1 Theory Of Differential Equations

Definitio 1.1 (Classification of Differential Equations)**.** Differential equations involves in relations of functions and their derivatives. We usually use t as independent variable with physical meaning of time.

If only ordinary derivatives (the function depends only in one variable) are involved, the equation is called an **ordinary differential equation** (ODE). If partial derivatives are involved, the equation is called a **partial differential equation** (PDE).

Order of a differential equation is the order of the highest derivative. A nth order ODE can be written as $F(t, x, x'', \cdots, x^{(n)}) = 0$. If F is a linear function, we call this a linear ODE.

A linear ODE is **homogeneous** if it can be written as $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$ for constants a_i.

Definitio 1.2 (Initial Value Problem)**.**

Theorema 1.1 (Solutions To linear Homogeneous ODE)**.** *Solutions to nth degree Linear homogeneous ODE of one function forms a vector space of n dimension.*

I.e., with ODE $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$, the set of solutions forms a vector space of dimension n.

Theorema 1.2 (Existance And Uniqueness Theore for Initial Value Problem)**.** *Assuming we have the system of ODE*

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \cdots, x_n) \\ x_2' &= f_2(t, x_1, x_2, \cdots, x_n) \\ &\cdots \\ x_n' &= f_n(t, x_1, x_2, \cdots, x_n) \end{aligned} \tag{1}$$

with initial condition $x_i(t_0) = x_{i0}$ for $i = 1, 2, \cdots, n$.

If f_i and $\partial_{x_j} f_i$ for all applicable i, j are continuous in a region R containing $(t_0, x_{10}, x_{20}, \cdots, x_{n0})$, then there exists a unique solution $x_i = x_i(t)$ defined on an interval I containing t_0.

2 Easy Differential Equations

Theorema 2.1. *The equation $\frac{dy}{dt} + p(t)y = g(t)$ is solved by $y(t) = \frac{1}{\mu(t)}(\int \mu(t)g(t)dt + C)$ where $\mu(t) = \exp \int p(t)dt$.*

3 Linear ODE

Theorema 3.1 (System of Homogeneous Linear ODE)**.** *To find solution to the linear homogeneous system with constant coefficients $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is to find the eiganvalue and eiganvector of $\boldsymbol{A}$. If there is eiganvectors forming a basis of $\mathbb{R}^n$, then the solution is*

$$\boldsymbol{x} = \sum_{i=1}^{n} c_i \boldsymbol{\xi}_i e^{\lambda_i t}$$

For $n = 2$. If there is one repeated eiganvalue ρ corresponding the the vector ξ. The first solution is $\boldsymbol{x}_1 = \boldsymbol{\xi} e^{\rho t}$, and the second solution is $\boldsymbol{x}_2 = \boldsymbol{\xi} t e^{\rho t} + \boldsymbol{\eta} e^{\rho t}$, where $(\boldsymbol{A} - \rho \boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}$

Definitio 3.1 (Fundamental Matrix)**.** $\boldsymbol{\psi}$ is fundamental matrix of $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ if each column of it is a solution to the system and $\det \boldsymbol{\psi} \neq 0$. (The space of its solutions)

Theorema 3.2 (Non-homogeneous linear ODE problem)**.** *Consider Non-homogeneous problem* $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}$

Diagonalisation *We can diagonlise* $\boldsymbol{A} = \boldsymbol{T}^{-1}\boldsymbol{D}\boldsymbol{T}$. *Let* $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$ *and solve* $\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y} + \boldsymbol{T}^{-1}\boldsymbol{g}$, *then* $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$ *is the solution.*

Variation of Parameters *Let* $\boldsymbol{\psi}$ *be the fundamental matrix of* $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. *The particular solution is* $\boldsymbol{\psi} \int \boldsymbol{\psi}^{-1}\boldsymbol{g} dt$.

4 Laplace Transform

Definitio 4.1 (Laplace Transform)**.**

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{2}$$

Theorema 4.1 (Tables of Laplace Transform)**.**

1. $\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
2. $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$
3. $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f\}$, *domain unchanged.* ($u_c(t) = t, t < c, 1$ *otherwise)*
4. $\mathcal{L}\{e^{ct}f(t)\} = \mathcal{L}\{f\}(s-c)$,$s - c > a$, *if* $\mathcal{L}\{f\}$ *is defined for* $s > a$
5. $\mathcal{L}\{\delta(t-a)\} = e^{-sa}$
6. $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)$
7. $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c})$
8. $\mathcal{L}\{1\} = \frac{1}{s}$
9. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ *for* $s > a$
10. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ *for* $s > 0$
11. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ *for* $s > 0$
12. $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$ *for* $s > a$
13. $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$ *for* $s > a$
14. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ *for* $s > 0$
15. $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$ *for* $s > |a|$
16. $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$ *for* $s > |a|$
17. $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$ *for* $s > a$

