# 1.1 Fourier Transforms of Bessel Functions

**1 Introduction**

**Question 1**

We let and equation (1) can be written as

Hence we can use Runge-Kutta 4 method to obtain results numerically. A fixture of forwards and backwards integrations are used to illustrate interesting behaviors of Bessel’s functions in case

Interesting behavior of (numerical) Bessel’s functions

* For Bessel’s functions are oscillating and the amplitudes of the oscillations are decreasing as (see figure 1 to 3)
* The amplitudes of oscillations increase as n increases. (see figure 1 to 3)
* It seems to be there are infinite many of zeroes for all Bessel’s functions (see figure 1 to 3)
* Two Bessel’s functions, with initial conditions and respectively, satisfy . (see figure 4)
* As increases (remains the same), Bessel’s function increases the amplitudes of oscillations, but remains in the same frequencies. (see figure 5 and 6)

Figures 1 to 3 contain 2 plots, with initial conditionandrespectively.



Figure 1: n=0 (forwards integration)



Figure 2: n=2 (backwards integration)



Figure 3: n=4 (forwards integration)

Figures 4 contains 2 plots, with initial condition andrespectively.



Figure 4: n=0 (forwards integration)

Figure 5 contains plots with initial conditions respectively.

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Figure 5: n=1 (backwards integration)

Figure 6 contains plots with initial conditions respectively.



Figure 6: n=0 (forwards integration)

The numerical methods do not work when . This is due to the ‘’ is involved in the difference equation of , which causes tends towards infinite at the second iteration.

**Question 2**

Figure 7 to 9 plots for for .

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Figure 7: n=0



Figure 8



Figure 9

For the summation method is not accurate. As becomes too large for Matlab to handle for large r, hence to series solution becomes inaccurate.

**2 The Discrete Fourier Transform**

**Question 3**

Consider the following

To investigated the the upper bound of , we need to use the mean value theorem:

A function f is continuous on where and differentiable on such that

Now, let such that then we have

Let real variables, which lies in Apply this inequality on interval

where , hence

We apply

Provided

Hence under then the DFT tend to the Fourier transform.

**3 Fourier Transforms of Bessel Functions**

**Question 4**

We write

Use a change of variable, on the second integral of and use we obtain

Hence

and

By discrete Fourier transform, we have

*and*

Hence contains and does not contain . Where as contains one only . Therefore by replace before calculating DFT would allow completetly.

If , then the change of variable on the second integral of would give us

Hence

and

Similarly, should be replaced by before calculating the DFT.

**Question 5**

1. Note: from the series solution for we see that is even if n is even and odd if n is odd. Figure 10 to 14 are plots of at range , for .



Figure 10



Figure 11



Figure 12



Figure 13



Figure 14

1. Figure 15 to 14 are plots of at range , for .



Figure 15



Figure 16



Figure 17



Figure 18

Comment

* The theoretical formula may give problems at . The FFT deals with by using the the results from its neighbourhood.
* The size of the spikes increases proportionally with the N (see figure 19 to 21)
* Increase X increases the size of the spikes and the gaps between the spikes, as they only apply at the edges and the origin (see figure 22)
* The error decreases as X increases, or as N decreases. Moreover, the error stays constant if X/N stays constant. Hence the results agree with our answer in Question 3. (see figure 23 to 25)



Figure 19 N=50



Figure 20 N=500



Figure 21 N=5000



Figure 22



Figure 23: N=50,



Figure 24



Figure 25



Figure 26

**Reference:**

1. https://en.wikipedia.org/wiki/Mean\_value\_theorem

**Appendix**

**Programs for Question 1**

%RK4

n=8;

x0=0.5;

xn=20;

y0=-3;

z0=1;

h=0.1;

m=round((xn-x0)/h);

f=@(x,y,z)(z);

g=@(x,y,z)(-(z\*x/x^2)-(x^2-n^2)\*y/x^2);

y=zeros(1,m);

x=zeros(1,m);

y(1,1)=y0;

x(1,1)=x0;

for n=1:m+1

x1=x0+h;

a1=f(x0,y0,z0);

b1=g(x0,y0,z0);

a2=f(x0+0.5\*h,y0+0.5\*h\*a1,z0+0.5\*h\*b1);

b2=g(x0+0.5\*h,y0+0.5\*h\*a1,z0+0.5\*h\*b1);

a3=f(x0+0.5\*h,y0+0.5\*h\*a2,z0+0.5\*h\*b2);

b3=g(x0+0.5\*h,y0+0.5\*h\*a2,z0+0.5\*h\*b2);

a4=f(x0+h,y0+h\*a3,z0+h\*b3);

b4=g(x0+h,y0+h\*a3,z0+h\*b3);

y1=y0+h\*(a1/6+a2/3+a3/3+a4/6);

z1=z0+h\*(b1/6+b2/3+b3/3+b4/6);

x(n+1)=x1;

y(n+1)=y1;

xlabel('x')

ylabel('y')

x0=x1;

y0=y1;

z0=z1;

end

plot(x,y)

hold on

%RK4 backwards

n=1;

x0=1;

xn=20;

y0=3;

z0=1;

h=0.1;

m=round((xn-x0)/h);

