23.6 Accretion Discs

1. Fluids Equation

Question 1

(i) By using equation (3), the RHS of equation (5) can be simplified as

$$RHS = -\frac{\partial}{\partial R}\left[\frac{R}{(R^2\Omega)'}\frac{\partial}{\partial t}(\Omega R^2 \Sigma) + \frac{1}{(R^2\Omega)'}\frac{\partial}{\partial R}(R\Sigma V_R R^2 \Omega)\right]$$

and hence by product rule

$$RHS = -\frac{\partial}{\partial R}\left[\frac{R}{(R^2\Omega)'}\frac{\partial}{\partial t}(\Omega R^2 \Sigma) + \frac{1}{(R^2\Omega)'}\left(R\nu\Sigma(R^2\Omega)' + R^2\Omega\frac{\partial(R\Sigma V_R)}{\partial R}\right)\right]$$

$$RHS = -\frac{\partial}{\partial R}\left[R\nu\Sigma + \frac{R^2\Omega}{(R^2\Omega)'}\left(\frac{\partial(R\Sigma V_R)}{\partial R} + R\frac{\partial \Sigma}{\partial t}\right)\right]$$

From equation (2), we can deduce

$$RHS = -\frac{\partial}{\partial R}(R\nu\Sigma) = -R\frac{\partial \Sigma}{\partial t} = LHS$$

As required.

(ii) Consider

$$\frac{\partial \Gamma}{\partial R} = 2\pi\frac{\partial}{\partial R}(R\nu\Sigma R^2\Omega')$$

Since $R^2\Omega' = -\frac{3}{2}(GM)^{\frac{1}{2}}R^{-\frac{1}{2}}$, we have

$$\frac{\partial \Gamma}{\partial R} = -3\pi(GM)^{\frac{1}{2}}\frac{\partial}{\partial R}\left(R^{\frac{1}{2}}\nu\Sigma\right)$$

Also since $(R^2\Omega)' = \frac{1}{2}(GM)^{\frac{1}{2}}R^{-\frac{1}{2}}$, the RHS of equation (5) can be written

$$RHS = -\frac{\partial}{\partial R}\left[\frac{-\pi(GM)^{\frac{1}{2}}}{\pi(GM)^{\frac{1}{2}}R^{-\frac{1}{2}}}\frac{\partial}{\partial R}\left(3\nu\Sigma R^{\frac{1}{2}}\right)\right] = \frac{\partial}{\partial R}\left[R^{\frac{1}{2}}\frac{\partial}{\partial R}\left(3\nu\Sigma R^{\frac{1}{2}}\right)\right]$$

Hence we can obtain equation (6) from equation (5)

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[R^{\frac{1}{2}}\frac{\partial}{\partial R}\left(3\nu\Sigma R^{\frac{1}{2}}\right)\right]$$

Question 2

(i) Substitute equation (2) into equation (6), we obtain

$$-\frac{\partial}{\partial R}(R\Sigma V_R) = \frac{\partial}{\partial R}\left[R^{\frac{1}{2}}\frac{\partial}{\partial R}\left(3\nu\Sigma R^{\frac{1}{2}}\right)\right]$$

hence

$$\frac{\partial}{\partial R}\left[R^{\frac{1}{2}}\frac{\partial}{\partial R}\left(3\nu\Sigma R^{\frac{1}{2}}\right) + R\Sigma V_R\right] = 0$$

therefore, we have

$$V_R = -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left(\nu\Sigma R^{\frac{1}{2}}\right)$$

(ii) In a steady state, we have $\dot{m} = constant = A$,
Substitute equation (8) into equation (7), we obtain

$$\frac{A}{6\pi}R^{-1/2} = \frac{\partial}{\partial R}\left(\nu\Sigma R^{\frac{1}{2}}\right)$$

integrate with respect to R,

$$v\Sigma = \frac{A}{6\pi} + BR^{-1/2}$$

Where A=$\dot{m}$ and B are arbitrary constants.
Now apply inner boundary condition

