1.1 Fourier Transforms of Bessel Functions

1 Introduction

Question 1

We let $y' = z$ and equation (1) can be written as

$$x^2 z' + xz + (x^2 - n^2)y = 0$$

Hence we can use Runge-Kutta 4 method to obtain results numerically. A fixture of forwards and backwards integrations are used to illustrate interesting behaviors of Bessel's functions in case $n = 0, 1\ and\ 4$.

Interesting behavior of (numerical) Bessel's functions

- For $n = 0,1,4$, Bessel's functions are oscillating and the amplitudes of the oscillations are decreasing as $x \to \infty$ (see figure 1 to 3)
- The amplitudes of oscillations increase as n increases. (see figure 1 to 3)
- It seems to be there are infinite many of zeroes for all Bessel's functions (see figure 1 to 3)
- Two Bessel's functions, $J_n^1\ and\ J_n^2$ with initial conditions (x_0, y_0, z_0) and $(x_0, -y_0, -z_0)$ respectively, satisfy $J_n^1 = -J_n^2$. (see figure 4)
- As $|y_0|$ increases ($x_0\ and\ z_0$remains the same), Bessel's function increases the amplitudes of oscillations, but remains in the same frequencies. (see figure 5 and 6)

Figures 1 to 3 contain 2 plots, with initial condition $x_0 = 0.1, y_0 = 1, z_0 = 1$ and $x_0 = 1, y_0 = -1, z_0 = 1$ respectively.

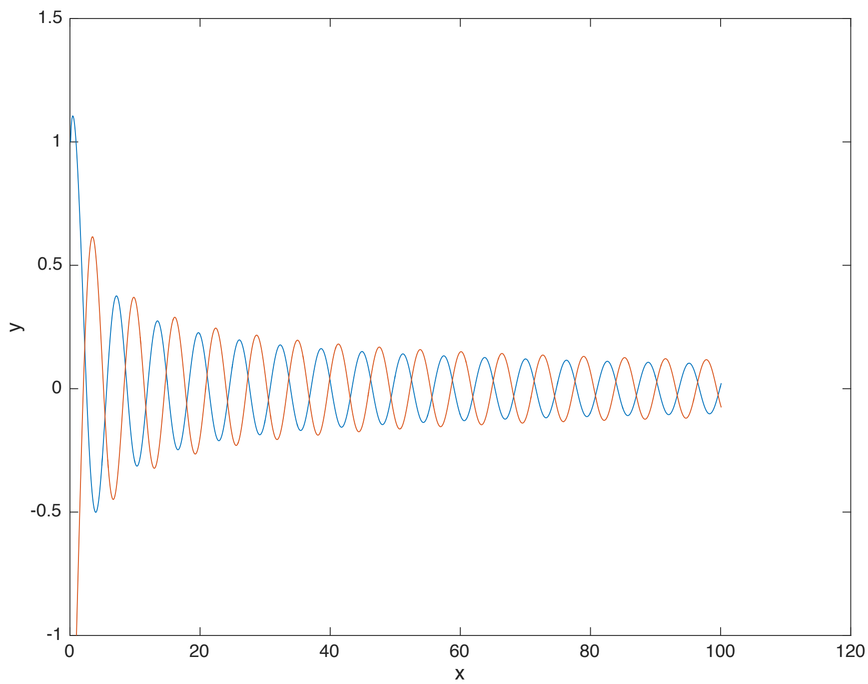

Figure 1: n=0 (forwards integration)

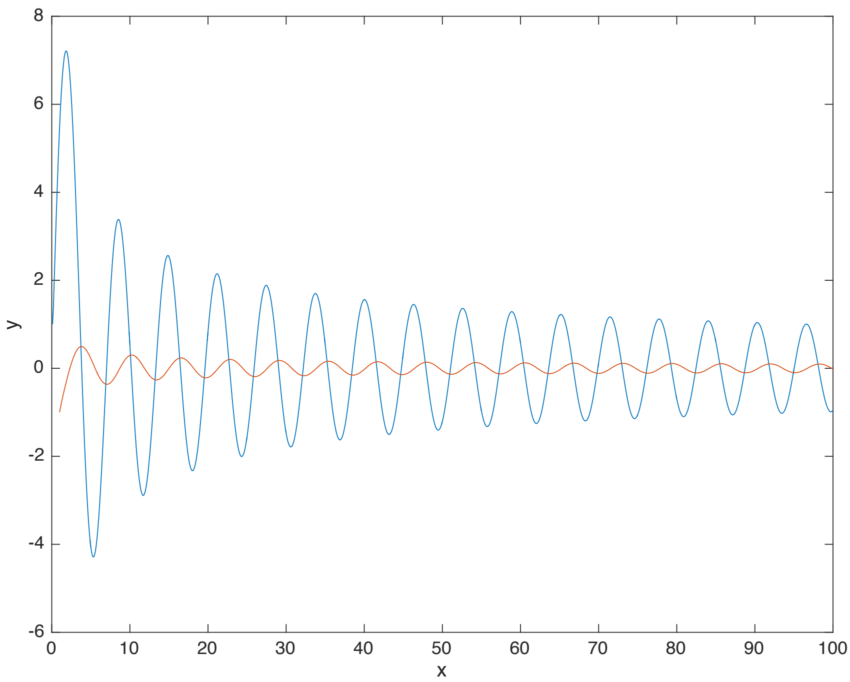

Figure 2: n=2 (backwards integration)

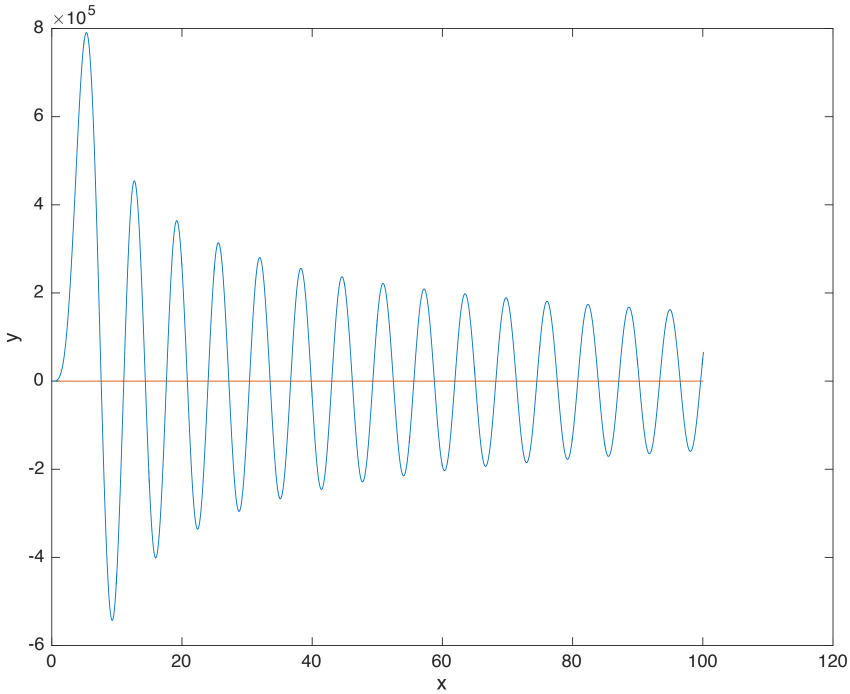

Figure 3: n=4 (forwards integration)

Figures 4 contains 2 plots, with initial condition $x_0 = 1, y_0 = 1, z_0 = 1$ and $x_0 = 1, y_0 = -1, z_0 = -1$ respectively.

