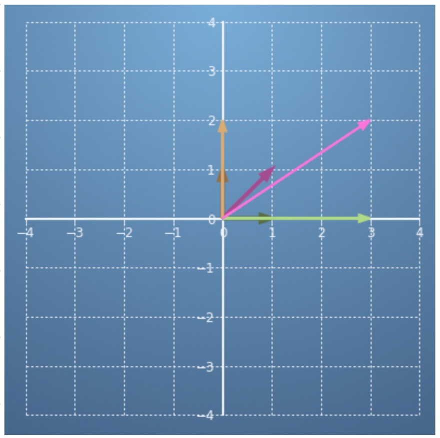


Apply $T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvectors: all original vectors

(all have eigenvalues of 2, as they double in size + stay in same direction)

2.

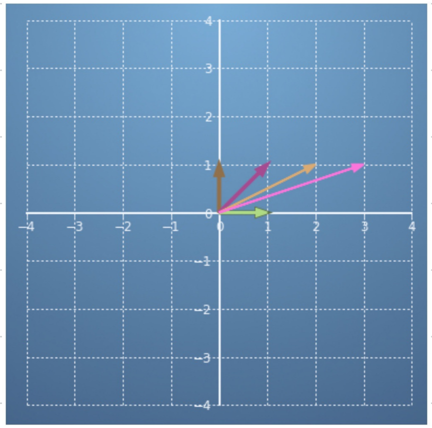


Apply $T = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

eigenvalue = 3 eigenvalue = 2

3.

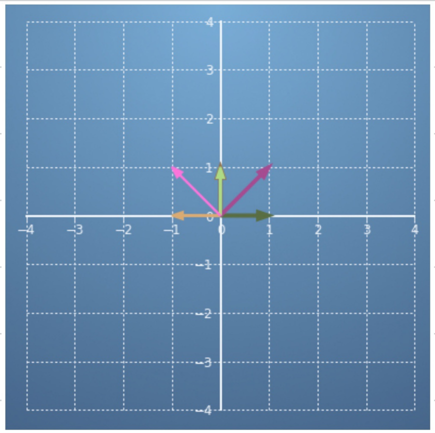


Apply $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenvalue = 1

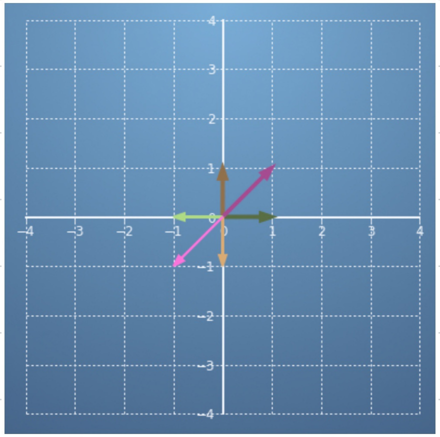
4.



Apply $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Has no eigenvectors in the plane

5.

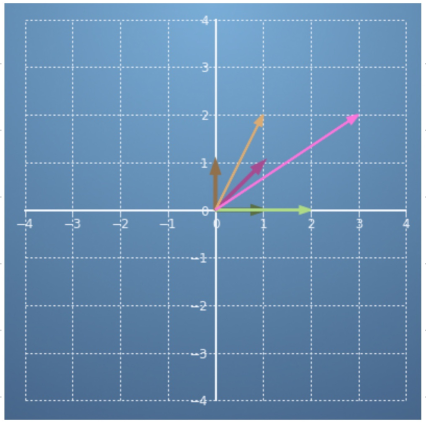


Apply $T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Eigenvectors : $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(all have eigenvalues of -1 as they reverse direction but have same size)

6.



