DE MATHEMATICA PURA

On Pure Mathematics

Harry Han

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Abstract

These are my notes when taking the class *Fundamentals of Pure Mathematics* at the University of Edinburgh. They are not a replicate of the lecture notes: they are my thoughts and explorations. Most importantly, all proofs presented in this document are of my own conception.

Terms like "Theorem, Proposition" are coined in Latin. As the English terms descended from Latin, most of them are self-explanatory.

Caput 1

Notation

- The `\mathbb{}` fonts are used to denote sets. ($\mathbb{S}, \mathbb{Y}$, etc.)
- $\mathbb{A} \succ \mathbb{B}$ denotes there exits a surjective function $f : \mathbb{A} \to \mathbb{B}$. $\prec$, $\asymp$ denotes injective, bijective, respectively.
- e is used to denote the identity of a group.
- When there is no ambiguity, the notation for the operation of group is ommited. (i.e., $a \odot b = ab$). a^{-1} is used to denote the inverse of a.
- Sequence and series are denoted as (s_n) and $\sum_{k=1}^{\infty} s_k$ respectively.
- $\mathcal{L}_s(s_n), \mathcal{L}_s(s_n)$ is the limit of supremum & infimum. See definition 2.2.4.

Caput 2

Analysis

2.1 Real Number

2.1.1 The Countable Sets

Axioma 2.1.1 (The "Smallest" Infinite Set)**.** A set $\mathbb{S}$ is infinite iff $\mathbb{S} \succ \mathbb{N}$.

Observatio 2.1.1. Although FPM is a pure mathematic class with emphasis on rigor, no rigorous definition for the infinite set has been proposed. This definition/axiom is of my own conception.

Definitio 2.1.1 (Countable Set)**.** A set $\mathbb{S}$ is countable iff $\mathbb{N} \asymp \mathbb{S}$ (there exists a bijection $f : \mathbb{N} \to \mathbb{S}$).

Corollarium 2.1.1 (At Most Countable)**.** *Let $\mathbb{A}$ be an infinite set.*
($\mathbb{A} \prec \mathbb{N}$) iff ($\mathbb{A} \asymp \mathbb{N}$).

Proof. We want to prove $\mathbb{A} \prec \mathbb{N}$ is equivalent to $\mathbb{A} \asymp \mathbb{N}$. $\mathbb{A} \asymp \mathbb{N} \to \mathbb{A} \prec \mathbb{N}$ is by definition. We only need to prove the other direction; i.e., provided $\mathbb{A} \prec \mathbb{N}$, find a bijective function $h : \mathbb{A} \to \mathbb{N}$.

Let $f : \mathbb{A} \to \mathbb{N}$ be an injective mapping. If f is bijective, we are done. If f is injective but not bijective, let $\mathbb{N}^-$ be the range of f. As $\mathbb{A}$ is infinite, $\mathbb{N}^-$ is also infinite. Let $f' : \mathbb{A} \to \mathbb{N}^-$ such that $f(a) = f'(a)$. f' is an bijective mapping.

Thus we only need to show there exists a mapping $g : \mathbb{N}^- \to \mathbb{N}$ that is bijective.

g can be constructed by such: sort $\mathbb{N}^-$ and $\mathbb{N}$ in ascending order. Let the first element in the sorted $\mathbb{N}^-$ maps to the first in the sorted $\mathbb{N}$, the secound to secound, etc. As $\mathbb{N}^-$ is infinite, g must be bijective.

Indeed $h = g \circ f' : \mathbb{A} \to \mathbb{N}$ is the bijective mapping we seek. Q.E.D.

