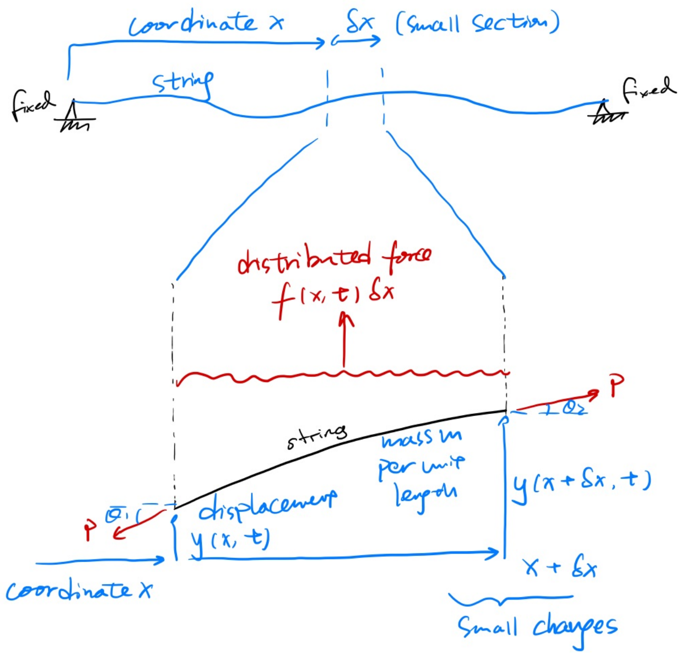
Vibration of Continuous Systems

Qianshuo (Harry) Ye

I. Vibration of 1D structures

1. Equations of motions of a stretched string



Resolve transversely:

$$f(x,t)\,\delta x + P\sin\theta_2 - P\sin\theta_1 = m\,\delta x\,\frac{\partial^2 y}{\partial t^2}$$

Small angle approximation:

$$\sin\theta_1 \simeq \theta_1 \approx \frac{dy}{dx}\Big|_x$$

$$\sin\theta_2 \simeq \theta_2 \approx \frac{dy}{dx}\Big|_{x+dx}$$

Substitute:

$$f(x,t)\,\delta x + P\left\{\frac{\partial y}{\partial x}\Big|_{x+dx} - \frac{\partial y}{\partial x}\Big|_x\right\} = m\,\delta x\,\frac{\partial^2 y}{\partial t^2}$$

Divide by δx and take limit:

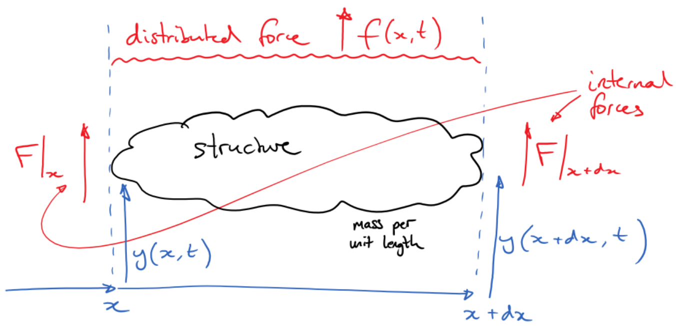
$$\boxed{m\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2} = f(x,t)}$$

Equation of motion of an ideal stretched string

Summary of obtaining equations of motion:

★ Equations of motions of other systems available in DB

1. Draw a big diagram of a small section of the structure



2. label the coordinates: the start and length of the small section
3. label the displacements from equilibrium: nominal values on the left, small changes on the right
4. identify any distributed external forces
5. work out expressions for the internal forces on both sides of the small section
6. apply $F = ma$ and take limits

2. Free vibration and travelling waves

Without external forcing:

$$m\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2} = 0$$

Wave equation:

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0 , \quad \boxed{c = \sqrt{P/m}}$$

c is the wave speed

* General solution from wave equation

$$y(x,t) = f(x-ct) + g(x+ct)$$

$f(x-ct)$: travel in $+x$ direction (forwards)

$g(x+ct)$: travel in $-x$ direction (backwards)

* Separation of variables

$$y(x,t) = U(x)\, e^{i\omega t}$$ → harmonic solution

$$\frac{\partial^2 y}{\partial t^2} = \ddot{y} = -\omega^2 U e^{i\omega t} \qquad \frac{\partial^2 y}{\partial x^2} = y'' = U'' e^{i\omega t} \Rightarrow PU'' + m\omega^2 U = 0$$

