Notes on Pure Mathematics

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Notation

- The `\mathbb{}` fonts are used to denote sets. ($\mathbb{S}, \mathbb{Y}$, etc.)
- $\mathbb{A} \succ \mathbb{B}$ denotes there exits a surjective function $f : A \to B$. $\prec$, $\asymp$ denotes injective, bijective, respectively.

Axiom 0.1 (The "Smallest" Infinite Set)**.** A set $\mathbb{S}$ is infinite iff $\mathbb{S} \succ \mathbb{N}$.

Remark 0.1. Although FPM is a pure mathematic class with emphasis on rigor, no rigorous definition for the infinite set has been proposed. This definition/axiom is of my own conception.

Definition 0.1 (Countable Set)**.** A set $\mathbb{S}$ is countable iff $\mathbb{N} \asymp \mathbb{S}$ (there exists a bijection $f : \mathbb{N} \to \mathbb{S}$).

Corollary 0.1 (At Most Countable)**.** *Let* $\mathbb{A}$ *be an infinite set.*
($\mathbb{A} \prec \mathbb{N}$*) iff (*$\mathbb{A} \asymp \mathbb{N}$*).*

Proof. We want to prove $\mathbb{A} \prec \mathbb{N}$ is equivalent to $\mathbb{A} \asymp \mathbb{N}$. $\mathbb{A} \asymp \mathbb{N} \to \mathbb{A} \prec \mathbb{N}$ is by definition. We only need to prove the other direction; i.e., provided $\mathbb{A} \prec \mathbb{N}$, find a bijective function $h : \mathbb{A} \to \mathbb{N}$.

Let $f : \mathbb{A} \to \mathbb{N}$ be an injective mapping. If f is bijective, we are done. If f is injective but not bijective, let $\mathbb{N}^-$ be the range of f. As $\mathbb{A}$ is infinite, $\mathbb{N}^-$ is also infinite. Let $f' : \mathbb{A} \to \mathbb{N}^-$ such that $f(a) = f'(a)$. f' is an bijective mapping.

Thus we only need to show there exists a mapping $g : \mathbb{N}^- \to \mathbb{N}$ that is bijective.

g can be constructed by such: sort $\mathbb{N}^-$ and $\mathbb{N}$ in ascending order. Let the first element in the sorted $\mathbb{N}^-$ maps to the first in the sorted $\mathbb{N}$, the secound to secound, etc. As $\mathbb{N}^-$ is infinite, g must be bijective.

Indeed $h = f' \circ g : \mathbb{A} \to \mathbb{N}$ is the bijective mapping we seek. QED. $\square$

Theorem 0.0.1 (List of Countable and Uncountable Sets)**.** *Any of the following sets are countable.*

1. $\mathbb{Z}$ *and any of its infinite subsets.*

2. $\mathbb{R}$ *and any of its infinite subsets.*

3. *If* $\mathbb{S}$ *is countable,* $\{\mathbb{S} \times \mathbb{S}\}, \{\mathbb{S} \times \mathbb{S} \times \cdots \times \mathbb{S})\}$ *are also countable.*