

# Notes on Pure Mathematics

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February 2, 2023

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## **Abstract**

This is my notes when taking the class *Fundamentals of Pure Mathematics* at the University of Edinburgh. These notes do not follow the course lecture notes; instead, they are my thoughts and works inspired by the class.

# Chapter 1

## Notation

- The `\mathbb{}` fonts are used to denote sets. ( $\mathbb{S}$ ,  $\mathbb{Y}$ , etc.)
- $\mathbb{A} \succ \mathbb{B}$  denotes there exists a surjective function  $f : A \rightarrow B$ .  $\prec$ ,  $\asymp$  denotes injective, bijective, respectively.
- $e$  is used to denote the identity of a group.
- When there is no ambiguity, the notation for the operation of group is omitted. (i.e.,  $a \odot b = ab$ ).  $a^{-1}$  is used to denote the inverse of  $a$ .

# Chapter 2

## Analysis

**Axiom 2.0.1** (The "Smallest" Infinite Set). A set  $\mathbb{S}$  is infinite iff  $\mathbb{S} \succ \mathbb{N}$ .

*Remark 2.0.1.* Although FPM is a pure mathematic class with emphasis on rigor, no rigorous definition for the infinite set has been proposed. This definition/axiom is of my own conception.

**Definition 2.0.1** (Countable Set). A set  $\mathbb{S}$  is countable iff  $\mathbb{N} \asymp \mathbb{S}$  (there exists a bijection  $f : \mathbb{N} \rightarrow \mathbb{S}$ ).

**Corollary 2.0.1** (At Most Countable). *Let  $\mathbb{A}$  be an infinite set.  $(\mathbb{A} \prec \mathbb{N})$  iff  $(\mathbb{A} \asymp \mathbb{N})$ .*

*Proof.* We want to prove  $\mathbb{A} \prec \mathbb{N}$  is equivalent to  $\mathbb{A} \asymp \mathbb{N}$ .  $\mathbb{A} \asymp \mathbb{N} \rightarrow \mathbb{A} \prec \mathbb{N}$  is by definition. We only need to prove the other direction; i.e., provided  $\mathbb{A} \prec \mathbb{N}$ , find a bijective function  $h : \mathbb{A} \rightarrow \mathbb{N}$ .

Let  $f : \mathbb{A} \rightarrow \mathbb{N}$  be an injective mapping. If  $f$  is bijective, we are done. If  $f$  is injective but not bijective, let  $\mathbb{N}^-$  be the range of  $f$ . As  $\mathbb{A}$  is infinite,  $\mathbb{N}^-$  is also infinite. Let  $f' : \mathbb{A} \rightarrow \mathbb{N}^-$  such that  $f(a) = f'(a)$ .  $f'$  is an bijective mapping.

Thus we only need to show there exists a mapping  $g : \mathbb{N}^- \rightarrow \mathbb{N}$  that is bijective.

$g$  can be constructed by such: sort  $\mathbb{N}^-$  and  $\mathbb{N}$  in ascending order. Let the first element in the sorted  $\mathbb{N}^-$  maps to the first in the sorted  $\mathbb{N}$ , the second to second, etc. As  $\mathbb{N}^-$  is infinite,  $g$  must be bijective.

Indeed  $h = f' \circ g : \mathbb{A} \rightarrow \mathbb{N}$  is the bijective mapping we seek. Q.E.D.

**Theorem 2.0.1** (List of Countable and Uncountable Sets). *Any of the following sets are countable.*

1.  $\mathbb{Z}$  and any of its infinite subsets.

2.  $\mathbb{R}$  and any of its infinite subsets.

3. If  $\mathbb{S}$  is countable,  $\{\mathbb{S} \times \mathbb{S}\}, \{\mathbb{S} \times \mathbb{S} \times \cdots \times \mathbb{S}\}$  are also countable.

# Chapter 3

## Algebra

**Definition 3.0.1** (Group). Group is a set  $\mathbb{S}$  with an operation  $\odot$  that fulfills the following four properties:

1. Closure
2. Associativity:  $(a \odot b) \odot c = a \odot (b \odot c)$ ;
3. Identity
4. Inverse

**Theorem 3.0.1** (Consequence of the Definition). *There are many non-obvious properties that directly follows the definition.*

1. *General Associativity: Parenthesis does not matter, as long as the order is the same:*  $a \odot b \odot c \odot d \odot e \odot f \odot g \cdots = (a \odot ((b \odot c) \odot e (\odot f \odot g) \cdots)) = \cdots$
2. *Order of Inverse:*  $(a \odot b)^{-1} = b^{-1} \odot a^{-1}$ .

Here are some examples of groups.

1.  $\mathbb{S} = \{e\}$
2.  $\mathbb{S} = \{e, a, b, c\}$ . With the following operation: 1. All elements are their own inverse; 2. The group is abelian. 2.  $a \odot b = c, a \odot c = b, b \odot c = a$ .

*Hypothesis 1.* These are some of my hypothesis and thoughts.

1. different properties of odd finite groups and even finite groups
2. If defining the revert of the operation  $\odot$  to be  $\oslash$  as such:  $a \odot b = a \oslash b^{-1}$ . What are the sets such that it would be a group under both  $\odot$  &  $\oslash$ ?
3. Can we have a set  $\mathbb{S}$ , such that under the operation  $\odot$  we have  $\forall a, b \in \mathbb{S}, a \odot b = b \odot a$  but without associativity? (Community without associativity?)