General solution:

$$U(x) = Ae^{-ikx} + Be^{ikx}$$

$$Pk^2 + m\omega^2 U = 0$$

$$\Rightarrow \boxed{\omega^2 = \frac{P}{m}k^2 = c^2k^2}$$

k is the wave-number 'spatial frequency'

* $T = 2\pi/\omega$, $\lambda = 2\pi/k$

* Overall solution

$$\begin{aligned} y(x,t) &= U(x)\, e^{i\omega t} \\ &= (Ae^{-ikx} + Be^{ikx})\, e^{i\omega t} \\ &= Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)} \\ &= Ae^{-ik(x - \frac{\omega}{k}t)} + Be^{ik(x + \frac{\omega}{k}t)} \\ &= f(x-ct) + g(x+ct) \end{aligned}$$

3. Modes of vibration

<u>Vibration mode</u>: a free motion in which all points in the structure <u>move</u> <u>sinusoidally</u> at a <u>particular frequency</u> with a <u>characteristic mode shape</u>.

natural frequency / resonant frequency

$$y(x,t) = U(x)\,\mathrm{Real}\{e^{i(\omega t+\phi)}\}$$
$$= U(x)\cos(\omega t+\phi)$$
$$U(x) = A'\cos(kx) + B'\sin(kx)$$

Set boundary conditions = both end fixed

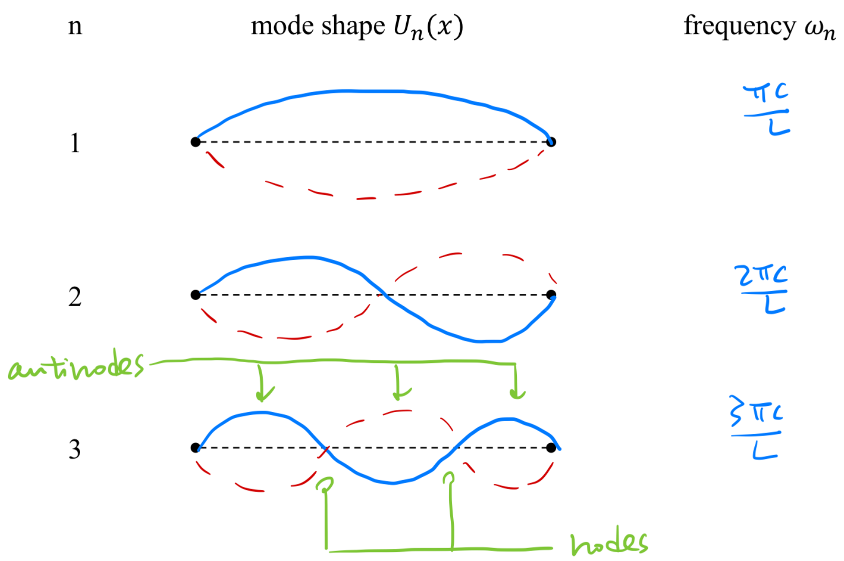
$y=0$ at $x=0$ and $y=0$ at $x=L$

$\Rightarrow$ $A'=0$ and $k=\frac{n\pi}{L}$

From $\omega^2 = c^2k^2$:

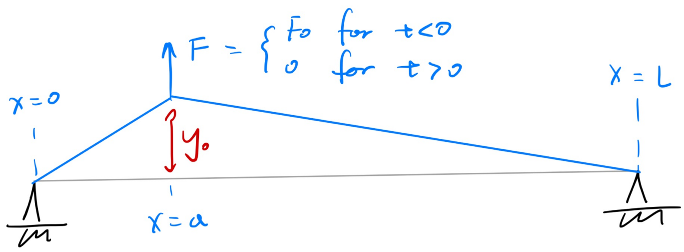
overall amplitude of vibration

$$\boxed{\omega_n = k_n\sqrt{\frac{P}{m}} = \frac{n\pi}{L}\sqrt{\frac{P}{m}}, \quad U_n = B'\sin\frac{n\pi x}{L}}$$



II. Transient response and transfer functions

1. Transient response of a plucked string



After release there is no external force → Satisfy equation of free motion

⇒ the motion should consist of a combination of vibration modes

$$\boxed{y(x,t) = \sum_n b_n U_n(x) \cos(\omega_n t + \phi) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t + \phi)}$$

At t=0 string is ① at rest and ② in the shape of a triangle

⇒ From ①: $\phi = 0$

From ②: $y(x,0) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} \frac{Y_0 x}{a} & 0 \le x \le a \\ \frac{Y_0 (L-x)}{L-a} & a \le x \le L \end{cases}$

↓

Half range Fourier series

Finding coefficients b_n (finding b'_n is easier):

