QF602 Derivatives

Lecture 3 - Option Pricing Models

Harry Lo
lo.harry@gmail.com

Singapore Management University

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Brownian Motion

- A standard Brownian Motion on $[0, T]$, is a stochastic process $W(t), 0 \leq t \leq T$ with the following properties:
  - BM starts at 0. $W(0) = 0$.
  - Continuous path. The mapping, $t \to W(t)$ is, with probability 1, a continuous function on $[0, T]$.
  - Independent increment. The increments $W(t_k) - W(t_{k-1})$ for all k are independent for any k. All you care is the distance between t_k and t_{k-1}.
  - Normally distributed increment. The Brownian increment has the following distribution: $W(t) - W(s) \sim N(0, t - s)$ for any $0 \leq s \leq t \leq T$.

Brownian Motion with Drift

- For a constant μ and $\sigma > 0$, we call a process $X(t)$ a Brownian motion with drift μ and volatility σ if

$$\frac{X(t) - \mu t}{\sigma}$$

  is a standard Brownian motion.
- We can construct X from a standard Brownian motion W by setting

$$X(t) = \mu t + \sigma W(t).$$

- It follows that $X(t) \sim N(\mu t, \sigma^2 t)$.
- Moreover, X solves the stochastic differential equation (SDE)

$$dX(t) = \mu dt + \sigma dW(t)$$

- The assumption that $X(0) = 0$ is a natural normalization, but we may construct a Brownian motion with parameters μ and σ and initial value x by simply adding x to each X(t).

Geometric Brownian Motion

- A stochastic process $S(t)$ is a geometric Brownian motion (GBM) if $\ln S(t)$ is a Brownian motion with inital value $\ln S(0)$.
- In other words, a GBM is an exponentiated Brownian motion.
- GBM is the most fundamental model of the value of a financial asset.
- In 1900, Louis Bachelier developed a model of stock prices that in retrospect we described as Brownian motion. The mathematics of Bownian motion had not yet been developed.
- In 1960s, Paul Samuelson pioneered the use of GBM as model in finance.

Geometric Brownian Motion

- Brownian motion can take negative values, an undersirable feature in a model of the price of a stock.
- GBM is always positive because the exponential function takes only positive values.
- The dynamics of a GBM can be specified by the following SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- To find the solution of the SDE, we set $f(S(t)) = \ln S(t)$, by Ito's lemma, we have

$$df(S(t)) = f^{'}(S(t))dS(t) + \frac{1}{2}f^{''}(S(t)) < dS(t) >$$

where, $f^{'}(S) = \frac{1}{S}$, $f^{''}(S) = -\frac{1}{S^2}$, $< dS >= \sigma^2 S^2 dt$.

Geometric Brownian Motion

- Substitute in all the terms, we have

$$d \ln S(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t).$$

- Express in the integral form, we have

$$\begin{aligned} \ln S(T) &= \ln S(0) + \int_0^T \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^T \sigma dW(u) \\ &= \ln S(0) + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}x, \end{aligned}$$

where $x \sim N(0,1)$. The solution is given as

$$S(T) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}x}$$

Geometric Brownian Motion

- What is the expectation of $S(T)$ in risk neutral measure? What is the numeraire asset associated to risk neutral measure?
- Let's start with something simpler. What is the expectation of the exponential of a normal random variable x?

$$E\left[e^{ax}\right] = e^{\frac{1}{2}a^2}$$

where a is a arbitrary number.
- This is the moment generating function of $x \sim N(0,1)$.
- All other terms in $S(T)$ are non-random, so

$$\begin{aligned} E[S(T)] &= S(0)e^{\left(\mu-\frac{1}{2}\sigma^2\right)T}E\left[e^{\sigma\sqrt{T}x}\right] \\ &= S(0)e^{\left(\mu-\frac{1}{2}\sigma^2\right)T}e^{\frac{1}{2}\sigma^2 T} \\ &= S(0)e^{\mu T} \end{aligned}$$

Black Scholes

- Assume r is the risk free rate, the money market account is given as $\beta(t) = e^{rt}$.
- The numeraire asset is $\beta(t)$, this is associated with the risk neutral measure.
- Under the risk neutral measure, the stock price dynamics for non-dividend paying stock as given by

$$dS(t) = rS(t)dt + \sigma S(t)dW(t).$$

where W is the standard Brownian motion in the risk neutral measure.

