

DE MATHEMATICA PURA
On Pure Mathematics

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Index Capitum

1	Notation	1
2	Analysis	2
2.1	Real Number	2
2.1.1	Axioms	2
2.1.2	The Countable Sets	2
2.2	Sequence and Series	3
2.2.1	Sequence	3
2.2.2	Series	4
2.2.3	Interstring Sequences and Series	7
2.2.4	Decimal Expansion	7
2.3	Real Functions	7
2.3.1	Continuity	7
2.4	Limits of Functions	9
2.4.1	Bizare Functions	10
2.5	Differentiability	11
3	Algebra	12
3.1	Group: Definition	12
3.2	Between Groups	16
3.2.1	Product and Polymorphism	17
3.3	Group Actions	17
I	On Polynomials	19
II	Latin and Abbreviations	20
III	Chronology of Proposed, Proved, and Disproved Hypotheses	21

Abstract

These are my notes when taking the class *Fundamentals of Pure Mathematics* at the University of Edinburgh. They are not a replicate of the lecture notes: they are my thoughts and explorations. Most importantly, all proofs presented in this document are of my own conception.

Terms like “Theorem, Proposition” are coined in Latin. As the English terms descended from Latin, most of them are self-explanatory.

Caput 1

Notation

- The `\mathbb{}` fonts are used to denote sets. (\mathbb{S} , \mathbb{Y} , etc.)
- $\mathbb{A} \succ \mathbb{B}$ denotes there exists a surjective function $f : \mathbb{A} \rightarrow \mathbb{B}$. \prec , \asymp denotes injective, bijective, respectively.
- e is used to denote the identity of a group.
- When there is no ambiguity, the notation for the operation of group is omitted. (i.e., $a \odot b = ab$). a^{-1} is used to denote the inverse of a .
- $\mathbb{H} \leq \mathbb{S}$ denotes that \mathbb{H} is a subgroup of \mathbb{S} . If $\mathbb{H} \neq \mathbb{S}$, it is a proper subgroup and is denoted as $\mathbb{H} < \mathbb{S}$. See [definition](#).
- Sequence and series are denoted as (s_n) and $\sum_{k=1}^{\infty} s_k$ respectively.
- $\mathcal{L}_s(s_n), \mathcal{L}_s(s_n)$ is the limit of supremum & infimum. See [definition 2.2.4](#).

Caput 2

Analysis

2.1 Real Number

2.1.1 Axioms

Axioma 2.1.1 (Archimedean Property). $\forall r \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > r$.

Axioma 2.1.2 (The Completeness of Real Number). Let $\mathbb{D} \subseteq \mathbb{R}$. If \mathbb{D} is bounded, there exists $s \& i, \in \mathbb{R}$ such that they are the supremum and infimum of \mathbb{D} .

2.1.2 The Countable Sets

Axioma 2.1.3 (The "Smallest" Infinite Set). A set \mathbb{S} is infinite iff $\mathbb{S} \succ \mathbb{N}$.

Observatio 2.1.1. Although FPM is a pure mathematic class with emphasis on rigor, no rigorous definition for the infinite set has been proposed. This definition/axiom is of my own conception.

Definitio 2.1.1 (Countable Set). Another A set \mathbb{S} is countable iff $\mathbb{N} \asymp \mathbb{S}$ (there exists a bijection $f : \mathbb{N} \rightarrow \mathbb{S}$).

Theorema 2.1.1 (At Most Countable). *Let \mathbb{A} be an infinite set.*
 $(\mathbb{A} \prec \mathbb{N})$ iff $(\mathbb{A} \asymp \mathbb{N})$.

Demonstratio. We want to prove $\mathbb{A} \prec \mathbb{N}$ is equivalent to $\mathbb{A} \asymp \mathbb{N}$. $\mathbb{A} \asymp \mathbb{N} \rightarrow \mathbb{A} \prec \mathbb{N}$ is by definition. We only need to prove the other direction; i.e., provided $\mathbb{A} \prec \mathbb{N}$, find a bijective function $h : \mathbb{A} \rightarrow \mathbb{N}$.

Let $f : \mathbb{A} \rightarrow \mathbb{N}$ be an injective mapping. If f is bijective, we are done. If f is injective but not bijective, let \mathbb{N}^- be the range of f . As \mathbb{A} is infinite, \mathbb{N}^-

is also infinite. Let $f' : \mathbb{A} \rightarrow \mathbb{N}^-$ such that $f(a) = f'(a)$. f' is an bijective mapping.

Thus we only need to show there exists a mapping $g : \mathbb{N}^- \rightarrow \mathbb{N}$ that is bijective.

g can be constructed by such: sort \mathbb{N}^- and \mathbb{N} in ascending order. Let the first element in the sorted \mathbb{N}^- maps to the first in the sorted \mathbb{N} , the second to second, etc. As \mathbb{N}^- is infinite, g must be bijective.

Indeed $h = g \circ f' : \mathbb{A} \rightarrow \mathbb{N}$ is the bijective mapping we seek. Q.E.D.

Theorema 2.1.2 (List of Countable and Uncountable Sets). *Any of the following sets are countable.*

1. \mathbb{Z}, \mathbb{Q}
2. *Any infinite subset of countable sets.*
3. *Any Unions of countable and finite sets.*
4. *Any products of countable sets and finite sets. i.e., if \mathbb{S}, \mathbb{T} are countable, $\{\mathbb{S} \times \mathbb{S}\}, \{\mathbb{S} \times \mathbb{T} \times \cdots \times \mathbb{S}\}$ are also countable.*

Coniectura 2.1.1. Is the product of countable number of countable sets countable? (Proposed Feb 6)

2.2 Sequence and Series

2.2.1 Sequence

Definitio 2.2.1 (Sequence).

