Notes on Pure Mathematics

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Abstract

This is my notes when taking the class *Fundamentals of Pure Mathematics* at the University of Edinburgh. These notes do not follow the course lecture notes; instead, they are my thoughts and works inspired by the class. Terms like "Theorem, Proposition" are coined in Latin. Here is the list: theorema for theorem, Corollarium for corollary, Propositio for proposition, Definitio for definition, Conietura for hypothesis (conjecture).

Chapter 1

Notation

- The `\mathbb{}` fonts are used to denote sets. ($\mathbb{S}, \mathbb{Y}$, etc.)
- $\mathbb{A} \succ \mathbb{B}$ denotes there exits a surjective function $f : A \to B$. $\prec$, $\asymp$ denotes injective, bijective, respectively.
- e is used to denote the identity of a group.
- When there is no ambiguity, the notation for the operation of group is ommited. (i.e., $a \odot b = ab$). a^{-1} is used to denote the inverse of a.

Chapter 2

Analysis

Axioma 2.0.1 (The "Smallest" Infinite Set)**.** A set $\mathbb{S}$ is infinite iff $\mathbb{S} \succ \mathbb{N}$.

Observatio 2.0.1. Although FPM is a pure mathematic class with emphasis on rigor, no rigorous definition for the infinite set has been proposed. This definition/axiom is of my own conception.

Definitio 2.0.1 (Countable Set)**.** A set $\mathbb{S}$ is countable iff $\mathbb{N} \asymp \mathbb{S}$ (there exists a bijection $f : \mathbb{N} \to \mathbb{S}$).

Corollarium 2.0.1 (At Most Countable)**.** *Let $\mathbb{A}$ be an infinite set.*
($\mathbb{A} \prec \mathbb{N}$) iff ($\mathbb{A} \asymp \mathbb{N}$).

Proof. We want to prove $\mathbb{A} \prec \mathbb{N}$ is equivalent to $\mathbb{A} \asymp \mathbb{N}$. $\mathbb{A} \asymp \mathbb{N} \to \mathbb{A} \prec \mathbb{N}$ is by definition. We only need to prove the other direction; i.e., provided $\mathbb{A} \prec \mathbb{N}$, find a bijective function $h : \mathbb{A} \to \mathbb{N}$.

Let $f : \mathbb{A} \to \mathbb{N}$ be an injective mapping. If f is bijective, we are done. If f is injective but not bijective, let $\mathbb{N}^-$ be the range of f. As $\mathbb{A}$ is infinite, $\mathbb{N}^-$ is also infinite. Let $f' : \mathbb{A} \to \mathbb{N}^-$ such that $f(a) = f'(a)$. f' is an bijective mapping.

Thus we only need to show there exists a mapping $g : \mathbb{N}^- \to \mathbb{N}$ that is bijective.

g can be constructed by such: sort $\mathbb{N}^-$ and $\mathbb{N}$ in ascending order. Let the first element in the sorted $\mathbb{N}^-$ maps to the first in the sorted $\mathbb{N}$, the secound to secound, etc. As $\mathbb{N}^-$ is infinite, g must be bijective.

Indeed $h = f' \circ g : \mathbb{A} \to \mathbb{N}$ is the bijective mapping we seek. Q.E.D.

Theorema 2.0.1 (List of Countable and Uncountable Sets)**.** *Any of the following sets are countable.*

1. *$\mathbb{Z}$ and any of its infinite subsets.*

2. $\mathbb{R}$ *and any of its infinite subsets.*

3. *If* $\mathbb{S}$ *is countable,* $\{\mathbb{S} \times \mathbb{S}\}, \{\mathbb{S} \times \mathbb{S} \times \cdots \times \mathbb{S})\}$ *are also countable.*

Chapter 3

Algebra

Definitio 3.0.1 (Group)**.** Group is a set $\mathbb{S}$ with an operation $\odot$ that fulfills the following four properties:

1. Closure
2. Associtivity: $(a \odot b) \odot c = a \odot (b \odot c)$;
3. Identity
4. Inverse

Theorema 3.0.1 (Consequence of the Definition)**.** *There are many non-obvious properties that directly follows the definition.*

1. *General Associtivity: Parenthesis does not matter, as long as the order is the same:* $a \odot b \odot c \odot d \odot e \odot f \odot g \cdots = (a \odot ((b \odot c) \odot e(\odot f \odot g) \cdots) = \cdots$
2. *Order of Inverse:* $(a \odot b)^{-1} = b^{-1} \odot a^{-1}$.

Here are some examples of groups.

1. $\mathbb{S} = \{e\}$
2. $\mathbb{S} = \{e, a, b, c\}$. With the following operation: 1. All elements are their own inverse; 2. The group is abelian. 2. $a \odot b = c, a \odot c = b, b \odot c = a$.

Coniectura 3.0.1. These are some of my hypothesis and thoughts.

1. different properties of odd finite groups and even finite groups
2. If defining the reverto of the operation $\odot$ to be $\oslash$ as such: $a \odot b = a \oslash b^{-1}$. What are the sets such that it would be a group under both $\odot \& \oslash$?

3. Can we have a set $\mathbb{S}$, such that under the operation $\odot$ we have $\forall a, b \in \mathbb{S}, a \odot b = b \odot a$ but without associtivity? (Community without associtivity?)

Definitio 3.0.2 (Order of Group and element)**.** The order of the group $\mathbb{S}$ is $|\mathbb{S}|$ (How many elements it has).
The order of an element $s \in \mathbb{S}$ is the smallest integer i such that $s^i = e$. (If such i exists)

Definitio 3.0.3 (Cyclic Group)**.** Let $\mathbb{G}$ be a group and g one of its element. Considering the set:

$$\mathbb{S} = \{\cdots g^{-2}, g^{-1}, e, g, g^1, g^2 \cdots\}$$

If $\mathbb{S}$ is finite, it is called a cyclic group. (It can be shown that it must be a subgroup of $\mathbb{G}$.