$$0 = \frac{A}{6\pi} + BR_{in}^{-1/2}$$

hence $B = -\frac{AR_{in}^{1/2}}{6\pi}$
Hence our original equation becomes

$$v\Sigma = \frac{A}{6\pi}\left(1 - \frac{R_{in}^{1/2}}{R^{1/2}}\right)$$

Figure 1 is a plot of $v\Sigma$ in units of $\dot{m}$ against $\frac{R}{R_{in}}$

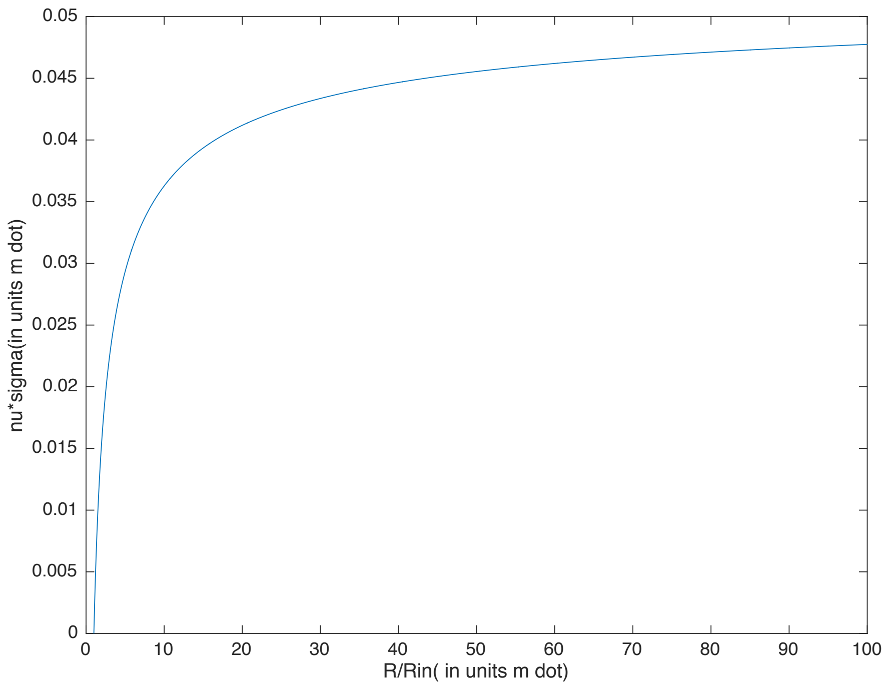

Figure 1

By substituting $r = \frac{R}{R_0}, \tau = \frac{t}{t_0}, \sigma(r,\tau) = \frac{\Sigma}{\Sigma_0}$ $and\ \eta(r,\sigma) = \frac{v}{v_0}$, we obtain

$$\frac{\Sigma_0}{t_0}\frac{\partial\sigma}{\partial\tau} = \frac{R_0 v_0 \Sigma_0}{R_0^3}\frac{1}{r}\left[r^{\frac{1}{2}}\frac{\partial\left(3\eta\sigma r^{1/2}\right)}{\partial r}\right]$$

Hence, we obtain the condition

$$t_0 = \frac{R_0^2}{v_0}$$

such that equation (6) remains the same for these dimensionless variables.

Question 3

(i) Substitute $X = r^{1/2}$and $2XdX = dr$ and into equation (6), (with dimensionless variables)

$$\frac{\partial\sigma}{\partial\tau} = \frac{1}{4X^3}\frac{\partial(3\eta\sigma X)}{\partial X}$$

As X is independent of τ, we have

$$\frac{\partial 4X^3\sigma}{\partial \tau} = \frac{\partial(3\eta\sigma X)}{\partial X}$$

Hence $f = 4X^3\sigma \ and \ g = 3\eta\sigma X.$

(ii) By definition of derivatives,

$$\frac{\partial f}{\partial \tau} = \frac{f(X_i, \tau_{n+1}) - f(X_i, \tau_n)}{\Delta\tau}$$
$$\frac{\partial g}{\partial X} = \frac{f(X_{i+1}, \tau_n) - f(X_i, \tau_n)}{\Delta X}$$

And applied the definition again for the second derivative, we obtain

$$\frac{\partial^2 g}{\partial X^2} = \frac{g(X_{i+1}, \tau_n) - 2g(X_i, \tau_n) + g(X_{i-1}, \tau_n)}{(\Delta X)^2}$$

Then equation (9) becomes

$$\frac{f(X_i, \tau_{n+1}) - f(X_i, \tau_n)}{\Delta\tau} = \frac{g(X_{i+1}, \tau_n) - 2g(X_i, \tau_n) + g(X_{i-1}, \tau_n)}{(\Delta X)^2}$$

which simplifies to

$$f_i^{n+1} = f_i^n + \frac{\Delta\tau}{(\Delta X)^2}(g_{i+1}^n - g_i^n + g_{i-1}^n)$$

Question 4

Choose $\Delta\tau = 0.001$, which is smaller than $\frac{1}{2}(\Delta X)^2\frac{f}{g'}$ we obtain time evolution of surface density.