Definitio 2.2.2 (Convergent and Divergent).

Definitio 2.2.3 (Increasing and Decreasing Sequence(Monotone)).

Definitio 2.2.4 (Limit of supremum & infimum). For a sequence (s_n) , let b_i denotes the supremum of $\{s_n | n > i\}$. If (b_n) converges, the value it converges to is called the limit of supremum of (s_n) , and is denoted as $\mathcal{L}_s(s_n)$. (b_n) is called the supremum sequence. Similarly infimum sequence and limit of infimum are defined, and the later denoted as $\mathcal{L}_i(s_n)$.

Observatio 2.2.1. Notice supremum and infimum sequences are monotone.

Theorema 2.2.1 (Convergence and Limit of supremum & infimum). *A sequence (s_n) converges if and only if $\mathcal{L}_s(s_n) = \mathcal{L}_i(s_n)$. (Proposed Feb 8 2023, proved Feb 9)*

Demonstratio. We want to prove that $(\mathcal{L}_i(s_n) = \mathcal{L}_s(s_n)) \iff (s_n)$ converges.

Forward direction: We shall show that $\lim_{n \rightarrow \infty} (s_n) = \mathcal{L}_s(s_n) = \mathcal{L}_i(s_n) = \lambda$. $\forall \epsilon > 0$, we know by our assumption that $(\exists N \in \mathbb{N})(\forall n > N)$ the set $\{s_n | n > N\}$ is bounded by $\lambda \pm \epsilon$. This is the definition for the convergent sequence.

We shall prove the contraposition of the backwards direction, i.e. $(\mathcal{L}_i(s_n) \neq \mathcal{L}_s(s_n)) \rightarrow (s_n)$ diverges. The contraposition can be proved by contradiction.

Assuming $(\lambda = \mathcal{L}_i(s_n) \neq \mathcal{L}_s(s_n))$ and (s_n) converges to l . S.D.U., let $\lambda > l$. Let $\epsilon = (\lambda - l)/2$. Since (s_n) converges to l , there exists $N \in \mathbb{N}$ such that $\forall n > N$, $|s_n - l| < \epsilon$. However, we know that $\mathcal{L}_i(s_n) = \lambda$, which means that there exists N' such that $\forall n > N'$ we have at least one element $s_i > \lambda - \epsilon$. Indeed $s_i - l > \epsilon$, contradicting with our assumption that (s_n) converges. Thus we conclude the backwards direction is also true. Q.E.D.

Definitio 2.2.5 (Cauchy Sequence). [1] A sequence (s_n) is a Cauchy Sequence iff $(\forall \epsilon > 0)(\exists N)(\forall n, m > N)(|s_n - s_m| < \epsilon)$

Theorema 2.2.2. A sequence converges if and only if it is a Cauchy Sequence.

Observatio 2.2.2. We are to outline our proof of (s_n) converges $\iff (s_n)$ is Cauchy Sequence.

The forward direction is obvious. To prove the backwards direction, notice: 1) All Cauchy Sequences are bounded; 2) the infimum and supremum sequence converge by monotone convergence theorem; 3) They must converge to the same value; 4) By theorem 2.2.1 the sequence must converge.

Observatio 2.2.3. We can define a pseudo Cauchy Sequence to be sequence (s_n) such that $(\forall \epsilon > 0)(\exists N)(\forall n > N)(|s_n - s_{n+1}| < \epsilon)$. Indeed all convergent sequence are pseudo Cauchy Sequence, but not all pseudo Cauchy Sequence are convergent. An example is the partial sum of harmonic series, i.e., $(\sum_{i=1}^n \frac{1}{i})$.

2.2.2 Series

Definitio 2.2.6 (Series). A series can be expressed as $\sum_{k=1}^{\infty} a_k$.

Definitio 2.2.7 (Convergent and Divergent). Consider the series : $(s_n) = \sum_{k=1}^n a_k$. (s_n) is called the partial sum of the series. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if its partial sum converges; otherwise it diverges.

Exempli Gratia 2.2.1. List of Convergent and Divergent series:

1. Harmonic Series.

Definitio 2.2.8 (Cauchy Criterion). A series befits Cauchy Criterion if and only if its partial sum is a Cauchy Sequence.

Definitio 2.2.9 (Absolute Convergent). A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if $\sum_{k=1}^{\infty} |a_k|$ converges. Otherwise it converges non-absolutely

Theorema 2.2.3 (Convergence Reveries).

1. For convergent series $\sum_{k=1}^{\infty} s_k$, $\sum_{k=1}^{\infty} s'_k$, and constant c , all of the following sequence converges: $\sum_{k=1}^{\infty} -s_k$, $\sum_{k=1}^{\infty} c \cdot s_k$, $\sum_{k=1}^{\infty} s_k + s'_k$, $\sum_{k=1}^{\infty} s_k \cdot s'_k$.
In particular, $\sum_{k=1}^{\infty} \frac{1}{s_k}$ diverges.
 $\sum_{k=1}^{\infty} \frac{s_k}{s'_k}$ may diverge or converge.
2. **Absolute Convergent:**
If a series converges absolutely, it converges. The converse is not true.
3. **Comparison Test:**
For convergent series $\sum_{k=1}^{\infty} s_k$, if $|b_k| \leq s_k$ for all k , $\sum_{k=1}^{\infty} b_k$ converges.
For divergent series $\sum_{k=1}^{\infty} d_k = \infty$, if $e_k \geq d_k$ for all k , $\sum_{k=1}^{\infty} e_k$ diverges. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converges, the followings also converge: $\sum_{k=1}^{\infty} (a_k + b_k)$, $\sum_{k=1}^{\infty} (a_k - b_k)$, $\sum_{k=1}^{\infty} (a_k \cdot b_k)$
4. **Ratio Test:**
For series $\sum_{k=1}^{\infty} s_k$, let $d = \lim_{k \rightarrow \infty} \left| \frac{s_k}{s_{k-1}} \right|$.
If $d < 1$, the series converges absolutely.
If $d > 1$, the series diverges.
If $d = 1$, the series may converge or diverge.
5. **Root Test:**
For series $\sum_{k=1}^{\infty} s_k$, let $d = \lim_{k \rightarrow \infty} |(s_k)^{1/k}|$.
If $d < 1$, the series converges absolutely.
If $d > 1$, the series diverges.
If $d = 1$, the series may converge or diverge.
6. **Alternating Series Test:**
For series in the form $\sum_{k=1}^{\infty} (-1)^k s_k$. If s_k is decreasing and $\lim_{k \rightarrow \infty} s_k = 0$, the series converge. (Copied from textbook on 14 Feb 2023, not proved.)