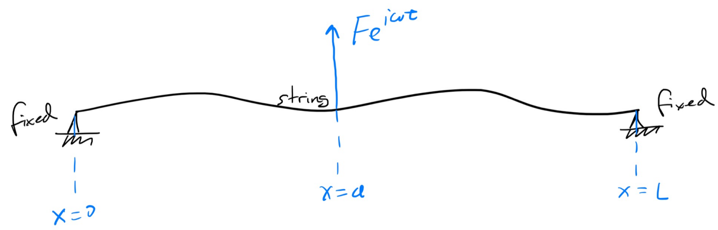
$$b_n = \frac{2 Y_0 L^2}{n^2 \pi^2 a (L-a)} \sin \frac{n\pi a}{L}$$

The motion is therefore:

recall that $\omega = ck$, $k = \frac{n\pi}{L}$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2 Y_0 L^2}{n^2 \pi^2 a (L-a)} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right)$$

2. Forced harmonic response and transfer function



→ Satisfy the free equation of motion everywhere except at the point where the force is applied

$F = P\frac{\partial y}{\partial x}\Big|_{a-} - P\frac{\partial y}{\partial x}\Big|_{a+}$

Applying continuity and force balance at $x=a$:

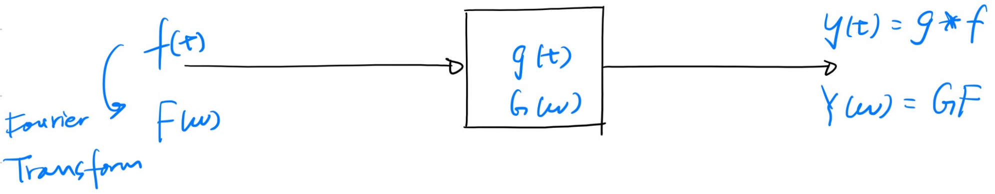
$$U(x) = \begin{cases} \dfrac{Fc}{\omega P} \dfrac{\sin(\omega(L-a)/c)}{\sin(\omega L/c)} \sin(\omega x/c) & \text{for } 0 < x \le a \\ \dfrac{Fc}{\omega P} \dfrac{\sin(\omega a/c)}{\sin(\omega L/c)} \sin(\omega(L-x)/c) & \text{for } a \le x < L \end{cases}$$

$$\boxed{G(a, x, \omega) = \frac{U(x)}{F(a)}}$$ Transfer function

✡ Remarks:

- Displacement U goes to infinity where $\sin(\omega L/c) = 0$ → resonance
  Appears as peaks in a plot of G vs ω
- No resonance peak where $\sin(\omega_n a/c) = 0$ or $\sin(\omega_n x/c) = 0$
  → the position of the input force or the measurement location are at a nodal point of the corresponding mode.
- The expression is symmetrical if we swap a and x
  → interchange the excitation point and the observation point gives the same result
- Resonant frequencies ω_n of the structure gives the system poles $(s = i\omega)$
  → marginally stable: when perturbed it will continue to vibrate indefinitely as there is no damping
  (in practice there is normally a bit of damping)
- If we choose input to be displacement and the output to be force, the transfer function is upside down:
  → zeros become poles
  → peaks become anti-resonances and vice-versa

3. Transient response for arbitrary loads



Convolve input force $f(t)$ with the transfer function $g(t)$ in time domain
OR multiply $G(\omega)$ and $F(\omega)$ in frequency domain.

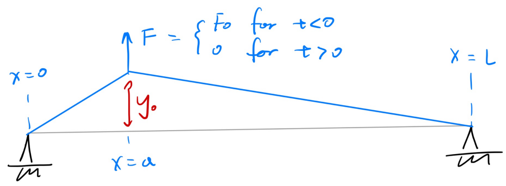
III. Wave propagation perspective

1. Transient response from D'Alembert's solution

General solution to wave equation:

$$y(x,t) = f(x-ct) + g(x+ct)$$

Consider the same example as in section II



✱ At $t=0$:

$$f(x) + g(x) = y_0(x)$$

$$\frac{\partial y}{\partial x} = 0 \rightarrow -cf'(x) + cg'(x) = 0 \rightarrow f(x) = g(x) + k$$