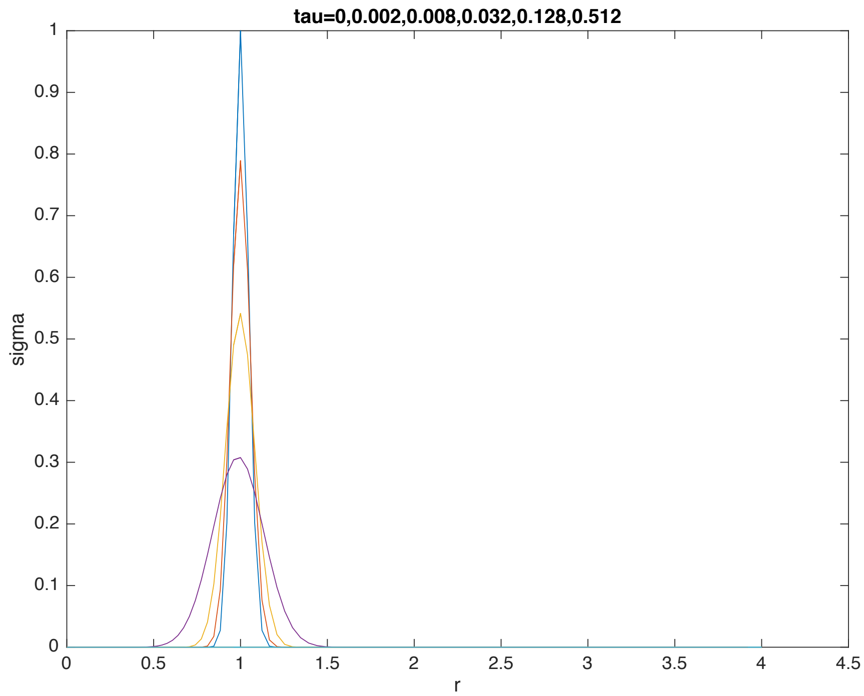

Figure 2

Question 5

(i) Program from Question 4 is adapted to produce table 1. It contains height and position (r) of the peak in surface density at the times used in Question 4.

Table 1

Time (τ)	Height (σ)	Position(r)
0	1.0000	1.0000
0.002	0.6231	1.0000
0.008	0.2933	1.0000
0.032	0.0946	1.0000
0.128	0	0.0004
0.512	0	0.0004

Furthermore, the program is also adapted to produce the time evolution of the total angular momentum. (Notes it is in units of $R_0^2(GM)^{1/2}$)

(ii)

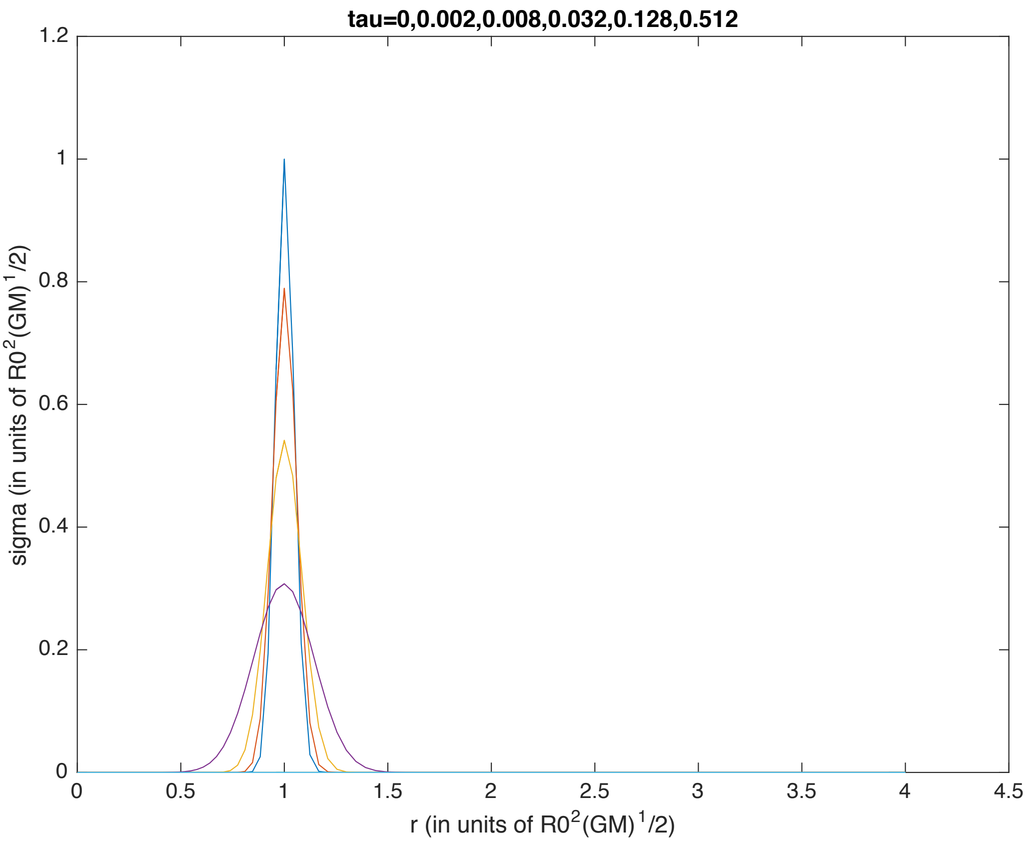

Figure 3

Figure 4 is the position of the peak angular momentum surface density as a function of time. Note that the position should be at $r = 1$ for all time. However, at later stage, the maximum height becomes computationally zero, which causes the position to drops to 0.0004.