Apply $T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

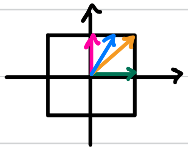
Eigenvectors : $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenvalue = 2

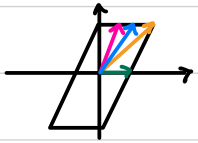
Special eigen-cases (geometrically)

Uniform Scaling : all vectors are eigenvectors (enlarged square)

180° rotation : all vectors are eigenvectors (eigenvalues of -1) (rotated square)



=> (horizontal shear + vertical scaling)

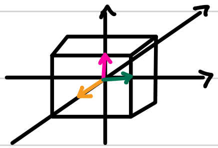


green vector : eigenvector with eigenvalue of 1

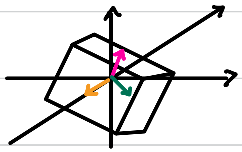
blue vector : eigenvector (less obvious)

apply inverse transform to see why blue vector is eigenvector

Tougher in ≥ 3D (in ML: can have many eigenvectors)



=> rotation



orange vector : eigenvector

finding eigenvector of 3D rotation : found axis of rotation

Calculating eigenvectors

$$Ax = \lambda x$$

transformation matrix (n-dimensional transform: n×n) → A; vector (n-dimensional) → x; scalar factor → λ; vector → x

(want to find x that makes both sides equal)

↳ for our eigenvectors, having A applied to them just scales their length or does nothing at all (same as scaling length by 1)

$$(A - \lambda I)x = 0$$

(x = 0 not interesting - no length or direction: trivial solution)

I: n×n identity matrix (subtracting scalars from matrices not defined, so I tidies the maths without changing meaning)

Want to find $A - \lambda I = 0$ (can test if matrix op. results in 0 output by calculating determinant)

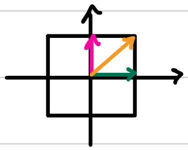
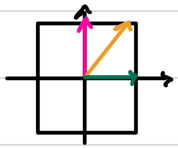
$\det(A - \lambda I) = 0$ e.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

↓

$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$ (evaluating this gives characteristic polynomial)

$\lambda^2 - (a+d)\lambda + ad - bc = 0$ (eigenvalues are solutions, plug into original expression for eigenvectors)

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

e.g.  vertical scaling $\times 2$ $\Rightarrow$ 

Transformation matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

must have solutions at $\lambda = 1$ and $\lambda = 2$

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) = 0$$

$$(A - \lambda I)x = 0$$

$$@\ \lambda = 1: \begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$$

$$@\ \lambda = 2: \begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0$$

when eigenvalue $= 1$: have eigenvector where x_2 must be 0

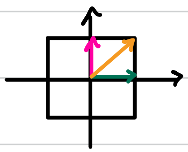
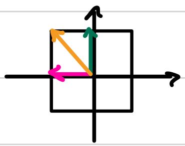
(but don't know about x_1 - any vector along horizontal axis could be eigenvector of this system)

anything on horizontal, as long as it's 0 in vertical

$$@\ \lambda = 1: x = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

$$@\ \lambda = 2: x = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

anything on vertical, as long as it's 0 in horizontal

e.g.  90° ↺ $\Rightarrow$ 

Transformation matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\det \begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

(has no real-numbered solutions, $\therefore$ no eigenvectors)

Iterative numerical methods used by computer for computing eigenvectors of high-dimensional problems

(2x2 matrix) $\quad (\det(A) - x\mathrm{Tr}(A) + x^2)$

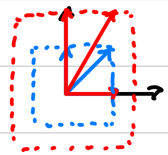
$$P_2(x) = (a_{11}a_{22} - a_{12}a_{21}) - x(a_{11} + a_{22}) + x^2$$

Characteristic polynomials, eigenvalues and eigenvectors quiz

1. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → calculate eigenvalues by solving characteristic polynomial: $\lambda^2 - \underbrace{(a+d)}_{\text{trace}}\lambda + \underbrace{(ad-bc)}_{\text{determinant}} = 0$

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ → $(A - \lambda I)x = 0$, $\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) = 2 - 3\lambda + \lambda^2$, $\lambda_1 = 1$ and $\lambda_2 = 2$

2. For matrix A, eigenvectors of A are vectors for which applying the matrix transformation is same as scaling by some constant



$(A - \lambda I)x = 0$

@ $\lambda = 1$: $\begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$ @ $\lambda = 1$: $x = \begin{pmatrix} t \\ 0 \end{pmatrix}$ ∴ eigenvectors: vectors 0 in vertical or horizontal