7. Cauchy's Condensation Test:

Consider series $\sum_{k=1}^{\infty} s_k$. If s_k is decreasing and greater than zero, the series converge if and only if $\sum_{k=1}^{\infty} s_{2^k} 2^k$ converges.

8. Integral Test For $s_k > 0$, the series $\sum_{k=1}^{\infty} s_k$ converge if and only if $\int_a^{\infty} S(k)dk$ converge for some constant a , provided $\forall k \in \mathbb{N}, S(k) = s_k$.
(Proposed Feb 14 2023, modified and proved 16 Feb)

9. Raabe's Test For series $\sum_{k=1}^{\infty} s_k$, let $l = n \left(1 - \frac{s_{n+1}}{s_n}\right)$. The series converge if $l > 1$, diverges if $l < 1$, and is inconclusive if $l = 1$.

Coniectura 2.2.1 (Inspired from the Integral Test). If the finite integral, with some constant a , $\int_a^{\infty} f(k)dk$, converges for function f , $\lim_{n \rightarrow \infty} \sum_{i=0}^n f(\Delta x_i i + a)$ converge for $\Delta x_i \in \mathbb{R}$, provided $\{\Delta x_i\}$ is bounded. (Proposed 15 Feb 2023)

Demonstratio.

- To prove 2 of theorem 2.2.3, Consider the convergent series $\sum_{k=1}^{\infty} |a_k|$. Split $\sum_{k=1}^{\infty} a_k$ into $\sum_{k=1}^{\infty} p_k$ and $\sum_{k=1}^{\infty} n_k$, where p_k, n_k are positive and negative, respectively. (We can safely ignore any 0)
As $\sum_{k=1}^{\infty} p_k \leq \sum_{k=1}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty} n_k \geq -\sum_{k=1}^{\infty} |a_k|$, both series are bounded. By Monotone convergence theorem, both series converge. Thus $\sum_{k=1}^{\infty} a_k$, as the sum of two convergent series, must converge.
- Entry NO. 7, Cauchy's Condensation Test, has two directions: for decreasing and positive s_k , $\sum_{k=1}^{\infty} s_k$ converges $\iff \sum_{k=1}^{\infty} s_{2^k} 2^k$ converges.

To prove the forward direction, consider the convergent series:

$$2 \cdot \sum_{k=1}^{\infty} s_k = 2 \cdot s_1 + 2 \cdot (s_2 + s_3) + 2 \cdot (s_4 + s_5 + s_6 + s_7) \cdots$$

And

$$\sum_{k=1}^{\infty} s_{2^k} 2^k = s_1 + \underbrace{2 \cdot s_2}_{< 2 \cdot s_1} + \underbrace{4 \cdot s_4}_{< 2 \cdot (s_2 + s_3)} + \underbrace{8 \cdot s_8}_{< 2 \cdot (s_4 + s_5 + s_6 + s_7)} + \cdots \quad (2.1)$$

Thus by comparison test we conclude (2.1) converges.

The backwards direction directly follows the comparison test as

$$\sum_{k=1}^{\infty} s_{2^k} 2^k \geq \sum_{k=1}^{\infty} s_k.$$

- Here we present an informal proof for 2.2.3.8, the integral test with an extra restriction that the function is strictly decreasing. (15 Feb 2023)

Consider the function S with the property $\int_a^\infty S(k)dk$ converges for some constant a and its correspondent series $\sum_{k=1}^\infty S(k)$. Consider the function $\sigma(x) = S(x-1)$ $\int_a^\infty \sigma(k)dk$ converges, and is greater than $\sum_{k=\lceil a \rceil}^\infty s_k$ (as the function is strictly decreasing), thus by comparison test it converges, thus $\sum_{k=1}^\infty s_k$, as the sum of a convergent series and a constant also converge.

Q.E.D.

2.2.3 Interstring Sequences and Series

Sequences

Series

1. $\sum_{n=1}^\infty \frac{n}{2^n} = 1$

2.2.4 Decimal Expansion

2.3 Real Functions

2.3.1 Continuity

Definitio 2.3.1 (Continuity of a Function). Function $f : \mathbb{D} \rightarrow \mathbb{R}$ (provided $\mathbb{D} \subseteq \mathbb{R}$) is continuous at a if and only if for all sequence (x_i) (provided $x_i \in \mathbb{D}$) that converges to a the sequence $(f(x_i))$ converges to $f(a)$.

If the function f is continuous for all $a \in \mathbb{A}$, we say f is continuous on the interval \mathbb{A} .

Theorema 2.3.1 (Equivalent Definition). *The definition 2.3.1 of function $f : \mathbb{D} \rightarrow \mathbb{R}$ is continuous at $a \in \mathbb{D}$ is equivalent to $\lim_{x \rightarrow a} f(x) = f(a)$, provided \mathbb{D} is a non-degenerate interval in \mathbb{R} ; i.e. $\forall \epsilon \in \mathbb{D}, \exists \delta$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.*

Definitio 2.3.2 (Extreme Value Theorem). $f : \mathbb{I} \rightarrow \mathbb{R}$ where \mathbb{I} is a closed interval of real number is bounded and would attain supremum and infimum on \mathbb{I} .