$$\Rightarrow \quad f(x) = \frac{y_0+k}{2}, \quad g(x) = \frac{y_0-k}{2}$$

, k can be taken to be 0 as only the sum matters

✱ At $x=0$ and $x=L$

$$f(-ct) + g(ct) = 0 \rightarrow f(\xi) = -g(-\xi)$$

$$f(L-ct) + g(L+ct) = 0$$

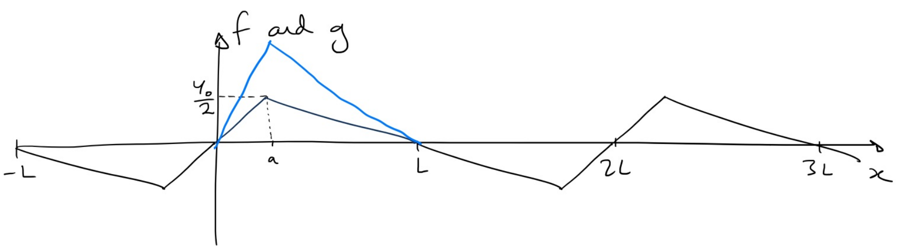
$$\Rightarrow \quad g(L+ct) = -f(L-ct) = g(ct-L)$$

$\Rightarrow$ f and g must be periodic over distance $2L$

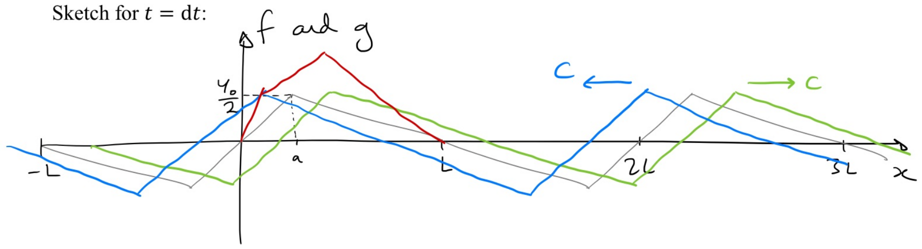
☆ Summary:

- At $t=0$: $f(x) = g(x) = y_0(x)/2$
- At $t=0$: $f(x) = -f(-x)$ → initially an odd function
- For all t: f and g repeat over $2L$
- f travels forward and g travels backward at speed c

Sketch for $t = 0$:



Sketch for $t = \mathrm{d}t$:



2. Dispersion equation

Consider two travelling sinusoidal waves

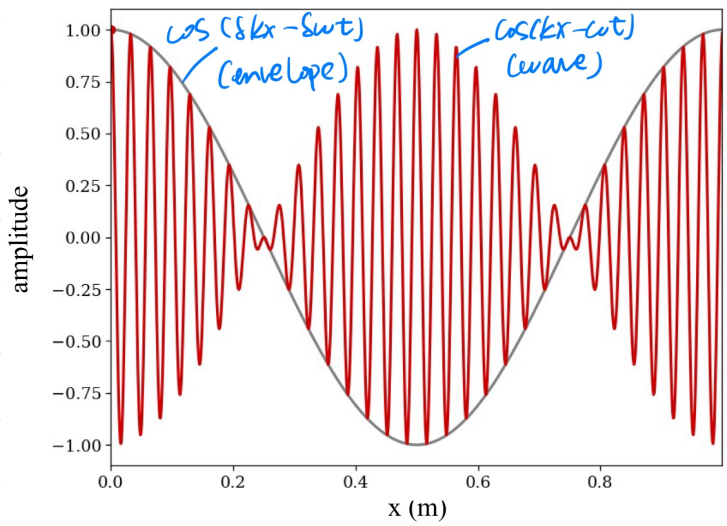
$$y(x,t) = A_0\cos(k_1x - \omega_1 t) + A_0\cos(k_2 x - \omega_2 t)$$

where $\omega_1, \omega_2 = \omega \mp \delta\omega$, $k_1, k_2 = k \mp \delta k$

$$y(x,t) = A_0\cos\left([k-\delta k]x - [\omega - \delta\omega]t\right) + A_0\cos\left([k+\delta k]x - [\omega+\delta\omega]t\right)$$

$$= 2A_0 \underbrace{\cos(kx-\omega t)}_{\text{wave}} \underbrace{\cos(\delta kx - \delta\omega t)}_{\text{envelope}}$$

$C_p = \omega/k$ $\qquad$ $C_g = \delta\omega/\delta k$



In the limit $\delta k \to dk$ and $\delta\omega \to d\omega$:

$$\boxed{\begin{aligned}&\text{phase velocity: } C_p = \omega/k \\ &\text{group velocity: } C_g = d\omega/dk\end{aligned}}$$

* For stretched string:

$$\omega = k\sqrt{P/m}$$

$$C_p = \frac{\omega}{k} = \sqrt{\frac{P}{m}} \qquad C_g = \frac{d\omega}{dk} = \sqrt{\frac{P}{m}}$$

* For bending beam:

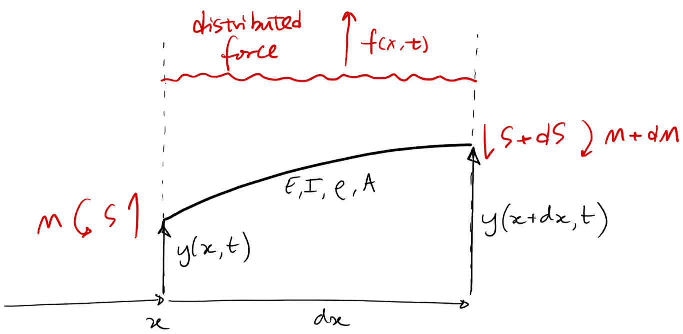
$$\omega = k^2\sqrt{\frac{EI}{\rho A}}$$

$$C_p = \frac{\omega}{k} = k\sqrt{\frac{EI}{\rho A}} = \sqrt{\omega}\sqrt[4]{\frac{EI}{\rho A}} \qquad C_g = \frac{d\omega}{dk} = 2k\sqrt{\frac{EI}{\rho A}} = 2C_p$$

high frequencies travel faster than slow

energy travels 2x as fast as peak of a given wave

IV. Vibration of bending beams



$$S = \frac{dM}{dx}, \quad M = EI\,dk, \quad dk = \frac{\partial^2 y}{\partial x^2}$$

$$dS = \frac{dS}{dx}dx = EI\frac{\partial^4 y}{\partial x^4}dx$$

$$-dS + f(x,t)dx = \rho A x \frac{\partial^2 y}{\partial t^2}$$

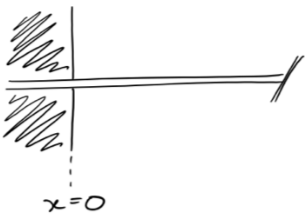
$$\boxed{\rho A \frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = f(x,t)}$$

Equation of motion for the Euler-Bernoulli bending beam

need 4 boundary conditions, 2 at each end

* Particular boundary conditions

(i) Clamped boundary
(or built-in)

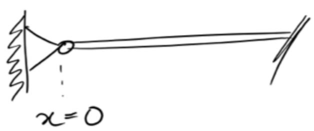


at $x = 0$: $y = 0$ (no displacement)

$\frac{\partial y}{\partial x} = 0$ (no rotation)

5.1

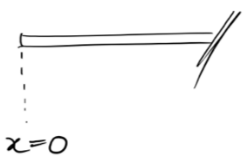
(ii) pinned (or hinged) boundary



at $x = 0$: $y = 0$ (no displacement)

$EI\frac{\partial^2 y}{\partial x^2} = 0$ (no bending moment)

(iii) free boundary



at $x = 0$: $EI\frac{\partial^2 y}{\partial x^2} = 0$ (no bending moment)

$EI\frac{\partial^3 y}{\partial x^3} = 0$ (no shear force)

* General solution:

$$y(x,t) = U(x)\, e^{i\omega t}$$

$$EI \frac{d^4 U}{dx^4} - \rho A \omega^2 U = 0$$

harmonic in space — exponential in space — 'evanescent wave'

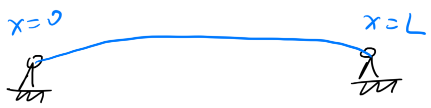
$$U(x) = C_1 e^{ikx} + C_2 e^{-ikx} + C_3 e^{kx} + C_4 e^{-kx}$$

with $k^4 = \omega^2 \frac{\rho A}{EI}$

Real form:

$$U(x) = D_1 \cos kx + D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx$$

* Modes of a pinned beam



$U(0)=0,\ U''(0)=0 \Rightarrow D_1 + D_3 = 0$

$U(L)=0,\ U''(L)=0 \Rightarrow D_2 \sin kL = D_4 \sinh kL = 0$

$D_2 \sin kL = 0$

$D_2 = 0$ OR $\boxed{\sin kL = 0 \quad kL = n\pi \text{ for } n = 1, 2 \ldots}$

(no response) — vibration modes

$D_4 \sinh kL = 0$

$D_4 = 0$ — $k = 0$ (beam at rest)

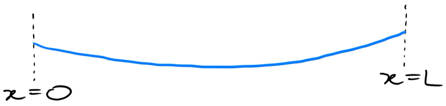
Natural frequencies:

$$\omega_n = k_n^2 \sqrt{\frac{EI}{\rho A}} = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

Mode shapes:

$$U_n(x) = \sin(k_n x) = \sin\left(\frac{n\pi x}{L}\right)$$ → identical to a stretched string

