

# Several Variable Calculus and Differential Equations Review

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# 1 Vectors and Geometry

**Theorem 1.0.1** (Vector Computation). *Basics computation rules for vectors:*

1.  $\underline{a} \cdot \underline{b} = \cos \theta |\underline{a}| |\underline{b}|$
2.  $|\underline{a} \times \underline{b}| = \sin \theta |\underline{a}| |\underline{b}|$
3.  $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$
4.  $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$
5.  $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$
6.  $\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ , moreover, this equal to the area of the parallelepiped specified by the vectors  $\underline{a}, \underline{b}, \underline{c}$

**Definition 1.1** (Line and Plane in 2D and 3D space).

**Definition 1.2** (Vector Function). We consider the vector function  $\underline{v}(t)$  whose domain is  $\mathbb{R}$  and whose range is  $\mathbb{R}^3$ . It has three components  $f(t), g(t)$  and  $h(t)$  and can be written in the form:

$$\underline{v}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix} = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$$

Vector function obey most of the integrating and differentiation rules of a normal function.

**Theorem 1.0.2** (Arc Length). *The length of the curve traced out by the position vector function  $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$  over the interval  $a \leq t \leq b$  is*

$$s(t) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt = \int_a^b |\underline{r}'(t)| dt$$

**Definition 1.3** (Unit Tangent Vector). For vector function  $\underline{r}(t)$ , its unit tangent vector is  $\underline{T} = \frac{\underline{r}'(t)}{|\underline{r}'(t)|}$

**Definition 1.4** (Curvature). For vector function  $\underline{r}(t)$ , its curvature is

$$\kappa = \left| \frac{\underline{T}'(t)}{s'(t)} \right| = \frac{|\underline{T}'(t)|}{|\underline{r}'(t)|} = \frac{|\underline{r}'(t) \times \underline{r}''(t)|}{|\underline{r}'(t)|^3}$$

**Definition 1.5** (Normal and binormal Vector). For vector function  $\underline{r}(t)$ , its unit tangent vector is  $\underline{T} = \frac{\underline{r}'(t)}{|\underline{r}'(t)|}$ , and its unit normal vector is  $\underline{N}(t) = \frac{\underline{T}'(t)}{|\underline{T}'(t)|}$ , and its binormal vector is  $\underline{T}(t) \times \underline{N}(t)$

## 2 Partial Derivatives

### 2.1 Limits

**Definition 2.1** (Limits of function of two variables). Let  $f(x, y)$  be a function of two variables. The limit of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$  is defined as

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\sqrt{(x-a)^2 + (y-a)^2} < \delta$  implies  $|f(x, y) - L| < \epsilon$ . Significantly, the limit of  $f(x, y)$  as  $(x, y) \rightarrow (a, b)$  is independent of the path of  $x$  and  $y$ . i.e. if assuming the relationship  $y = Y_1(x)$  and  $y = Y_2(x)$ , provided  $Y(a) = b$ , we arrives to different value of the limit of  $f(x, Y(x))$  as  $x \rightarrow a$ , then limit of  $f(x, y)$  as  $(x, y)$  approach  $(a, b)$  does not exist.

It is sometimes easier to evaluate the limit using polar coordinates (or other kinds of transformation). Let  $(x, y) = (r \cos \theta, r \sin \theta)$ , then  $(x, y) \rightarrow 0 \iff r \rightarrow 0$ . All limit law for function of one variable also apply to function of multiple variables.

**Definition 2.2** (Continuity). Let  $f(x, y)$  be a function of two variables.  $f(x, y)$  is continuous at  $(a, b) \iff \lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

### 2.2 Partial Derivatives

**Definition 2.3** (Partial Derivative).

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

More generally, for  $n$  variables, we have

$$f_{x_1}(x_1, x_2, \dots, x_n) = \frac{\partial f}{\partial x_1} = \lim_{\Delta x_1 \rightarrow 0} \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1}$$

NB: the notation of partial derivative:

$$f_x(x, y) = \frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x} f(x, y)$$

**Theorem 2.2.1** (Clairaut's Theorem). *Let  $f(x, y)$  be a function of two variables. Then*

$$f_{xy} = f_{yx}$$

*More generally the order of the partial derivative does not matter.*

*Remark 2.1.* NB: Tangent plane and linear approximation.