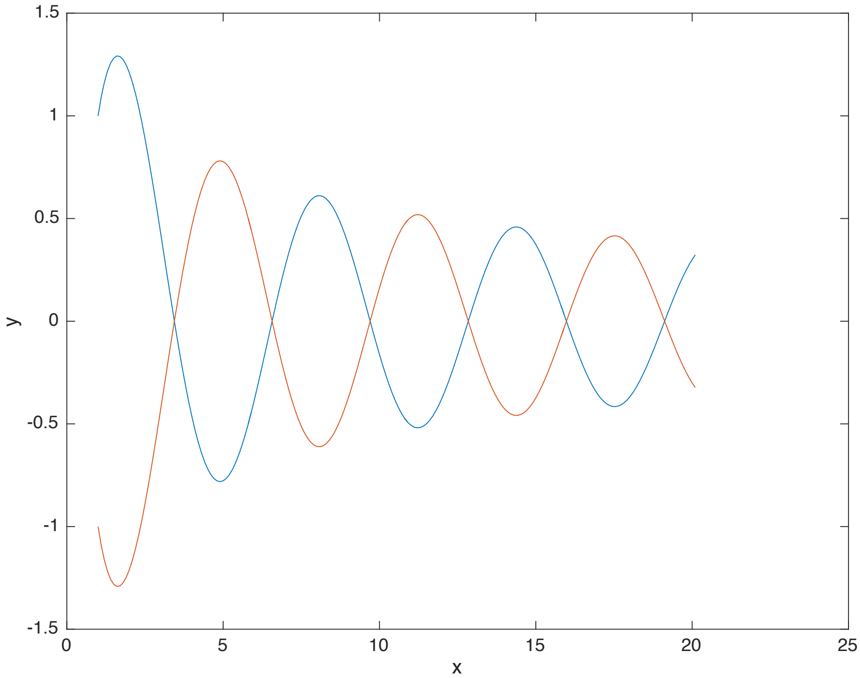

Figure 4: n=0 (forwards integration)

Figure 5 contains plots with initial conditions $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\\1\end{pmatrix}$ respectively.

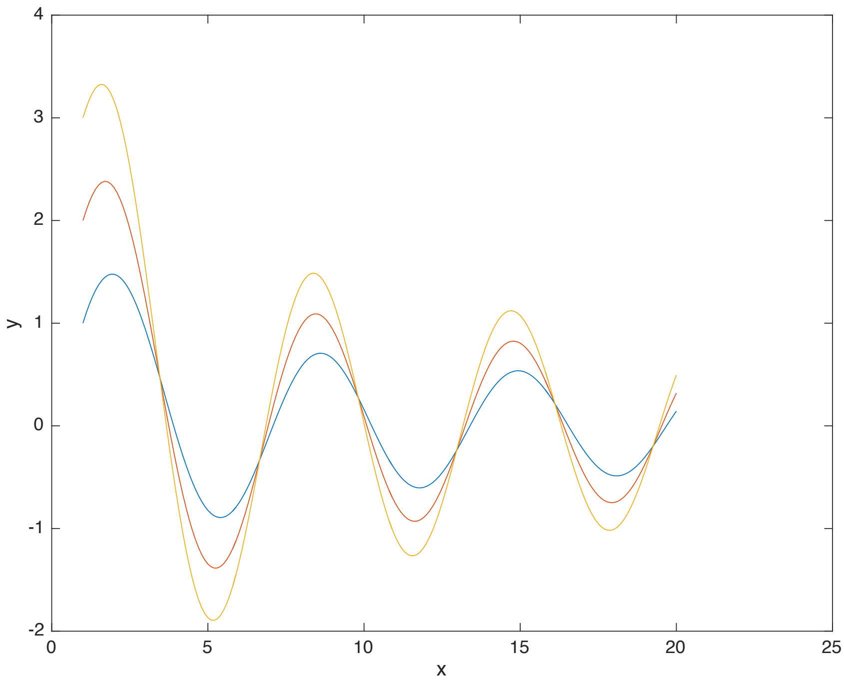

Figure 5: n=1 (backwards integration)

Figure 6 contains plots with initial conditions $\begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\1\end{pmatrix}$ respectively.

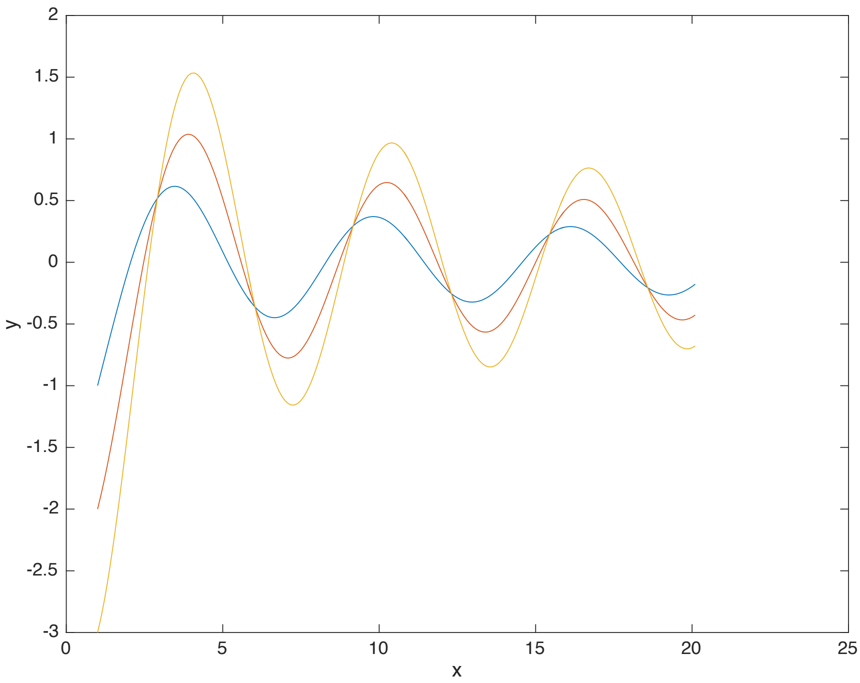

Figure 6: n=0 (forwards integration)

The numerical methods do not work when $x_0 = 0$. This is due to the '$\div\, x_i$' is involved in the difference equation of z_i, which causes z_i tends towards infinite at the second iteration.

Question 2

Figure 7 to 9 plots J_n for $n = 0, 1$ and 4 for $0 \leq x \leq 20$.