@ $\lambda = 2$: $\begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0$ @ $\lambda = 2$: $x = \begin{pmatrix} 0 \\ t \end{pmatrix}$

3. $A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$ → characteristic polynomial: $(A - \lambda I)x = 0$, $\det\begin{pmatrix} 3-\lambda & 4 \\ 0 & 5-\lambda \end{pmatrix} = (3-\lambda)(5-\lambda) - 4 \times 0$

$= 15 - 8\lambda + \lambda^2$, ∴ $\lambda_1 = 5$, $\lambda_2 = 3$

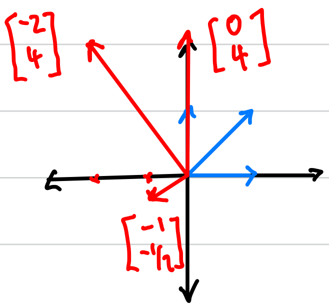
4. Eigenvectors of A: $(A - \lambda_i I)x_i = 0$ $\quad -x_1 + 2x_2 = 0$, $x_1 = 2x_2$ $\begin{pmatrix} -x_1+2x_2 \\ t \end{pmatrix}$ (eigenvalues)

@ $\lambda_1 = 5$: $\begin{pmatrix} 3-5 & 4 \\ 0 & 5-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + 4x_2 \\ 0 \end{pmatrix} = 0$ eigenvectors: $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1/2 \end{bmatrix}$

@ $\lambda_2 = 3$: $\begin{pmatrix} 3-3 & 4 \\ 0 & 5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_2 \\ 2x_2 \end{pmatrix} = 0$ eigenvectors: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = 0 \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

(if a vector is an eigenvector of a matrix: so is any non 0 multiple of that vector)

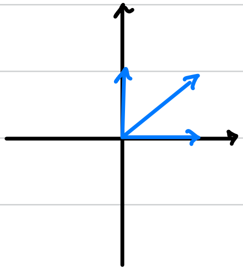


5. $A = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - 5\lambda + 4$, $(\lambda - 4)(\lambda - 1)$, $\therefore \lambda = 4,\ \lambda = 1$

$x_1 = 1 \quad x_1 = 2 \quad x_1 = 3 \quad x = -1$

6. Eigenvectors: @ $\lambda = 4$: $\begin{pmatrix} 1-4 & 0 \\ -1 & 4-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_1 \\ -x_1 \end{pmatrix} = 0$, $\therefore$ eigenvectors: $\begin{pmatrix} -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -6 \\ -2 \end{pmatrix}, \begin{pmatrix} -9 \\ -3 \end{pmatrix}, \ldots \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\uparrow x_1 = 0$ $\qquad x_1 = 1, x_2 = 2$



@ $\lambda = 1$: $\begin{pmatrix} 1-1 & 0 \\ -1 & 4-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_1 + 3x_2 \end{pmatrix} = 0$, $\begin{pmatrix} t \\ -x_1 + 3x_2 \end{pmatrix}$ $\therefore$ eigenvectors: $\begin{pmatrix} t \\ 5 \end{pmatrix}$,

$x_1 = 3x_2$ $\qquad x_1 = 1, x_2 = 3$ $\begin{pmatrix} t \\ 8 \end{pmatrix}, \ldots$

↳ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ an eigenvector? $x_1 = 3x_2$, $3 = 3 \times 1$, $\therefore$ YES

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$?: $x_1 = 3 \times 2 = 6 \neq 3$, $\therefore$ NO (doesn't satisfy any equations)

$x_1 = 0, x_2 = 1$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?: $x_1 = -3 \times 0 = 0$, $-0 \times 0 = 0$, $\therefore$ YES

$\begin{bmatrix} 3 \\ -1 \end{bmatrix}$?: $x_1 = 3 \times -1 = -3 \neq 3$, $\therefore$ NO

7. $A = \begin{bmatrix} -3 & 8 \\ 2 & 3 \end{bmatrix}$, characteristic polynomial $= \lambda^2 - 25$, $\lambda_1 = 5$, $\lambda_2 = -5$