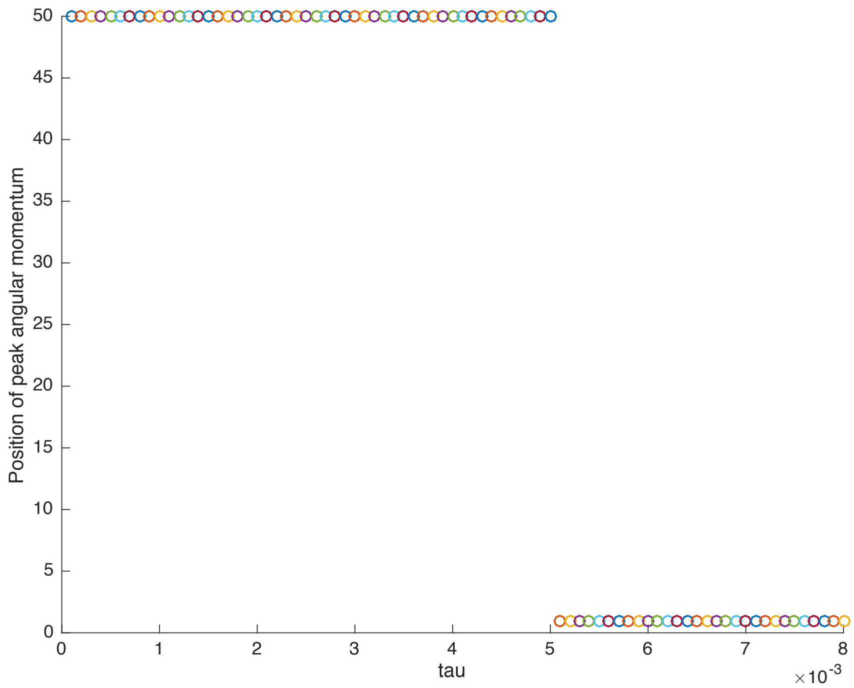

Figure 4

(iii) Figure 5 is a (zoomed in) plot of angular momentum and surface density

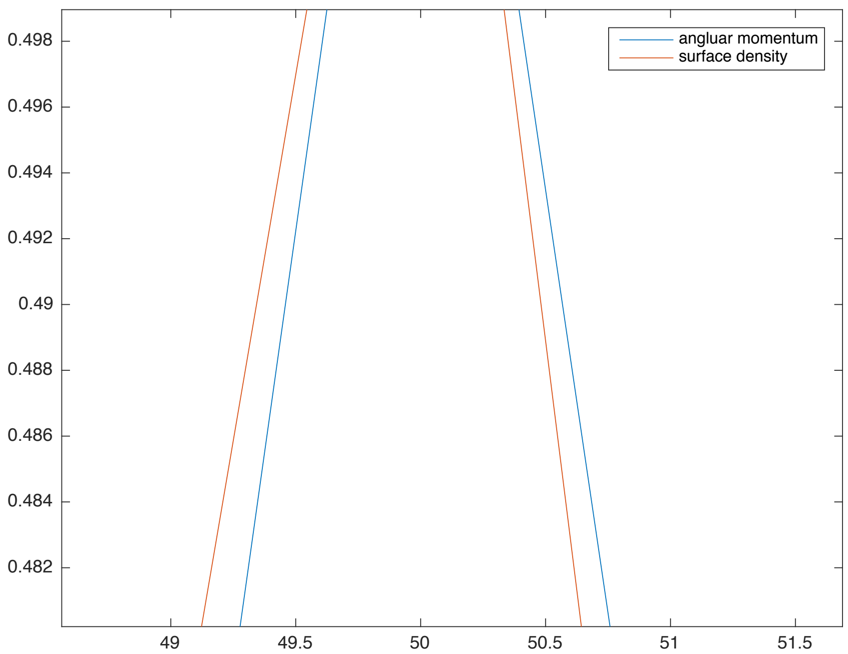

Figure 5

Hence the angular momentum is slightly shifted to the right in the x-axis compared with the surface density.

Question 6

Substitute $r = \frac{R}{R_0}, \tau = \frac{t}{t_0}, \sigma(r,\tau) = \frac{\Sigma}{\Sigma_0}\ \eta(r,\sigma) = \frac{\nu}{\nu_0}\ \ and \frac{dR}{dt} = V_R(t)$ into equation (8),
The condition for t_0 to satisfies, such that equation (8) remains the same for these dimensionless variables is still

$$t_0 = \frac{R_0^2}{\nu_0}$$

Hence we have

$$\frac{\partial r}{\partial \tau} = -\frac{3}{\sigma r^{1/2}}\frac{\partial}{\partial r}\left(\eta\sigma r^{1/2}\right)$$

By inverse chain rule, we obtain

$$\frac{\partial r}{\partial \tau} = -\frac{3}{\sigma r^{1/2}}\frac{\partial}{\partial r}\left(\eta\sigma r^{1/2}\right)$$

Let $X = r^{1/2}$, we obtain

$$\frac{\partial X}{\partial \tau} = -\frac{1}{4\sigma X^3}\frac{\partial}{\partial X}(3\eta\sigma X)$$

$$\frac{\partial X}{\partial \tau} = -\frac{1}{f}\frac{\partial g}{\partial r}$$

Where $f = 4X^3\sigma\ and\ g = 3\eta\sigma X$ as in Question 3.
Using the similar set up as in Question 4, we have

$$X_i^{n+1} = X_i^n - \frac{\Delta\tau}{\Delta X}\frac{1}{f_i^n}\left(g_{i+1}^n - g_i^n\right)$$

The radial velocity (in term of dimensionless variables) may be discretised as

$$v_i^n = \frac{r_i^{n+1} - r_i^n}{\Delta\tau}$$

Hence, by adopting the program in Question 4, we obtain a plot of radial velocity as a function of radius in the time steps.

