Vibration of Discrete Systems

Qianshuo (Harry) Ye

I. Normal modes of undamped systems

1. Key results from Part IA

① A N-degree-of-freedom system has N natural modes, each with its own natural frequency

② In each mode, all points of the system will oscillate in the same phase or 180° out of phase

③ Each mode on its own behaves like a single DoF system

④ Total system response to excitation is the sum (superposition) of the modal responses

⑤ A mode with zero frequency corresponds to rigid body displacement or rotation of system

⑥ If the system is symmetric, each mode is either symmetric or anti-symmetric

⑦ Modes at lower frequencies involve motion with larger length-scales

2. Mass and stiffness matricies by energy

* Lagrange's equation

Lagrangian $L = T - V$

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot q_j}}_{\text{momentum}}\Big) - \Big(\underbrace{\frac{\partial L}{\partial q_j}}_{\text{force}}\Big) = 0$$

Expand:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_j} - \cancelto{0}{\frac{\partial V}{\partial \dot q_j}}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = \underbrace{Q_j}_{\text{external force}}$$

* Lagrange for small motions

Consider small vibrations about a stable equilibrium position

Taylor expansion of $V(q_j)$:

$$V(q_1, q_2, \dots q_n) = V_0 + \frac{\partial V}{\partial q_1}\cdot q_1 + \frac{\partial V}{\partial q_2}\cdot q_2 + \cdots$$
$$+ \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_1^2} q_1 + \frac{\partial^2 V}{\partial q_2^2} q_2^{\,2} + \cdots + \frac{\partial^2 V}{\partial q_1 \partial q_2} q_1 q_2 + \cdots\right) + \cdots$$

Notes:

① The constant term V_0 can be ignored

② The equilibrium pose is a stationary point in U

$$\Rightarrow \left.\frac{\partial V}{\partial q_i}\right|_{equ.} = 0$$

③ Ignores terms of order higher than 2 because of small motions assumption

$$V = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \underbrace{\frac{\partial^2 V}{\partial q_i \cdot \partial q_k}}_{\text{"Stiffness coefficient" } k_{jk}} \cdot q_j \, q_k$$

* Quadratic form

$$\boxed{V = \frac{1}{2} \underline{q}^T [K] \underline{q}}$$

Symmetric stiffness matrix

$$\boxed{T = \frac{1}{2} \underline{\dot{q}}^T [M] \underline{\dot{q}}}$$

Symmetric mass matrix

Note: m_{jk} may depend on $\underline{q}$ but can be approximating by evaluating at the equilibrium position

3. The eigenvalue problem

$$[M] \underline{\ddot{q}} + [K] \underline{q} = \underline{Q}$$

Assuming solution of the form $\underline{q} = \underline{u} e^{i\omega t}$ and set $\underline{Q} = 0$

$$\Rightarrow \boxed{([K] - \omega^2 [M]) \underline{u} = 0}$$

* Procedure

① Solve $|[K] - \omega^2 [M]| = 0$ for ω_n

② For each ω_n, Solve the simultaneous equation

$$[K] \underline{u}^{(n)} = \omega_n^2 [M] \underline{u}^{(n)}$$

$[K]$, $[M]$ symmetric $\rightarrow$ eigenvalues are real and eigenvectors are orthogonal

* Orthogonality Condition

$$\underline{u}^{(n)T}[M]\underline{u}^{(m)} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

$$\underline{u}^{(n)T}[K]\underline{u}^{(m)} = \begin{cases} 0 & \text{if } n \neq m \\ \omega_n^2 & \text{if } n = m \end{cases}$$

4. Normal Coordinates

$$\underline{q} = [U]\underline{\eta} \longrightarrow \text{normal coordinates } (\underline{y} = [U]\underline{q} \text{ in DB})$$

$[U]$: $N \times N$ matrix of normalised eigenvectors

$$[M][U]\ddot{\underline{\eta}} + [K][U]\underline{\eta} = \underline{Q}$$

$$\underbrace{[U]^T[M][U]}\ddot{\underline{\eta}} + \underbrace{[U]^T[K][U]}\underline{\eta} = [U]^T\underline{Q}$$

from orthogonality

$$\boxed{[I]\ddot{\underline{\eta}} + [\mathrm{diag}(\omega_n^2)]\underline{\eta} = [U]^T\underline{Q}}$$ modal response formula