The following proof is wrong, see 2.4.1; but I think it may be remedied, so I will leave it here for now.

Demonstratio. We shall first show that a continuous function whose domain is a closed interval must have finite number of local maxima, and thus bounded above. Its maximum is the maximum of the set consisted of all local maxima and its value at the end points.¹

In this proof let $f(x)$ be a continuous function defined in a closed interval $[l, h]$.

A real number $r = f(a)$ is a local maximum of function f if there exists $\delta > 0$ such that for all $x \in (a - \delta, a + \delta)$, $f(a) > f(x)$.

Moreover, we claim that for all local maxima $f(a)$ attained at a , there exists a interval (s, a) such that the function is strictly increasing in the interval (s, a) , but not so for any interval (s', a) with $s' < s$. There also exists another interval (a, d) such that the function is strictly decreasing in the interval but not so for any interval (a, d') with $d < d'$.

The interval (s, d) is called the characteristic interval of local maxima attained at a , and $\lambda = (d - a)$ is its characteristic length. We claim the characteristic intervals partitions $[l, h]$. Define function $\gamma : [l, h] \rightarrow \mathbb{R}$ as $\gamma(x) = \lambda$ if $x \in (a, d)$. Importantly, $\exists \beta > 0$ such that $\exists \zeta \in [x - \lambda, x + \lambda]$ and $|f(x) - f(\zeta)| > \beta$.

If $f(x)$ attains infinitely many local maxima, we claim that $\exists \alpha \in [l, h]$ such that $\lim_{x \rightarrow \alpha} \gamma(x) = 0$. Thus there exists $\beta > 0$ such that for all $\delta > 0$ there exists $0 < \lambda < \delta$ such that there exists $\zeta \in [\alpha - \lambda, \alpha + \lambda]$ and $|f(\alpha) - f(\zeta)| > \beta$, which means f is discontinuous at α , contradicting our assumption, and we can conclude that f is indeed bounded above.

So far we have proved that a function that is continuous in a closed interval must be bounded above. It is similar to prove that it is bounded below. Q.E.D.

Observatio 2.3.1. We have not restrict the domain of function f to be real number to show it is bounded, although it may be required if we need to show it would attains sup and inf.

Notice, extreme value theorem does not hold when \mathbb{I} is a open interval. For example, function $f(x) = \frac{1}{x}$ is continuous but unbounded in the open interval $(0, 1)$.

Coniectura 2.3.1 (Valid Domain for EVT). Investigate the valid domain for EVT under the definition of continuity.

¹Indeed a continuous function in a closed interval may have 0 local maximum or minimum: in such cases the statement still holds, though it requires more justifications.

Theorema 2.3.2 (Intermediate Value Theorem). *Let $f : \mathbb{D} \rightarrow \mathbb{R}$ be continuous on the interval $[a, b] \in \mathbb{D}$. $\forall c$ such that $f(a) < c < f(b)$ $\exists d \in [a, b]$ such that $f(d) = c$.*

Demonstratio.[Proof of Intermediate Value Theorem] Construct a set $\mathbb{E} = \{e \in [a, b] | f(e) < c\}$. Since $f(a) < c$, \mathbb{E} is non empty. As \mathbb{E} is bounded, by the completeness of real number, there exists a supremum, which shall be denoted as $\sup \mathbb{E} = s$. As \mathbb{E} is bounded by a close interval, s is also bounded by the same interval. Our aim is to show that $f(s) = c$.

By our assumption f is continuous at s . Assuming $f(s) < c$. Since f is continuous at s , there exists $\delta \in \mathbb{R}$ such that for all $s < x < s + \delta$, $|f(x) - f(s)| < c - f(s)$, i.e., $f(s) < f(x) < c$, contradicting our assumption that s is the supremum of \mathbb{E} . Thus we conclude $f(s) \geq c$. $f(s) \leq c$ can be proved similarly. Q.E.D.

Observatio 2.3.2. The sequence definition of continuity sets no restraints on the domains of the function. Indeed by this definition all discrete functions are continuous: contradicting our intuition about continuity.

The more important consequence of continuity relies heavily on the properties of real number, i.e, the completeness of real number.

2.4 Limits of Functions

Definitio 2.4.1 (Sequence Definition of Limit). Let $f : \mathbb{D} \rightarrow \mathbb{R}$ be a function. We say $\lim_{x \rightarrow a} f(x) = L$ if and only if for every sequence (x_n) , provided $x_i \in \mathbb{D}$, which converges to a we have $(f(x_n))$ that converges to L .

Definitio 2.4.2 (Sided Limit). Limit may be two sided or one sided.

1. Limit from left: $\lim_{x \rightarrow a^-} f(x) = L$ if and only if for every sequence (x_n) , provided $x_i \in \mathbb{D}$ and $x_i < a$, which converges to a we have $(f(x_n))$ that converges to L .
2. Limit from right: $\lim_{x \rightarrow a^+} f(x) = L$ if and only if for every sequence (x_n) , provided $x_i \in \mathbb{D}$ and $x_i > a$, which converges to a we have $(f(x_n))$ that converges to L .

Definitio 2.4.3 (Infinit Limit).

Theorema 2.4.1 (Limit is Unique). *Let $f : \mathbb{D} \rightarrow \mathbb{R}$ be a function.*

$\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$.