**Definition 2.4** (Differentiability). Let  $z = f(x, y)$ ,  $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ , then the function is differentiable if and only if  $(\epsilon_1, \epsilon_2) \rightarrow (0, 0)$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ , where  $\epsilon_1, \epsilon_2$  are defined as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

Moreover, it can be shown that  $z$  is differentiable at  $(a, b)$  if and only if the partial derivative  $f_x, f_y$  exists and are continuous near  $(a, b)$

**Definition 2.5** (Differentials).  $w = f(x, y, z)$  the total differential  $dw$  is defined as

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

**Theorem 2.2.2** (Chain Rule). *Suppose  $z = f(x, y)$  and  $x = g(t)$ ,  $y = h(t)$  then*

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

*Suppose  $z = f(x, y)$  and  $x = g(t, u)$ ,  $y = h(t, u)$  then*

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}, \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

**Theorem 2.2.3** (Implicit Differentiation). *Suppose  $F(x, y, z) = 0$ , and  $z$  is a implicit function of  $x$ , then:*

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z},$$

*If  $z$  is a implicit function of  $y$ , then:*

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

**Definition 2.6** (Gradient). Let  $f(x, y)$  be a function of two variables, then the gradient of  $f$  is defined as a vector function:

$$\nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

**Definition 2.7** (Directional Derivative). For function  $f(x, y)$ , the directional derivative at  $(a, b)$  at the direction of the unit vector  $\underline{u} = (u_1, u_2)$  in  $x - y$  plane is:

$$\nabla f \cdot \underline{u} = f_x(a, b)u_1 + f_y(a, b)u_2$$

Note it is a constant.

**Theorem 2.2.4** (Maximum Directional Derivative). Suppose  $f(x, y)$  is differentiable at  $(a, b)$ , then the directional derivative of  $f$  at  $(a, b)$  in the direction of  $\underline{u}$  is maximized if and only if  $\underline{u}$  is parallel to the gradient of  $f$  at  $(a, b)$ .

*Proof.* Notice the directional derivative equals to

$$\nabla f \cdot \underline{u} = \cos \theta |\nabla f| |\underline{u}|$$

Which is maximized when  $\theta = 0$ . □

*Remark 2.2.* Tangent plane to level surface and normal line: see lecture notes page 27.

## 2.3 Maximum and Minimum

**Definition 2.8** (Critical Point). Let  $f(x, y)$  be a function of two variables, then  $(a, b)$  is a critical point of  $f$  if  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ , or one of its partial derivatives does not exist.

**Theorem 2.3.1.** If  $(a, b)$  is a local maximum or minimum of  $f(x, y)$ , then  $(a, b)$  is a critical point of  $f$ .

**Theorem 2.3.2** (Second Derivative Test). Suppose the second partial derivative of  $f(x, y)$  exists at  $(a, b)$  and is continuous around it, and  $f_x(a, b) = f_y(a, b) = 0$ . Let

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2 = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

Then

1. If  $D(a, b) > 0$  and  $f_{xx}(a, b) > 0$ , then  $(a, b)$  is a local minimum of  $f(x, y)$
2. If  $D(a, b) > 0$  and  $f_{xx}(a, b) < 0$ , then  $(a, b)$  is a local maximum of  $f(x, y)$
3. If  $D(a, b) < 0$ , then  $(a, b)$  is a saddle point of  $f(x, y)$
4. If  $D(a, b) = 0$ , the test is inconclusive

