Handin Week 6

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March 1, 2023

Q1 We are to find all subgroups of $\mathbb{D}_3$, and determine which of them are *normal*. (A subgroup N of the group G is normal if and only if every left coset of N equals to the corresponding right coset, i.e. $\forall k \in G, kN = Nk$)

Before our investigation of subgroups, recall for $\mathbb{D}_3, h^2 = e, gh = hg^2, hg = g^2h,$ and $g^3 = 1$. Each one of these statament can be checked easily.

To find all subgroups of the dihedral group $\mathbb{D}_3$, Lagrange theorem is required which claims that its subgroup must have the order of 1,2,3, or 6.

The subgroup with 1 element is the trivial subgroup $\{e\}$, the subgroup with 6 element is $\mathbb{D}_3$ itself.

Theorem 2.4.6 states that any groups with 2, or 3 elements must be cyclic(as 2 and 3 are prime). Thus we shall consider all possible cyclic subgroup:

1. $\langle e \rangle = \{e\}$;
2. $\langle g \rangle = \{e, g, g^2\}$;
3. $\langle g^2 \rangle = \{e, g^2, g\} = \langle g \rangle$;
4. $\langle h \rangle = \{e, h\}$ (as $h^2 = e$);
5. $\langle hg \rangle = \{e, hg\}$ (as $(hg)^2 = e$);
6. $\langle hg^2 \rangle = \{e, hg^2\}$ (as $(hg^2)^2 = e$);

And we conclude there are only 6 subgroup for $\mathbb{D}_3$, namely:
$\{e\}, \{e, g, g^2\}, \{e, h\}, \{e, hg\}, \{e, hg^2\}$ and $\mathbb{D}_3$ itself.

Next, we are to find all *normal* subgroups.

1. $\mathbb{D}_3$ itself; This is a normal group as $\forall a \in \mathbb{D}_3, a\mathbb{D}_3 = \mathbb{D}_3 a = \mathbb{D}_3$
2. $\{e\}$; This is a normal group as $\forall a \in \mathbb{D}_3,\ ae = ea$;

3. $\mathbb{S}_3 = \{e, g, g^2\} = \langle g \rangle$; This is also a normal group, which can be shown by brute force: the first three cases are obvious: $e\mathbb{S}_3 = \mathbb{S}_3 e; g\mathbb{S}_3 = \mathbb{S}_3 g = \mathbb{S}_3; g^2\mathbb{S}_3 = \mathbb{S}_3 g^2 = \mathbb{S}_3$.

   $h\mathbb{S}_3 = \{h, hg, hg^2\} = \{h, hg^2, gh\} = \{h, gh, gh^2\} = \mathbb{S}_3 h$

   $hg\mathbb{S}_3 = \{hg, hg^2, hg^3\} = \{hg, h, hg^2\} = \{hg, ghg, g^2hg\} = \mathbb{S}_3 h$

   $hg^2\mathbb{S}_3 = \{hg^2, h, hg\} = \{hg^2, hg, h\} = \{hg^2, ghg^2, g^2hg^2\} = \mathbb{S}_3 h$

4. $\mathbb{S}_4 = \{e, h\} = \langle h \rangle$ is not a normal group as $g\mathbb{S}_4 = \{g, gh\} \neq \{g, hg\} = \mathbb{S}_4 g$.

5. $\mathbb{S}_5 = \{e, hg\} = \langle hg \rangle$ is not a normal group as $g\mathbb{S}_5 = \{g, ghg\} = \{g, h\} \neq \{g, hgg\} = \mathbb{S}_5 g$.

6. $\mathbb{S}_6 = \{e, g^2h\} = \langle hg^2 \rangle$ is not a normal group as $g\mathbb{S}_5\{g, h\} \neq \{g, gh\}\{g, g^2hg\} = \mathbb{S}_5 g$.

Q2 We are to prove the series

$$\sum_{n=1}^{\infty} \left(\frac{an+2}{n+1}\right)^{n^2} \tag{1}$$

converges when $0 < a < 1$, and diverges when $a \geq 1$.

Proof. We shall first prove that the series converge when $0 < a < 1$, and it diverges when $a > 1$. Ratio test shall be applied (which is valid, as all terms of series are positive):

$$\begin{aligned}\rho = \lim_{n\to\infty} \left(\left(\frac{an+2}{n+1}\right)^{n^2}\right)^{1/n} &= \lim_{n\to\infty} \left(\frac{an+2}{n+1}\right)^{n} \\ &= \lim_{n\to\infty} \left(\frac{a+2/n}{1+1/n}\right)^{n} = a^n\end{aligned} \tag{2}$$

Thus

$$0 < a < 1 \implies a^n < 1 \implies \rho < 1 \implies \text{series (1) converges;}$$

and $a > 1 \implies a^n > 1 \implies \rho > 1 \implies$ series (1) diverges.

When $a = 1$, consider:

$$\lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^{n^2} = \lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^{n^2} = e^n > 0 \tag{3}$$

Thus we conlude the series diverges when $a = 1$ by the divergence test. Q.E.D.