# 4.5 Euler’s Equations

**2 Project Work**

**2.1 Project requirements**

Program 1 implemented 4 Stage Runge-Kutta methods to solve Euler’s equations (5) numerically. To test the program, we chose and .



Figure 1: Plot 0f against with t

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Figure 2: Plot 0f against with t



Figure 3: Plot 0f against with t



Figure 4: 3-D phase plots 0f

Equations (6) and (7) are used to check the accuracy of the numerical results. Table 1 displays the value of and at the beginning and end of runs.

Table 1: Display of E and

|  |  |  |
| --- | --- | --- |
|  |  |  |
| Beginning of runs | 1.5000 | 3.5000 |
| End of the runs | 1.5000 | 3.5000 |

**Question 1** Let respectively. Moreover, , and be the middle constant, i.e.

Now, we define , and . Hence, without loss of generality, we have

Hence constants can always be taken in a way that , without loss of generality.

In case , Euler equation becomes

This can be solved analytically. We have

Where . Hence, in the case, the body processes about the OZ axis with frequency . Furthermore, if the direction of procession is clockwise, and if the direction of procession is anticlockwise.

In case , are all constants. Hence the rigid body continue to spin with the same axis.

Explanation of Let , then Euler’s equations become

Hence without loss of generality, we can chose take and .

The scaling factor Let where

Consider

Hence with the re-scaling, is equivalent to choosing .

**2.2 Results requirements**

**Question 2** Let us set the initial conditions as , and run the program. Figure 5 is a 3-D phase space plot of of , which choose the OX axis for without loss of generality. Thus, we obtain a stable solution that ­ is rotating around the OX axis with small amplitude deviation from (1,0,0) before scaling.

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Figure 5: 3-D plots of

Similarly, a small amplitude deviation from gives us stable solutions near the OZ axis. (see Figure 6) 

Figure 6

**Question 3**

1. In case , the other two Euler’s equations become

(1)

(2)

By differentiating (1) and substitute it into (2), we can get a 2nd order differential equation,

Let , and we have

The period T satisfies hence the analytic expression for the period is

Figure 7 is a plot of against *t*, with conditions . We can deduce that the period is around 8.



Figure 7: Plots of against t

1. Similarly, in case , we have

and the period satisfies

From Figure 8, we can deduce that the period is also around 10.



Figure 8: Plots of against t

In comparison of (i) and (ii), the case has slightly longer period.

Consistency with Question 2

**Question 4** In both case, the numerical results are consistent with equation (6) and (7), as shown in Table 2 and 3

Table 2

|  |  |  |
| --- | --- | --- |
|  |  |  |
| Beginning of runs | 1.0001 | 2.8001 |
| End of the runs | 1.0001 | 2.8001 |

Table 3

|  |  |  |
| --- | --- | --- |
|  |  |  |
| Beginning of runs | 1.0001 | 1.4003 |
| End of the runs | 1.0001 | 1.4003 |

**Question 5** When is very close to OY axis, we can set to investigate the solutions. Figure 9 shows 3D plots of against *t* in the same graph.



Figure 9: the 3D plot

Physical description The motion is stable and the largest amplitude oscillation is around .

**Question 6**

Inspired from Question 5, let us set and (w.o.l.g) and vary to seek for such solution. From the computed results, we can discover that as ; . See Figure 10 and 11 for few iterations.



Figure 10:



Figure 11:

Hence such solution exists.

Simulate the solution numerically The numerical solution only works for finite time. When *t* is very large, jumps periodically. (see Figure 12) This is plausible as this solution is not stable. Also from the numerical results, we discover that hence H must satisfies .



Figure 12: :

**Question 7**

We separation all possible qualitative types of motion in 3 cases:

Noted: for rang of behaviour of each type try different values of A, B and C, we can see the range of behaviour for each type.



The body oscillates at a near constant rate around a particular axis.