* Modes of a free-free beam



$U''(0) = 0 \;, \quad U'''(0) = 0$

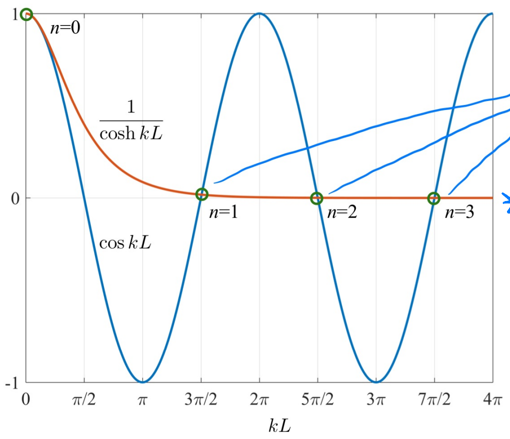
$U''(L) = 0 \;, \quad U'''(L) = 0$

$$\Rightarrow \begin{bmatrix} -\cos kL + \cosh kL & -\sin kL + \sinh kL \\ \sin kL + \sinh kL & -\cos kL + \cosh kL \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Non-trivial solution occurs when the determinant is zero

$$\Rightarrow (-\cos kL + \cosh kL)^2 - (\sinh^2 kL - \sin^2 kL) = 0$$

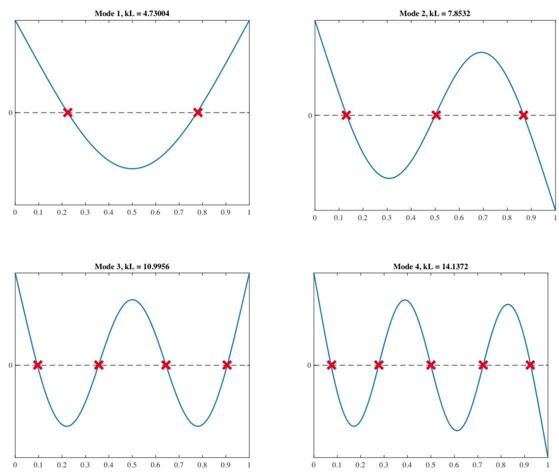
$$\cos kL \cosh kL = 1 \quad \rightarrow \quad \cos kL = \frac{1}{\cosh kL}$$



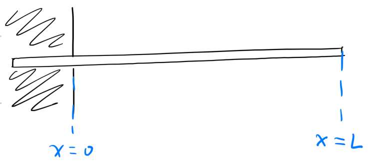
$kL \approx (n + \frac{1}{2})\pi$

$n = 1, 2, 3 \ldots$

* the n=0 solution corresponds to rigid body motion of the free-free beam: the wavenumber and natural frequency are both zero, and the structure can translate or rotate freely