Martingale pricing

- Let $V(t)$ be a tradable asset price and $N(t)$ be a strictly positive asset, for $t < T$, we have

$$\frac{V(t)}{N(t)} = E_t^N \left[\frac{V(T)}{N(T)} \right]$$

- The subscript t denotes the expectation is taken at time t.
- The superscript N denotes the expectation is taken under the measure induced by the numeraire asset N.

Pricing a call option in BS

- Let $V(T) = (S(T) - K)^+$ and $N(t) = \beta(T)$.
- We want to compute the pricing of the call option at time $t = 0$.
- Apply the martingale pricing formula, we have
$$V(0) = \beta(0) E_0^N \left[\frac{(S(T) - K)^+}{\beta(T)} \right]$$
.
- Recall that $\beta(t) = e^{rt}$, it yields $\frac{\beta(0)}{\beta(T)} = e^{-rT}$ and it is non-random in the BS model, so that we can take it out of the expectation:
$$V(0) = e^{-rT} E_0^N \left[(S(T) - K)^+ \right]$$
.
- Compute the expectation which is a standard BS type computation and we get the celebrated BS formula for the call option.

Another formulation

- Instead of starting from the SDE of the spot S, we can start from the SDE of the forward price $F(t, T)$ at time t which matures at T.
- Note that $F(T, T) = S(T)$ and the SDE of the forward price is a martingale under the T-forward measure:

$$dF(t, T) = \sigma F(t, T) dW(t)$$

  where W is a standard BM in the T-forward measure.
- The numeraire asset is the zero coupon bond $Z(t, T)$ matures at time T, which can be computed as

$$Z(t, T) = E\left[e^{-\int_t^T r(u) du}\right]$$

- If we assume constant risk free rate then

$$Z(t, T) = e^{-r(T-t)}.$$

Another formulation

- Apply the martingale pricing formula, we have

$$V(0) = N(0)E_t^N\left[\frac{F(T,T)-K)^+}{N(T)}\right] = N(0)E_t^N\left[\frac{S(T)-K)^+}{N(T)}\right]$$

.

- $N(T) = Z(T,T) = 1$.
- $N(0) = Z(0,T) = e^{-rT}$, so we have

$$V(0) = e^{-rT}E_t^N\left[S(T)-K)^+\right].$$

- This is equivalent to the result that we have before. This is the so-called the Black 76 formulation.

Implied volatility

- In the Black Scholes call option formula, volatility σ is one of the input parameters:

$$Call\_price = BSCall(F, K, r, \sigma, T).$$

- In practice, option prices are observable in the market along side with other parameters except for σ.
- Implied volatility is defined as the value of σ of such that the Black Schloes call option formula produces the same price as the observable option price:

$$\sigma_{implied} = BSCall^{-1}(F, K, r, T, Call\_price).$$

- In other words, one would need to invert the Black Scholes call option formula subject to the implied volatility.

Implied volatility

- Recall the Black Scholes call option formula

$$BSCall(\sigma) = e^{-rT}(F(0,T)N(d_1) - KN(d_2))$$

$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

$$N(a) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{a} e^{-u^2/2du}$$

- There is no known closed form solution to the implied volatility by inverting the Black Scholes formula.
- One needs to use numerical methods.

Root searching algorithm

- Problem statement: given a function $f(x)$, can you find x such that $f(x) = 0$.
- If the function f is differentiable with respect to x then we can use Newton Raphson's method (NR), x_0 is the initial guess,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Root searching algorithm

- A toy problem: can you find x such that $x^2 = 7$?
- We set the function $f(x) = x^2 - 7$, $f^{'}(x) = 2x$.
- Initial guess $x_0 = 2.5$, $f(x_0) = 2.5^2 - 7 = -0.75$, $f^{'}(x_0) = 5$.
- Apply NR, we have $x_1 = 2.5 - \frac{-0.75}{5} = 2.5$.
- It converges at the third iteration, $x = 2.6457513111$

iteration	x	f(x)	f'(x)
0	2.5000000000	-0.7500000000	5.0000000000
1	2.6500000000	0.0225000000	5.3000000000
2	2.6457547170	0.0000180224	5.2915094340
3	2.6457513111	0.0000000000	5.2915026221
4	2.6457513111	0.0000000000	5.2915026221
5	2.6457513111	0.0000000000	5.2915026221
6	2.6457513111	0.0000000000	5.2915026221
7	2.6457513111	0.0000000000	5.2915026221
8	2.6457513111	0.0000000000	5.2915026221