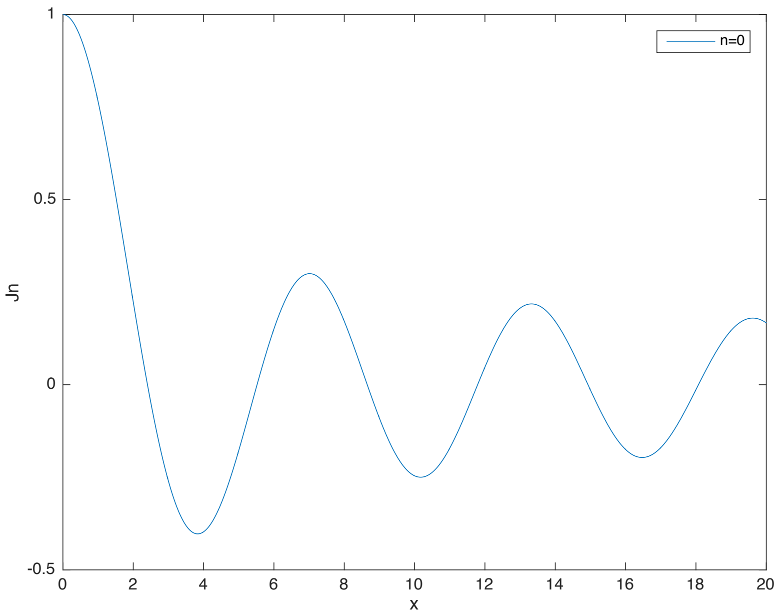

Figure 7: n=0

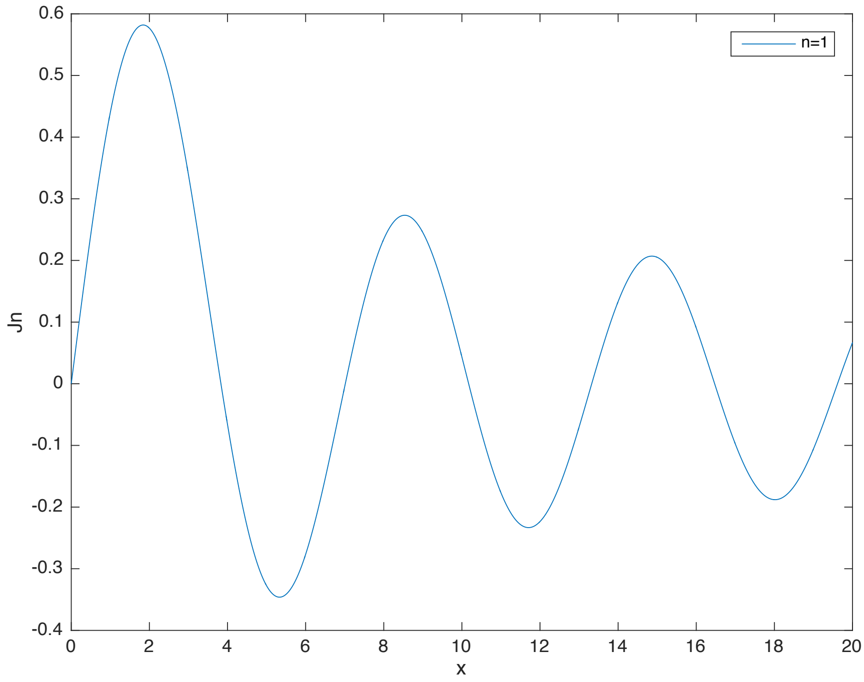

Figure 8

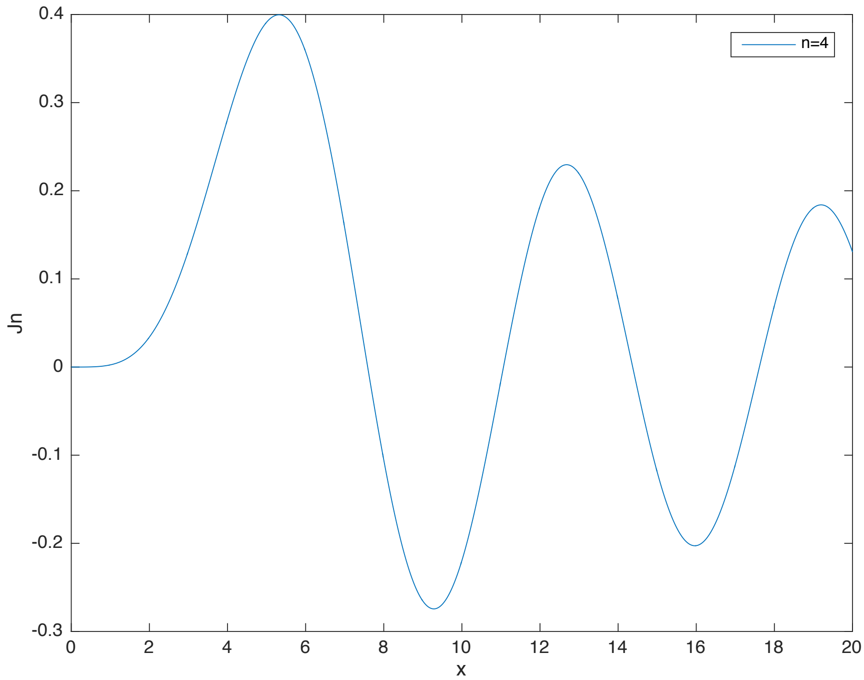

Figure 9

For $x > 100$, the summation method is not accurate. As $\left(\frac{1}{2}x\right)^{2r+n}$ becomes too large for Matlab to handle for large r, hence to series solution becomes inaccurate.

2 The Discrete Fourier Transform

Question 3
Consider the following

$$\left|\hat{F}(k)-\hat{F}_s\right| = \lim_{X\to\infty}\left|\int_{-X}^{+X} F(x)\exp(-2\pi ikx)\,dx - \frac{X}{N}\sum_{r=0}^{N-1} F_r\omega_N^{-rs}\right|$$

$$\left|\hat{F}(k)-\hat{F}_s\right| = \lim_{X\to\infty} 2\times\sum_{r=0}^{N-1}\left|\int_{r\Delta x}^{(r+1)\Delta x} F(x)\exp(-2\pi ikx)\,dx - \frac{X}{N}F_r\omega_N^{-rs}\right| \quad (*)$$

To investigated the the upper bound of $\left|\hat{F}(k)-\hat{F}_s\right|$, we need to use the mean value theorem:
A function f is continuous on $[a,b]$, where $a<b$ and differentiable on (a,b), such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Now, let $d\in[a,b]$, such that $f(d) = \max_{x\in[a,b]} f'(x)$, then we have

$$(b-a)f'(d) \geq |f(b)-f(a)|$$

Let x, y real variables, which lies in $[a,-b]$ Apply this inequality on interval $[a,b]$

$$(b-a)f(d) \geq \left|\int_a^b f(x)-f(y)dx\right|$$

where $\int_b^a f(y)dx \geq 2Nf(x)$, hence

$$(b-a)f(d) \geq \left|\int_a^b f(x)dx - (b-a)f(x)\right| \quad (**)$$

We apply $(**)\ on\ (*)$

$$\left|\hat{F}(k)-\hat{F}_s\right| = \lim_{X\to\infty} 2\times\sum_{r=0}^{N-1}\left|\int_{r\Delta x}^{(r+1)\Delta x} F(x)\exp(-2\pi ikx)\,dx - \frac{X}{N}F_r\omega_N^{-rs}\right|$$

$$\left|\hat{F}(k)-\hat{F}_s\right| \leq \lim_{X\to\infty} 2\times\sum_{r=0}^{N-1}\left|\frac{X}{N}F(d)\right| \to 0$$

Provided $\frac{X}{N}\to 0$.
Hence under $X\to\infty, N\to\infty\ and\ \frac{X}{N}\to 0,$ then the DFT tend to the Fourier transform.

3 Fourier Transforms of Bessel Functions

Question 4
We write

$$I_1 = \int_0^X F(x)\cos(2\pi kx)dx - i\int_0^X F(x)sin(2\pi kx)\,dx$$

$$I_2 = \int_0^X F(x)[\cos(2\pi kx) - i\sin(2\pi kx)]\,dx + \int_{-X}^0 F(x)[\cos(2\pi kx)+i\sin(2\pi kx)]dx$$

Use a change of variable, $x\to -x$ on the second integral of I_2 and use $F(x)=F(-x)$, we obtain

$$I_2 = \int_0^X F(x)[\cos(2\pi kx) - i\sin(2\pi kx)]\,dx + \int_0^X F(x)[\cos(2\pi kx)+i\sin(2\pi kx)]dx$$

Hence

$$Im(I_2) = 0$$

and

$$Re(I_2) = 2\int_0^X F(x)\cos(2\pi kx)\,dx = 2Re(I_1)$$

By discrete Fourier transform, we have

$$I_1 = \frac{X}{N}\sum_{r=0}^{N-1} F_r\omega_N^{-rs}$$

and

$$I_2 = \frac{X}{N}\sum_{r=-N}^{N-1} F_r\omega_N^{-rs}$$

Hence $2I_1$contains $2F_0$ and does not contain F_N. Where as I_2 contains one only F_0 and F_N. Therefore by replace F_o $with\frac{1}{2}(F_o + F_N)$ before calculating DFT would allow $2I_1$ $matches$ $with$ I_2completetly.

If $F(-x) = -F(x)$, then the change of variable $x \to -x$ on the second integral of I_2 would give us

$$I_2 = \int_0^X F(x)[\cos(2\pi kx) - i\sin(2\pi kx)]\,dx - \int_0^X F(x)[\cos(2\pi kx) + i\sin(2\pi kx)]dx$$

Hence

$$Re(I_2) = 0$$

and

$$Im(I_2) = -2\int_0^X F(x)\sin(2\pi kx)\,dx = 2Im(I_1)$$

Similarly, F_0 should be replaced by $\frac{1}{2}(F_0 - F_N)$ before calculating the DFT.

Question 5

(i) Note: from the series solution for $J_n(x)$, we see that $J_n(x)$ is even if n is even and odd if n is odd. Figure 10 to 14 are plots of $\hat{J}_n(k)$ at range $-\frac{1}{\pi} \leq k \leq \frac{1}{k}$, for $n = 0, 1, 2, 4$ and 8.