Example:



Figure 13

Range of behaviour For the body oscillates at a very rapidly. For , the body oscillates at a slower rate.

1. (See Question 6) which is the equation of boundaries of these regions. Hence as thee boundaries are approached, the solutions believe like:

*begins away from OY axis, but tends towards the steady unstable solution, parallel to OY as (as described in Question 6)*

Example:



Figure 14

the body oscillates at an slower rate compared with case , and no components of oscillates at constant rate.

The choice of determines the which axis that the body oscillates (approximately) a constant rate. E.g. if , then the oscillation around at (approximately) a constant rate, vice versa (see Figure 13 and 14)

**Question 8**

Case

* Effect on Question 5 (obtained from Program 11)



Figure 15

Affect on Question 6



Due to damping takes place, the condition will not hold.

Affect on Question 7: follows from Question 6, the classifications would not be valid. In general, the affect acts like a damping to the motion of the body, hence the amplitudes of oscillations decrease against with time. (see figure 16)



Figure 16

**Question 9**

The classification in Question 7 would still be useful. There is still a division of the solution space into regions. However, we need to change the boundaries from to a non-linear boundaries with the damping factor taken into account.

**Reference:**

1. Tong, D. Lecture notes on Classical Dynamics (2015)
2. Cowley S.J. Lecture notes on Numerical Analysis (2014)

**Appendix**

Program 1 (to produce figure 1-3)

%Solve Euler's Equation numerically

n=1;

A=1.5;

B=1;

C=0.5;

t0=0;

tn=10;

%step size h

step=0.1;

U0=1;

V0=1;

W0=1;

m=(tn-t0)/step;

f=@(t,V,W)((B-C)\*V\*W/A);

g=@(t,U,W)((C-A)\*W\*U/B);

h=@(t,U,V)((A-B)\*U\*V/C);

w1=zeros(1,m);

t=zeros(1,m);

E=0.5\*A\*U0^2+0.5\*B\*V0^2+0.5\*C\*W0^2;

H=sqrt(A^2\*U0^2+B^2\*V0^2+C^2\*W0^2);

display([E,H^2])

%4-stage explicit RK methods

while n<m+1

t1=t0+step;

a1=f(t0,V0,W0);

b1=g(t0,U0,W0);

c1=h(t0,U0,V0);

a2=f(t0+0.5\*step,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

b2=g(t0+0.5\*step,U0+0.5\*step\*a1,W0+0.5\*step\*c1);

c2=h(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1);

a3=f(t0+0.5\*step,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

b3=g(t0+0.5\*step,U0+0.5\*step\*a2,W0+0.5\*step\*c2);

c3=h(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2);

a4=f(t0+step,V0+step\*b3,W0+step\*c3);

b4=g(t0+step,U0+step\*a3,W0+step\*c3);

c4=h(t0+step,U0+step\*a3,V0+step\*b3);

U1=U0+step\*(a1/6+a2/3+a3/3+a4/6);

V1=V0+step\*(b1/6+b2/3+b3/3+b4/6);

W1=W0+step\*(c1/6+c2/3+c3/3+c4/6);

t(n)=t1;

w1(n)=U1;

plot(t,w1)

xlabel('time')

ylabel('w1')

%disp([t1,U1])

t0=t1;

U0=U1;

V0=V1;

W0=W1;

n=n+1;

end

E=0.5\*A\*U1^2+0.5\*B\*V1^2+0.5\*C\*W1^2;

H=sqrt(A^2\*U1^2+B^2\*V1^2+C^2\*W1^2);

display([E,H^2])

Program 4 (to produce figure 4)

%Solve Euler's Equation numercailly

n=1;

A=1.5;

B=1;

C=0.5;

t0=0;

tn=10;

%step size h

step=0.1;

U0=1;

V0=1;