8. @ $\lambda = 5$: $\begin{bmatrix} -3-5 & 8 \\ 2 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 & 8 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $-8x_1 + 8x_2 = 0$, $x_1 = x_2$, $\therefore$ eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$2x_1 - 2x_2 = 0$, $x_1 = x_2$

@ $\lambda = -5$: $\begin{bmatrix} -3--5 & 8 \\ 2 & 3--5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $2x_1 + 8x_2 = 0$, $x_1 + 4x_2 = 0$, $\therefore$ eigenvectors: $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

9. $A = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}$, $\lambda^2 - 2\lambda + 1$, $\lambda_1 = \lambda_2 = 1$

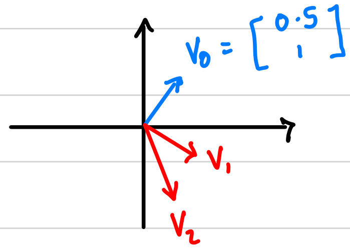
$4x_1 + 4x_2 = 0$, $x_1 = -x_2$, $\therefore$ eigenvectors: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ etc.

@ $\lambda = 1$: $\begin{bmatrix} 5-1 & 4 \\ -4 & -3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $-4x_1 - 4x_2 = 0$, $x_1 = -x_2$

10. $A = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$, $\lambda^2 + \lambda + 1$, $-b \pm \sqrt{b^2 - 4ac}/2a = \dfrac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2}$, $\therefore$ no real solutions

Changing to the eigenbasis

$$T = \begin{pmatrix} 0.9 & 0.8 \\ -1 & 0.35 \end{pmatrix}$$ ← change in location of particle after 1 timestep



$v_1 = Tv_0$

$v_2 = Tv_1$
$= T(Tv_0)$
$= T^2 v_0$

$v_n = T^n v_0$

(n could be large)

May need to apply same matrix transformations many times

Diagonal matrix: all values 0 except diagonal

↳ make things easier with matrix to power

e.g. pure scaling matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

$T^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$ (what if T isn't diagonal? - eigenanalysis)

$$\begin{tikzcd} v \arrow[r, "T^n"] \arrow[d, "C^{-1}"'] & T^n v \\ {[v]_E} \arrow[r, "D^n"'] & {[T^n v]_E} \arrow[u, "C"'] \end{tikzcd}$$

Change to basis where transformation T becomes diagonal : eigenbasis

Each column of transform matrix : represents new location of transformed unit vectors

$C = \begin{pmatrix} x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \end{pmatrix}$ ← eigenvectors $\quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad T^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$

$T^n = CD^nC^{-1}$

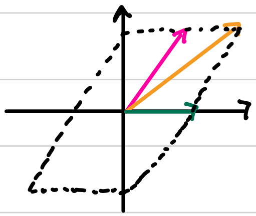
(apply T many times without large computational costs)

Applying transformation T : same as converting to eigenbasis, applying diagonalised matrix, then convert back again

$T = CDC^{-1}$

$T^2 = CDC^{-1}CDC^{-1} = CDDC^{-1} = CD^2C^{-1}$, $T^n = CD^nC^{-1}$

e.g. $T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$



$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$, @ $\lambda = 1$. $\begin{bmatrix} 1-1 & 1 \\ 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

@ $\lambda = 2$: $\begin{bmatrix} 1-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$x_1 = x_2$

$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $C^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

(conversion matrix built from eigenvectors)

$T^2 = CD^2C^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (same result as before)

Diagonalisation applications quiz

cols are eigenvectors of T

1. $T = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, change of basis matrix $C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, diagonal matrix $D = C^{-1}TC = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 10 & -5 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$ (diagonal matrix)

2. $D = C^{-1}TC = \frac{1}{3}\begin{bmatrix} 0 & -1 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 \\ 1 & 7/3 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ 2 & 14/3 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

3. $D = C^{-1}TC = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

4. $T = CDC^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 2\alpha \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$

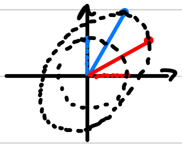
5. $T = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $T^3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 125 & 0 \\ 0 & 64 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 125 & 64 \\ 125 & 128 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 186 & -61 \\ 122 & 3 \end{bmatrix}$