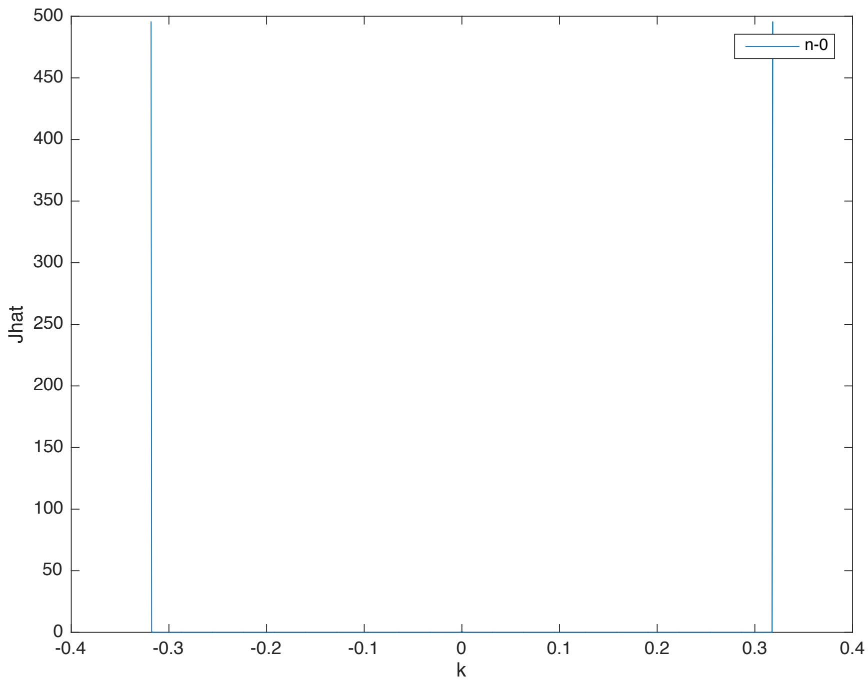

Figure 10

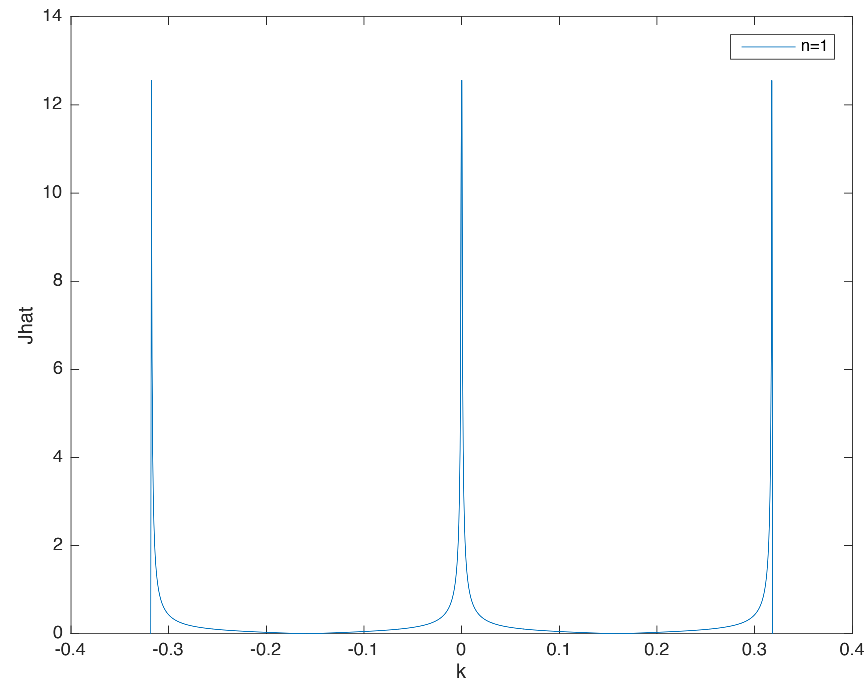

Figure 11

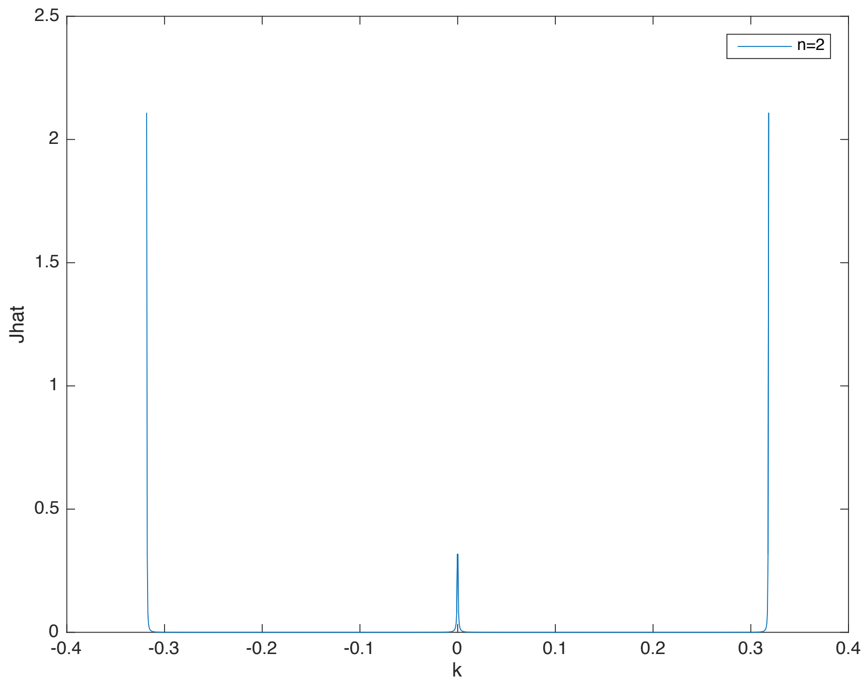

Figure 12

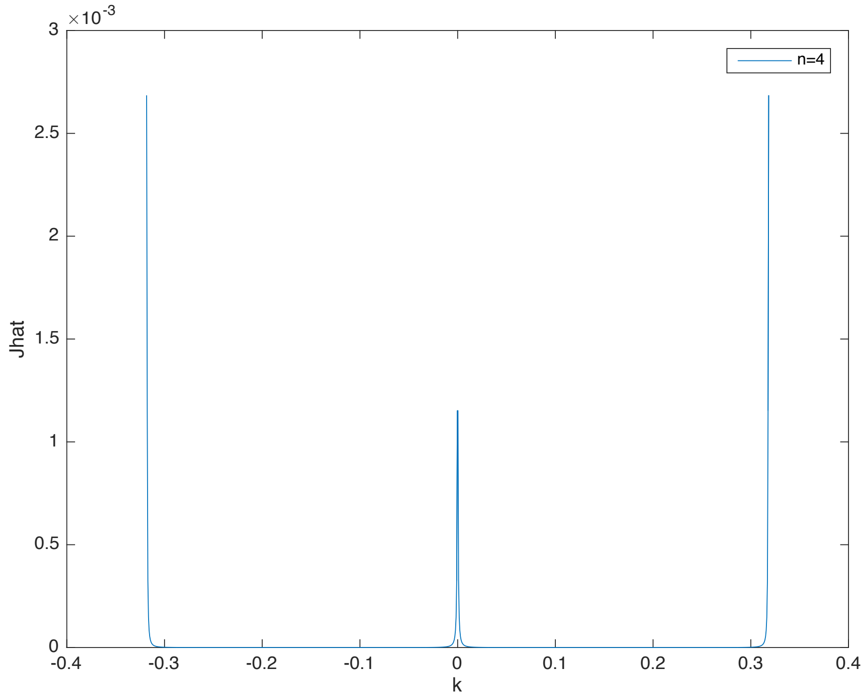

Figure 13

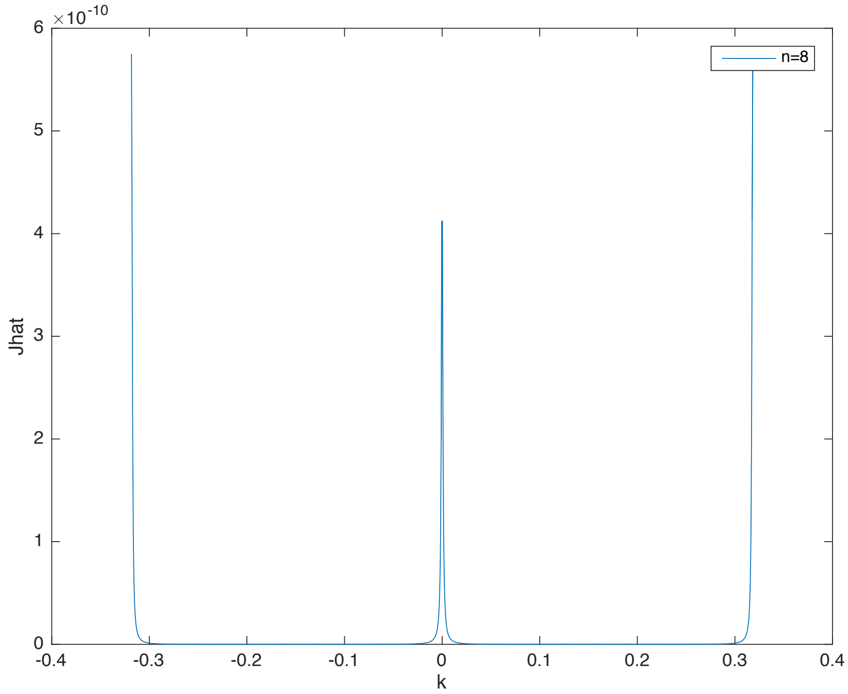

Figure 14

(ii) Figure 15 to 14 are plots of $\hat{J}_n(k)$ at range $-\frac{1}{\pi} \leq k \leq \frac{1}{k}$, for $n = 0, 1, 2$ and 4.