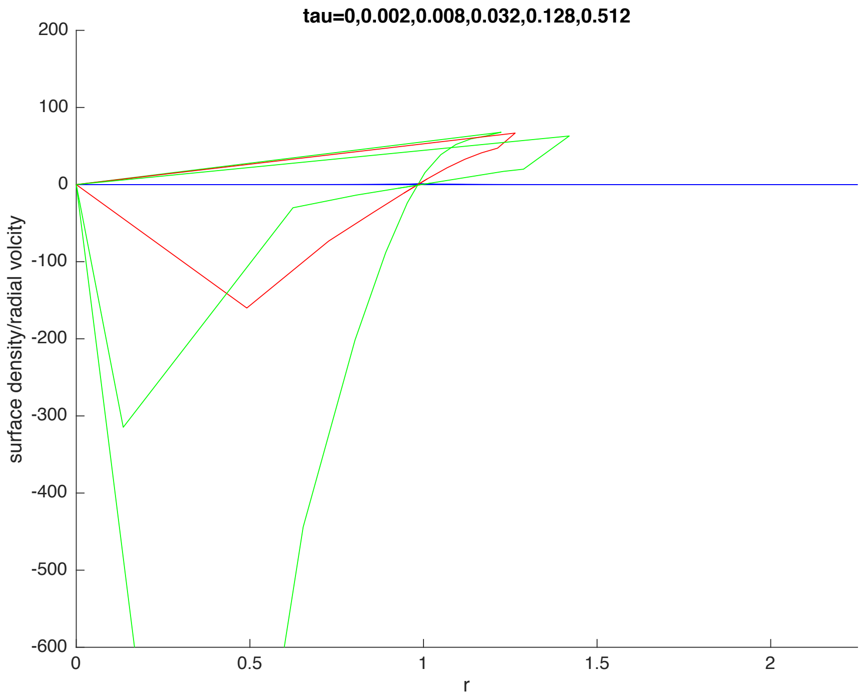

Figure 6

Question 7

By using the radial velocities in Question 6, we obtain figure 7, a plot of the evolution of particles' orbits for $r_0 = 0.9, 0.95, 1.0, 1.05, and\ 1.1$.

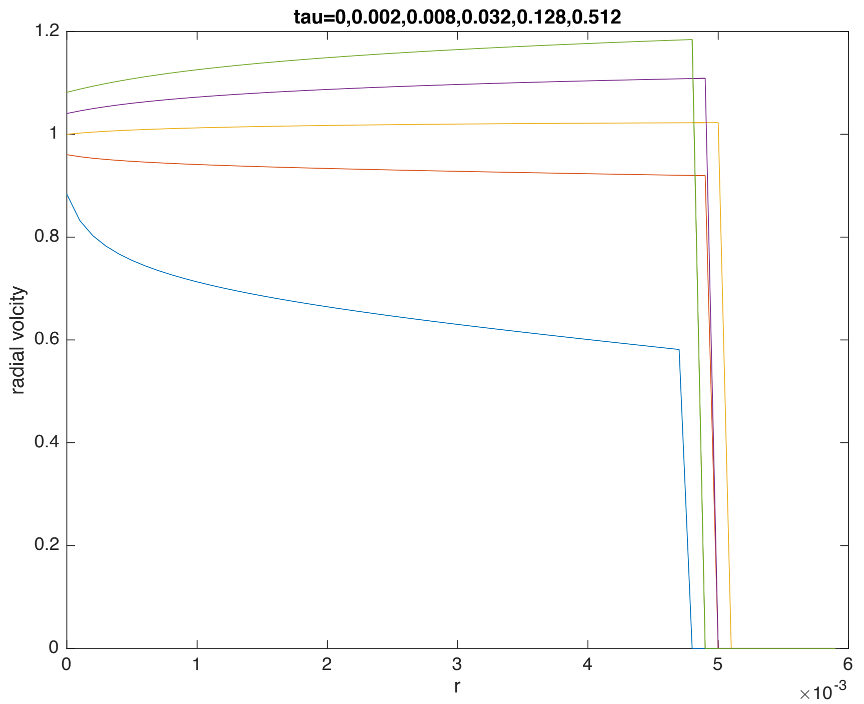

Figure 7

Figure 8 is a plot of maximum radius attained by the particles as a function of its initial radius.