* Modes of a clamped-free beam

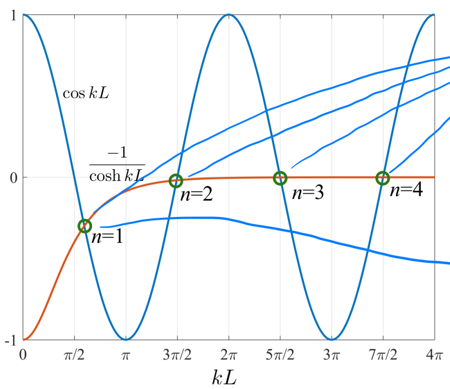


$U(0) = 0 \;, \quad U'(0) = 0$

$U''(L) = 0 \;, \quad U'''(L) = 0$

Put into matrix form and set the determinant to zero

$$\Rightarrow \cos kL \cosh kL = -1$$

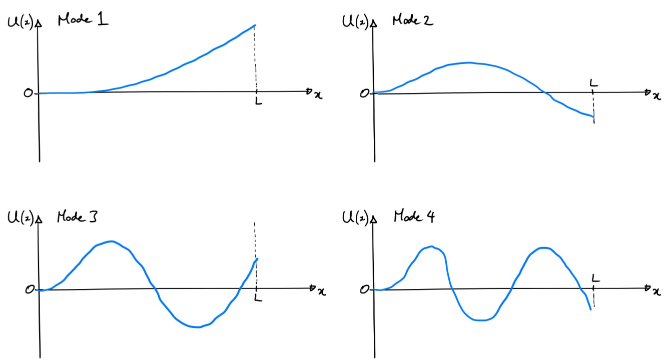


$kL \approx (n - \frac{1}{2})\pi$

$n = 1, 2, 3$

there is no n=0 solution this time, as there are no rigid body modes

$kL = 1.8751$



V. Continuous vs discrete systems

* Analogous concepts

Discrete	Continuous
Vector of displacements, **u** (or generalised coordinates)	Continuous function of displacements, $u(x)$ (or generalised coordinates)
Matrix equations ('coupled oscillators') $\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{F}$	Partial differential equations (PDE's) $\frac{\partial^2 y}{\partial t^2} + \mathcal{D}_x\{y\} = f(x,t)$
Finite number of modes	Infinite number of modes
Modes found by finding eigenvalues and eigenvectors, i.e. solve $\det\{\mathbf{K} - \omega^2\mathbf{M}\} = 0$ and $\mathbf{K}\mathbf{y} = \omega_n^2\mathbf{M}\mathbf{y}$	Modes found by solving unforced PDE together with boundary conditions: assume separation of variables

* Common ways of discretizing continuous systems

- 'lumped mass' model
- 'modal reduction'
- 'Galerkin' approach
- 'finite element' method

1. Transfer function

For discrete system:

$$G(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

For continuous system:

input position

measure position

$$\boxed{G(x,y,\omega) = \frac{U(y)}{F(x)} = \sum_{n=1}^{\infty} \frac{u_n(x)\,u_n(y)}{\omega_n^2 - \omega^2}}$$

modes, n

* Impulse response in time domain can be found by inverse Fourier / Laplace transform

Corresponding impulse response:

$$g(x,y,\omega) = \sum_n \frac{u_n(x)\,u_n(y)}{\omega_n} \sin(\omega_n t)$$

☆ The result relies on the orthogonality of modes and the mode shape must be suitably normalised

Orthogonality:

For discrete system:

$$\underline{u}_a^T M \underline{u}_b = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$$

For continuous system:

$$\int u_a(x) u_b(x) \underbrace{dm}_{\text{mass}} \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$$

2. Damping

Transfer function including damping:

$$\boxed{G(x,y,\omega) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i \zeta_n \omega_n \omega - \omega^2}}$$

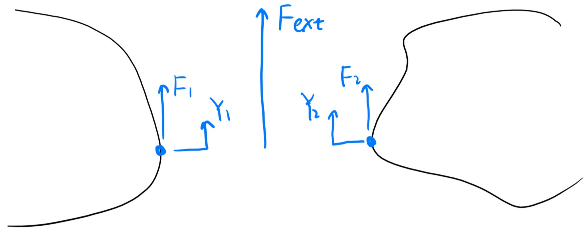
ζ_n: modal damping factor (dimensionless)

Corresponding impulse response:

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} e^{-\zeta \omega_n t} \sin \omega_n t$$

VI. Coupling systems

1. Using transfer functions



Uncoupled transfer functions:

$$G_1 = \frac{Y_1(\omega)}{F_1(\omega)} \qquad G_2 = \frac{Y_2(\omega)}{F_2(\omega)}$$