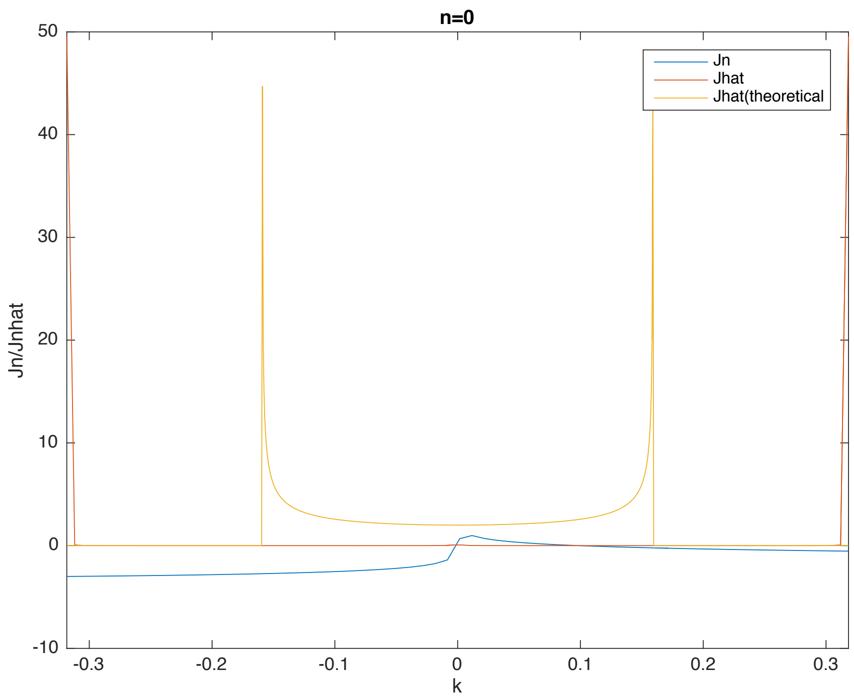

Figure 15

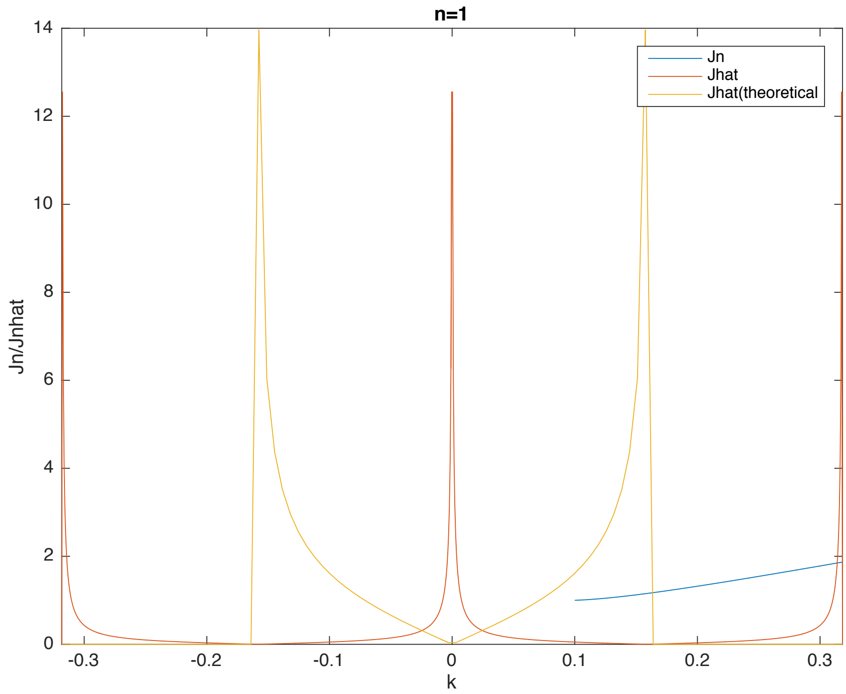

Figure 16

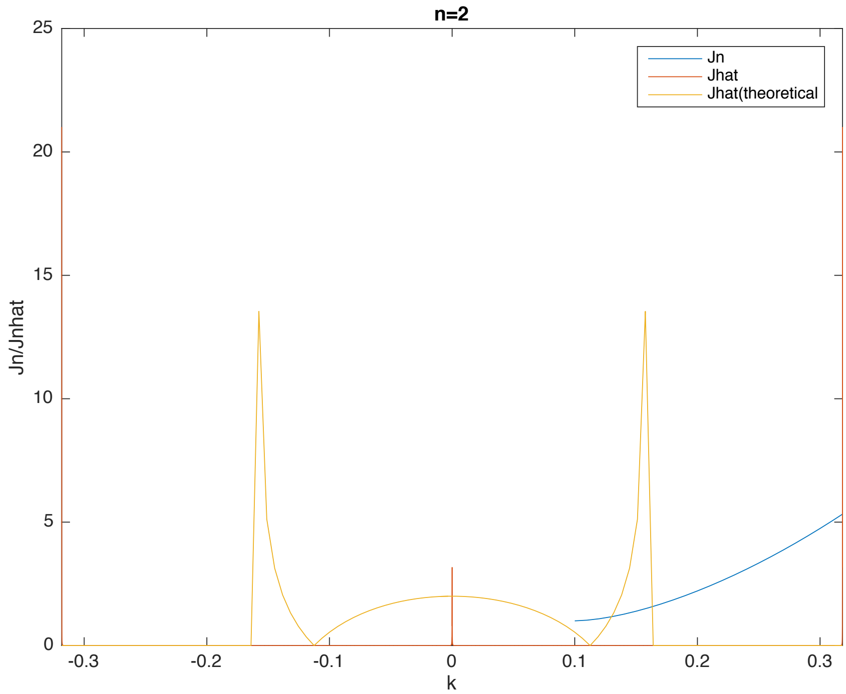

Figure 17

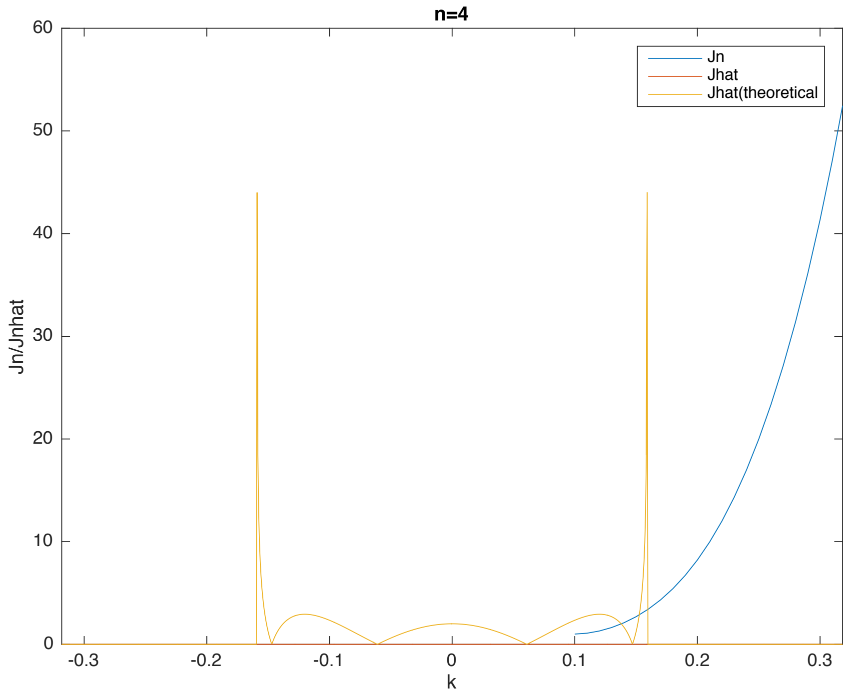

Figure 18

<u>Comment</u>

- The theoretical formula may give problems at $k = \frac{1}{2\sqrt{\pi}}$. The FFT deals with by using the the results from its neighbourhood.
- The size of the spikes increases proportionally with the N (see figure 19 to 21)
- Increase X increases the size of the spikes and the gaps between the spikes, as they only apply at the edges and the origin (see figure 22)
- The error decreases as X increases, or as N decreases. Moreover, the error stays constant if X/N stays constant. Hence the results agree with our answer in Question 3. (see figure 23 to 25)