Theorema 2.4.2 ($\epsilon - \delta$ Definition). *The sequence definition of limit is equivalent to the $\epsilon - \delta$ definition.*

1. $\lim_{x \rightarrow a} f(x) = L$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x such that $|x - a| < \delta \implies |f(x) - L| < \epsilon$.
2. $\lim_{x \rightarrow a^-} f(x) = L$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x such that $a - \delta < x < a \implies |f(x) - L| < \epsilon$.
3. $\lim_{x \rightarrow a^+} f(x) = L$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x such that $a < x < a + \delta \implies |f(x) - L| < \epsilon$.

2.4.1 Bizare Functions

Coniectura 2.4.1.

1. Is there a function defined in close interval $[a, b]$ but is not strictly increasing nor decreasing for any interval in its domain?
2. Is there a function that is continuous everywhere but not differentiable at any point?
- 3.

A Continuous Function in Finite Domain with Infinite Local Maxima Define function $f : \mathbb{R} \rightarrow \mathbb{R}$ as such:

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It has infinitely many local maxima in the interval $[-1, 1]$.

Notice the function is continuous at 0. Moreover, for $\forall \delta > 0$, $[-\delta, \delta]$ contains infinitely many local maxima. Denominate points with such property as transedental points. It is possible to have a function defined in domain $[a, b]$ while at same time have $\forall \epsilon \in [a, b], \epsilon$ is a transedental point? Can such function be continuous?

Non Increasing Nor Decreasing Let function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as such:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

As infinitely many number of rational numbers and irrational number are contained in any domain $[a, b]$, such function is not increasing nor decreasing for any intervals of real number.

Unbounded Functions Commonly, unbounded functions are defined in a open interval, e.g., $\frac{1}{x}$ in the interval $(0, 1]$. By extreme value theorem we know that there is no continuous function in close interval that is unbounded. However, there is function that is non-continuous that shows unbounded behaviour in close interval.

Consider the function $f : [0, 1] \rightarrow (\infty, 1]$ defined as thus:

$$f(x) = \begin{cases} 2^{n+1}, & \text{if } x = 2^{-n} \text{ for some natural number } n \\ x, & \text{otherwise} \end{cases}$$

2.5 Differentiability

Definitio 2.5.1 (Differentiability). A real function f is differntiable at point $a \in \mathbb{R}$ if f is defined at some open interval containing a and

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (2.2)$$

exists. $f'(a)$ is called the derivative of f at the point a .

Theorema 2.5.1 (Differentiability Implies Continuity). *Let f be a real function. If f is differentiable at point a , then f is continuous at a .*

Demonstratio. aa

Q.E.D.

Caput 3

Algebra

3.1 Group: Definition

Definitio 3.1.1 (Group). Group is a set \mathbb{S} with an operation \odot that fulfills the following four properties:

1. Closure: $\forall a, b \in \mathbb{S}, a \odot b \in \mathbb{S}$.
2. Associativity: $\forall a, b, c \in \mathbb{S}, (a \odot b) \odot c = a \odot (b \odot c)$;
3. Identity: $\exists e \in \mathbb{S}$ such that $a \odot e = e \odot a = a$ for all $a \in \mathbb{S}$;
4. Inverse: $\forall a \in \mathbb{S}, \exists a^{-1} \in \mathbb{S}$ such that $a \odot a^{-1} = e$.

Observatio 3.1.1. A quick definition for operation, \odot , within the set \mathbb{S} is that $\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$; moreover, we can denote $\odot(a, b) = a \odot b$.

The definition of identity can not be simplified to $\exists e \in \mathbb{S}$ such that $a \odot e = a$ for all $a \in \mathbb{S}$. Nothing has prevented us from arbitrating that for a certain set \mathbb{S} and $a, b \in \mathbb{S}$, $a \odot b = a$ while $b \odot a \neq a$.

In contrast, the definition of inverse needs not such emphasis. If a set \mathbb{S} has the property of closure, associativity, and identity, $\forall a \in \mathbb{S} \exists a^{-1} \in \mathbb{S}$ such that $a \odot a^{-1} = e$ would implies that $a^{-1}a = e$. Here is a quick proof.

Let $b = a^{-1}$. $ab = e \implies abb = bab = b$ (multiply both sides of e with b). We know b itself has an inverse, denoted as c ; thus $abbc = babc = bc$, and by associativity, $ab(bc) = abbc = babc = ba(bc) \implies ab = ba = e$.

Theorema 3.1.1 (Consequence of the Definition). *There are many non-obvious properties that directly follows the definition.*

1. *General Associativity: Parenthesis does not matter, as long as the order is the same: $a \odot b \odot c \odot d \odot e \odot f \odot g \cdots = (a \odot ((b \odot c) \odot e (\odot f \odot g) \cdots)) = \cdots$*

2. *Order of Inverse:* $(a \odot b)^{-1} = b^{-1} \odot a^{-1}$.

Coniectura 3.1.1. These are some of my hypothesis and thoughts.

1. different properties of odd finite groups and even finite groups
2. If defining the revert of the operation \odot to be \oslash as such: $a \odot b = a \oslash b^{-1}$. What are the sets such that it would be a group under both \odot & \oslash ?
3. Can we have a set \mathbb{S} , such that under the operation \odot we have $\forall a, b \in \mathbb{S}, a \odot b = b \odot a$ but without associativity? (Community without associativity?)

Definitio 3.1.2 (Order of Group and element). The order of the group \mathbb{S} is $|\mathbb{S}|$ (How many elements it has).

The order of an element $s \in \mathbb{S}$ is the smallest integer i such that $s^i = e$. (If such i exists)

Here are some examples of groups.

1. $\mathbb{S} = \{e\}$
2. $\mathbb{S} = \{e, a, b, c\}$. With the following operation: 1. All elements are their own inverse; 2. The group is abelian. 2. $a \odot b = c, a \odot c = b, b \odot c = a$.

