1 Linear Algebra

Definitio 1.1 (Norm). p norm of vector: $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$. ($\|\boldsymbol{x}\|_\infty = \max_i |x_i|$)

Matrix norm: $\|A\|_p = \sup_{\boldsymbol{x}\neq 0} \frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p}$. $\|A\|_1$ is maximum absolute column sum, $\|A\|_\infty$ is maximum absolute row sum. $\|A\|_2 = \sqrt{\rho(A^TA)}$ where ρ is the spectral radius.

We have these inequalities for all norms:

- $\|\boldsymbol{A}\boldsymbol{y}\| \leq \|\boldsymbol{A}\| \, \|\boldsymbol{y}\|$
- $\|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\| \, \|\boldsymbol{B}\|$
- $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
- $\|\boldsymbol{A}+\boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$
- $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$
- $\|\alpha\boldsymbol{A}\| = |\alpha|\|\boldsymbol{A}\|$

For orthogonal matrices $\|\boldsymbol{Q}\| = 1$, and $\|\boldsymbol{A}\boldsymbol{Q}\| = \|\boldsymbol{A}\|$

Theorema 1.1 (vectors).

1. ***Projection***: *$proj_y(x) = \frac{x\cdot y}{y \cdot y} y$*

Theorema 1.2 (Matices).

- ***pseudo inverse***: *$\boldsymbol{A}^\dagger = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$*
- ***Condition number*** *(Squre matrices): $\kappa_p(\boldsymbol{A}) = \|\boldsymbol{A}\|_p \, \|\boldsymbol{A}^{-1}\|_p$ for invertible $\boldsymbol{A}$, ∞ for singular $\boldsymbol{A}$.*
- *Let $\boldsymbol{A}$ be square matrix with eiganvalue $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$, then $\kappa_2(\boldsymbol{A}) = \|\boldsymbol{A}\| \, \|\boldsymbol{A}^{-1}\| = \frac{|\lambda_1|}{|\lambda_n|}$*

2 Solve Linear System

Theorema 2.1 (LU). *LU factorization: $A = LU$ where L is lower triangular and U is upper triangular.*

Algorithm 1 LU factorization

$L = I, U = A$
for $k = 1, \ldots, n-1$ **do**
  for $i = k+1, \ldots, n$ **do**
    $l_{ik} = \frac{u_{ik}}{u_{kk}}$
    $(u_{ik}, \cdots, u_{in}) = (u_{ik}, \cdots, u_{in}) - l_{ik}(u_{kk}, \cdots, u_{kn})$
  end for
end for

Theorema 2.2. *Gaussian Elimination To solve $Ax = b$. Let $A = LU$. We solve $Ly = b$ and $Ux = y$.*

Theorema 2.3 (GEPP)**.** ***Gaussian Elimination with Partial Pivoting***: $\boldsymbol{PA} = \boldsymbol{LU}$ *where $\boldsymbol{P}$ is a permutation matrix.*

Algorithm 2 GEPP

$\boldsymbol{L} = \boldsymbol{I}, \boldsymbol{U} = \boldsymbol{A}, \boldsymbol{P} = \boldsymbol{I}$
for $k = 1, \ldots, n-1$ **do**
    Choose $i \in \{k, \cdots, n\}$ that maximizes $|u_{ik}|$
    swap row $(u_{kk}, \cdots, u_{kn})$ with $(u_{ik}, \cdots, u_{in})$
    swap row $(l$
    swap p_{ik} and p_{kk} in P
    for $i = k+1, \ldots, n$ **do**
        $l_{ik} = \frac{u_{ik}}{u_{kk}}$
        $(u_{ik}, \cdots, u_{in}) = (u_{ik}, \cdots, u_{in}) - l_{ik}(u_{kk}, \cdots, u_{kn})$
    end for
end for

To solve $\boldsymbol{Ax} = \boldsymbol{b}$: find $\boldsymbol{PA} = \boldsymbol{LU}$, solve $\boldsymbol{Ly} = \boldsymbol{Pb}$, solve $\boldsymbol{Ux} = \boldsymbol{y}$,

Theorema 2.4 (Iterative Methods)**.** *To solve $\boldsymbol{Ax} = \boldsymbol{b}$. Let $\boldsymbol{A} = \boldsymbol{M} + \boldsymbol{N}$. Start with guess $\boldsymbol{x}_0$, and compute $\boldsymbol{x}_{k+1} = \boldsymbol{M}^{-1}(\boldsymbol{b} - \boldsymbol{N}\boldsymbol{x}_k)$. This method converge if $\|\boldsymbol{M}^{-1}\boldsymbol{N}\| < 1$.*

Theorema 2.5 (Least Squre)**.** *Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^n$. The vector $\boldsymbol{x} \in \mathbb{R}^m$ that minimise $\|\boldsymbol{Ax} - \boldsymbol{b}\|_2$ satisfies $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$*

Via QR: Compute reduced QR: $\boldsymbol{A} = \boldsymbol{QR}$ where $\boldsymbol{Q} \in \mathbb{R}^{m \times n} \boldsymbol{R} \in \mathbb{R}^{n \times n}$. Compute $\boldsymbol{y} = \boldsymbol{Q}^T\boldsymbol{b}$, solve $\boldsymbol{Rx} = \boldsymbol{y}$

Via SVD: Compute SVD: $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$ where $\boldsymbol{U} \in \mathbb{R}^{m \times m} \boldsymbol{\Sigma} \in \mathbb{R}^{m \times n} \boldsymbol{V} \in \mathbb{R}^{n \times n}$. Compute $\boldsymbol{z} = \boldsymbol{U}^T\boldsymbol{b}$, solve $\boldsymbol{\Sigma y} = \boldsymbol{z}$, $\boldsymbol{x} = \boldsymbol{Vy}$

Theorema 2.6 (eiganvalue problem)**.** *Suppose we have computed approximate eiganvalue and eiganvector by power iteration. Suppose, $|\lambda_1| > |\lambda_2|$, and $x_1^T z^0 \neq 0$. We have, for the kth iteration, $\sigma_k = \pm 1$, $|\lambda_1| > |\lambda_2|$ is the greatest eiganvalue, corresponding to $\boldsymbol{x}_1$: $\left\|\sigma_k \boldsymbol{z}^k - \boldsymbol{x}_1\right\| \leq \alpha_1 \left(\frac{|\lambda_2|}{|\lambda_1|}\right)^k$, and $\|\lambda_k - \lambda_1\| \leq \alpha_2 \left(\frac{|\lambda_2|}{|\lambda_1|}\right)^{2k}$*

Theorema 2.7 (Computation Cost)**.**

1. *LU factorization: $C(n) = \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n$*
2. *Forward substitution: $C(n) = n^2$*
3. *Gaussian elimination: $C(n) = \frac{2}{3}n^3 + \frac{5}{2}n^2 - \frac{7}{6}n$*
4. *Gram-Schmidt: $C(n) = 2n^3 + n^2 + n$*
5. *Householder: $C(n) = \frac{4}{3}n^3 + O(n^2)$*