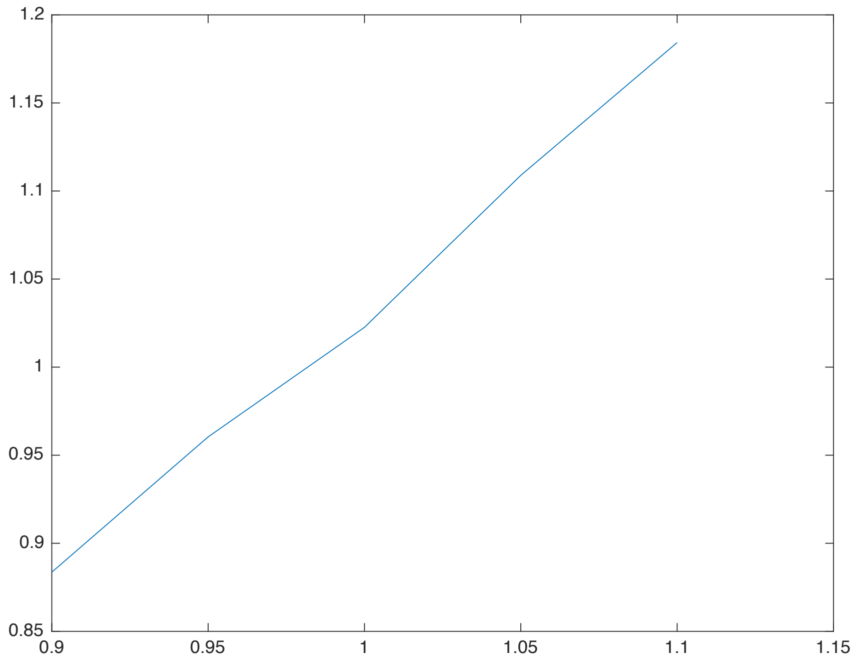

Figure 8

Figure 9 is a plot of time took to reach the maximum radius as a function of its initial radius.

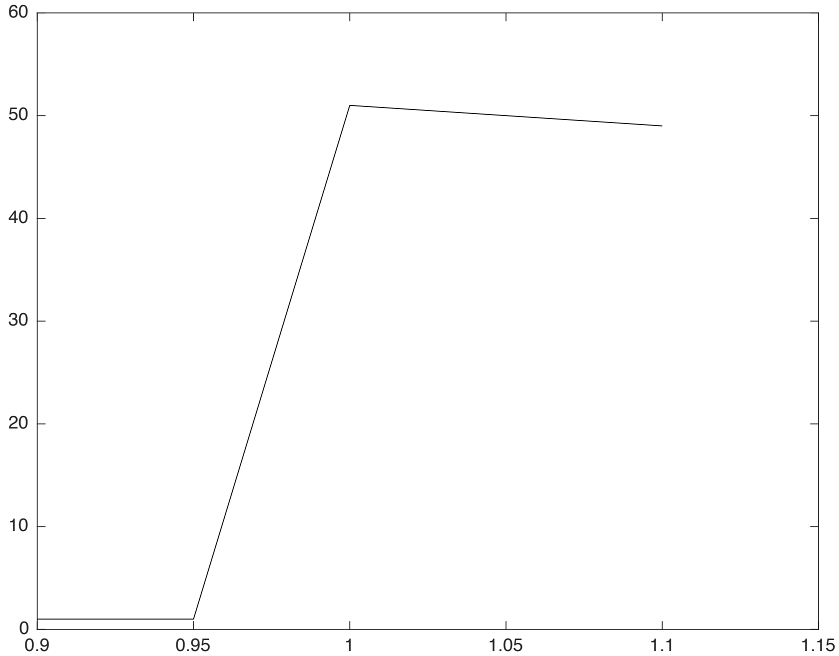

Figure 9

Figure 10 is a pot of the time it takes to reach a boundary (at the lower bound) as a function of its initial radius.

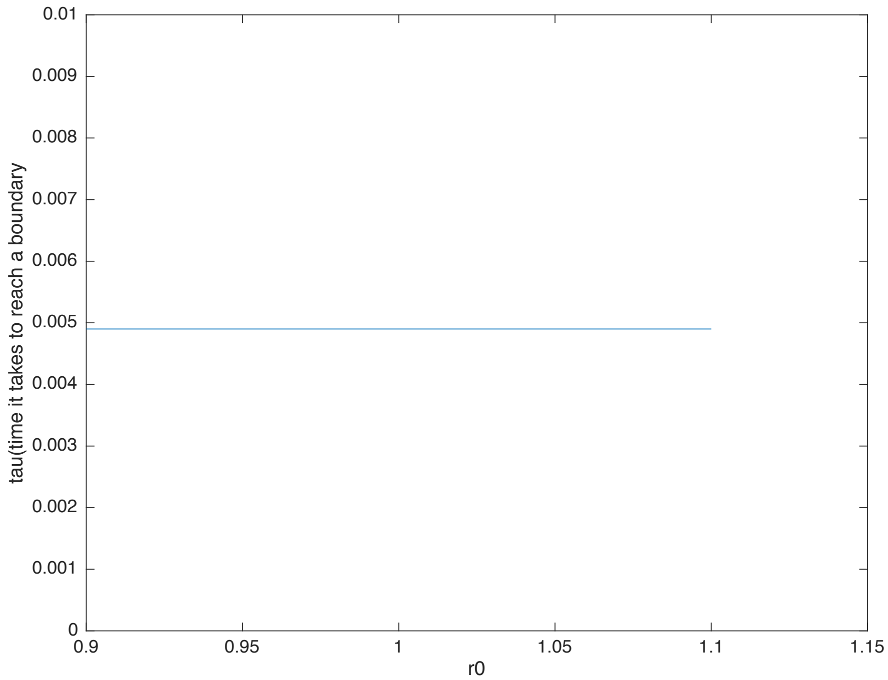

Figure 10

Question 8
From figure 10 in Question 7, all the initial radii $r_0 = 0.9 - 1.1$ have reached the inner boundary by $\tau = 0.512$.
Hence, by using equation (12), we deduce that

$$M_r = \int_{0.9}^{1.1} \pi r^2 \exp\left(-\frac{(r^{\frac{1}{2}} - 1)^2}{0.001}\right) dr = 0.344$$

$$M_T = \int_{0.004}^{4} \pi r^2 \exp\left(-\frac{(r^{\frac{1}{2}} - 1)^2}{0.001}\right) dr = \quad 0.354$$

where M_r is the initial mass that has reached the inner boundary by $\tau = 0.512$ and M_T is the total initial mass. Numerical results are obtained from program 'Question 8', by using composite Simpson's rule.
Hence, the fraction of the initial mass that reached the inner boundary by $\tau = 0.512$ is approximately 97%.