Each row is a simple 2nd DE of the form

$$\ddot{\eta}_j + \omega_j^2\eta_j = f_j, \quad j = 1, 2, 3, \dots N$$

5. Response to initial excitation

For free vibration, the solution to the 2nd DE above is:

$$\eta_j(t) = C_j\cos(\omega_j t - \phi_j) \quad j = 1, 2, \dots N$$

Transform $\underline{q}$ into normal coordinates

$$\underline{q}(t) = [U]\underline{\eta}(t) = \sum_{j=1}^{N}\eta_j(t)\,\underline{u}^{(j)}$$

$$= \sum_{j=1}^{N} C_j\,\underline{u}^{(j)}\cos(\omega_j t - \phi_j)$$

II. Vibration transfer functions

1. The response formula

Assume that the input is sinusoidal : $\underline{Q} = \bar{\underline{Q}} e^{i\omega t}$

→ response in form of $\underline{q} = \bar{\underline{q}} e^{i\omega t}$

$$-\omega^2 [M] \bar{\underline{q}} e^{i\omega t} + [K] \bar{\underline{q}} e^{i\omega t} = \bar{\underline{Q}} e^{i\omega t}$$

$$-\omega^2 [I] \bar{\underline{\eta}} e^{i\omega t} + [\mathrm{diag}(\omega_n^2)] \bar{\underline{\eta}} e^{i\omega t} = [U]^T \bar{\underline{Q}} e^{i\omega t}$$

$$\Rightarrow \bar{\underline{\eta}} = \left[\mathrm{diag}\left(\frac{1}{\omega_n^2 - \omega^2}\right)\right] [U]^T \bar{\underline{Q}}$$

Transfer back to the original coordinates:

$$\boxed{\bar{\underline{q}} = [U] \left[\mathrm{diag}\left(\frac{1}{\omega_n^2 - \omega^2}\right)\right] [U]^T \bar{\underline{Q}}}$$

★ For an sinusoidal forcing applied to the j-th coordinate, the response at coordinate k is given by the transfer function:

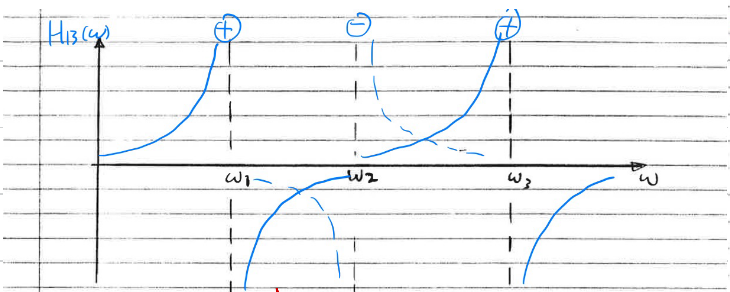
$$\boxed{H_{jk}(\omega) = \frac{q_k}{F_j} = \sum_{n=1}^{N} \frac{u_k^{(n)} u_j^{(n)}}{\omega_n^2 - \omega^2}}$$

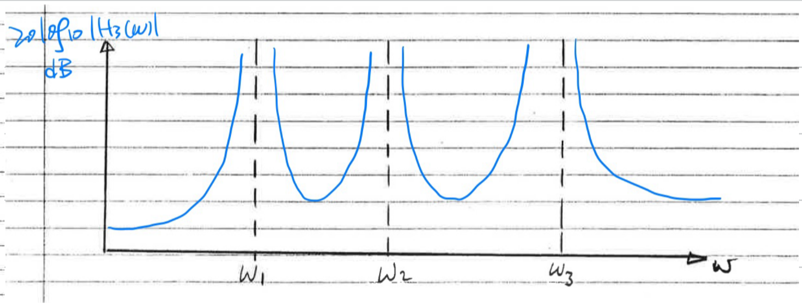
excitation point j

measurement point k

* Notes:

- Antiresonance when $H(\omega) = 0$
- When 'modal constant' ($u_j^{(n)} u_k^{(n)}$) change sign, no antiresonance between ω_n and ω_{n+1}





Same sign

hence do not cancel

2. Modal damping

$$\boxed{H_{jk}(\omega) = \frac{q_k}{F_j} = \sum_{n=1}^{N} \frac{u_k^{(n)} u_j^{(n)}}{\omega_n^2 + i2\zeta_n \omega\omega_n - \omega^2}}$$

ζ_n is the dimensionless damping ratio

The modal damping assumption is acceptable provided the damping does not alter the eigenvectors significantly ($\zeta \leq 0.1$)

3. Impulse response

"Each separate 'modal oscillator'" responds according to the single DoF formula"

Undamped system:

$$h_{jk} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

Lightly damped system:

$$h_{jk} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\zeta_n \omega_n t} \sin \omega_n t$$

for $\zeta \ll 1$

4. Reciprocal theorem

The response (harmonic or transient) at one point to driving at another point is identical to that obtained when the drive and response points are interchanged

$$H_{jk}(\omega) = H_{kj}(\omega)$$

5. Frequency response curves

Consider ω close to one mode frequency ω_n

"Receptance" — transfer function for position

$$H_q(\omega) = \frac{\omega_n^2}{a_n}\frac{q_k}{F_j} = \frac{1}{1 - \omega^2/\omega_n^2 + 2i\zeta_n\, \omega/\omega_n}$$

$$a_n = u_k^{(n)} u_j^{(n)}$$

→ case (a) in the Meachanics DB

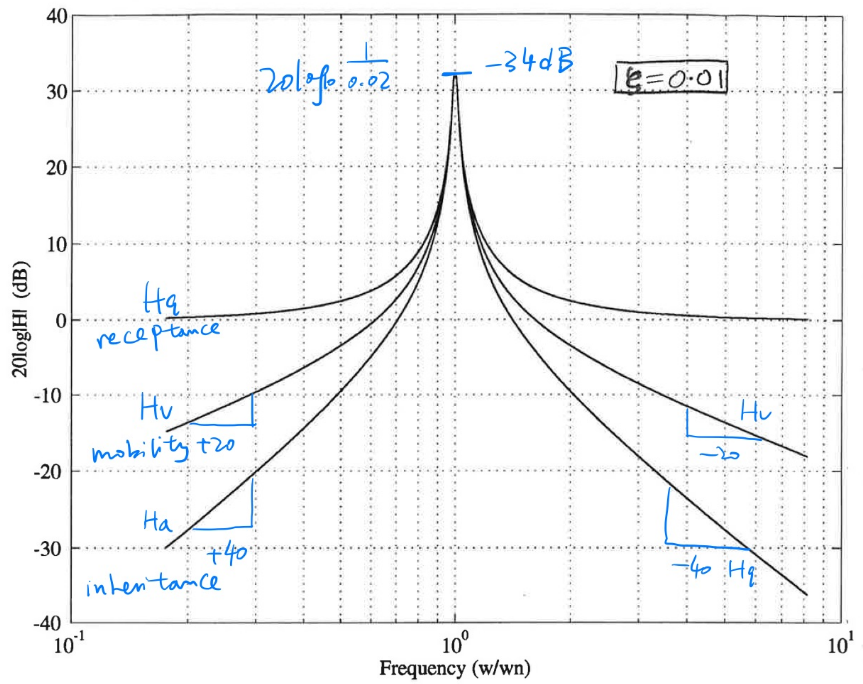
"Mobility" — transfer function for velocity

$$H_v(\omega) = \left(\frac{i\omega}{\omega_n}\right) H_q(\omega) = \frac{i\omega/\omega_n}{1 - \omega^2/\omega_n^2 + 2i\zeta_n\, \omega/\omega_n}$$

"Inheritance" — transfer function for acceleration

$$H_a(\omega) = \left(\frac{i\omega}{\omega_n}\right)^2 H_q(\omega) = \frac{-\omega^2/\omega_n}{1 - \omega^2/\omega_n^2 + 2i\zeta_n\, \omega/\omega_n}$$