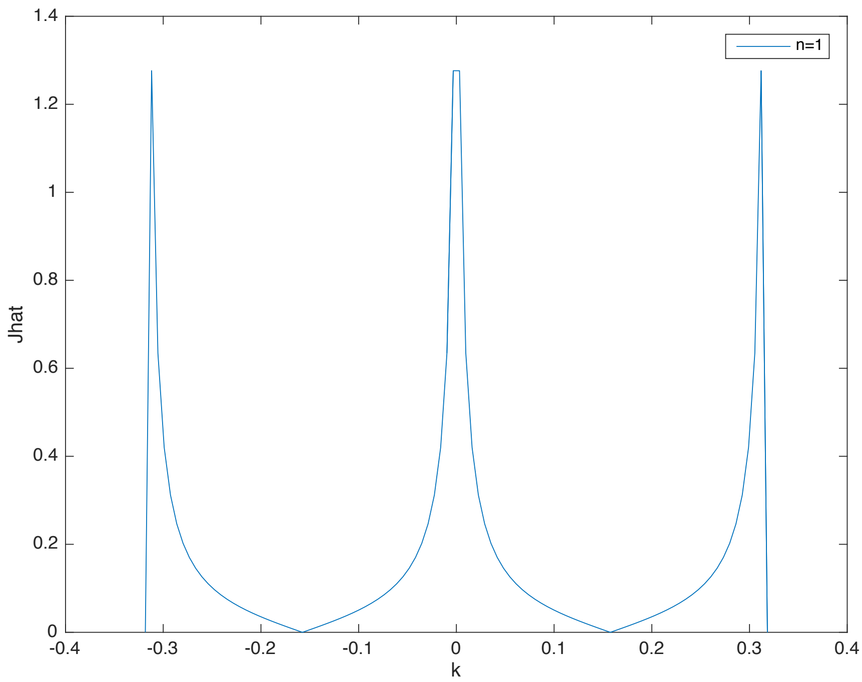

Figure 19 N=50

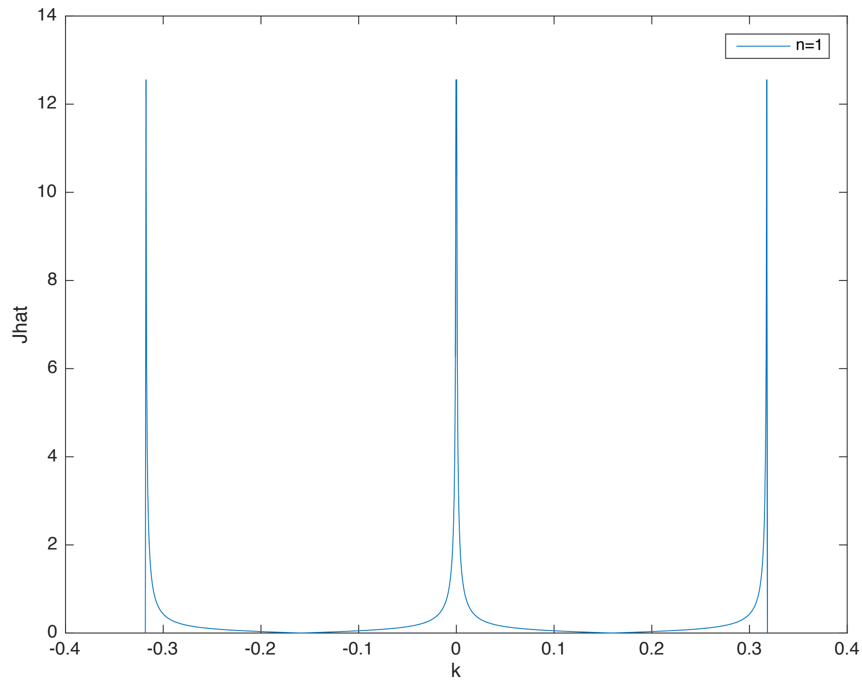

Figure 20 N=500

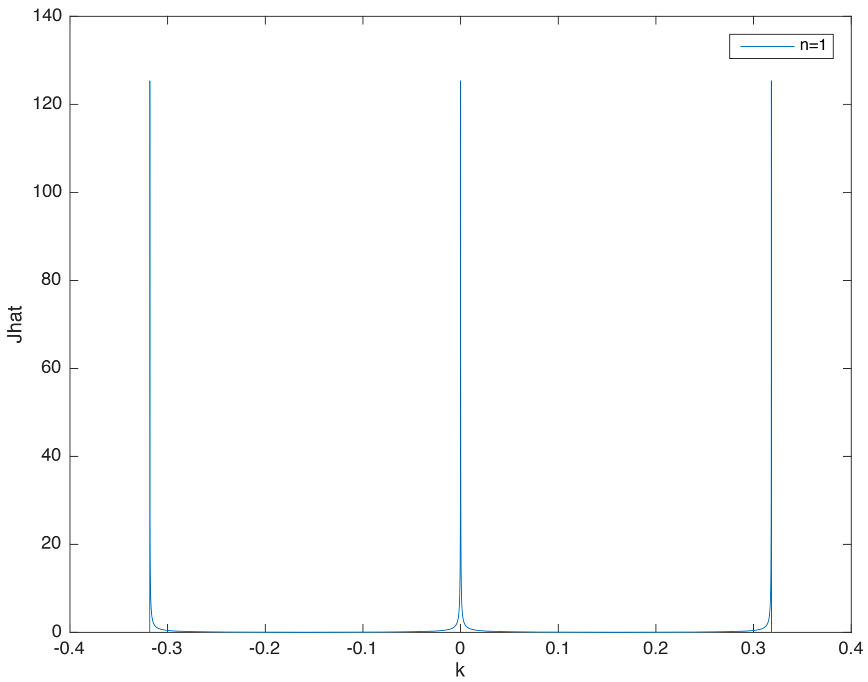

Figure 21 N=5000

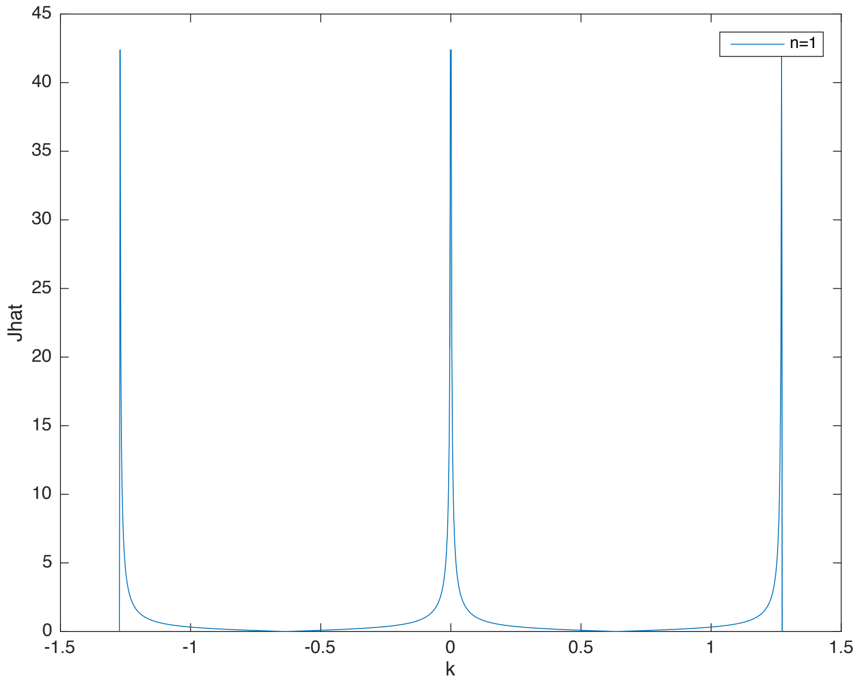

Figure 22 $X = \frac{4}{\pi}, N = 500$

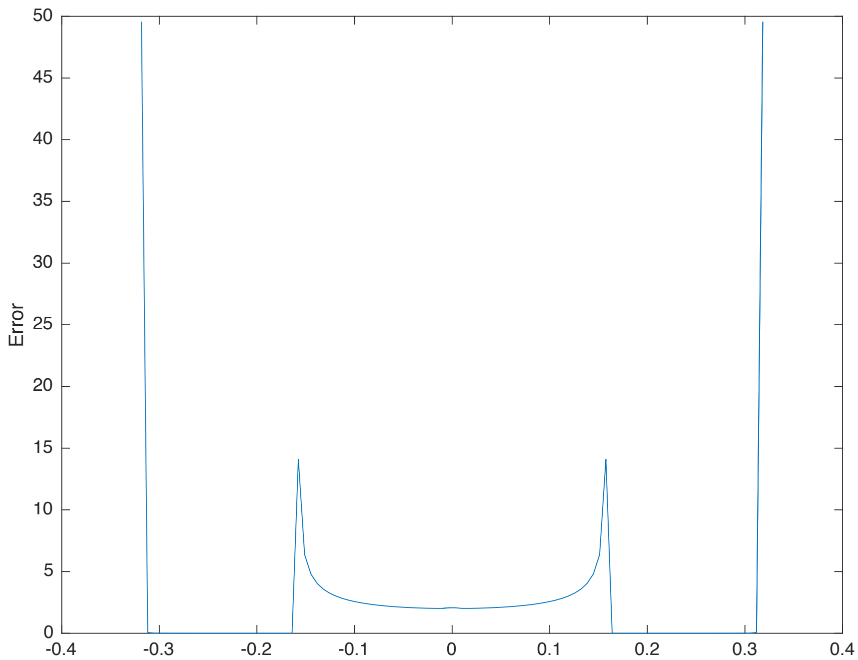

Figure 23: N=50, $X = \frac{1}{\pi}$

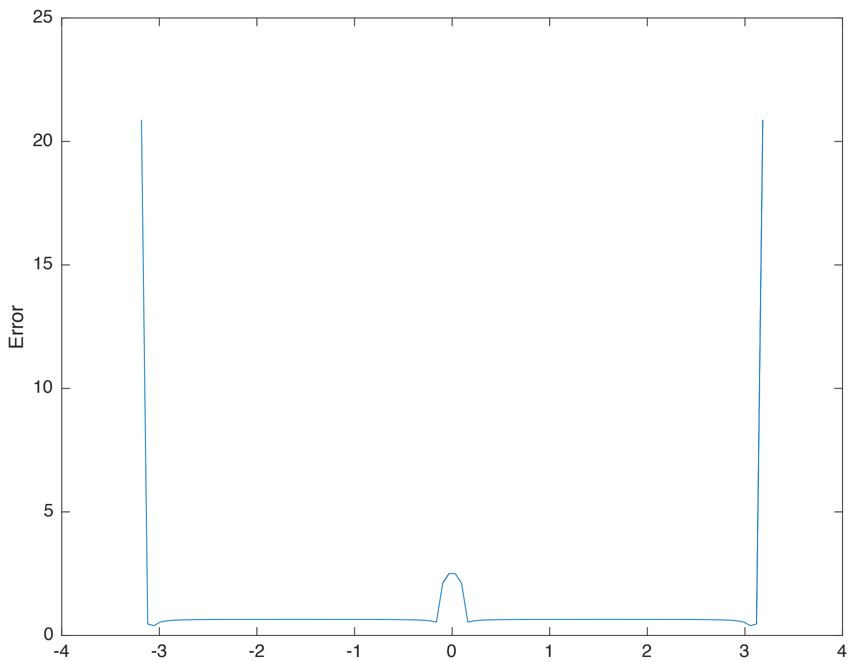

Figure 24 $N = 50, X = \frac{10}{\pi}\left(\frac{X}{N} = \frac{1}{5\pi}\right)$

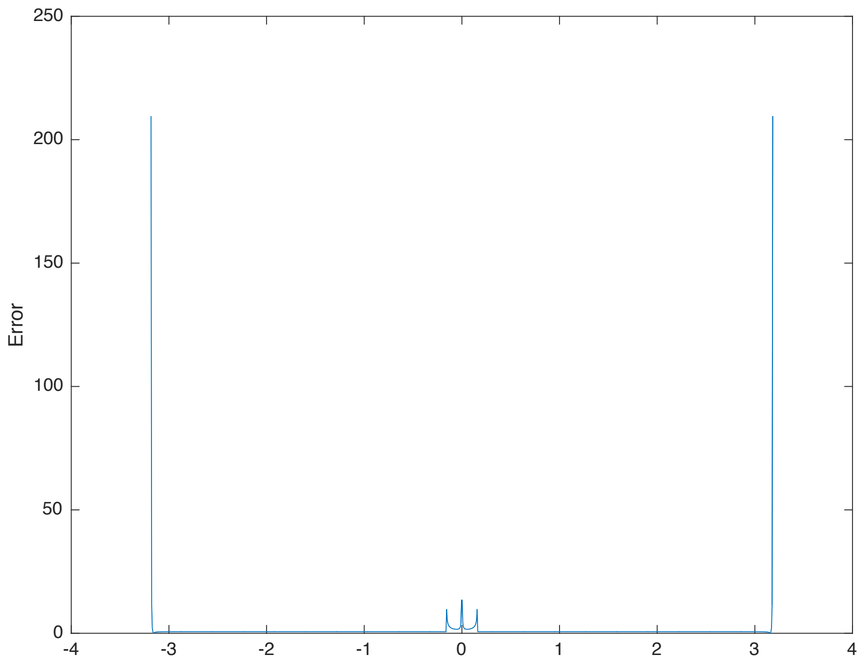

Figure 25 $N = 500,\ X = \frac{10}{\pi}$

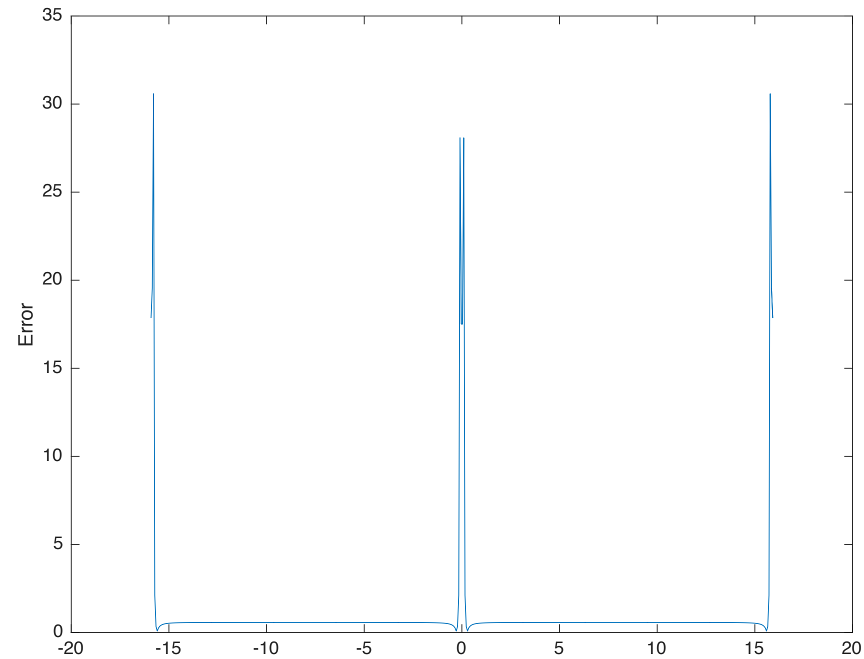

Figure 26 $N = 250, X = \frac{50}{\pi}\ \left(\frac{X}{N} = \frac{1}{5\pi}\right)$

Reference:

1. https://en.wikipedia.org/wiki/Mean_value_theorem

Appendix

Programs for Question 1

```
%RK4
n=8;
x0=0.5;
xn=20;
y0=-3;
z0=1;
h=0.1;
m=round((xn-x0)/h);
f=@(x,y,z)(z);
g=@(x,y,z)(-(z*x/x^2)-(x^2-n^2)*y/x^2);

y=zeros(1,m);
x=zeros(1,m);
y(1,1)=y0;
x(1,1)=x0;


for n=1:m+1
    x1=x0+h;
    a1=f(x0,y0,z0);
    b1=g(x0,y0,z0);

    a2=f(x0+0.5*h,y0+0.5*h*a1,z0+0.5*h*b1);
    b2=g(x0+0.5*h,y0+0.5*h*a1,z0+0.5*h*b1);

    a3=f(x0+0.5*h,y0+0.5*h*a2,z0+0.5*h*b2);
    b3=g(x0+0.5*h,y0+0.5*h*a2,z0+0.5*h*b2);

    a4=f(x0+h,y0+h*a3,z0+h*b3);
    b4=g(x0+h,y0+h*a3,z0+h*b3);

    y1=y0+h*(a1/6+a2/3+a3/3+a4/6);
    z1=z0+h*(b1/6+b2/3+b3/3+b4/6);

    x(n+1)=x1;
    y(n+1)=y1;
     xlabel('x')
   ylabel('y')

