1 Notation

Definitio 1.1.

1. Open ϵ disk centered at z_0: $D(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$
2. Closed ϵ disk centered at z_0: $\overline{D}(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$
3. Punctured ϵ disk centered at z_0: $D'(z_0, \epsilon) = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$
4. Annulus centered at z_0: $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$
5. For a meromorphic function f, with zeros z_i and poles p_i, $N_0(f) = \sum \text{order of } z_j$ and $N_\infty(f) = \sum \text{order of } p_j$

2 Holomorphic Function

Definitio 2.1 (Argument)**.**

1. $\arg(z) = \{\theta : z = |z|e^{i\theta}\} = \{\text{Arg}(z) + 2k\pi\}$, where $-\pi < \text{Arg}(z) \leq \pi$

Theorema 2.1 (Cauchy Riemann Equation)**.** *Let $f(z) = u(x, y) + iv(x, y)$ be holomorphic complex function. The following equation holds $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$. That is:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Only Partial Converse: If u_x, u_y, v_x, v_y exists in a neighborhood if z_0 and is continuous, and f satisfy Cauchy Riemann Equation, then it is holomorphic.

Definitio 2.2 (Harmonic)**.** Let $h : R^2 \to R$. h is harmonic if

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

N. B. 2.1. For holomorphic function $f = u(x, y) + iv(x, y)$. Both u and v are harmonic. We say v is harmonic conjugate of u.

Definitio 2.3 (Some Holomorphic Functions)**.**

1. $\sin z = \frac{\exp iz - \exp -iz}{2i}$
2. $\cos z = \frac{\exp iz + \exp -iz}{2}$
3. $\sinh z = \frac{\exp z - \exp -z}{2}$
4. $\cosh z = \frac{\exp z + \exp -z}{2}$

N. B. 2.2. Note that $\tan z = \frac{\sin z}{\cos z}$, $\sec z = \frac{1}{\cos z}$, $\csc z = \frac{1}{\sin z}$, $\cot z = \frac{1}{\tan z}$, $\tanh z = \frac{\sinh z}{\cosh z}$, $\text{sech} z = \frac{1}{\cosh z}$, $\text{csch} z = \frac{1}{\sinh z}$, $\coth z = \frac{1}{\tanh z}$

Also note that $\sinh(iz) = i\sin(z)$

Definitio 2.4 (Logarithm)**.**

1. $\text{Log} z = \log|z| + i\text{Arg}(z)$
2. $\log z = \log|z| + i\arg(z)$

Note that $\log z = \{w : \exp(w) = z\} = \{\ln|z| + i\theta + i2\pi\}$

Let us define $\phi < \text{Arg}_\phi(z) \leq \phi + 2\pi$, and $\text{Log}_\phi = \ln|z| + i\text{Arg}_\phi(z)$.

Definitio 2.5 (Complex Power)**.** $z^a = \exp wa$, where $w \in \log(z)$

Definitio 2.6 (Branch Cut)**.** Branch cut is a subset of $\mathbb{C}$: $L_{z_0,\phi} = \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \geq 0\}$. I.e., a ray starting at z_0 with argument ϕ.

Notice that Log_ϕ is holomorphic on $\mathbb{C} \backslash L_{0,\phi}$

N. B. 2.3. $\mathrm{Log}(z-1)$ is holomorphic on $\mathbb{C}\backslash L_{1,-\pi}$ $\log(z^2-1) = \log(z+1)+\log(z-1)$. We can pick two diffent branch of log.

Theorema 2.2 (Conformal Map)**.** *A map is conformal if it preserves angle and orientation.*

A holomorphic map is conformal in domain D if $f' \neq 0$ in D.

Theorema 2.3 (Mobius Transformation)**.** *Mobius transformation is function of the form $f(z) = \frac{az+b}{cz+d}$, where $ad \neq bc$. f_M can be represented by matrix of SL_2 such that $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Notice that $f_{MN} = f_M \circ f_N$*

Mobius Transformation is conformal, and map circline to circlines.

Mobius Transformation can be deconstructed into four kinds:

1. ***translation**: $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$*
2. ***dilation**: $\begin{bmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{bmatrix}$*
3. ***rotation**: $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$*
4. ***inversion**: $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$*

Theorema 2.4 (Cross Ratio)**.**

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

Mobius Transformation preserves cross ratio: $[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$.