From figure 8, we can see that the maximum radius attained by the particle is below the outer boundary. Hence the fraction of the initial mass that reached the outer boundary by $\tau = 0.512$ is 0.

From the results in Question 4, we obtain table 2

Table 2

Initial radius (reached inner boundary by $\tau = 0.512$)	Surface density (σ) at $\tau = 0.512$	Mass($\pi r_i^2 \sigma_i$) at $\tau = 0.512$
0.9	0	0
0.95	0	0
1.0	0	0
1.05	0	0
1.1	0	0

From table 2, we conclude that the fraction of the remain value of the initial mass that has reached the inner boundary is 0 at $\tau = 0.512$. And similarly, the fraction of the initial mass that has reached the outer boundary is also 0 at $\tau = 0.512$.
Hence, we can obtain table 3 to compare these values

Table 3

in terms of fraction of the initial mass			
reached the inner boundary by $\tau = 0.512$	97%	the remain value that has reached the inner boundary $\tau = 0.512$	0%
reached the outer boundary by $\tau = 0.512$	0%	the remain value that has reached the outer boundary $\tau = 0.512$	0%

Hence, we conclude that the result for the inner boundary is fair accuracy. The results for the outer boundary are both zero, due to the effect of diffusion (see equation 9).

Appendix

Program for Question 2

```
x = linspace(1,100,1000);
y=(1-x.^(-1/2))/(6*pi);
plot(x,y)
xlabel('R/Rin(in units m dot)')
ylabel('nu*sigma(in units m dot)')
```

Program for Question 4

```
X1=0.02;
Xn=2;
dX=2/100;
m=100;
X=zeros(m,1);
X(1)=dX;
for i=2:m
    X(i)=X(i-1)+dX;
end
dtau=0.0001;
n=(0.52/dtau);
sigma=zeros(m,n+1);
f=zeros(m,n+1);
g=zeros(m,n+1);
for i=1:m
sigma(i,1)=exp(-((X(i)-1)^2)/0.001);
end
for i=1:m
        g(i,1)=3*sigma(i,1)*X(i);
        f(i,1)=4*(X(i)^3)*sigma(i,1);
end
for j=1:n+1
    for i=j+1:m-j
        f(i,j+1)=f(i,j)+(dtau/(dX^2))*(g(i+1,j)-2*g(i,j)+g(i-1,j));
        sigma(i,j+1)=f(i,j+1)/(4*X(i)^3);
        g(i,j+1)=3*sigma(i,j+1)*X(i);
    end
end
    plot(X.^2,sigma(:,1))
    hold on
    plot(X.^2,sigma(:,3))
    plot(X.^2,sigma(:,9))
    plot(X.^2,sigma(:,33))
    plot(X.^2,sigma(:,129))
    plot(X.^2,sigma(:,513))
    xlabel('r')
ylabel('sigma')
title('tau=0,0.002,0.008,0.032,0.128,0.512')
    hold off
```

Program for Question 5

```
X1=0.02;
Xn=2;
dX=2/100;
m=100;
X=zeros(m,1);
X(1)=dX;
for i=2:m
    X(i)=X(i-1)+dX;
end
```

```
dtau=0.0001;
n=(0.52/dtau);
sigma=zeros(m,n+1);
f=zeros(m,n+1);
g=zeros(m,n+1);
for i=1:m
sigma(i,1)=exp(-((X(i)-1)^2)/0.001);
end
for i=1:m
        g(i,1)=3*sigma(i,1)*X(i);
        f(i,1)=4*(X(i)^3)*sigma(i,1);
end
for j=1:n+1
    for i=j+1:m-j
        f(i,j+1)=f(i,j)+(dtau/(dX^2))*(g(i+1,j)-2*g(i,j)+g(i-1,j));
        sigma(i,j+1)=f(i,j+1)/(4*X(i)^3);
        g(i,j+1)=3*sigma(i,j+1)*X(i);
    end
end
[M0,I0]=max(sigma(:,1))
    [M1,I1]=max(sigma(:,3));
    hold on
    [M2,I2]=max(sigma(:,9));
   [M3,I3]=max(sigma(:,33));
    [M4,I4]=max(sigma(:,129));
    [M5,I5]=max(sigma(:,513));

    hold off
a=[M0^2,I0*dX;M1^2,I1*dX;M2^2,I2*dX;M3^2,I3*dX;M4^2,I4*dX;M5^2,I5*dX]
```