f=@(x,y,z)(z);

g=@(x,y,z)(-(z/x)-(x^2-n^2)\*y/x^2);

y=zeros(1,m);

x=zeros(1,m);

y(1,1)=y0;

x(1,1)=x0;

d1=10;

d2=10;

for n=1:m

x1=x0+h;

a1=f(x0,y0,z0);

b1=g(x0,y0,z0);

a2=f(x0+0.5\*h,y0+0.5\*h\*a1,z0+0.5\*h\*b1);

b2=g(x0+0.5\*h,y0+0.5\*h\*a1,z0+0.5\*h\*b1);

a3=f(x0+0.5\*h,y0+0.5\*h\*a2,z0+0.5\*h\*b2);

b3=g(x0+0.5\*h,y0+0.5\*h\*a2,z0+0.5\*h\*b2);

a4=f(x0+h,y0+h\*a3,z0+h\*b3);

b4=g(x0+h,y0+h\*a3,z0+h\*b3);

y1=y0+h\*(a1/6+a2/3+a3/3+a4/6);

z1=z0+h\*(b1/6+b2/3+b3/3+b4/6);

while abs(d1)>0.01

while abs(d2)>0.01

y2=y0+h\*(f(x0,y0,z0)+f(x1,y1,z1))/2;

z2=z0+h\*(g(x0,y0,z0)+g(x1,y1,z1))/2;

d1=y2-y1;

d2=z2-z1;

y1=y2;

z1=z2;

end

end

x0=x1;

y0=y1;

z0=z1;

x(n+1)=x1;

y(n+1)=y1;

end

xlabel('x')

ylabel('y')

plot(x,y)

hold on

**Programs for Question 2**

N=100;

n=1;

m=1000;

A=zeros(m,N);

x=linspace(0,20,m);

for r=0:N-1

A(:,r+1)=(((-1)^(r))\*((0.5\*(x)).^(2\*r+n))/((factorial(r))\*(factorial(n+r))));

end

J=sum(A,2);

plot(x,J)

xlabel('x')

ylabel('Jn')

legend('n=4')

**Programs for Question 5**

n=2;

m=10000;

x=linspace(0,1/pi,m/2);

k=linspace(-1/pi,1/pi,m);

J2hat=zeros(m,1);

J=besselj(n,x);

J(1)=(besselj(n,0)+besselj(n,1/pi))/2;

J1hat=(fft(J));

for j=1:m/2

J2hat(j)=real(J1hat(j));

J2hat(m-j+1)=real(J1hat(j));

end

plot(k,abs(J2hat));

xlabel('k')

ylabel('Jhat')

legend('n=2')

%RK4

n=4;

x0=0.1;

xn=10/pi;

y0=1;

z0=1;

h=0.01;

m=1000;

f=@(x,y,z)(z);

g=@(x,y,z)(-(z\*x/x^2)-(x^2-n^2)\*y/x^2);

y=zeros(1,m);

x=zeros(1,m);

y(1,1)=y0;

x(1,1)=x0;

for j=1:m+1

x1=x0+h;

a1=f(x0,y0,z0);

b1=g(x0,y0,z0);

a2=f(x0+0.5\*h,y0+0.5\*h\*a1,z0+0.5\*h\*b1);

b2=g(x0+0.5\*h,y0+0.5\*h\*a1,z0+0.5\*h\*b1);

a3=f(x0+0.5\*h,y0+0.5\*h\*a2,z0+0.5\*h\*b2);

b3=g(x0+0.5\*h,y0+0.5\*h\*a2,z0+0.5\*h\*b2);

a4=f(x0+h,y0+h\*a3,z0+h\*b3);

b4=g(x0+h,y0+h\*a3,z0+h\*b3);

y1=y0+h\*(a1/6+a2/3+a3/3+a4/6);

z1=z0+h\*(b1/6+b2/3+b3/3+b4/6);

x(j+1)=x1;

y(j+1)=y1;

xlabel('x')

ylabel('y')

x0=x1;

y0=y1;

z0=z1;

end

plot(x,y)

hold on

%FFT

m=1000;

x=linspace(0,1/pi,m/2);

k=linspace(-1/pi,1/pi,m);

J2hat=zeros(m,1);

J=besselj(n,x);

J(1)=(besselj(n,0)+besselj(n,1/pi))/2;

J1hat=(fft(J));

for j=1:m/2

J2hat(j)=real(J1hat(j));

J2hat(m-j+1)=real(J1hat(j));

end

plot(k,abs(J2hat));

%Jhat (theoritical)

m=1000;

k=linspace(-1/pi,1/pi,m);

u=2\*pi\*k;

T=cos(n\*acos(u));

l1=find(u>1);

l2=find(u<-1);

[d1 d2]=size(l2);

for j=l1:m;

T(j)=0;

end

for j=1:d2

T(j)=0;

end

Jthat=2\*((-i)^n)\*((1-4\*pi^2\*k.^2).^(-1/2)).\*T;

plot(k,abs(Jthat))

xlabel('k')

ylabel('Jn/Jnhat')

legend('Jn','Jhat','Jhat(theoretical')

title('n=4')

xlim([-1/pi 1/pi])