Implied vol using NR

- To find implied volatility, one can set the function to be

$$f(\sigma) = BSCall(\sigma) - price$$

$$f^{'}(\sigma) = vega(\sigma)$$

- where is the observable call option price.
- Have a sensible intital guess for σ_0, say 20%.
- Then we are ready to use the NR iteration formula

$$\sigma_1 = \sigma_0 - \frac{BSCall(\sigma_0 - price}{vega(\sigma_0)}$$

Example

- The observable option price is 21.9078.
- $S(0) = 100, K = 80, r = q = 0, T = 1.$
- It converges after 3 iterations.

S	100		iteration	ivol	f(ivol)	f'(ivol)
K	80		0	0.20000000	-0.72191	19.05342
input vol	0.23456		1	0.23788877	0.07550	22.82026
r	0		2	0.23458037	0.00046	22.54126
q	0		3	0.23456000	0.00000	22.53951
ttm	1		4	0.23456000	0.00000	22.53951
			5	0.23456000	0.00000	22.53951
call price	21.9078		6	0.23456000	0.00000	22.53951

Note that for deep out or deep in the money options, the NR method will fail. One of the reasons is that the vega is too small. If you are going to implement this at work, refer to the works by Peter Jaeckel on this topic (see, http://www.jaeckel.org/).

Bachelier model

- Under the Bachelier model, the forward price of an underlying security F follows a Brownian motion with volatility σ

$$dF(t, T) = \sigma dW(t)$$

- The solution of the SDE can be computed as

$$F(T, T) = F(0, T) + \sigma\sqrt{T}x$$

- Similar to Black Scholes, there are closed form formula for call and put options in the Bachelier model:

$$BachCall = (F(0, T) - K)N(d) + \sigma\sqrt{T}n(d)$$

$$BachPut = (K - F(0, T)N(-d) + \sigma\sqrt{T}n(-d)$$

$$d = \frac{F(0, T) - K}{\sigma\sqrt{T}}$$

where $n(.)$ is the probability density function of the standard normal distribution.

Bachelier vs Black: Lognormal Implied vol skew

- Consider the case that, for a maturity T, only an ATM call option is traded in the market at price P.
- Forward is 2%, K is 2%, maturity is 1 year.
- Assuming one trader uses Black model, the other uses Bachelier model.
- Since there is only one option traded in the market, they can only calibrate to that price.
- If they completely trust their models and a client ask them to quote the options with strikes at 1.5% and 2.5%. How much will they quote?

Bachelier vs Black: Lognormal Implied vol skew

- For a trader who uses Black model, the calibrates his model and get a Black implied vol, aka lognormal implied vol or just lognormal vol.

$$\sigma_{lognormal} = BSCall^{-1}(F, K, r, T, Price).$$

- For a trader who uses a Bachelier model, he calibrates his model and get a Bachelier implied vol, aka normal vol.

$$\sigma_{normal} = BachCall^{-1}(F, K, r, T, Price).$$

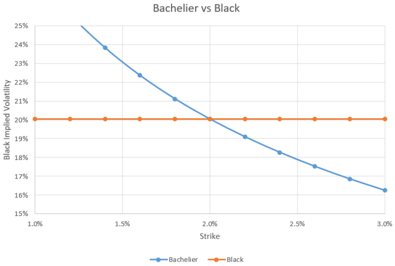
- A rough formula to connect the implied vols (for ATM option)

$$\sigma_{lognormal} \approx \sigma_{normal} F.$$

- For the exact solution, one would need to resort to numerical methods.

Bachelier vs Black: Lognormal Implied vol skew

- Assume the calibrated vol is 20%. Since the Black model assumes constant lognormal vol for all strikes, if we use the calibrated Black model to price options for all strikes, the corresponding implied vols will be flat at the level which the Black model is calibrated to.
- For the trader who uses Bachelier model, since he is calibrated to the ATM option price, his model must produce the same lognormal implied vol for ATM. That's where the blue line crosses the orange line.