Continuity: $Y_1 = Y_2 = Y$

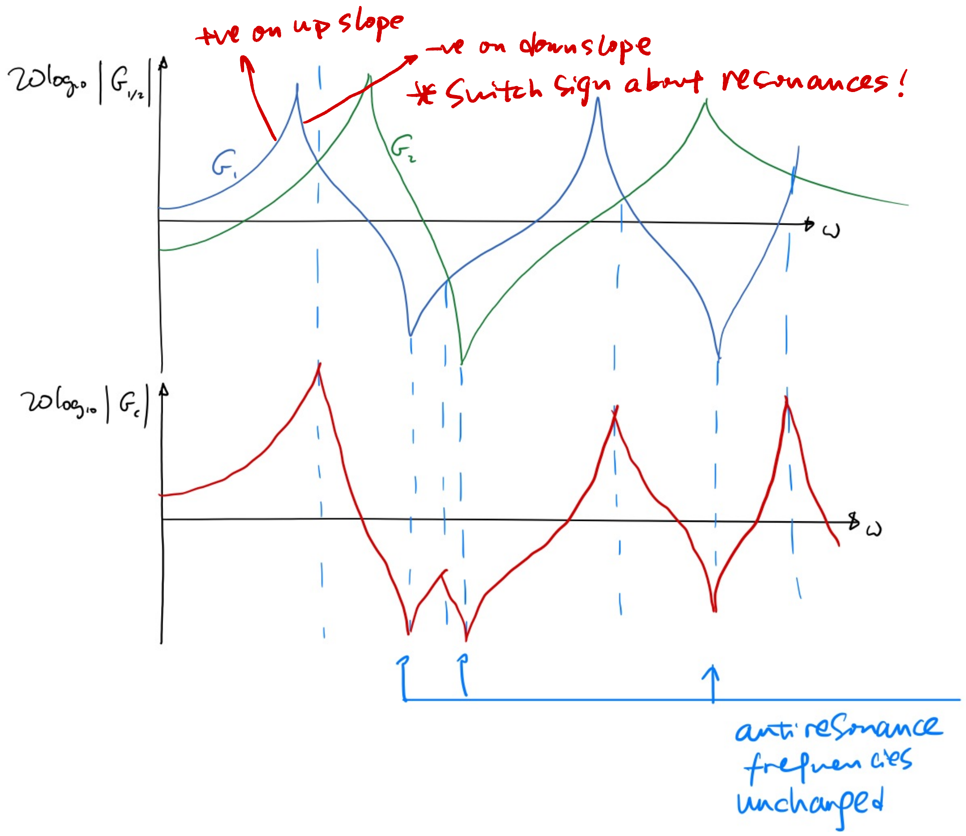
Force balance: $F_1 + F_2 = F_{ext}$

$$\Rightarrow Y\left(\frac{1}{G_1} + \frac{1}{G_2}\right) = F_{ext}$$

$$\boxed{G_c = \frac{Y}{F_{ext}} = \frac{G_1 G_2}{G_1 + G_2}}$$

* Remarks:

- Peak occurs when $G_1 = -G_2$ → uncoupled transfer functions have equal magnitude and opposite phase
- Anti-resonance frequencies occur when either G_1 or G_2 are zero
  → at the same frequencies as the anti-resonances of each uncoupled system



* If $|G_1| \ll |G_2|$

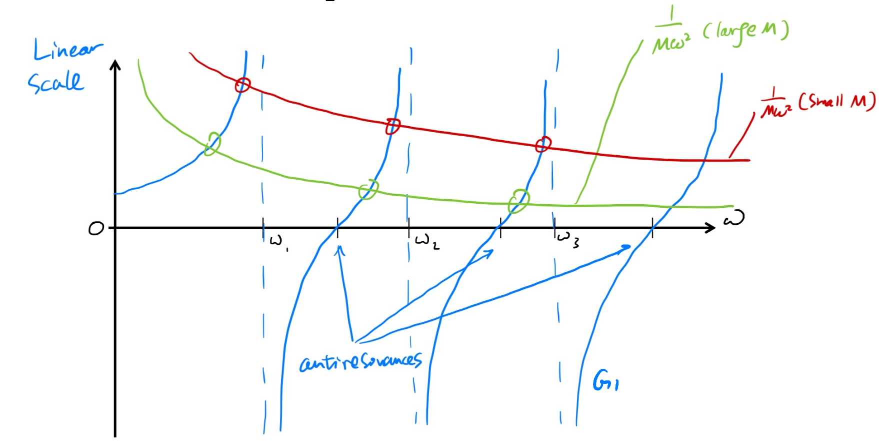
$$G_C = \frac{G_1 G_2}{G_2} = G_1$$

* If $G_1 = G_2$

$$G_C = \frac{G_1^2}{2G_1} = \frac{G_1}{2} \rightarrow 2\times \text{Stiffness}$$

* if the structure is symmetric, the modes are symmetric or antisymmetric

Example: Coupling with a point mass



At natural frequencies:

$$\frac{1}{G_1} - M\omega^2 = 0 \rightarrow G_1 = \frac{1}{M\omega^2}$$

From plot, smaller mass gives smaller shift in natural frequencies

2. Using free vibration solutions

* Method:

1. For each subsystem, write down or find the free vibration solution
2. Apply continuity and balance of forces at the junction
3. Solve to find the response of the coupled system

VII. Rayleigh's principle

* Rayleigh's Quotient:

$$\boxed{R = \frac{V}{\tilde{T}}}$$

$V \rightarrow$ potential energy — listed in DB

$\tilde{T} \rightarrow$ kinetic energy with the time derivatives ignored $= \frac{1}{2} m \int y^2 dx$

* Derivation of Rayleigh's quotient for continuous systems

$$T_{max} = \frac{1}{2} m \int \left(\frac{\partial u}{\partial t}\right)^2 dx$$

$$= \frac{1}{2} m \int (\omega_n u)^2 dx$$

$$= \omega_n^2 \cdot \frac{1}{2} m \int u^2 dx$$

$$= \omega_n^2 \tilde{T}_{max}$$

Due to energy conservation, $V + T = \text{constant}$

$$\Rightarrow V_{max} = T_{max}$$

$$= \omega_n^2 \tilde{T}_{max}$$

$$\omega_n^2 = \frac{V_{max}}{\tilde{T}_{max}}$$