W0=1;

m=(tn-t0)/step;

f=@(t,V,W)((B-C)\*V\*W/A);

g=@(t,U,W)((C-A)\*W\*U/B);

h=@(t,U,V)((A-B)\*U\*V/C);

w1=zeros(1,m);

w2=zeros(2,m);

w3=zeros(1,m);

t=zeros(1,m);

E=0.5\*A\*U0^2+0.5\*B\*V0^2+0.5\*C\*W0^2;

H=sqrt(A^2\*U0^2+B^2\*V0^2+C^2\*W0^2);

display([E,H^2])

%4-stage explicit RK methods

while n<m+1

t1=t0+step;

a1=f(t0,V0,W0);

b1=g(t0,U0,W0);

c1=h(t0,U0,V0);

a2=f(t0+0.5\*step,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

b2=g(t0+0.5\*step,U0+0.5\*step\*a1,W0+0.5\*step\*c1);

c2=h(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1);

a3=f(t0+0.5\*step,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

b3=g(t0+0.5\*step,U0+0.5\*step\*a2,W0+0.5\*step\*c2);

c3=h(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2);

a4=f(t0+step,V0+step\*b3,W0+step\*c3);

b4=g(t0+step,U0+step\*a3,W0+step\*c3);

c4=h(t0+step,U0+step\*a3,V0+step\*b3);

U1=U0+step\*(a1/6+a2/3+a3/3+a4/6);

V1=V0+step\*(b1/6+b2/3+b3/3+b4/6);

W1=W0+step\*(c1/6+c2/3+c3/3+c4/6);

t(n)=t1;

w1(n)=U1;

w2(n)=V1;

w3(n)=W1;

plot3(w1,w2,w3)

xlabel('w1')

ylabel('w2')

zlabel('w3')

t0=t1;

U0=U1;

V0=V1;

W0=W1;

n=n+1;

end

E=0.5\*A\*U1^2+0.5\*B\*V1^2+0.5\*C\*W1^2;

H=sqrt(A^2\*U1^2+B^2\*V1^2+C^2\*W1^2);

display([E,H^2])

Program 6 (for figure 5,6)

%Answer Q2 with 3D plots

n=1;

A=input('please input your A=');

C=input('please input your C=');

B=1;

t0=0;

tn=500;

%step size h

step=0.01;

U0=0.01;

V0=1;

W0=0.01;

m=(tn-t0)/step;

w1=zeros(1,m);

w2=zeros(2,m);

w3=zeros(1,m);

t=zeros(1,m);

E=0.5\*A\*U0^2+0.5\*B\*V0^2+0.5\*C\*W0^2;

H=sqrt(A^2\*U0^2+B^2\*V0^2+C^2\*W0^2);

display([E,H^2])

f=@(t,V,W)(E\*(B-C)\*V\*W/A);

g=@(t,U,W)(E\*(C-A)\*W\*U/B);

h=@(t,U,V)(E\*(A-B)\*U\*V/C);

%4-stage explicit RK methods

while n<m+1

t1=t0+step;

a1=f(t0,V0,W0);

b1=g(t0,U0,W0);

c1=h(t0,U0,V0);

a2=f(t0+0.5\*step,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

b2=g(t0+0.5\*step,U0+0.5\*step\*a1,W0+0.5\*step\*c1);

c2=h(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1);

a3=f(t0+0.5\*step,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

b3=g(t0+0.5\*step,U0+0.5\*step\*a2,W0+0.5\*step\*c2);

c3=h(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2);

a4=f(t0+step,V0+step\*b3,W0+step\*c3);

b4=g(t0+step,U0+step\*a3,W0+step\*c3);

c4=h(t0+step,U0+step\*a3,V0+step\*b3);

U1=U0+step\*(a1/6+a2/3+a3/3+a4/6);

V1=V0+step\*(b1/6+b2/3+b3/3+b4/6);

W1=W0+step\*(c1/6+c2/3+c3/3+c4/6);

t(n)=t1;

w1(n)=U1;

w2(n)=V1;

w3(n)=W1;