### 2.3.1 Function of Three Variables

**Theorem 2.3.3** (Second Derivative Test). *Suppose the second partial derivative of  $f(x, y, z)$  exists at  $(a, b, c)$  and is continuous around it, and  $f_x(a, b, c) = f_y(a, b, c) = f_z(a, b, c) = 0$ . Let*

$$D_1(a, b, c) = f_{xx}(a, b, c)$$

$$D_2(a, b, c) = \begin{vmatrix} f_{xx}(a, b, c) & f_{xy}(a, b, c) \\ f_{xy}(a, b, c) & f_{yy}(a, b, c) \end{vmatrix}$$

$$D_3(a, b, c) = \begin{vmatrix} f_{xx}(a, b, c) & f_{xy}(a, b, c) & f_{xz}(a, b, c) \\ f_{xy}(a, b, c) & f_{yy}(a, b, c) & f_{yz}(a, b, c) \\ f_{xz}(a, b, c) & f_{yz}(a, b, c) & f_{zz}(a, b, c) \end{vmatrix}$$

Then

1. If  $D_1(a, b, c) > 0$  and  $D_2(a, b, c) > 0$  and  $D_3(a, b, c) > 0$ , then  $(a, b, c)$  is a local minimum of  $f(x, y, z)$
2. If  $D_1(a, b, c) < 0$  and  $D_2(a, b, c) > 0$  and  $D_3(a, b, c) < 0$ , then  $(a, b, c)$  is a local maximum of  $f(x, y, z)$
3. If  $D_3 \neq 0$  and  $(a, b, c)$  is neither a maximum nor a minimum, then  $(a, b, c)$  is a saddle point of  $f(x, y, z)$
4. If  $D_3(a, b, c) = 0$ , the test is inconclusive

### 2.3.2 Lagrange Multiplier

**Theorem 2.3.4** (Lagrange Multiplier). *Let  $f(x, y)$  be a function of two variables and differential along the smooth curve  $g(x, y) = 0$  where  $\nabla g(x, y) \neq 0$ . If  $f$  attains minimum value along  $g = 0$  at  $(a, b)$ , then there exists  $\lambda \in \mathbb{R}$  such that*

$$\nabla f(a, b) = -\lambda \nabla g(a, b)$$

For function  $f(x, y, z)$  subject to constrain  $g(x, y, z) = 0$ , the same result holds.

## 3 Multiple Integrals

### 3.1 Double And Triple Integrals

**Definition 3.1** (Double Integral). Omitted.

**Definition 3.2** (Average Value). The average value of a function  $f(x, y)$  over a region  $D$  is defined as

$$f_{avg} = \frac{1}{A(D)} \iint_D f(x, y) dA$$

**Theorem 3.1.1** (Fubini's Theorem). If  $f(x, y)$  is continuous on the rectangle  $R = (x, y)\mathbb{R}^2, a \leq x \leq b, c \leq y \leq d$ , then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

**Corollary 3.1.2.** Suppose  $f(x, y) = g(x)h(y)$  on  $R = [a, b] \times [c, d]$ , then

$$\iint_R f(x, y) dA = \int_a^b g(x) dx \int_c^d h(y) dy$$

*Remark 3.1.* Triple integral, see example 50 page 47.

## 3.2 Change of Variables

### 3.2.1 Double Integrals: To Polar Coordinates

**Theorem 3.2.1** (Change of Variables). For  $(x, y)$  in Cartesian coordinates, recall the polar coordinate  $(r, \theta)$  is defined by:  $r^2 = x^2 + y^2, x = r \cos \theta, y = r \sin \theta$

If the area  $R$  is given by  $r_0 \leq r \leq r_1, \theta_0 \leq \theta \leq \theta_1$ , then:

$$\iint_R f(x, y) dA = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta = \int_{r_0}^{r_1} \int_{\theta_0}^{\theta_1} f(r \cos \theta, r \sin \theta) r dr d\theta$$

Notice the extra  $r$  in the integrand.