Definitio 3.1.3 (Subgroup). A subgroup of a group \mathbb{S} is a subset \mathbb{H} of \mathbb{S} that is also a group under the same operation \odot . It is denoted as $\mathbb{H} \leq \mathbb{S}$. If $\mathbb{H} \neq \mathbb{S}$, we call it a proper subgroup with notation $\mathbb{H} < \mathbb{S}$.

Theorema 3.1.2 (Test For Subgroup). *For group \mathbb{S} and its subset $\mathbb{H} \subseteq \mathbb{S}$, \mathbb{H} is a subgroup of \mathbb{S} if and only if*

1. \mathbb{H} is not empty
2. $\forall h, k \in \mathbb{H}, h \odot k^{-1} \in \mathbb{H}$.

Graphs can help us to construct/discover more examples of groups.

Definitio 3.1.4 (Graph). A graph is a finite set of vertices and edges connecting the vertices; or, with abstraction of set theory, a graph consists of two sets \mathbb{V} and \mathbb{E} , where each element of \mathbb{E} is an element of $\mathbb{V} \times \mathbb{V}$.

Definitio 3.1.5 (Isomorphism). Isomorphism of a graph is a bijection of vertices that preserves all edges; or, \exists bijective $f : \mathbb{V} \rightarrow \mathbb{V}$, such that $\mathbb{E}' = \{(f(a), f(b)) | a, b \in \mathbb{V}\} = \mathbb{E}$.

Definitio 3.1.6 (Dihedral Group). The dihedral group of order $2n$ is the group of symmetries of a regular n -gon. It is the direct product of two copies of the cyclic group of order n .

Definitio 3.1.7 (Cyclic Group). Let \mathbb{G} be a group and g one of its element. Considering the set:

$$\mathbb{S} = \{\cdots g^{-2}, g^{-1}, e, g, g^1, g^2 \cdots\}$$

If \mathbb{S} is finite, it is called a cyclic group. Most importantly, all sets of such form with inheritted operation must be a group.

Theorema 3.1.3 (Properties of Cyclic Group). *Here are some properties immediately follows the definition.*

1. All set in the form of the defintion of the cyclic group is a subgroup.
2. Any subgroup of a cyclic group is also cyclic.

Demonstratio.

Q.E.D.

Exempli Gratia 3.1.1. An example of cyclic group is \mathbb{Z}_n . (Integer under modular n addition);

Propositio 3.1.3.1 (Number of Subgroup for \mathbb{Z}_n). *The number of the subgroup of cyclic group \mathbb{Z}_n is equal to the number of the divisor of n . (Proposed 27 Feb 2023)*

Definitio 3.1.8 (Left Coset).

Theorema 3.1.4 (Lagrange Theorem). *Consider finite group \mathbb{G} and its subgroup \mathbb{S} . $|\mathbb{S}|$ divides $|\mathbb{G}|$.*

Propositio 3.1.4.1. *Lagrange theorem implies that: for a group \mathbb{G} and $g \in \mathbb{G}$:*

1. $\omega(g)$ divides $|\mathbb{G}|$;
2. $g^{|\mathbb{G}|} = e$;

Exempli Gratia 3.1.2. The followings demonstrate Lagrange Theorem.

1. \mathbb{Z}_{10} under addition modula 10 and its subgroup $\mathbb{S} = \{0, 2, 4, 6, 10\}$.
 $|\mathbb{Z}_{10}| = 10, |\mathbb{S}| = 5$.

Demonstratio.[Proof of Lagrange Theorem] Let $\mathbb{G} = \{g_1, g_2, g_3, \dots\}$ be a group and $\mathbb{S} = \{s_0, s_1, s_2, \dots\}$ (let $s_0 = e$) be its subgroup. If $\mathbb{S} = \mathbb{G}$, we are done. If not, sine detrimento universalitatis (without loss of generality), let $g_i \notin \mathbb{S}$. Consider the set: $\mathbb{D}_1 = \{g_1 s | s \in \mathbb{S}\}$. The set \mathbb{D}_1 has the following properties:

1. $g_1 s \in \mathbb{D}_1 \rightarrow g_1 s \in \mathbb{G}$
2. $|\mathbb{D}_1| = |\mathbb{S}|$.
3. $(\forall d \in \mathbb{D}_1)$ the set $\mathbb{D}'_1 = \{ds | s \in \mathbb{S}\} = \mathbb{D}_1$
4. $g_1 s \in \mathbb{D}_1 \rightarrow g_1 s \notin \mathbb{S}$.

Property I is true because \mathbb{G} is a group with the property closure.

By claiming that $g_1 s_i \neq g_1 s_j$ for $i \neq j$ it is sufficently to show property II is true.

To prove property III, we shall prove statement 1) $\mathbb{D}_1 \subseteq \mathbb{D}'_1$ and 2) $\mathbb{D}'_1 \subseteq \mathbb{D}_1$. To prove statement 1), consider $a \in \mathbb{D}_1$, $\exists s_1 \in \mathbb{S}$ such that $a = g_1 s_1$. Let \mathbb{D}'_1 be defined as $\mathbb{D}'_1 = \{bs | s \in \mathbb{S}\}$. and b can be written in the form of $g_1 s_2$. Indeed $bs_2^{-1} s_1 = a \rightarrow a \in \mathbb{D}'_1 \rightarrow \mathbb{D}_1 \subseteq \mathbb{D}'_1$. Statement 2) can be proved similarly.

Property IV can be proved by contradiction. Assuming $\exists g_1 s \in \mathbb{D}_1$ and $g_1 s \in \mathbb{S}$. We have $g_1 s s^{-1} \in \mathbb{S}$ (by Inverse and Closure property of group) $\rightarrow g_1 \in \mathbb{S}$, (by associativity property of group) contradicting our assumption that $g \notin \mathbb{S}$.