%plot(t,w3)

plot3(w1,w2,w3,'r');

xlabel('w1');

ylabel('w2');

zlabel('w3');

t0=t1;

U0=U1;

V0=V1;

W0=W1;

n=n+1;

end

E=0.5\*A\*U1^2+0.5\*B\*V1^2+0.5\*C\*W1^2;

H=sqrt(A^2\*U1^2+B^2\*V1^2+C^2\*W1^2);

display([E,H^2])

Program 9 (for figure 9)

%ans to Q5

%Solve Euler's Equation numercailly

n=1;

A=1.4;

B=1;

C=2.5;

t0=0;

tn=20;

%step size h

step=0.01;

U0=1;

V0=0;

W0=0.820;

m=(tn-t0)/step;

f=@(t,V,W)((B-C)\*V\*W/A);

g=@(t,U,W)((C-A)\*W\*U/B);

h=@(t,U,V)((A-B)\*U\*V/C);

w1=zeros(1,m);

w2=zeros(1,m);

w3=zeros(1,m);

t=zeros(1,m);

E=0.5\*A\*U0^2+0.5\*B\*V0^2+0.5\*C\*W0^2;

H=sqrt(A^2\*U0^2+B^2\*V0^2+C^2\*W0^2);

display([E,H^2])

%4-stage explicit RK methods

while n<m+1

t1=t0+step;

a1=f(t0,V0,W0);

b1=g(t0,U0,W0);

c1=h(t0,U0,V0);

a2=f(t0+0.5\*step,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

b2=g(t0+0.5\*step,U0+0.5\*step\*a1,W0+0.5\*step\*c1);

c2=h(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1);

a3=f(t0+0.5\*step,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

b3=g(t0+0.5\*step,U0+0.5\*step\*a2,W0+0.5\*step\*c2);

c3=h(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2);

a4=f(t0+step,V0+step\*b3,W0+step\*c3);

b4=g(t0+step,U0+step\*a3,W0+step\*c3);

c4=h(t0+step,U0+step\*a3,V0+step\*b3);

U1=U0+step\*(a1/6+a2/3+a3/3+a4/6);

V1=V0+step\*(b1/6+b2/3+b3/3+b4/6);

W1=W0+step\*(c1/6+c2/3+c3/3+c4/6);

t(n)=t1;

w1(n)=U1;

w2(n)=V1;

w3(n)=W1;

plot3(w1,w2,w3)

xlabel('w1')

ylabel('w2')

zlabel('w3')

t0=t1;

U0=U1;

V0=V1;

W0=W1;

n=n+1;

end

E=0.5\*A\*U1^2+0.5\*B\*V1^2+0.5\*C\*W1^2;

H=sqrt(A^2\*U1^2+B^2\*V1^2+C^2\*W1^2);

display([E,H^2])

Program 10 (for Question 7)

% Sovling Q6

n=1;

A=0.25;

B=0.27;

C=0.30;

t0=0;

tn=50;

%step size h

step=0.2;

U0=1;

V0=1;

W0=1;

m=(tn-t0)/step;

f=@(t,V,W)((B-C)\*V\*W/A);

g=@(t,U,W)((C-A)\*W\*U/B);

h=@(t,U,V)((A-B)\*U\*V/C);

w1=zeros(1,m);

w2=zeros(1,m);

w3=zeros(1,m);

t=zeros(1,m);

E=0.5\*A\*U0^2+0.5\*B\*V0^2+0.5\*C\*W0^2;

H=sqrt(A^2\*U0^2+B^2\*V0^2+C^2\*W0^2);

display([E,H^2])