### 3.2.2 Triple Integrals: To Cylindrical Coordinates

**Theorem 3.2.2** (Change of Variables To Cylindrical Coordinates). Recall the cylindrical coordinate  $(r, \theta, z)$  is defined by:  $r^2 = x^2 + y^2, x = r \cos \theta, y = r \sin \theta$  If the volume  $V$  is given by  $r_0 \leq r \leq r_1, \theta_0 \leq \theta \leq \theta_1, z_0 \leq z \leq z_1$ , then:

$$\iiint_V f(x, y, z) dV = \iiint_D f(r \cos \theta, r \sin \theta, z) r dz dr d\theta = \int_{r_0}^{r_1} \int_{\theta_0}^{\theta_1} \int_{z_0}^{z_1} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

Notice the extra  $r$  in the integrand.



### 3.2.3 Triple Integrals: To Spherical Coordinates

**Theorem 3.2.3** (Change of Variables To Spherical Coordinates). *Recall the spherical coordinate  $(r, \theta, \phi)$  is defined by:  $r^2 = x^2 + y^2 + z^2$ ,  $x = r \sin \phi \cos \theta$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \phi$*

*If the volume  $V$  is given by  $r_0 \leq r \leq r_1$ ,  $\theta_0 \leq \theta \leq \theta_1$ ,  $\phi_0 \leq \phi \leq \phi_1$ , then:*

$$\iiint_V f(x, y, z) dV = \iiint_D f(r \sin \phi \cos \theta, r \sin \theta \sin \phi, r \cos \phi) r^2 \sin \phi dr d\theta d\phi$$

*Notice the extra  $r^2 \sin \phi$  in the integrand.*

*Remark 3.2.* More excercises needed.

### 3.2.4 Change of Variable in General: Jacobian

**Definition 3.3** (Jacobian). Consider the transformation from  $u, v$  plane to  $x, y$  plane, where  $x = x(u, v)$ ,  $y = y(u, v)$  For bijective transformation, the jacobian is defined as:

$$J_2 = \frac{\partial x \partial y}{\partial u \partial v} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The jacobian for transformation from  $u, v, w$  plane to  $x, y, z$  plane, where  $x = x(u, v, w)$ ,  $y = y(u, v, w)$ ,  $z = z(u, v, w)$  is defined as:

$$J_3 = \frac{\partial x \partial y \partial z}{\partial u \partial v \partial w} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

**Theorem 3.2.4** (Change of Variable in General). *If the transformation from  $u, v$  plane to  $x, y$  plane is bijective, then:*

$$\iint_D f(x, y) dA = \iint_D g(u, v) |J_2| du dv$$

Where  $g(u, v) = f(x(u, v), y(u, v))$ .

*If the transformation from  $u, v, w$  plane to  $x, y, z$  plane is bijective, then:*

$$\iiint_V f(x, y, z) dV = \iiint_V g(u, v, w) |J_3| du dv dw$$

Where  $g(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w))$ .

*Notice the Jacobian is taken as absolute value.*

*Remark 3.3.* More excercises needed. See Example 54, lecture note page 51.

## 4 Vector Calculus

### 4.1 Vector Fields

**Definition 4.1** (Vector Field). A vector field is a function  $\underline{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . The input and output are  $n$  dimensional vectors.

#### 4.1.1 Conservative Vector Fields

**Theorem 4.1.1** (Conservative Vector Fields). Vector field  $\underline{F}(x, y)$  is conservative, provided that it is defined on all  $\mathbb{R}^2$ , if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

For vector field  $\underline{G}(x, y, z)$ , provided it is defined on all  $\mathbb{R}^3$ , it is conservative if and only if:

$$\underline{\nabla} \times \underline{G} = 0$$

**Definition 4.2** (Conservative Vector Field). A vector field  $\underline{F}$  is conservative if there exists a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\underline{\nabla} f = \underline{F}$ .