If $\mathbb{G} = \mathbb{S} \cup \mathbb{D}_1$, we are done, as $|\mathbb{G}| = 2|\mathbb{S}|$.

If $\exists g_2 \in \mathbb{G} \setminus \mathbb{S}, \mathbb{D}_1$. Construct the set $\mathbb{D}_2 = \{g_2 s | s \in \mathbb{S}\}$. All elements in \mathbb{D}_2 have properties I, II of \mathbb{D}_1 , and a stronger IV property: $g_2 s \in \mathbb{D}_2 \rightarrow g_2 s \notin \mathbb{S}, \mathbb{D}_1$.

Thus by same reasoning, if $\mathbb{G} = \mathbb{S} \cup \mathbb{D}_1 \cup \mathbb{D}_2$, $|\mathbb{G}| = 3|\mathbb{S}|$. If not, we can constuct more disjointed sets $\mathbb{D}_3, \mathbb{D}_4, \dots, \mathbb{D}_n$ until the union of them and \mathbb{S} forms \mathbb{G} . This can always be done as \mathbb{G} is finite, and will have an order of $(n+1) \cdot |\mathbb{S}|$. Q.E.D.

Propositio 3.1.4.2 (Some Application of Lagrange Theorem).

1. For a group \mathbb{G} with order p and $k \in \mathbb{G}$ with order q ; then q divides p .
2. For a group \mathbb{G} with prime order (i.e., $|\mathbb{G}|$ is prime), it must be a cyclic group.

Demonstratio. 3.1.4.2.1 implies 3.1.4.2.2. To prove proposition 3.1.4.2.1, let \mathbb{G} be a group and $\langle g \rangle$ be a cyclic group containing $g \in \mathbb{G}$, by langrange theorem $|\langle g \rangle|$ must divides $|\mathbb{G}|$. Q.E.D.

3.2 Between Groups

Definitio 3.2.1 (Homomorphism). Group \mathbb{G} & \mathbb{P} are isomorphic to each other if there exists a function (shall we add the following[proposed 14 mar] defined for all $a \in \mathbb{G}$) $\phi : \mathbb{G} \rightarrow \mathbb{P}$ such that $a, b \in \mathbb{G} \implies \phi(ab) = \phi(a)\phi(b)$.

The function ϕ is denominated as a homomorphism of the group \mathbb{G} & \mathbb{P} .

Definitio 3.2.2 (Isomorphism). A homomorphism $\phi : \mathbb{G} \rightarrow \mathbb{P}$ that is also a bijection is a isomorphism of group \mathbb{G} & \mathbb{P} , and we denote two such isomorphic group with $\mathbb{G} \cong \mathbb{P}$.

Propositio 3.2.0.1 (Consequence of homomorphism). *Let $\phi : \mathbb{G} \rightarrow \mathbb{P}$ denote a homomorphism.*

1. $\phi(e_{\mathbb{G}}) = e_{\mathbb{P}}$.
2. $\phi(a^{-1}) = \phi(a)^{-1}$.
3. *If ϕ is injection, the order of $\phi(a)$ is the same as the order of a .*
4. $\{\phi(c) | c \in \mathbb{G}\} \leq \mathbb{P}$.

Definitio 3.2.3 (Image). The image of a homomorphism $\phi : \mathbb{G} \rightarrow \mathbb{P}$ is the set $\{\phi(c) | c \in \mathbb{G}\}$.

Definitio 3.2.4 (Kernel). The kernel of a homomorphism $\phi : \mathbb{G} \rightarrow \mathbb{P}$ is the set $\{c \in \mathbb{G} | \phi(c) = e_{\mathbb{P}}\}$.

Definitio 3.2.5 (Normal Group). A subgroup \mathbb{H} of a group \mathbb{G} is called normal if $\forall a \in \mathbb{G}$, the left and right coset of \mathbb{H} are equal, i.e. $a\mathbb{H} = \mathbb{H}a$.

Theorema 3.2.1. *Let $\phi : G \rightarrow H$ be a group homomorphism.*

1. $\ker \phi$ is a normal subgroup of G .
2. $\phi : G \rightarrow H$ is injective if and only if $\ker \phi = \{e_G\}$.
3. *If ϕ is injective, it gives an isomorphism between G and im_{ϕ} .*

Coniectura 3.2.1 (Proposed 20 Mar 2023). Let $\phi : G \rightarrow H$ be a homomorphism. Define kernel of ϕ respect to $a \in H$ as $\ker_a \phi = \{g \in G | \phi(g) = a\}$. For all a, b such that $\ker_a \phi, \ker_b \phi$ is not an empty set, $\ker_a \phi = \ker_b \phi$

Coniectura 3.2.2 (Proposed 21 Mar 2023). Let $\phi : G \rightarrow H$ be a homomorphism. Then there exists a subgroup $S < G$ such that ϕ is a isomorphism between S and im_{ϕ} .

