

Hyperplan Arrangements

Harry Han

July 3, 2024

1 Hyperplane Arrangement

1.1 Hyperplane Arrangement in \mathbb{C}^n

We can consider \mathbb{C}^n as a n dimensional vector space over \mathbb{C} . Hyperplane in it are all the points that satisfy the equaiton: $c_0 + c_1x_1 + c_2x_2 + c_3x_3 + \cdots c_nx_n = 0$, where c_i are not all zero. That is, a hyperplane, P , is the algebraic variety $V(c_1x_1 + c_2x_2 + c_3x_3 + \cdots c_nx_n + c_0)$. We can write P into a $1 \times (n + 1)$ matrix:

$$P = [c_1 \ c_2 \ c_3 \ \cdots \ c_0]$$

We can define mapping $\phi : P \rightarrow \mathbb{C}^{n-1}$ thus: let c_i be none zero, clearly the point $p \in P$ if and only if p can be written as

$(a_1, a_2, \cdots, a_{i-1}, \frac{c_0 + c_1a_1 + \cdots + c_na_n}{-c_i}, a_{i+1}, \cdots, a_n)$. We let $\phi(p) = (a_1, a_2, \cdots, a_{i-1}, a_{i+1}, a_n F_p^n)$. This is a homeomorphism.

Consider a hyperplane arrangement, Σ , with two different hyperplanes P, Q written in matrix form:

$$\mathcal{M} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,n} & c_{1,0} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,n} & c_{2,0} \end{bmatrix}$$

Provided P, Q are different, they will intersect if and only if Σ is full ranked. In such case we call the arrangement **simple** and the intersection is a $n - 2$ dimension affine space.

Similarly we we can describe any hyperplane arrangement in \mathbb{C}^n with a $m \times (n + 1)$ matrix. We define simple arrangement thus:

Definitio 1.1 (Simple Arrangement). A hyperplane arrangment Σ with hyperplanes P_1, P_2, \cdots, P_m is simple if the interestion of k hyperplanes, if exists, is a $n - k$ dimension affine space. We regard a single point as 0 dimension and the empty set as negative dimension.

If an arrangement is simple, any $k \times n + 1$ submatrix of its matrix form shall either: (1) full ranked, or (2) have no solution when considered as a system of linear equations.

1.2 Hyperplane Arrangement in F_p^n

Considering F_p^n , where F_p is the field $\mathbb{N} \setminus p\mathbb{N}$ for a prime integer p . We can consider it as an n dimension vector space over the field F_p .

Our theory in the previous section can be extended to F_p^n : A hyperplane in F_p^n is a $n - 1$ dimension affine space represented as the algebraic variety $V(c_0 + c_1x_1 + c_2x_2 + c_3x_3 \cdots + c_nx_n)$, where at least one of the c_i is non-zero. All hyperplanes of F_p^n are $n - 1$ dimension affine space and are homeomorphic to F_p^{n-1} . In a simple arrangement, the intersection of k hyperplanes is a $n - k$ dimension affine space.

As a finite field, we can count number of points in any affine subspace of F_p^n .

Theorem 1.2.1 (Affine Spaces in F_p^n). *Any m dimensional affine subspace of F_p^n defined as $V(c_0 + c_1x_1 + c_2x_2 + \cdots + c_nx_n; d_0 + d_1x_1 + \cdots + d_nx_n; \cdots)$ are homeomorphic to F_p^m . In particular, they have p^m element.*

It is thus natural to ask how many points of intersections there are in a hyperplane arrangement. Specifically, in an arrangement with m hyperplanes we are interested in how many points of intersections are strictly on $0, 1, 2, \cdots m$ hyperplanes. By **strictly intersection** of k hyperplanes we mean a point lays on k hyperplanes but not on $k + 1$ hyperplanes.

Definitio 1.2 (Strictly Intersection). If p belongs to the strictly intersection of k hyperplanes if it is on k hyperplanes but not on $k + 1$ hyperplanes.

This is how to count:

Theorem 1.2.2 (The Counting Theorem). *Assuming Σ is a simple hyperplane arrangement in F_p^n consists of m hyperplane, and can be represented in the matrix \mathcal{M} .*

Let f_l denote the number of full ranked $l \times (n + 1)$ submatrices of \mathcal{M} .

Since the arrangement is simple, no points are in the intersection of more than n hyperplanes.

The number of the points that are in the intersection of n hyperplane equal to the number of non-singular $n \times (n + 1)$ submatrices of \mathcal{M} . Let this number be \mathfrak{N}_n . Since there are no points in the intersection of more than n hyperplanes, \mathfrak{N}_n is the number of points strictly intersected by n hyperplanes.

The number of the points strictly intersected by $n-2$ hyperplanes (denoted by \mathfrak{N}_{n-1}) is number of full ranked $(n-1) \times n$ sub-matrices of \mathcal{M} times p minus $n\mathfrak{N}_n$. That is, if there are f_{n-1} full ranked $(n-1) \times n$ submatrices, $\mathfrak{N}_{n-1} = p \cdot f_{n-1} - n \cdot C(n, n-1)\mathfrak{N}_n$.

The formulae goes on.

$$\mathfrak{N}_{n-2} = f_{n-2} \cdot p^2 - \cdot C(n-1, n-2)\mathfrak{N}_{n-1} - \cdot C(n, n-2)\mathfrak{N}_n.$$

In general,

$$\mathfrak{N}_k = f_k \cdot p^{n-k} - \sum_{i=k+1}^n C(i, k)\mathfrak{N}_i \quad (1)$$

2 Unimodular Matrix

Notation

1. $C(n, k) = \frac{n!}{k!(n-k)!}$, that is n choose k .