Theorema 2.1.1 (List of Countable and Uncountable Sets)**.** *Any of the following sets are countable.*

1. $\mathbb{Z}, \mathbb{Q}$
2. *Any infinite subset of countable sets.*
3. *Any Unions of countable and finite sets.*
4. *Any products of countable sets and finite sets. i.e., if* $\mathbb{S}, \mathbb{T}$ *are countable,* $\{\mathbb{S} \times \mathbb{S}\}, \{\mathbb{S} \times \mathbb{T} \times \cdots \times \mathbb{S}\}$ *are also countable.*

Coniectura 2.1.1. Is the product of countable number of countable sets countable? (Proposed Feb 6)

2.2 Sequence and Series

2.2.1 Sequence

Definitio 2.2.1 (Sequence)**.**

Definitio 2.2.2 (Convergent and Divergent)**.**

Definitio 2.2.3 (Increasing and Decreasing Sequence(Monotone))**.**

Definitio 2.2.4 (Limit of supremum & infimum)**.** For a sequence (s_n), let b_i denotes the supremum of $\{s_n | n > i\}$. If (b_n) converges, the value it converges to is called the limit of supremum of (s_n), and is denoted as $\mathcal{L}_s(s_n)$. (b_n) is called the supremum sequence. Similarly infimum sequence and limit of infimum are defined, and the later denoted as $\mathcal{L}_i(s_n)$.

Observatio 2.2.1. Notice supremum and infimum sequences are monotone.

Theorema 2.2.1 (Convergence and Limit of supremum & infimum)**.** *A sequence* (s_n) *converges if and only if* $\mathcal{L}_s(s_n) = \mathcal{L}_i(s_n)$. *(Proposed Feb 8 2023, proved Feb 9)*

Proof. We want to prove that $(\mathcal{L}_i(s_n) = \mathcal{L}_s(s_n)) \iff (s_n)$ converges.
Forward direction: We shall show that $\lim_{n\to\infty}(s_n) = \mathcal{L}_s(s_n) = \mathcal{L}_i(s_n) = \lambda$. $\forall \epsilon > 0$, we know by our assumption that $(\exists N \in \mathbb{N})(\forall n > N)$ the set $\{s_n | n > N\}$ is bounded by $\lambda \pm \epsilon$. This is the definition for the convergent sequence.

We shall prove the contraposition of the backwards direction, i.e. $(\mathcal{L}_i(s_n) \neq \mathcal{L}_s(s_n)) \to (s_n)$ diverges. The contraposition can be proved by contradiction.

Assuming ($\lambda = \mathcal{L}_i(s_n) \neq \mathcal{L}_s(s_n)$) and (s_n) converges to l. S.D.U., let $\lambda > l$. Let $\epsilon = (\lambda - l)/2$. Since (s_n) converges to l, there exists $N \in \mathbb{N}$ such that $\forall n > N$, $|s_n - l| < \epsilon$. However, we know that $\mathcal{L}_i(s_n) = \lambda$, which means that there exists N' such that $\forall n > N'$ we have at least one element $s_i > \lambda - \epsilon$. Indeed $s_i - l > \epsilon$, contradicting with our assumption that (s_n) converges. Thus we conclude the backwards direction is also true. Q.E.D.

Definitio 2.2.5 (Cauchy Sequence)**.** [1] A sequence (s_n) is a Cauchy Sequence iff $(\forall \epsilon > 0)(\exists N)(\forall n, m > N)(|s_n - s_m| < \epsilon)$

Theorema 2.2.2. *A sequence converges if and only if it is a Cauchy Sequence.*

Observatio 2.2.2. We are to outline our proof of (s_n) converges $\iff$ (s_n) is Cauchy Sequence.
The forward direction is obvious. To prove the backwards direction, notice: 1) All Cauchy Sequences are bounded; 2) the infimum and supremum sequence converge by monoteon convergence theorem; 3) They must converge to the same value; 4) By theorem 2.2.1 the sequence must converge.

Observatio 2.2.3. We can define a pseudo Cauchy Sequence to be sequence (s_n) such that $(\forall \epsilon > 0)(\exists N)(\forall n > N)(|s_n - s_{n+1}| < \epsilon)$. Indeed all convergent sequence are pseudo Cauchy Sequence, but not all pseudo Cauchy Sequence are convergent. An example is the partial sum of harmonic series, i.e, $(\sum_{i=1}^{n} \frac{1}{i})$.

2.2.2 Series

Definitio 2.2.6 (Series)**.** A series can be expressed as $\sum_{k=1}^{\infty} a_k$.