3.2.1 Product and Polymorphism

Theorem 3.2.2. *Let $\mathbb{H}, \mathbb{K} \leq \mathbb{G}$ be subgroups of \mathbb{G} with the union of $\mathbb{H}, \mathbb{K} = \{e\}$*

1. *The map $\phi : \mathbb{H} \times \mathbb{K} \rightarrow \mathbb{H}\mathbb{K}$ given by $\phi(h, k) = hk$ is a bijection.*
2. *$\mathbb{H}\mathbb{K}$ is a subgroup of \mathbb{G} if and only if for all $h, k : \exists h'k'$ such that $hk = k'h'$.*
3. *$\mathbb{H}\mathbb{K}$ is isomorphic to $\mathbb{H} \times \mathbb{K}$ if and only if every element of \mathbb{H} commutes with every element of \mathbb{K} , i.e. $\forall h, k : hk = kh$.*

3.3 Group Actions

Definitio 3.3.1 (Group Action). A group action between a group \mathbb{G} and a set S is a function $f : \mathbb{G} \times S \rightarrow S$. Let $g \in \mathbb{G}$ & $s \in S$. Let $f(g, s)$ be denoted as $g \cdot s$. Group action must fulfill the two following property:

1. $e \cdot s = s$ for all $s \in S$, with e being the identity of the group \mathbb{G}
2. $g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s$ for all $g_1, g_2 \in \mathbb{G}$ & $s \in S$

Theorem 3.3.1 (Immediate Consequence of the Definition). *Let G be a group acting on the set S . Let Greek letters $\alpha, \beta \dots \in G$, while Latin letter $a, b, c, \dots \in S$.*

1. $\beta a = b \implies \beta^{-1} b = a$
- 2.

Definitio 3.3.2 (Orbit). Let G act on S and $s \in S$. The orbit of s is the set $\{g \cdot s | g \in G\}$.

Definitio 3.3.3 (Stabilizer). Let G act on S and $s \in S$. The stabilizer of s is the set $\{g \in G | g \cdot s = s\}$.

Theorem 3.3.2 (Orbit Equivalence Relation). *Let G be a group acting on the set S . Define the relationship $a \sim b$ if $b \in \text{Orb}_G(a)$. This is a equivalent relation; thus the orbits of the element of S partitions S .*

Theorem 3.3.3 (Orbit Stabilizer Theorem). *Let G act on X and $x \in X$. The stabilizer of x is the set $\{g \in G | g \cdot x = x\}$. $|\text{Orb}_G(x)| |\text{Stab}_G(x)| = |G|$*

To prove Orbit Stabilizer Theorem, we need lemma 3.3.4.

Lemma 3.3.4. *Let G be a group acting on the set S .*

1. *For a and $b \in \text{Orb}_G(a)$, $|\text{Stab}_G(a)| = |\text{Stab}_G(b)|$.*
2. *Let set $S_b = \{g \in G | ga = b\}$. For all $b_1, b_2 \in \text{Orb}_G(a)$, $|S_{b_1}| = |S_{b_2}|$.*

To prove the first part of the lemma, consider $\{a, b \dots\} \in \text{Orb}_G(a)$. By definition of orbit, we know $\exists \beta \in G$ such that $\beta a = b$. Let $\text{Stab}_G(a) = \{e_g, e_1, e_2 \dots\}$. Notice that $\forall \epsilon \in \text{Stab}_G(a)$, $\beta \epsilon \beta^{-1} \beta a = \beta \epsilon \beta^{-1} b = b$; i.e., $\beta \epsilon \beta^{-1} \in \text{Stab}_G(b)$. Since $\epsilon_1 \neq \epsilon_2 \implies \beta \epsilon_1 \beta^{-1} \neq \beta \epsilon_2 \beta^{-1}$, we conclude that $|\text{Stab}_G(a)| \leq |\text{Stab}_G(b)|$.

Such argument can be repeated for all $a', b' \in \text{Orb}_G(a)$ such that $|\text{Stab}_G(a')| \leq |\text{Stab}_G(b')|$; the only possibility that is true is that indeed they are all equal to each other.

Demonstratio.[Orbit Stabilizer Theorem] Let G be a group acting on the set S . Consider the pseudo set (that allows the repetition of the elements) $L = \{ag | g \in G\}$. By lemma 3.3.4 we claim $L = \text{Orb}_G(a) \times |\text{Stab}_G(a)|$, i.e., L is made up of all element of orbit of a repeated $|\text{Stab}_G(a)|$ times. Thus $|G| = |L| = |\text{Stab}_G(a)| |\text{Orb}_G(a)|$. Q.E.D.

Theorema 3.3.5 (Cauchy's Theorem). *Let G be a group. Let p be prime. If p divides $|G|$, then G contains a element of order p .*

Appendix I

On Polynomials

Coniectura I.0.1. Provided $n + 1$ points in 2 dimensional spaces, there are exactly one n degree polynomials that passes through all the points, provided that those $n + 1$ points are not passed through by another polynomial of lower degree.

Extension: how about $n + 1$ points in d dimensional spaces? (Proposed 1 Mar 2023)

Exempli Gratia I.0.1. For points $(0, 0), (1, 1), (2, 4)$, the only 2 degree polynomials that pass through it is $y = x^2$.

However, for points $(0, 0), (1, 1), (2, 2)$, there are no degree 2 polynomials that pass through it.

This hypothesis is definitely related to linear independence.

Demonstratio.[A Quick proof using Fundamental Theorem of Algebra]
Q.E.D.

Appendix II

Latin and Abbreviations

De Mathematica Pura	On Pure Mathematics
Caput	Chapter
Index Capitis	Index of Chapters
Theorema, Theorematae	Theorem
Definitio, Definitiones	Definition
Propositio, Propositiones	Proposition
Coniectura, Coniecturae	Conjecture
Demonstratio, Demonstrationes	Proof
Q.E.D.	Quod Erat Demonstrandum
	Which was to be demonstrated, signify end of proof
Exempli gratia	For (the sake of) example
SDU(sine detrimento universalitatis)	without any loss of generosity

Appendix III

Chronology of Proposed, Proved, and Disproved Hypotheses

Hypothesis/Theorem	Date of Proposition	Date of Resolution	Outcome
Theorem 2.2.1	Feb 8, 2023	Feb 9	PROVED
Theorem 2.2.3.8	Feb 14, 2023	Feb 17	PROVED ¹
Theorem 2.2.3.6	Feb 14, 2023		
Hypothesis 2.2.1	Feb 16, 2023		
Theorem 2.2.3.9	Feb 17, 2023		

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