6. $T = \begin{bmatrix} 2 & 7 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1/3 \\ 1 & 7/3 \end{bmatrix}$, $T^3 = \begin{bmatrix} 7 & 1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 0 & -1/3 \\ 1 & 7/3 \end{bmatrix} = \begin{bmatrix} -7 & 8 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1/3 \\ 1 & 7/3 \end{bmatrix} = \begin{bmatrix} 8 & 21 \\ 0 & -1 \end{bmatrix}$

7. $T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $T^5 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

Visualising matrices and eigen

Matrices : represent linear transformations (list of vectors which tell us where to move a set of basis vectors to during a transformation)

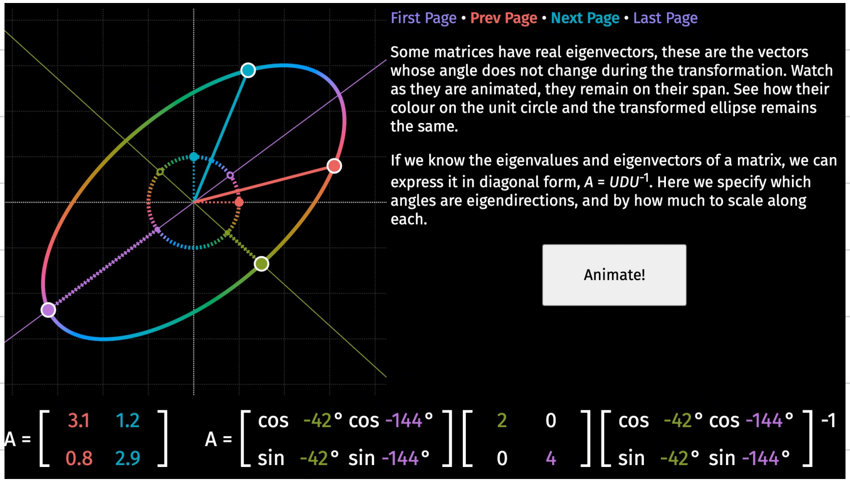


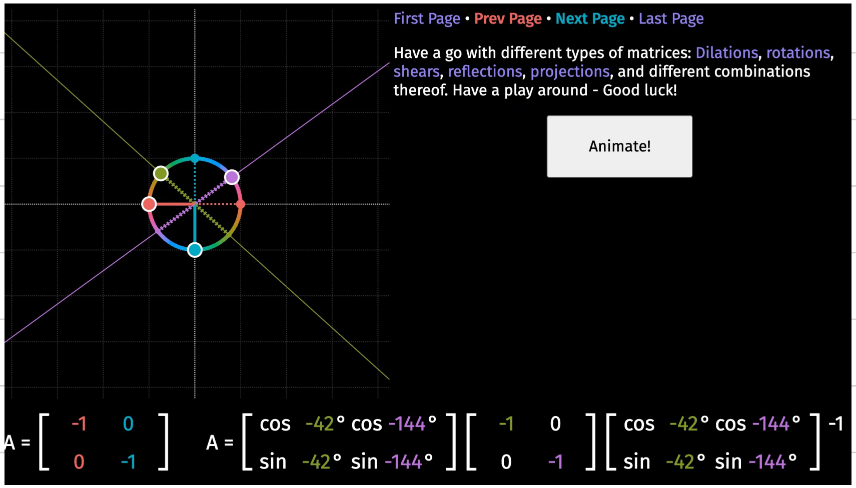
↳ 1st col: where basis vector i maps to

↳ 2nd col: where basis vector j maps to

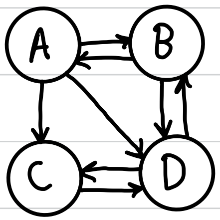
e.g. $A = \begin{bmatrix} 1.8 & 1.2 \\ 1.3 & 2.9 \end{bmatrix}$

↳ unit circle transforms to ellipse





Intro to PageRank



Network of pages linking to each other

Which pages are most relevant to search?