Let g be the unique mobius transformation that maps z_2, z_3, z_4 to $1, 0, \infty, x$. Then $g(z_1) = [z_1, z_2, z_3, z_4]$

Theorema 2.5 (Riemman Sphere)**.** *Projection of z onto Riemann sphere is: $\phi(z) = \phi(x+iy) = (2x/(r+1), 2y/(r+1), (r-1)/(r+1))$ for $r = |z|$. $\phi^{-1} = \frac{X+iY}{1-Z}$*

3 Integral

Theorema 3.1 (Path Independent Lemma)**.** *Let $D \in \mathbb{C}$ be a domain and f continuous on D. The followings are equivalent:*

1. *f has an antiderivative on D*
2. *$\int_\gamma f(z)dz = 0$ for all closed path γ in D*
3. *All contour integral of f is path independent (only depends on endpoint.)*

Theorema 3.2 (Cauchy Integral Theorem)**.** *Let f be holomorphic on the loop γ and inside the loop. Then $\int_\gamma f(z)dz = 0$*

Theorema 3.3 (Cauchy Integral Formula)**.** *Let f be holomorphic on the loop γ and inside the loop. Let z be in the interior of the loop. Then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw$$

Also:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} dw$$

That is $\int_\gamma \frac{1}{z-z_0} dz = 2\pi i$ if z_0 is inside γ.

3.1 Properties of Holomorphic Functions

Theorema 3.4. *Let f be holomorphic on D and $z_0 \in D$. We have $f(z_0)2\pi = \int_0^{2\pi} f(z_0 + Re^{iz})dz$*

Theorema 3.5 (Liouville's Theorem)**.** *Let f be holomorphic and bounded on $\mathbb{C}$. Then f is constant.*

Theorema 3.6 (Maximum Modulus Principle)**.** *Let f be holomorphic on a domain D. Then $|f(z)|$ has no maximum in D unless f is constant.*

Theorema 3.7 (Maximum Modulus Principle for Harmonic Function)**.** *Let $D \subset \mathbb{R}^2$ be a domain and ϕ be harmonic. If ϕ is bounded above or below by $M \neq 0$, then it is constant.*

Theorema 3.8 (Morera's Theorem)**.** *Let f be continuous on a domain D. If $\int_\gamma f(z)dz = 0$ for all closed path γ in D, then f is holomorphic.*

Theorema 3.9 (Open Mapping Theorem)**.** *Let f be holomorphic on a domain D. If f is not constant, then $f(D)$ is open.*

Theorema 3.10 (Identity Theorem)**.** *Let f and g be holomorphic on a domain D. If $f = g$ on a set with a limit point in D, then $f = g$ on D.*

4 Series

Theorema 4.1 (Weierstrass M-Test)**.** *Let f_n be a sequence of functions on D such that $|f_n(z)| \leq M_n$ for all $z \in D$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on D.*

Theorema 4.2 (Laurent Series)**.** *Let f be holomorphic on $A_{r,R}(z_0)$, then f can be represented by Laurent Series for any loop γ in $A_{r,R}(z_0)$:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} dw$

4.1 Zero and Singularities

Definitio 4.1 (Zero)**.** Let f be holomorphic on D. z_0 is a zero of f if $f(z_0) = 0$. z_0 is a zero of order n if $f(z_0) = f^1(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$.

All zeros of finite order are isolated.

Definitio 4.2 (Singularity)**.** Let f be holomorphic on $A_{0,R}(z_0)$ but not holomorphic at z_0. Suppose $f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$. Then z_0 is ***Removable Singularity*** if $a_j = 0$ for all $j < 0$. ***Pole*** of order n if $a_j = 0$ for all $j < -n$ and $a_{-n} \neq 0$. ***Essential Singularity*** if $a_j \neq 0$ for infinitely many $j < 0$.

5 Residue Calculus

Definitio 5.1. Residue Let f be holomorphic on $A_{0,R}(z_0)$, possible not at z_0. Then the residue of f at z_0 is a_{-1} (coefficient of $(z - z_0)^{-1}$ term) in the Laurent Series of f at z_0. It is denoted as $\text{Res}(f, z_0)$.