Theorema 4.2 (Convolution Theorem)**.**

$$f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

1. $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
2. $f*g = g*f$
3. $f*(g*h) = (f*g)*h$
4. $f*(g+h) = f*g + f*h$

5 Autonumous System

Considering a system of ODE: $\frac{dx}{dt} = F(x,y), \frac{dy}{dt} = G(x,y)$. F, G are independent of t. This system is **autonomous**.

Definitio 5.1 (Critical Point)**.** (x_0, y_0) is a critical point if $F(x_0,y_0) = G(x_0,y_0) = 0$.

The nature of critical is easy to determine for linear system.

Definitio 5.2 (Almost Linear System)**.** Consider autonomous system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x})$. If $\mathbf{0}$ is a critical point, and $\frac{||\boldsymbol{g}||}{||\boldsymbol{x}||} \to \mathbf{0}$ as $\boldsymbol{x} \to \mathbf{0}$, this is an almost linear system. Significantly, if F, G are twice differentiable. The system is almost linear, and is approximated by $\frac{d}{dt}\boldsymbol{u} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \boldsymbol{u}$, where $\hat{\boldsymbol{x}}$ is a critical point, $F_x = F_x(\hat{\boldsymbol{x}}), \cdots$, $\boldsymbol{u} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. The stability of $\hat{\boldsymbol{x}}$ for the almost linear system is the same as linear system, except when the linear system has purly imaginary eiganvalues or equal real eiganvalues.

Theorema 5.1 (Lyapunov)**.** *For a linear system $x' = F(x,y)$, $y' = G(x,y)$. We can find a Lyapunov function $V(x(t), y(t))$ such that V is positive definite and $V' = \frac{dV}{dx}\frac{dx}{dt} + \frac{dV}{dy}\frac{dy}{dt}$ is negative definite in a domain containing $(0,0)$, then $(0,0)$ is asymptotic stable critical point. IF V' semi negative definite, $(0,0)$ is stable.*

If For all interval containing $(0,0)$ there is a point such that V is positive, while V' is positive definite in a domain containing $(0,0)$, $\mathbf{0}$ is unstable.

Theorema 5.2. *If F, G has continuous partial derivative is domain D, a close trajectory must contain a critical point that is not a saddle point.*

Theorema 5.3. *If function F, G has continuous first partial derivative in simply connected domain D. If $F_x + G_y$ has the same sign in D, then there is no close trajectory in D.*

Theorema 5.4 (Poincare-Bendixson)**.** *Let F, G have continuous first partial derivative. Let D be a closed set containing no critical point. If there exists a trajectory in D for all $t > t_0$, there must be a closed trajectory (periodic solution) in D.*

6 Fourier Series

Theorema 6.1 (Fourier Convergence Theorem)**.** *Suppose f and f' are piecewise continuous on interval $-L < x < L$, while f is periodic with period $2L$. Then,*

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$,

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$.

7 Sturm Liouville Theory

Theorema 7.1 (Homogeneous Sturm Liouville Problem)**.**

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0 \tag{3}$$

With Boundary Conditions $\alpha_1 y(0) + \alpha_2 y'(0) = 0, \beta_1 y(1) = \beta_2 y'(1) = 0$. *Which can be written as, with* $L(y) = -[p(x)y']' + q(x)y$:

$$L[y] = \lambda r(x)y$$

For u, v satisfying the boundary conditions, we have: $(u, L(v)) = (L(u), v)$.

λ is the eigenvalues, ϕ are the eigenfunctions.

A function, f, can be expanded as $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$, *where* $c_n = \int_0^1 r(x)f(x)\phi_m(x)dx$

Theorema 7.2 (Non-homogeneous Sturm Liouville Problem)**.**

$$[p(x)y']' - q(x)y + \mu r(x)y + f(x) = 0 \iff L[y] = \mu r(x)y + f(x) \tag{4}$$

Have solution $y = \phi(x) = \sum_{n=1}^{\infty}\frac{c_n}{\lambda_n - \mu}$. *Where λ, ϕ are eigenvalues and function to the problem* $L[y] = \lambda r(x)y$. $c_n = \int_0^1 r(x)f(x)\phi_n(x)dx$