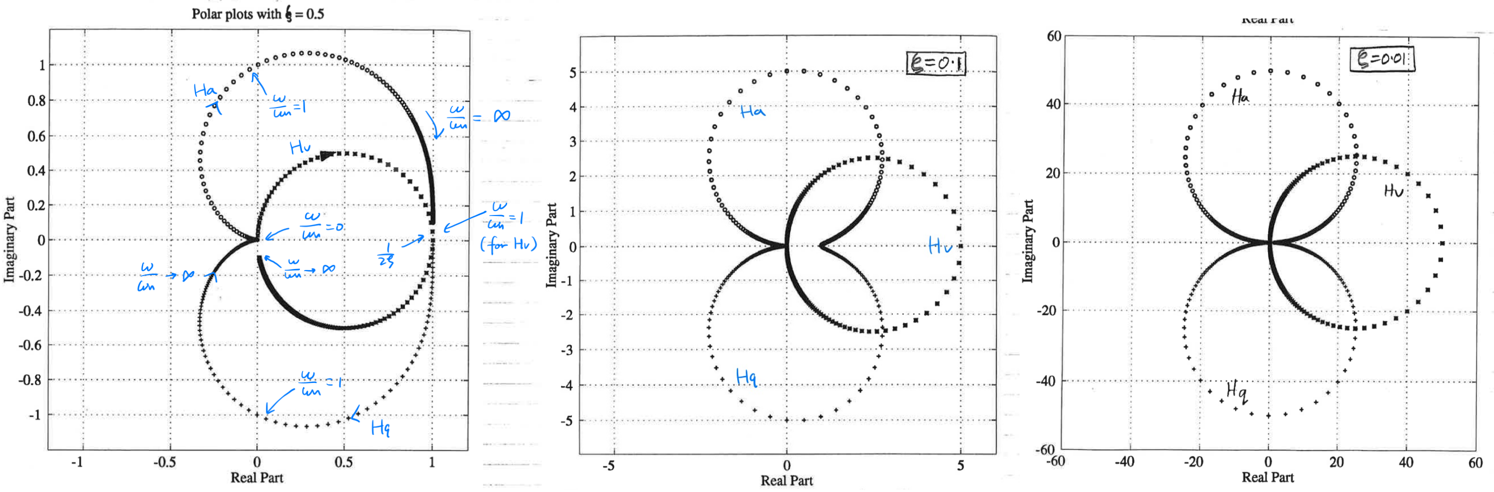
		$\omega/\omega_n \to 0$	$\omega/\omega_n = 1$	$\omega/\omega_n \to \infty$
RECEPTANCE	$20\log_{10}\lvert H_q\rvert$	0	$20\log_{10}\frac{1}{2\zeta_n}$	$-40\log_{10}\left(\frac{\omega}{\omega_n}\right)$
	ϕ_q	0	$-\pi/2$	$-\pi$
MOBILITY	$20\log_{10}\lvert H_v\rvert$	$20\log_{10}\left(\frac{\omega}{\omega_n}\right)$	$20\log 1/2\xi$	$-20\log_{10}\left(\frac{\omega}{\omega_n}\right)$
	ϕ_v	$+\pi/2$	0	$-\pi/2$
INERTANCE	$20\log_{10}\lvert H_a\rvert$	$40\log_{10}\left(\frac{\omega}{\omega_n}\right)$	$20\log 1/2\xi$	0 $(\lvert H_q\rvert = 1)$
	ϕ_a	$+\pi$	$+\pi/2$	0



* Half-power band-width

$$\boxed{\Delta\omega = 2\zeta_n\omega_n \quad \text{for } \zeta \ll 1}$$

Polar plots:



- the mobility plot is a circle for any ξ
- the receptance and inheritance are circles for $\xi_n \ll 1$
- all rise to a value of $1/\xi$ at $\omega/\omega_n = 1$
- can fit modal circles to measured frequency response curves to find ω_n, ξ_n and $a_n = u_j^{(n)} u_k^{(n)}$ for light damping