x0=x1;
y0=y1;
z0=z1;
end
 plot(x,y)
 hold on

%RK4 backwards
n=1;
x0=1;
xn=20;
y0=3;
z0=1;
h=0.1;
m=round((xn-x0)/h);

f=@(x,y,z)(z);
g=@(x,y,z)(-(z/x)-(x^2-n^2)*y/x^2);
```

```
y=zeros(1,m);
x=zeros(1,m);
y(1,1)=y0;
x(1,1)=x0;
d1=10;
d2=10;
for n=1:m
    x1=x0+h;
      a1=f(x0,y0,z0);
    b1=g(x0,y0,z0);
       a2=f(x0+0.5*h,y0+0.5*h*a1,z0+0.5*h*b1);
    b2=g(x0+0.5*h,y0+0.5*h*a1,z0+0.5*h*b1);
       a3=f(x0+0.5*h,y0+0.5*h*a2,z0+0.5*h*b2);
    b3=g(x0+0.5*h,y0+0.5*h*a2,z0+0.5*h*b2);
         a4=f(x0+h,y0+h*a3,z0+h*b3);
    b4=g(x0+h,y0+h*a3,z0+h*b3);
     y1=y0+h*(a1/6+a2/3+a3/3+a4/6);
    z1=z0+h*(b1/6+b2/3+b3/3+b4/6);

    while abs(d1)>0.01
    while abs(d2)>0.01

  y2=y0+h*(f(x0,y0,z0)+f(x1,y1,z1))/2;
  z2=z0+h*(g(x0,y0,z0)+g(x1,y1,z1))/2;
    d1=y2-y1;
    d2=z2-z1;
    y1=y2;
    z1=z2;
    end
    end

x0=x1;
y0=y1;
z0=z1;
  x(n+1)=x1;
  y(n+1)=y1;

end
xlabel('x')
ylabel('y')
 plot(x,y)
hold on
```

Programs for Question 2

```
N=100;
n=1;
m=1000;
A=zeros(m,N);
 x=linspace(0,20,m);

for r=0:N-1
    A(:,r+1)=(((-
1)^(r))*((0.5*(x)).^(2*r+n))/((factorial(r))*(factorial(n+r))));
end
  J=sum(A,2);
 plot(x,J)
    xlabel('x')
    ylabel('Jn')
    legend('n=4')
```

Programs for Question 5

```
n=2;
m=10000;
x=linspace(0,1/pi,m/2);
k=linspace(-1/pi,1/pi,m);
J2hat=zeros(m,1);
J=besselj(n,x);
J(1)=(besselj(n,0)+besselj(n,1/pi))/2;
J1hat=(fft(J));
for j=1:m/2