↙ normalise by no. of outgoing connections

link vectors
$$\begin{cases} L_A = (0, 1/3, 1/3, 1/3) \\ L_B = (1/2, 0, 0, 1/2) \\ L_C = (0, 0, 0, 1) \\ L_D = (0, 1/2, 1/2, 0) \end{cases} \Rightarrow \quad L = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{pmatrix}$$

(want probability of ending up on each of the pages)

↘ rows are inward links, normalised to page of origin

rank

$$r_A = \sum_{j=1}^{n} L_{A,j} r_j$$

(sum of ranks of all pages linked to A, weighted by link probability from L)

$$r^{i+1} = L r^i$$ (iterative until r stops changing)

$$r = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

(most entries will be 0, i.e. no inter-links : sparse matrices)

	A	B	C	D
rank =	0.1250	0.2083	0.2083	0.4583
	0.1354	0.2118	0.2118	0.4410
	0.1084	0.2611	0.2611	0.3694
	0.1084	0.2544	0.2544	0.3791
	0.1249	0.2311	0.2311	0.4129
	0.1227	0.2350	0.2350	0.4072
	0.1186	0.2425	0.2425	0.3963
	0.1194	0.2412	0.2412	0.3983
	0.1203	0.2395	0.2395	0.4007
	0.1201	0.2398	0.2398	0.4003
	0.1200	0.2401	0.2401	0.3999
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000
	0.1200	0.2400	0.2400	0.4000

as you randomly click around network: expect ≈ 40% of time on page D, 12% of time on page A etc.

r is eigenvector of matrix L, eigenvalue of 1

Multiply L by r many times, use diagonalisation method (but need all eigenvectors)

Repeatedly multiplying randomly selected initial guess vector by matrix: power method

↳ only gives 1 eigenvector, eigenvalue of 1

$r^{i+1} = d(Lr^i) + \frac{1-d}{n}$ (d : 0-1, damping factor : 1 - prob. of you suddenly randomly types in web address instead of clicking link on current page)

↳ finds compromise between speed + stability of iterative convergence process

PageRank Lab

Generate ranked list of web pages

Based on ideal random web surfer who, when reaching a page, goes to the next page by clicking on a link (see lab notebook)

$M = dL + \frac{1-d}{n}J$ (J : n×n matrix of 1's)

$Lr = r$

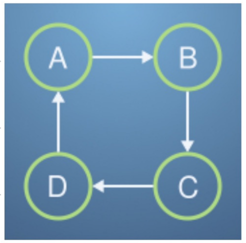
$\frac{1-x}{4} = 0.7$

Eigenvalues and Eigenvectors Quiz

1. Eigenvectors of $\begin{bmatrix} 4 & -5 & 6 \\ 7 & -8 & 6 \\ 3/2 & -1/2 & -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, others = $\begin{bmatrix} -2/\sqrt{9} \\ -2/\sqrt{9} \\ 1/\sqrt{9} \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix}$

2. In PageRank : want eigenvector of link matrix L with eigenvalue of 1 (can find using power iteration method, as it will be the largest eigenvalue)

PageRank can sometimes get into trouble if closed-loop structures appear. A simplified example might look like this,



With link matrix, $L = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Use the calculator in Q1 to check the eigenvalues and vectors for this system.

- Eigenvectors are complex (some are), but isn't an issue as long as eigenvector with value 1 is purely real
- Even though system is small, PageRank can still be calculated
- Proc. Pats will go around in a cycle rather than settling on a webpage
  ↳ if all sites started out equally, incoming Pats would equal outgoing, but system will not converge to this result by applying power iteration
- Other eigenvalues are not small compared to 1, so don't decay away with each power iteration

3. Loop can be remedied by damping

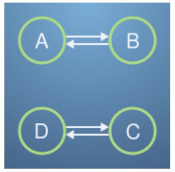
If we replace the link matrix with the damped, $L' = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.7 \\ 0.7 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.7 & 0.1 \end{bmatrix}$, how does this help?

↳ the other eigenvalues get smaller
(so their eigenvectors will decay away on power iteration)

↳ there is now a prob. to move to any website
(this helps power iteration settle down as it will spread out distribution of Pats)

4.

Another issue that may come up, is if there are disconnected parts to the internet. Take this example,



with link matrix, $L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

This form is known as block diagonal, as it can be split into square blocks along the main diagonal, i.e., $L = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, with $A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in this case.

