Hyperplan Arrangements

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1 Hyperplane Arrangement

1.1 Hyperplane Arrangement in $\mathbb{C}^n$

We can consider $\mathbb{C}^n$ as a n dimensional vector space over $\mathbb{C}$. Hyperplane in it are all the points that satisfy the equaiton: $c_0+c_1x_1+c_2x_2+c_3x_3+\cdots c_nx_n = 0$, where c_i are not all zero. That is, a hyperplane, P, is the algebraic variety $V(c_1x_1 + c_2x_2 + c_3x_3 + \cdots c_nx_n + c_0)$. We can write P into a $1 \times (n+1)$ matrix:

$$P = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix}$$

We can define mapping $\phi : P \to \mathbb{C}^{n-1}$ thus: let c_i be none zero, clearly the point $p \in P$ if and only if p can be written as $(a_1, a_2, \cdots, a_{i-1}, \frac{c_0+c_1a_1+\cdots+c_na_n}{-c_i}, a_{i+1}, \cdots, a_n)$. We let $\phi(p) = (a_1, a_2, \cdots, a_{i-1}, a_{i+1}, a_nF_p^n)$. This is a homeomorphism.

Consider a hyperplane arragnement, Σ, with two different hyperplanes P, Q written in matrix form:

$$\mathcal{M} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,n} & c_{1,0} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,n} & c_{2,0} \end{bmatrix}$$

Provided P, Q are different, they will intersect if and only if Σ is full ranked. In such case we call the arrangement ***simple*** and the intersection is a $n-2$ dimension affine space.

Similarly we we can descripe any hyperplane arrangement in $\mathbb{C}^n$ with a $m \times (n+1)$ matrix. We define simple arrangement thus:

Definitio 1.1 (Simple Arrangement)**.** A hyperplane arrangment Σ with hyperplanes $P_1, P_2, \cdots, P_m$ is simple if the interestion of k hyperplanes, if exists, is a $n-k$ dimension affine space. We regard a single point as 0 dimension and the empty set as negative dimension.

The following theorem determines condiftion for a hyperplane arrangement to be simple:

Theorema 1.1.1 (Simple Arrangement)**.** *An hyperplane arrangement, represented by the matrix $\mathcal{M}$, is simple if and only if any $k \times (n+1)$ submatrix of $\mathcal{M}$, when reduced to row-echelon form, shall contain one row like $[0,0,0,\cdots,0,1]$ (This row contains all zero entries except on the last coloumn).*

This means, any submatrix shall be either:

1. *be full ranked, or*
2. *have no solution when considered as a system of linear equations.*

Here is a corollary:

Corollarium 1.1.2. *Let matrix $\mathcal{M}$ with integer entries represents a hyperplane arrangement. If the arrangement is simple respect to $\mathbb{R}^n$, it will be simple respect to any F_p^n. However, if it is simple respect to F_p^n, it may not be simple respect to $\mathbb{R}^n$, or any other $F_{p'}^n$.*

1.2 Hyperplane Arrangement in F_p^n

Considering F_p^n, where F_p is the field $\mathbb{N} \setminus p\mathbb{N}$ for a prime integer p. We can consider it as an n dimension vector space over the field F_p.

Our theory in the previous section can be extended to F_p^n: A hyperplane in F_p^n is a $n-1$ dimension affine space represented as the algebraic variety $V(c_0 + c_1x_1 + c_2x_2 + c_3x_3 \cdots + c_nx_n)$, where at least one of the c_i is non-zero. All hyperplanes of F_p^n are $n-1$ dimension affine space and are homeomorphic to F_p^{n-1}. In a simple arrangement, the intersection of k hyperplanes is a $n-k$ dimension affine space.

As a finite field, we can count number of points in any affine subspace of F_p^n.

Theorema 1.2.1 (Affine Spaces in F_p^n)**.** *Any m dimensional affine subspace of F_p^n defined as $V(c_0 + c_1x_1 + c_2x_2 + \cdots + c_nx_n; d_0 + d_1x_1 + \cdots d_nx_n; \cdots)$ are homeomorphic to F_p^m. In particular, they have p^m element.*

It is thus natural to ask how many points of intersections there are in a hyperplane arrangement. Specifically, in an arrangement with m hyperplanes we are interested in how many points of intersections are strictly on $0, 1, 2, \cdots m$ hyperplanes. By ***strictly intersection*** of k hyperplanes we mean a point lays on k hyperplanes but not on $k+1$ hyperplanes.

Definitio 1.2 (Strictly Intersection)**.** If p belongs to the strictly intersection of k hyperplanes if it is on k hyperplanes but not on $k+1$ hyperplanes.

This is how to count:

Theorema 1.2.2 (The Counting Theorem)**.** *Assuming Σ is a simple hyperplane arrangement in F_p^n consists of m hyperplane, and can be represented in the matrix $\mathcal{M}$.*

Let f_l denote the number of full ranked $l \times (n+1)$ submatrices of $\mathcal{M}$.

Since the arrangement is simple, no points are in the intersection of more than n hyperplanes.

The number of the points that are in the intersection of n hyperplane equal to the number of non-singular $n \times (n+1)$ submatrices of $\mathcal{M}$. Let this number be $\mathfrak{N}_n$. Since there are no points in the intersection of more then n hyperplanes, $\mathfrak{N}_n$ is the number of points strictly intersected by n hyperplanes.

The number of the points strictly intersected by $n-2$ hyperplanes (denoted by $\mathfrak{N}_{n-1}$) is number of full ranked $(n-1) \times n$ sub-matrices of $\mathcal{M}$ times p minus $n\mathfrak{N}_n$. That is, if there are f_{n-1} full ranked $(n-1) \times n$ submatrices, $\mathfrak{N}_{n-1} = p \cdot f_{n-1} - n \cdot C(n, n-1)\mathfrak{N}_n$.

The formulae goes on.

$\mathfrak{N}_{n-2} = f_{n-2} \cdot p^2 - \cdot C(n-1, n-2)\mathfrak{N}_{n-1} - \cdot C(n, n-2)\mathfrak{N}_n$.

In general,

$$\mathfrak{N}_k = f_k \cdot p^{n-k} - \sum_{i=k+1}^{n} C(i,k)\mathfrak{N}_i \tag{1}$$

2 Unimodular Matrix

A unimodular matrix is a matrix such that all of its square submatrix has determinant 1 or -1. In F_p^n, we are to investigate hyperplane arrangements which can be represented as a unimodular matrix $\mathcal{M}$.

$\mathcal{M}$ being a unimodular matrix does not garantees that the arrangement is simple. If this additional condition is imposed, we can have this theorem:

Theorema 2.0.1 (Number of 0 Dimension Intersection)**.** *Let $\mathcal{M}$ represents a simple hyperplane arrangement in F_p^n. Assume the matrix is unimodular respect to $\mathbb{R}$, it will also be unimodular for any prime p in F_p.*

Then the number of 0 dimension hyperplane intersections stay unchanged for all prime p regarded in F_p^n.

Notation

1. $C(n,k) = \frac{n!}{k!(n-k)!}$, that is n choose k.