Theorema 5.1 (Calculating Residue)**.** *Let $z_0 \in \mathbb{C}$, and f holomorphic on the puctured disk, with pole of order m at z_0. Then*

$$Res(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z)$$

Moreover, if $f = \frac{g}{h}$ and f as a simple pole, then

$$Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

Theorema 5.2 (Cauchy Residue Theorem)**.** *Let γ be a loop, and f has finite singularities $z_1, \cdots z_k$ in the interior of the loop. Then*

$$\int_\gamma f = 2\pi i \sum Res(f, z_i)$$

6 Meromorphic Function

Definitio 6.1 (Meromorphic Function)**.** A function f is meromorphic on D if for every $z \in D$, f is holomorphic on z or f has a zero of finite order.

Theorema 6.1 (Argument Principle)**.** *Let γ be a loop in $\mathbb{C}$, f meromorphic on interior of γ and holomorphic and non-zero on γ, then we have:*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = N_0(f) - N_\infty(f)$$

Theorema 6.2 (Rouche's Theorem)**.** *Loet γ be a loop, and f, g holomorphic on and inside γ, such that $|f(z) - g(z)| \leq |f(z)$. We have $N_0(f) = N_0(g)$.*

6.1 Techniques of Integration

Theorema 6.3 (Trignometry Integration)**.** *For $R(\cos(\theta), \sin(\theta))$, define $f(z) = \frac{1}{iz}R(\frac{z+1/z}{2}, \frac{z-1/z}{2i})$ We have $\int_{C_1(0)} f(z)dz = \int_0^{2\pi} R(\cos\theta, \sin(\theta))$.*

Theorema 6.4 (Jordan's Lemma)**.** *Let P, Q be polynomial and $deg(Q) \leq deg(P) + 1$. We have*

$$\lim_{R\to\infty}\int_{C_R^+}\frac{P(z)}{Q(z)}e^{iaz}dz = 0 \textit{ if } a > 0$$

$$\lim_{R\to\infty}\int_{C_R^-}\frac{P(z)}{Q(z)}e^{iaz}dz = 0 \textit{ if } a < 0$$

Theorema 6.5 (Partial Circle)**.** *Let f be meromorphic on D with poles at c. Let S be a partial circle with center c, parametrize by $c + Re^{i\theta}$, for $\alpha \leq \theta \leq \beta$. Then we have*

$$\int_S f(z)dz = i(\beta - \alpha)Res(f, c)$$

6.1.1 P/Q

Consider Integral of the form $\int_{-\infty}^{\infty}\frac{P}{Q}$, where $deg(Q) - deg(P) \geq 2$. Let C_R be the close contour consists of line segment L, from $-R$ to R, and upper half circleσ, of radius R.

It can be shown that $\int_{C_R}\frac{P}{Q} \to 0$ as $R \to \infty$. Thus $\int_{-\infty}^{\infty}\frac{P}{Q} = 2\pi i \sum \text{Res}(f, c)$, where c are poles of f in the upper half plane.

6.1.2 half line

For $deg(Q) - deg(P) \geq 2$, $\int_0^\infty \frac{P}{Q} = -\sum \text{Res}(\log(z)P/Q, z_k)$, for all poles z_k.

6.1.3 R(x)sin(x)

The integral $\int_{-\infty}^{\infty} R(x)\sin(x)$, where $R = P/Q$ and $deg(Q) > deg(P)$, can be evaluated as the imaginary part of $\int_{-\infty}^{\infty} R(x)e^{ix}$.

6.2 Series

$\phi(z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)}$ has simple pole at $z = n$, for $n \in \mathbb{Z}$, and residue 1, $\xi(z) = \pi\frac{1}{\sin(\pi z)}$ has residue $(-1)^n$ at $z = n$.

$\sum_{n=-\infty}^{\infty} f(n) = -\sum \text{Res}(f(z)\pi\cot(\pi z), z)$, for all poles z, and $\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\sum \text{Res}(f(z)\pi\csc(\pi z), z)$, for all poles z.

$C(n, k) = \frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}$, for simple curve C enclosing $z = 0$.