- 2 eigenvalues of 1 (eigensystem is degenerate, any linear combo of eigenvectors with same eigenvalue is also an eigenvector)
- There isn't a unique PageRank (power iteration could settle to multiple values, depending on its starting conditions)
- Loops in the system

5. Apply damping to previous link matrix
   ↳ no longer have 2 eigenvalues of 1
   ↳ system will settle, but not into a loop structure
   ↳ settles the system to a single value
   ↳ only 1 eigenvalue of 1, PageRank will settle it's eigenvector through repeating the power iteration method

6. $A = \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - 2\lambda + \frac{1}{4}$

7. Eigenvalues: $\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-4\times1\times1/4}}{2} = \frac{2 \pm \sqrt{3}}{2}$, $\therefore \lambda_1 = 1 + \frac{\sqrt{3}}{2}$, $\lambda_2 = 1 - \frac{\sqrt{3}}{2}$

8. @ $\lambda_1 = 1 + \frac{\sqrt{3}}{2}$: $\begin{pmatrix} 3/2 - (1+\frac{\sqrt{3}}{2}) & -1 \\ -1/2 & 1/2 - (1+\frac{\sqrt{3}}{2}) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{3}}{2} & -1 \\ -1/2 & -\frac{1+\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\frac{1-\sqrt{3}}{2})x_1 - x_2 \\ -1/2\, x_1 - (\frac{1+\sqrt{3}}{2})x_2 \end{pmatrix} = 0$

$-1/2\, x_1 = \frac{1+\sqrt{3}}{2} x_2$

@ $\lambda_2 = 1 - \frac{\sqrt{3}}{2}$: $\begin{pmatrix} 3/2 - (1-\frac{\sqrt{3}}{2}) & -1 \\ -1/2 & 1/2 - (1-\frac{\sqrt{3}}{2}) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{3}}{2} & -1 \\ -1/2 & \frac{-1+\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\frac{1+\sqrt{3}}{2})x_1 - x_2 \\ -1/2\, x_1 - (\frac{1+\sqrt{3}}{2})x_2 \end{pmatrix} = 0$

$x_1 = -(1+\sqrt{3})x_2$

$x_2 = 1$,

$\therefore x_1 = -1-\sqrt{3}$, $\begin{pmatrix} -1-\sqrt{3} \\ 1 \end{pmatrix}$ (v_1)

$\frac{x_1}{2} = \frac{-(1+\sqrt{3})}{2} x_2$

$x_1 = -1+\sqrt{3}\, x_2$, $x_2 = 1$, $\begin{pmatrix} -1+\sqrt{3} \\ 1 \end{pmatrix}$ (v_2)

9. $C = \begin{array}{cc} v_1 & v_2 \end{array}$ $\begin{pmatrix} -1-\sqrt{3} & -1+\sqrt{3} \\ 1 & 1 \end{pmatrix}$

$D = C^{-1}AC = \frac{-1}{2\sqrt{3}}\begin{pmatrix} 1 & 1+\sqrt{3} \\ -1 & -1-\sqrt{3} \end{pmatrix}\begin{pmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} -1-\sqrt{3} & -1+\sqrt{3} \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} \frac{3-2\sqrt{3}}{12} & \frac{-3+\sqrt{3}}{12} \\ \frac{-3+2\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \end{pmatrix}\begin{pmatrix} -1-\sqrt{3} & -1+\sqrt{3} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+\frac{\sqrt{3}}{2} & 0 \\ 0 & 1-\frac{\sqrt{3}}{2} \end{pmatrix}$

diagonal entries are eigenvalues of matrix

10. $A^2 = CD^2C^{-1} = \begin{pmatrix} -1-\sqrt{3} & -1+\sqrt{3} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\frac{\sqrt{3}}{2} & 0 \\ 0 & 1-\frac{\sqrt{3}}{2} \end{pmatrix}^2 \begin{pmatrix} \frac{3-2\sqrt{3}}{12} & \frac{-3+\sqrt{3}}{12} \\ \frac{-3+2\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \end{pmatrix} = \begin{pmatrix} 11/4 & -2 \\ -1 & 3/4 \end{pmatrix}$