### 4.2 Curl and Divergence

**Definition 4.3** (Curl). The curl of a vector field  $\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k}$  is defined as:

$$\underline{\nabla} \times \underline{F} = \underline{i} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \underline{j} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \underline{k} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Moreover, curl of  $\underline{F}$  can be rewritten as below:

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

**Definition 4.4** (Divergence). The divergence of a vector field  $\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k}$  is defined as:

$$\underline{\nabla} \cdot \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

**Corollary 4.2.1.** Two facts about curl and divergence:

1.  $\text{div} (\text{curl } \underline{F}) = 0$
2.  $\text{curl} (\underline{\nabla} f) = 0$

### 4.3 Line Integral

**Definition 4.5** (Line Integral). If  $f$  is a function of three variable, the line integral along the curve  $\gamma$  is defined as:

$$\int_{\gamma} f(x, y, z) ds = \int_{t_0}^{t_1} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Where  $\gamma$  is parametrized by  $x = x(t), y = y(t), z = z(t)$ , from  $t_0$  to  $t_1$ .

Moreover

$$\int_{\gamma} F dx = \int_{t_0}^{t_1} F(x(t), y(t), z(t)) \frac{dx}{dt} dt$$

Line integral alone  $y$  and  $z$  are defined similarly.

**Definition 4.6** (Line Integral of a Vector Field). The line integral of a vector field  $\underline{F}$  along a curve  $\gamma$  is described by the vector function  $\underline{r}$  is defined as:

$$\int_{\gamma} \underline{F} \cdot d\underline{r} = \int_{\gamma} \underline{F}(\underline{r}) \cdot \underline{r}' dt = \int_{\gamma} P_x dx + P_y dy + P_z dz$$

Where  $\underline{F} = iP_x + jP_y + kP_z$

**Theorem 4.3.1** (Fundamental Theorem of Line Integral). *If  $\underline{F}$  is conservative, and  $\nabla f = \underline{F}$ . Let  $\gamma$  be a curve which starts from  $(x_0, y_0)$  and ends at  $(x_1, y_1)$ . Then*

$$\int_{\gamma} \underline{F} \cdot d\underline{r} = f(x_1, y_1) - f(x_0, y_0)$$

### 4.4 Surface Integral

**Theorem 4.4.1** (Surface Area). *The surface area of a parametrized surface  $\underline{r}(u, v) = i x(u, v) + j y(u, v) + k z(u, v)$ , for  $(u, v) \in \mathbb{D}$  is:*

$$\iint_{\mathbb{D}} |\underline{r}_u \times \underline{r}_v| dA$$

Where  $\underline{r}_u = \frac{\partial \underline{r}}{\partial u}$  and  $\underline{r}_v = \frac{\partial \underline{r}}{\partial v}$ .

Notice  $\underline{r}_u$  and  $\underline{r}_v$  are tangent vectors to the surface, and  $||$  operator returns their magnitude.

**Corollary 4.4.2** (When  $z$  is a function of  $x$  and  $y$ ). *For surface parametrized as  $\underline{r} = i x + j y + k f(x, y)$ , the surface area of the surface for  $(x, y) \in \mathbb{D}$  is*

$$\iint_{\mathbb{D}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA$$

**Definition 4.7** (Surface Integral). Consider function  $f(x, y, z)$  defined on the surface  $\mathbb{S}$  which is described by the vector function:

$$\underline{r}(u, v) = \underline{i}x(u, v) + \underline{j}y(u, v) + \underline{k}z(u, v), (u, v) \in \mathbb{D}$$

Then the surface integral of  $f$  over the surface  $\mathbb{S}$  is defined as:

$$\iint_{\mathbb{S}} f(x, y, z) dS = \iint_{\mathbb{D}} f(\underline{r}(u, v)) |\underline{r}_u \times \underline{r}_v| dA$$

Where  $\underline{r}_u = \frac{\partial \underline{r}}{\partial u}$ ,  $\underline{r}_v = \frac{\partial \underline{r}}{\partial v}$ , and  $f(\underline{r}(u, v)) = f(x(u, v), y(u, v), z(u, v))$

*Remark 4.1.* More excercises needed.

