Q1 We are to prove the sequence $\left(\dfrac{5n^7}{7^n}\right)_{n\in\mathbb{N}}$ converges.

Proof. We shall first prove the statement (1): $(n > 11) \to (5n^7 < 5^n)$
Statement 1 can be proved by induction.
Base case: for $n = 12$, $5n^7 = 179159040 < 5^n = 244140625$.
Induction: Assuming $5n^7 < 5^n$. $(\forall n > 11)$ $(\frac{n+1}{n} < \frac{12}{11} \approx 1.0909 < 1.2585 \approx 5^{\frac{1}{7}}) \to (\frac{n+1}{n})^7 < 5 \to 5(n+1)^7 < 5 \cdot 5n^7 < 5 \cdot 5^n = 5^{n+1}$. And statement 1 is proved.

We also need statement 2, which is obvious: $(0 < a < 1, 0 < n < m) \to (a^m < a^n)$

$\forall \epsilon > 0$, let $l_1 = \dfrac{\ln \epsilon}{\ln \frac{5}{7}}$ and $l_2 = 12$. Let N be the greater of l_1 and l_2.

$$(n > N) \to \underbrace{\frac{5n^7}{7^n}}_{\alpha} < \underbrace{\frac{5^n}{7^n}}_{\beta} < \underbrace{\frac{5^N}{7^N}}_{\gamma} \leq \underbrace{\left(\frac{5}{7}\right)^{\frac{\ln \epsilon}{\ln \frac{5}{7}}}}_{\delta} = \epsilon$$

$\alpha < \beta$ is of statement 1. $\beta < \gamma \leq \delta$ is of statement 2.

Thus we have proved that $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(\dfrac{5n^7}{7^n} < \epsilon)$.
Q.E.D.

Q2 Let $\mathbb{S}$ be a countable set, and let $\mathbb{E}_s$ be a countable sets for all $s \in \mathbb{S}$. Define the set

$$\mathbb{E} = \bigcup_{s \in \mathbb{S}} \mathbb{E}_s$$

We are to prove that $\mathbb{E}$ is countable by constructing a bijective mapping $\mathfrak{C} : \mathbb{N} \to \mathbb{E}$.

Proof. By definition there exists a bijective function $f : \mathbb{N} \to \mathbb{S}$, and bijective function $g_s : \mathbb{N} \to \mathbb{E}_s$. Moreover, let us denote each element of the set $\mathbb{E}_s$ to be $E_s^1, E_s^2, \cdots$. As there exists a bijective function g between $\mathbb{N}$ and $\mathbb{E}_s$, $\{E_s^i | i \in \mathbb{N}\} = \mathbb{E}_s$

We also know that there exists a bijective mapping $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

Now consider the function defined thus: $m : \mathbb{N} \times \mathbb{N} \to E$ such that $m(a, b) = E^a_{f(b)}$. m is bijective.

The function $\mathfrak{C} = m \circ h : \mathbb{N} \to \mathbb{E}$ is bijective, as all of the function used to construct it are bijective.

Q.E.D.