%4-stage explicit RK methods

while n<m+1

t1=t0+step;

a1=f(t0,V0,W0);

b1=g(t0,U0,W0);

c1=h(t0,U0,V0);

a2=f(t0+0.5\*step,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

b2=g(t0+0.5\*step,U0+0.5\*step\*a1,W0+0.5\*step\*c1);

c2=h(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1);

a3=f(t0+0.5\*step,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

b3=g(t0+0.5\*step,U0+0.5\*step\*a2,W0+0.5\*step\*c2);

c3=h(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2);

a4=f(t0+step,V0+step\*b3,W0+step\*c3);

b4=g(t0+step,U0+step\*a3,W0+step\*c3);

c4=h(t0+step,U0+step\*a3,V0+step\*b3);

U1=U0+step\*(a1/6+a2/3+a3/3+a4/6);

V1=V0+step\*(b1/6+b2/3+b3/3+b4/6);

W1=W0+step\*(c1/6+c2/3+c3/3+c4/6);

t(n)=t1;

w1(n)=U1;

w2(n)=V1;

w3(n)=W1;

plot (t,w1,'r',t,w2,'b',t,w3,'g-.')

xlabel('time')

ylabel('w1/w2/w3')

legend('w1','w2','w3')

t0=t1;

U0=U1;

V0=V1;

W0=W1;

n=n+1;

end

E=0.5\*A\*U1^2+0.5\*B\*V1^2+0.5\*C\*W1^2;

H=sqrt(A^2\*U1^2+B^2\*V1^2+C^2\*W1^2);

display([E,H^2])

Program 11 (for Question 8)

%Question 8

n=1;

A=1.4;

B=1;

C=0.7;

t0=0;

tn=40;

k=0.01;

%step size h

step=0.1;

U0=1;

V0=1;

W0=1;

m=(tn-t0)/step;

f=@(t,U,V,W)(((B-C)/A)\*V\*W -k\*U);

g=@(t,U,V,W)(((C-A)/B)\*W\*U-k\*V);

h=@(t,U,V,W)(((A-B)/C)\*U\*V-k\*W);

w1=zeros(1,m);

w2=zeros(1,m);

w3=zeros(1,m);

t=zeros(1,m);

E=0.5\*A\*U0^2+0.5\*B\*V0^2+0.5\*C\*W0^2;

H=sqrt(A^2\*U0^2+B^2\*V0^2+C^2\*W0^2);

display([E,H^2])

%4-stage explicit RK methods

while n<m+1

t1=t0+step;

a1=f(t0,U0,V0,W0);

b1=g(t0,U0,V0,W0);

c1=h(t0,U0,V0,W0);

a2=f(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

b2=g(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

c2=h(t0+0.5\*step,U0+0.5\*step\*a1,V0+0.5\*step\*b1,W0+0.5\*step\*c1);

a3=f(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

b3=g(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

c3=h(t0+0.5\*step,U0+0.5\*step\*a2,V0+0.5\*step\*b2,W0+0.5\*step\*c2);

a4=f(t0+step,U0+step\*a3,V0+step\*b3,W0+step\*c3);

b4=g(t0+step,U0+step\*a3,V0+step\*b3,W0+step\*c3);

c4=h(t0+step,U0+step\*a3,V0+step\*b3,W0+step\*c3);

U1=U0+step\*(a1/6+a2/3+a3/3+a4/6);

V1=V0+step\*(b1/6+b2/3+b3/3+b4/6);

W1=W0+step\*(c1/6+c2/3+c3/3+c4/6);

t(n)=t1;

w1(n)=U1;

w2(n)=V1;

w3(n)=W1;

plot (t,w1,'r',t,w2,'b',t,w3,'g-.')

xlabel('time')

ylabel('w1/w2/w3')

legend('w1','w2','w3')

%plot3(w1,w2,w3)

t0=t1;

U0=U1;

V0=V1;

W0=W1;

n=n+1;

end

E=0.5\*A\*U1^2+0.5\*B\*V1^2+0.5\*C\*W1^2;

H=sqrt(A^2\*U1^2+B^2\*V1^2+C^2\*W1^2);

display([E,H^2])