Program for Question 6

```
dX=2/100;
m=100;
dtau=0.0001;
n=(0.52/dtau);
X=zeros(m,n+1);
X(1,1)=dX;
for i=2:m
    X(i,1)=X(i-1,1)+dX;
end
f=zeros(m,n+1);
g=zeros(m,n+1);
for i=1:m
        g(i,1)=log(3*sigma(i,1)*X(i,1));
        f(i,1)=-3/(4*X(i,1).^2);
end

for j=1:n+1
    for i=1:m-j
        if sigma(i,j)<(10^-40)
            X(i,j+1)=0;
        else
           X(i,j+1)=X(i,j)+(dtau/dX)*f(i,j)*(g(i+1,j)-g(i,j));
        f(i,j+1)= -3/(4*(X(i,j+1).^2));
        g(i,j+1)=log(3*sigma(i,j+1)*X(i,j+1));
        end
    end
end
v=zeros(m,n+1);
for j=1:n
   for i=1:m
    v(i,j)=(X(i,j+1)^2-X(i,j)^2)/dtau;
```

```
   end
end
 xlabel('r')
ylabel('surface density/radial volcity')
title('tau=0,0.002,0.008,0.032,0.128,0.512')
   hold on

    plot(X(:,3).^2,v(:,3),'g')
    plot(X(:,9).^2,v(:,9),'r')
    plot(X(:,33).^2,v(:,33),'g')
    plot(X(:,129).^2,v(:,129),'r')
    plot(X(:,513).^2,v(:,513),'g')
    hold off
```

Program for Question 7

```
dX=2/100;
m=100;
X=zeros(m,1);
X(1)=dX;
for i=2:m
    X(i)=X(i-1)+dX;
end
dtau=0.0001;
n=(1/dtau);
sigma=zeros(m,n+1);
f=zeros(m,n+1);
g=zeros(m,n+1);
for i=1:m
sigma(i,1)=exp(-((X(i)-1)^2)/0.001);
end
for i=1:m
        g(i,1)=3*sigma(i,1)*X(i);
        f(i,1)=4*(X(i)^3)*sigma(i,1);
end
for j=1:n+1
    for i=j+1:m-j
        f(i,j+1)=f(i,j)+(dtau/(dX^2))*(g(i+1,j)-2*g(i,j)+g(i-1,j));
        sigma(i,j+1)=f(i,j+1)/(4*X(i)^3);
        g(i,j+1)=3*sigma(i,j+1)*X(i);
    end
end

clear m n X f g dX dtau

dX=2/100;
m=100;
dtau=0.0001;
n=(1/dtau);
X=zeros(m,n+1);
X(1,1)=dX;
for i=2:m
    X(i,1)=X(i-1,1)+dX;
end
f=zeros(m,n+1);
g=zeros(m,n+1);
for i=1:m
        g(i,1)=3*sigma(i,1)*X(i,1);
        f(i,1)=4*(X(i,1)^3)*sigma(i,1);
end

for j=1:n+1
```

```
    for i=1:m-j
        if f(i,j)<(10^-4)
            X(i,j+1)=0;
        else
        X(i,j+1)=X(i,j)-(dtau/dX)*(1/f(i,j))*(g(i+1,j)-g(i,j));
        f(i,j+1)=4*(X(i,j+1)^3)*sigma(i,j+1);
        g(i,j+1)=3*sigma(i,j+1)*X(i,j+1);
        end
    end
end

tau=zeros(1,n+1);
tau(1,1)=0;
for i=2:n+1
    tau(1,i)=tau(1,i-1)+dtau;
end
   plot(tau(1,1:80),X(round(sqrt(0.9)/dX),1:80).^2)
   hold on
  plot(tau(1,1:60),X(round(sqrt(0.95)/dX),1:60).^2)
   plot(tau(1,1:60),X(round(sqrt(1.0)/dX),1:60).^2)
   plot(tau(1,1:60),X(round(sqrt(1.05)/dX),1:60).^2)
   plot(tau(1,1:60),X(round(sqrt(1.1)/dX),1:60).^2)
   hold off
    xlabel('r')
ylabel('radial volcity')
title('tau=0,0.002,0.008,0.032,0.128,0.512')
    hold off
M=zeros(5,1);
I=zeros(5,1);
N=zeros(5,1);
for i=1:5
    N(i)=0.85+i*0.05
end

[M(1) I(1)]=max(X(round(sqrt(0.9)/dX),:).^2);
[M(2) I(2)]=max(X(round(sqrt(0.95)/dX),:).^2);
[M(3) I(3)]=max(X(round(sqrt(1.0)/dX),1:60).^2);
[M(4) I(4)]=max(X(round(sqrt(1.05)/dX),1:60).^2);
[M(5) I(5)]=max(X(round(sqrt(1.1)/dX),1:60).^2);
%plot(N,M)
%plot(N,I,'k')
```

Program for Question 8

```
x0=0.0004;
x1=4;
m=10000;
h=(x1-x0)/m;
x=x0:h:x1;
f=pi*(x.^2).*exp(-((x.^(1/2)-1).^2)/0.001);

I1=f(2);
for i=4:2:m-1
    I1=I1+f(i);
end
I2=f(3);
for i=5:2:m-2
    I2=I2+f(i);
end

I=(h/3)*(f(1)+4*I1+2*I2+f(m))
```