Definitio 2.2.7 (Convergent and Divergent)**.** Consider the seires :
$(s_n) = \sum_{k=1}^{n} a_k$. (s_n) is called the partial sum of the series. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if its partial sum converges; otherwise it diverges.

Exampli Gratia 2.2.1. List of Convergent and Divergent series:

1. Harmonic Series.

Definitio 2.2.8 (Cauchy Criterion)**.** A series befits Cauchy Criterion if and only if its partial sum is a Cauchy Sequence.

Definitio 2.2.9 (Absolute Convergent)**.** A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if $\sum_{k=1}^{\infty} |a_k|$ converges. Otherwise it converges non-absolutely

Theorema 2.2.3 (Convergence Reveries)**.**

1. *Absolute Convergent If a series converge absolutely, it converges. The converse is not true.*

2. *Addition, Substraction, Multiplication, Division If* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *converges, the followings also converge:* $\sum_{k=1}^{\infty}(a_k + b_k)$, $\sum_{k=1}^{\infty}(a_k - b_k)$ $\sum_{k=1}^{\infty}(a_k \cdot b_k)$

Caput 3

Algebra

3.1 Group

Definitio 3.1.1 (Group)**.** Group is a set $\mathbb{S}$ with an operation $\odot$ that fulfills the following four properties:

1. Closure
2. Associtivity: $(a \odot b) \odot c = a \odot (b \odot c)$;
3. Identity
4. Inverse

Theorema 3.1.1 (Consequence of the Definition)**.** *There are many non-obvious properties that directly follows the definition.*

1. *General Associtivity: Parenthesis does not matter, as long as the order is the same:* $a \odot b \odot c \odot d \odot e \odot f \odot g \cdots = (a \odot ((b \odot c) \odot e(\odot f \odot g) \cdots) = \cdots$
2. *Order of Inverse:* $(a \odot b)^{-1} = b^{-1} \odot a^{-1}$.

Here are some examples of groups.

1. $\mathbb{S} = \{e\}$
2. $\mathbb{S} = \{e, a, b, c\}$. With the following operation: 1. All elements are their own inverse; 2. The group is abelian. 2. $a \odot b = c, a \odot c = b, b \odot c = a$.

Coniectura 3.1.1. These are some of my hypothesis and thoughts.

1. different properties of odd finite groups and even finite groups

2. If defining the reverto of the operation $\odot$ to be $\oslash$ as such: $a \odot b = a \oslash b^{-1}$. What are the sets such that it would be a group under both $\odot \& \oslash$?

3. Can we have a set $\mathbb{S}$, such that under the operation $\odot$ we have $\forall a, b \in \mathbb{S}, a \odot b = b \odot a$ but without associtivity? (Community without associtivity?)

Definitio 3.1.2 (Order of Group and element)**.** The order of the group $\mathbb{S}$ is $|\mathbb{S}|$ (How many elements it has).
The order of an element $s \in \mathbb{S}$ is the smallest integer i such that $s^i = e$. (If such i exists)

Definitio 3.1.3 (Cyclic Group)**.** Let $\mathbb{G}$ be a group and g one of its element. Considering the set:

$$\mathbb{S} = \{\cdots g^{-2}, g^{-1}, e, g, g^1, g^2 \cdots\}$$

If $\mathbb{S}$ is finite, it is called a cyclic group. (It can be shown that it must be a subgroup of $\mathbb{G}$.

Theorema 3.1.2 (Properties of Cyclic Group)**.** *Here are some properties immediately follows the definition.*

1. *Any subgroup of a cyclic group is also cyclic.*

Theorema 3.1.3 (Lagrange Theorem)**.** *Consider finite group* $\mathbb{G}$ *and its subgroup* $\mathbb{S}$*.* $|\mathbb{S}|$ *divides* $|\mathbb{G}|$*.*

Exampli Gratia 3.1.1. The followings demonstrate Lagrange Theorem.

1. $\mathbb{Z}_{10}$ under addition modula 10 and its subgroup $\mathbb{S} = \{0, 2, 4, 6, 10\}$. $|\mathbb{Z}_{10}| = 10, |\mathbb{S}| = 5$.