**Definition 4.8** (Oriented Surface). Let  $\underline{r}(u, v)$  be a vector function on  $\mathbb{D}$  that describes the surface  $S$ .  $\frac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|}$  is its unit normal vector to  $S$ .

Moreover, for surface  $z = f(x, y)$  the unit normal vector is:

$$\frac{-\underline{i} \frac{\partial f}{\partial x} - \underline{j} \frac{\partial f}{\partial y} + \underline{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

**Definition 4.9** (Surface Integral Over Vector Field). Let  $\underline{F}(x, y, z)$  be a vector field defined on the surface  $S$ . Then the surface integral of  $\underline{F}$  over  $S$  is defined as:

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_S \underline{F} \cdot \hat{n} dS = \iint_{\mathbb{D}} \underline{F}(\underline{r}(u, v)) \cdot \frac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|} \cdot |\underline{r}_u \times \underline{r}_v| dA$$

Which equals to:

$$\iint_{\mathbb{D}} \underline{F}(\underline{r}(u, v)) \cdot (\underline{r}_u \times \underline{r}_v) dA$$

Moreover, for surface  $z = g(x, y)$ , we have:

$$\underline{F} \cdot |\underline{r}_u \times \underline{r}_v| = (F_x \underline{i} + F_y \underline{j} + F_z \underline{k}) \cdot \left( -\frac{\partial g}{\partial x} \underline{i} - \frac{\partial g}{\partial y} \underline{j} + \underline{k} \right)$$

And the surface integral is:

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_{\mathbb{D}} -F_x \frac{\partial g}{\partial x} - F_y \frac{\partial g}{\partial y} + F_z dA$$

## 4.5 Theoremae Trinitatis

**Theorem 4.5.1** (Green's Theorem). *Let  $\gamma$  be positively oriented curve, simply close, and piecewise smooth curve in  $x, y$  plane. Let  $\Gamma$  be the region bounded by  $\gamma$ . If  $P(x, y)$  and  $Q(x, y)$  are continuous functions on  $\Gamma$ , then*

$$\int_{\gamma} Pdx + Qdy = \iint_{\Gamma} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

**Theorem 4.5.2** (Stokes' Theorem). *Let  $\gamma$  be positively oriented, simply close, and piecewise smooth curve in  $x, y$  plane. Let  $\Gamma$  be the region bounded by  $\gamma$ . If  $\underline{F}(x, y, z)$  is a continuous vector field on  $\Gamma$ , then*

$$\iint_{\Gamma} \nabla \times \underline{F} \cdot d\underline{S} = \int_{\gamma} \underline{F} \cdot d\underline{r}$$

**Theorem 4.5.3** (Divergence Theorem). *Let  $E$  be a simple solid region bounded by a positively oriented surface (outward)  $\mathbb{S}$ . Let  $\underline{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a vector field that has continuous partial derivative in  $E$ , then:*

$$\iint_{\mathbb{S}} \underline{F} \cdot d\underline{S} = \iiint_E \nabla \cdot \underline{F} dV$$

## 5 Differential Equations

### 5.1 Classification

**Definition 5.1** (Classification of Differential Equation). Classification of Differential Equation.

1. Ordinary differential equation (ODE) involves only differentiation of one variable
2. Partial differential equation (PDE) involves differentiation of more than one variable.
3. Order of differential equation is the highest order of differentiation in the equation.

The  $n^{\text{th}}$  order linear differential equation has the general form:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = a_0(x)$$

Where  $a_i$  is a functions of  $x$ . The equation is called homogeneous if  $a_0(x) = 0$ .

If  $S_1(x), S_2(x) \dots$  are solution to the homogeneous equation, then any of its linear combinations are also a solution.

