1 Basics

Definitio 1.1 (Graph)**.** Let n, m, f, k denote number of vertices, edges, faces, components respectively.

1. **Simple Graph** No loops, no multiple edges.
2. **Walk** Finite sequence of connected edges.
3. **Trail** walk with no repeated edges.
4. **Path** walk with no repeated vertices.
5. **Cycle** Closed plath with no repeated edges. (any loop or multiple edges are cycles)
6. **Disconnecting Set** A set of edges whose removal disconnects the graph (increase the number of components).
7. **Cut Set** smallest disconnecting set.
8. **Edge Connectivity** The size of the smallest cut set, $\lambda(G)$. We say G is k connected if $\lambda(G) \geq k$.
9. **Bridge** An edge whose removal increases the number of components.
10. **Separating Set** A set of vertices whose removal disconnects the graph.
11. **Vertex Connectivity** The size of the smallest separating set, $\kappa(G)$. We say G is k connected if $\kappa(G) \geq k$. We have $\kappa(G) \leq \lambda(G)$
12. **Cut Vertex** A vertex whose removal increases the number of components.
13. **Eulerian** A graph with a closed trail containing all edges.
14. **Semi-Eulerian** A graph with a trail containing all edges.
15. **Hamiltonian** A graph with a closed trail passing through exactly once all vertices. This is Hamiltonian cycle. (No repeated edges or vertices, passing through all vertices)
16. **Forest** A graph with no cycles.
17. **Tree** A connected forest.
18. **Cutset Rank** Number of edges in a spanning tree. $n - k$.
19. **Cycle Rank** Number of edges removed to get a spanning forest. $m - (n - k)$.
20. **Connected Digraph** A digraph whose corresponding undirected graph is connected.
21. **Strongly Connected Digraph** A digraph such that for all $u, v \in V$, there is a $u - v$ path and a $v - u$ path.
22. **Orientable** A graph is orientable if it is possible to assign a direction to each edge such that the resulting digraph is strongly connected.

23. **Tournament** A digraph such that two vertices are joined by one directed edge.+

24. **Chromatic Number** Coloring index.

25. **Chromatic Polynomial** Coloring edge with minimum number of colors.

Theorema 1.1 (Adjacency matrix). *Let A be adjacency matrix. Trace of A^2 is twice the number of edges. Trace of A^3 is 6 times the number of triangles.*

Theorema 1.2 (Bound for edges). *G be a simple graph with n vertices, k components, m edges. We have*

$$n - k \leq m \leq \frac{(n-k)(n-k+1)}{2}$$

Theorema 1.3 (Ore). *If G is simple graph with $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for all non-adjacent vertices u, v, then G is Hamiltonian.*

Definitio 1.2 (Algorithms).

1. The shortest path problem: Dijkstra's algorithm.

2. The minimum spanning tree problem: Greedy algorithm.

3. Chinese postman problem: find the shortest closed walk containing all edges. (If the graph is Eulerian, then it is the Eulerian cycle. If Semi-Eulerian, this is the Semi-Eulerian path along the path of least length between the two odd vertices.

4. Travelling salesman problem: find the shortest hamiltonian cycle. (Passing through all vertices exactly once and returning to the starting vertex) (Lower bound: find minimum spanning tree of $G - v$. Add the sum with two minimum edges of v).

Theorema 1.4 (Counting Trees). *There are n^{n-2} labeled trees on n vertices.*

Theorema 1.5 (Planarity).

1. *Every planar graph has a vertex of degree at most 5.*

2. *For planar graphs, $m \leq 3n - 6$. For plane graphs with no triangles, $m \leq 2n - 4$.*

Theorema 1.6 (Genus). *For simple graph, $g(G) \geq ceil\{(m - 3n)/6 + 1\}$. The equality holds if G is complete graph.*

Theorema 1.7 (Euler).

$$n - m + f = 2 - 2g \tag{1}$$

Theorema 1.8 (Coloring).

1. *Every planar graph is 4-colorable.*

2. *Four color theorem for maps is equivalent to four color theorem for planar graphs.*