J2hat(j)=real(J1hat(j));
J2hat(m-j+1)=real(J1hat(j));
end
plot(k,abs(J2hat));
xlabel('k')
ylabel('Jhat')
legend('n=2')
```

```
%RK4
n=4;
x0=0.1;
xn=10/pi;
y0=1;
z0=1;
h=0.01;
m=1000;
f=@(x,y,z)(z);
g=@(x,y,z)(-(z*x/x^2)-(x^2-n^2)*y/x^2);
y=zeros(1,m);
x=zeros(1,m);
y(1,1)=y0;
x(1,1)=x0;
for j=1:m+1
    x1=x0+h;
    a1=f(x0,y0,z0);
    b1=g(x0,y0,z0);

    a2=f(x0+0.5*h,y0+0.5*h*a1,z0+0.5*h*b1);
    b2=g(x0+0.5*h,y0+0.5*h*a1,z0+0.5*h*b1);

    a3=f(x0+0.5*h,y0+0.5*h*a2,z0+0.5*h*b2);
    b3=g(x0+0.5*h,y0+0.5*h*a2,z0+0.5*h*b2);

    a4=f(x0+h,y0+h*a3,z0+h*b3);
    b4=g(x0+h,y0+h*a3,z0+h*b3);

    y1=y0+h*(a1/6+a2/3+a3/3+a4/6);
    z1=z0+h*(b1/6+b2/3+b3/3+b4/6);

    x(j+1)=x1;
    y(j+1)=y1;
     xlabel('x')
   ylabel('y')

x0=x1;
y0=y1;
z0=z1;
end
 plot(x,y)
```

```
 hold on

%FFT
m=1000;
x=linspace(0,1/pi,m/2);
k=linspace(-1/pi,1/pi,m);
J2hat=zeros(m,1);
J=besselj(n,x);
J(1)=(besselj(n,0)+besselj(n,1/pi))/2;
J1hat=(fft(J));
for j=1:m/2
J2hat(j)=real(J1hat(j));
J2hat(m-j+1)=real(J1hat(j));
end
plot(k,abs(J2hat));

%Jhat (theoritical)
m=1000;
k=linspace(-1/pi,1/pi,m);
u=2*pi*k;
T=cos(n*acos(u));
l1=find(u>1);
l2=find(u<-1);
[d1 d2]=size(l2);
for j=l1:m;
    T(j)=0;
end
for j=1:d2
    T(j)=0;
end
Jthat=2*((-i)^n)*((1-4*pi^2*k.^2).^(-1/2)).*T;
plot(k,abs(Jthat))
xlabel('k')
ylabel('Jn/Jnhat')
legend('Jn','Jhat','Jhat(theoretical')
title('n=4')
xlim([-1/pi 1/pi])
```