**Theorem 5.1.1** (Uniqueness of the solution to Initial Value Problem). *Consider the general  $n^{\text{th}}$  order linear differential equation:*

$$f(y^n, y^{n-1}, \dots, y, x) = 0$$

*Provided with initial values*

$$y(x_0) = a_0, y'(x_1) = a_1 \dots y^{(n)}(x_n) = a_n$$

*There is one unique solution to this Initial Value Problem.*

## 5.2 First Order Ordinary Differential Equation

**Theorem 5.2.1** (Solutions to Linear First Order ODE). *The linear ODE of the form:*

$$y' + P(x)y = Q(x)$$

*has the general solution:*

$$y(x) = e^{-\int P(x)dx} \left( \int Q(x)e^{\int P(x)dx} dx + C \right)$$

*Where  $C$  is an arbitrary constant.*

**Theorem 5.2.2** (Separable ODE). *ODE are separable if it can be written in the form:*

$$\frac{dy}{dx} h(y) = g(x)$$

*And the solution is given by simply integration:*

$$\int h(y)dy = \int g(x)dx$$

**Theorem 5.2.3** (Homogeneous ODE (Another Homogeneous)). *An ODE is called homogeneous if it can be written in the form:*

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

*It can be solved by introducing a new variable  $u = \frac{y}{x}$ . Notice  $\frac{dy}{dx} = u + x \frac{du}{dx}$ .*

**Theorem 5.2.4** (Exact ODE). *The ODE is exact if it can be written in the form:*

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0$$

Where  $\psi$  is a function of  $x$  and  $y$ .

And its general the solution is  $\psi(x, y) = C$ .

Moreover, a equation in the form of  $M(x, y) + \frac{dy}{dx}N(x, y) = 0$  is if and only if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ . c.f. the determination of conservative vector field.

**Remark 5.1** (On Integration Factor). The process of finding the solution of a differential equation many times involves heavily upon the creative application of algebraic manipulation and the computation capabilities of the problem solver. For a particular kind of equation whose general solution is unknown, there are many times tricks that may be useful. One of them is integration factor.

For example, consider the following equation:

$$3xy + y^2 + (x^2 + xy)y' = 0$$

This equation can be turned into an exact equation by multiplying both sides by  $x$ , and the  $\psi$  function can be found to be:  $x^3y + \frac{1}{2}x^2y^2$ .

### 5.3 Second Order Ordinary Differential Equation

Consider second order ODE of the form:

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t)$$

Existence and Uniqueness Theorem (5.1.1) says that for the initial value problem  $y(t_0) = \alpha, y'(t_0) = \beta$ , there exists exactly one solution.

**Definition 5.2** (Wroskian). The Wroskian of two functions,  $f(x)$  and  $g(x)$  is defined as:

$$W(f, g) = \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$$

**Definition 5.3** (Linearly Independent Functions). Two functions,  $f_1(t)$  and  $f_2(t)$  are linearly independent if if their Wronskian is not zero. More significantly, it means that  $\frac{f_1}{f_2} = g(t)$ , where  $g(t)$  is not a constant function for  $t$  in their domain.

**Theorem 5.3.1** (Nature of Solutions to second order homogeneous ODE).  
*Consider the second order homogeneous ODE:*

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0$$

*Assuming  $y_1(t)$  and  $y_2(t)$  are two linearly independent solutions, then function  $y_r$  is a solution to the ODE if and only if  $y_r$  is linear combination of  $y_1$  and  $y_2$ .*

**Theorem 5.3.2** (Abel's Theorem). *Consider the second order homogeneous ODE:*

$$y'' + p(t)y' + q(t)y = 0$$

*with solutions  $y_1(t)$  and  $y_2(t)$ , then:  $W(y_1, y_2) = c \exp \left[ - \int p(t) dt \right]$*