6. Modal overlap

For a system with many resonances, the <u>modal overlap factor</u> is defined as

$$\boxed{\beta = \frac{\Delta\omega}{\omega_{n+1} - \omega_n}}$$

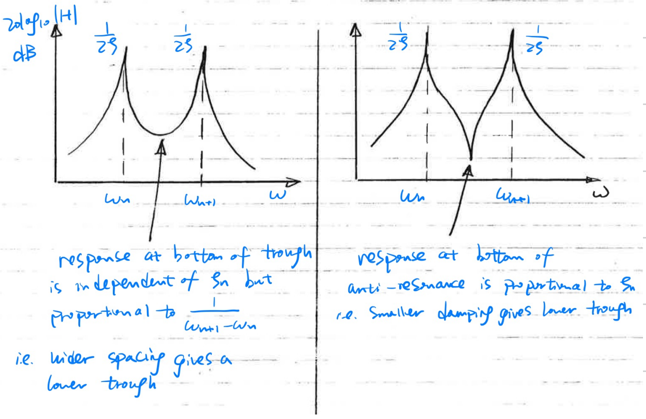
$\Delta\omega$ is the half-power bandwidth

For $\beta \ll 1$

- modal peaks are well separated
- responses are well approximated

For $\beta \gg 1$

- modal peaks overlap in frequency
- several resonances contribute to response



7. Interpreting response curves

* Analysis checklist

(1) Are the resonance peaks well separated
i.e. is the modal overlap low?

(2) How does the model density balance?

(3) Are the peaks spaced regularly or irregularly?

(4) Is there an "Structure" in the pattern of peak spacings?
(e.g. do peaks occur in pairs or clusters

(5) What is the pattern of anti-resonances?

III. Rayleigh's principle

1. Rayleigh's quotient

For vibration of a system with zero damping the maximum potential energy in the vibration cycle must equal the maximum kinetic energy.

PE : $V = \frac{1}{2} \underline{q}^T [K] \underline{q}$

KE : $T = \frac{1}{2} \dot{\underline{q}}^T [M] \dot{\underline{q}}$

$\underline{q}(t) = \underline{q} e^{i\omega t}, \quad \dot{\underline{q}}(t) = i\omega \underline{q} e^{i\omega t}$

$T = \omega^2 \frac{1}{2} \underline{q}^T [M] \underline{q}$

$= \omega^2 \tilde{T}$

Rayleigh's quotient

$$\boxed{R = \omega^2 = \frac{V}{\tilde{T}} = \frac{\underline{q}^T [K] \underline{q}}{\underline{q}^T [M] \underline{q}}}$$

Since $\omega_n^2 [M] \underline{u}^{(n)} = [K] \underline{u}^{(n)}$

$$\omega_n^2 = \frac{\underline{u}^{(n)T} [M] \underline{u}^{(n)}}{\underline{u}^{(n)T} [K] \underline{u}^{(n)}}$$

$\Rightarrow$
- if $\underline{q}$ is an eigenvector (i.e. $\underline{u}^{(n)}$), the Rayleigh's quotient gives the <u>exact</u> natural frequency corresponding to that eigenvector
- if $\underline{q}$ is a reasonable estimate of the eigenvector, then R is a very good estimate of the corresponding eigenvalue
- $R(\underline{q})$ is stationary at each node

☆ If the quantity $R = \frac{V}{\tilde{T}} = \frac{\underline{q}^T [M] \underline{q}}{\underline{q}^T [K] \underline{q}}$ is evaluated with any vector $\underline{q}$

$$\boxed{\begin{aligned} &R(\underline{q}) \geq \omega_1^2 \\ &R(\underline{q}) \leq \omega_N^2 \\ &R(\underline{q}) \approx \omega_n^2 \text{ if } \underline{q} \approx \underline{u}^{(n)} \end{aligned}}$$

2. Two general theorem

① If the inertia of any point of the system is increased without changing the stiffness, then all the natural frequencies will go down. (in special cases, e.g. extra mass is added to the nodal point)

② If the stiffness of any point of the system is increased without changing the inertia, all natural frequencies will go up.