Proof of Lagrange Theorem. Let $\mathbb{G} = \{g_1, g_2, g_3, \cdots\}$ be a group and $\mathbb{S} = \{s_0, s_1, s_2, \cdots\}$(let $s_0 = e$) be its subgroup. If $\mathbb{S} = \mathbb{G}$, we are done. If not, sine detrimento universalitatis(without loss of generality), let $g_i \notin \mathbb{S}$. Consider the set: $\mathbb{D}_1 = \{g_1 s | s \in \mathbb{S}\}$. The set $\mathbb{D}_1$ has the following properties:

1. $g_1 s \in \mathbb{D}_1 \to g_1 s \in \mathbb{G}$

2. $|\mathbb{D}_1| = |\mathbb{S}|$.

3. $(\forall d \in \mathbb{D}_1)$ the set $\mathbb{D}_1' = \{ds | s \in \mathbb{S}\} = \mathbb{D}_1$

4. $g_1 s \in \mathbb{D}_1 \to g_1 s \notin \mathbb{S}$.

Property I is true because $\mathbb{G}$ is a group with the property closure.
By claiming that $g_1 s_i \neq g_1 s_j$ for $i \neq j$ it is sufficently to show property II is true.

To prove property III, we shall prove statement 1) $\mathbb{D}_1 \subseteq \mathbb{D}_1'$ and 2) $\mathbb{D}_1' \subseteq \mathbb{D}_1$. To prove statement 1), consider $a \in \mathbb{D}_1$, $\exists s_1 \in \mathbb{S}$ such that $g_1 s_1$. Let $\mathbb{D}_1'$ be defined as $\mathbb{D}_1' = \{bs | s \in \mathbb{S}\}$. and b can be written in the form of $g_1 s_2$. Indeed $bs_2^{-1} s_1 = a \to a \in \mathbb{D}_1' \to \mathbb{D}_1 \subseteq \mathbb{D}_1'$. Statement 2) can be proved similarly.

Property IV can be proved by contradiction. Assuming $\exists g_1 s \in \mathbb{D}_1$ and $g_1 s \in \mathbb{S}$. We have $g_1 s s^{-1} \in \mathbb{S}$ (by Inverse and Closure property of group) $\to g_1 \in \mathbb{S}$,(by associtivity property of group) contradicting our assumption that $g \notin \mathbb{S}$.

If $\mathbb{G} = \mathbb{S} \cup \mathbb{D}_1$, we are done, as $|\mathbb{G}| = 2|\mathbb{S}|$.

If $\exists g_2 \in \mathbb{G} \vee g_2 \notin \mathbb{S}, \mathbb{D}_1$. Construct the set $\mathbb{D}_2 = \{g_w s | s \in \mathbb{S}\}$. All elements in $\mathbb{D}_2$ have properties I, II of $\mathbb{D}_1$, and a stronger IV property: $g_1 s \in \mathbb{D}_2 \to g_1 s \notin \mathbb{S}, \mathbb{D}_1$..

Thus by same reasoning, if $\mathbb{G} = \mathbb{S} \cup \mathbb{D}_1 \cup \mathbb{D}_2$, $|\mathbb{G}| = 3|\mathbb{S}|$. If not, we can constuct more disjoined sets $\mathbb{D}_3, \mathbb{D}_4, \cdots \mathbb{D}_n$ until the union of them and $\mathbb{S}$ forms $\mathbb{G}$. This can always be done as $\mathbb{G}$ is finite, and will have an order of $(n+1) \cdot |\mathbb{S}|$. Q.E.D.

Appendix I

Latin and Abbreviations

Theorema	Theorem
SDU(sine detrimento universalitatis)	without any of generosity

Appendix II

Chronology of Proposed, Proved, and Disproved Hypotheses

Hypothesis/Theorem	Date of Proposition	Date of Resolvation	Outcome
Theorem 2.2.1	Feb 8, 2023	Feb 9	PROVED

Bibliography

[1] Kenneth A. Ross. *Elementary Analysis, The Theory of Calculus, Second Edition*. Springer, 2013. ISBN: 9781461462705.