### 5.3.1 General Solutions to Second Order Homogeneous ODE with Constant Coefficients

**Theorem 5.3.3** (General Solutions to Second Order Homogeneous ODE with Constant Coefficients). *Let  $y'' + a_1y' + a_2y = 0$  be a second order homogeneous ODE. Substitute the function  $T(x) = e^{rx}$ , where  $r$  is a constant:*

$$r^2e^{rx} + a_1re^{rx} + a_2e^{rx} = 0 \iff r^2 + a_1r + a_2 = 0 \quad (1)$$

*Here we have three cases.*

**Case 1:** *equation 1 has two distinct real roots  $r_1$  and  $r_2$ .  
Then the general solution is:*

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}$$

**Case 2:** *equation 1 has two repeated real roots  $r_1$ .  
Then the general solution is:*

$$y(x) = C_1e^{r_1x} + C_2xe^{r_1x}$$

**Case 3:** *equation 1 has two complex roots  $r_1 + im_1$  and  $r_1 - im_1$ , where  $i$  is the imaginary unit.*

*Recall Euler's formula:  $e^{ix} = \cos x + i \sin x$ . Then the general solution is:*

$$y(x) = C_1e^{r_1x}e^{im_1x} + C_2e^{r_1x}e^{-im_1x}$$

*Which is equivalent to:*

$$y(x) = C_3 \cos(m_1x)e^{r_1x} + C_4 \sin(m_1x)e^{r_1x}$$



### 5.3.2 Solutions to Certain Second Order Non-Homogeneous ODE

**Theorem 5.3.4** (Method of Variation of Parameters). *Consider the linear non-homogeneous second order ODE:*

$$ay'' + by' + cy = g(t) \quad (2)$$

*Suppose the homogeneous equation  $(ay'' + by' + cy = 0)$  has the general solution:*

$$y_c(t) = C_1y_1(t) + C_2y_2(t)$$

*Then the particular solution of the non-homogeneous equation 2 is:*

$$y_p(t) = -y_1(t) \int \frac{y_2(t)g(t)}{aW(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t)g(t)}{aW(y_1, y_2)} dt$$

Where  $W$  is Wronskian operator and  $W(y_1, y_2) = y_1y_2' - y_2y_1'$ .  
See Boyce 4.4.

**Remark 5.2** (Method of Undetermined Coefficients). Method of Undetermined Coefficients may well be termed as method of educated guess. Consider the linear non-homogeneous second order ODE:

$$ay'' + by' + cy = g(t) \quad (3)$$

For certain  $g(t)$ , such as  $e^t$ , we can assume the solution is of the form  $ae^t$ , where  $a$  is a constant. Then we can simply plug in and solve for  $a$ .

### 5.3.3 Euler's Equation

**Theorem 5.3.5** (Solution to Euler's Equation). *Consider the equation:*

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = 0$$

*For constants  $a, b, c$ . The solution of can be found by introducing a new variable  $t = \ln(x)$*

*Notice  $\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{1}{x}$  and  $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \frac{1}{x^2} - \frac{dy}{dt} \frac{1}{x^2}$   
Then the equation becomes:*

$$a \frac{d^2y}{dt^2} + (b - a) \frac{dy}{dt} + cy = 0$$

*Which can be solved by theorem 5.3.3.*

## 6 Quaedam Formulae Utilissima

### 6.1 Trigonometric Formulae

1.  $\sin^2 \theta + \cos^2 \theta = 1$
2.  $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1$

### 6.2 Common Integrals

1.  $\int \cos^2 \theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}, \int_0^\pi \cos^2 \theta = \frac{\pi}{2}$
2.  $\int \sin^2 \theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}, \int_0^\pi \sin^2 \theta = \frac{\pi}{2}$

### 6.3 Other Common Computations

1. For  $\underline{r}(\phi, \theta) = a(\sin \phi \cos \theta \underline{i} + \sin \phi \sin \theta \underline{j} + \cos \phi \underline{k}), |\underline{r}_\phi \times \underline{r}_\theta| = a^2 \sin \phi$ .

$$[SDS](mol \cdot dm^{-3}) = 0.01 \frac{g}{100cm^{-3}} \div 288.38 \frac{g}{mol} \frac{1000cm^{-3}}{1dm^{-3}} = 3.5 \cdot 10^{-4} mol dm^{-3}$$