Bachelier vs Black: Lognormal Implied vol skew

- But for non-ATM options, the two models will produce different option prices.
- For options with strike lower than the forward, the Bachelier model gives higher prices than Black.
- For options with strike higher than the forward, the Bacheleir model gives lower prices than Black.

Shifted lognormal model

- In 1983, Mark Rubinstein proposed the follwoing SDE for ooption pricing

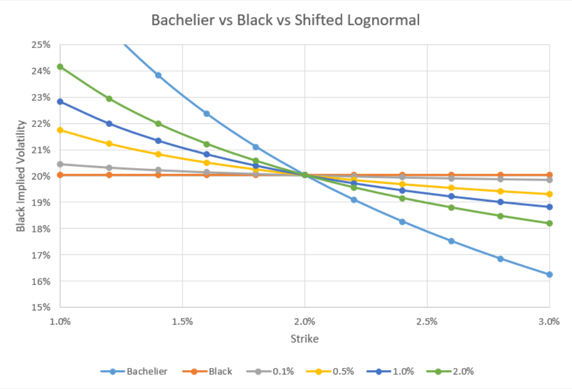
$$dF(t) = \sigma_{SLN}(F(t) + \alpha)dW(t)$$

- Notation is simplified as $F(t, T) \equiv F(t)$.
- σ_{SLN} is called the shifted lognormal vol.
- When α is small, the SDE ehaves like a Black model.
- When α is large, the SDE behaves like a Bachelier model.
- The call option price with strike K under shifted lognormal model is

$$SLNCall = BSCall(F(0) + \alpha, K + \alpha, \sigma_{SLN}, T)$$

Shifted lognormal model

- We can see that as α increases, the skew is closer to the Bachelier implied vol skew.
- But his formulation is difficult to control the skew in general setting as we need different α for different value of forward price $F(0)$



Another formulation

- Another forumulation of the shifted lognormal process is

$$dF(t) = \sigma_{SLN}(\beta F(t) + (1-\beta)F(0))dW(t).$$

- As $\beta \to 1$, the SDE becomes a Black model

$$dF(t) = \sigma_{SLN}F(t)dW(t).$$

- As $\beta \to 0$, the SDE become a Bachelier model

$$dF(t) = \sigma_{SLN}F(0)dW(t).$$

- The call option price under shifted lognormal model is

$$SLNCall = BSCall\left(\frac{F(0)}{\beta}, K + \frac{1-\beta}{\beta}F(0), \sigma_{SLN}\beta, T\right)$$

Forward Starting Option and Forward Implied Volatility

- Forward starting option is just like an European option but the strike is not set until some time in the future.
- The payoff of a forward starting call option is given as:

$$(S(T_2) - \alpha S(T_1))^+$$

  where $T_1 < T_2$ and α is the parameter which determines the moneyness.
- If we assume the spot price follows a lognormal process, there is a closed form formula.
- Let S follows:

$$dS(t) = rS(t)dt + \sigma_{fs} S(t) dW(t)$$

  in the risk neutral measure.

Forward Starting Option and Forward Implied Volatility

- By the martingale pricing formula:

$$\begin{aligned} V(0) &= \frac{N(0)}{N(T_2)} E_0\left[(S(T_2) - \alpha S(T_1))^+\right] \\ &= e^{-rT_2} E_0\left[(S(T_2) - \alpha S(T_1))^+\right] \end{aligned}$$

- By the tower law of expectation, we have

$$V(0) = e^{-rT_2} E_0\left[E_1\left[(S(T_2) - \alpha S(T_1))^+\right]\right] \tag{1}$$

- Let's concentrate on the expectation taken at T_1 and let $T_{1,2} := T_2 - T_1$:

$$E_1\left[(S(T_2) - \alpha S(T_1))^+\right] = S(T_1)e^{rT_{1,2}} N(d_1) - \alpha S(T_1) N(d_2)$$

where $d_1 = \frac{-\ln\alpha + rT_{1,2} + \frac{1}{2}\sigma_{fs}^2 T_{1,2}}{\sigma_{fs}\sqrt{T_{1,2}}}$, $d_2 = d_1 - \sigma_{fs}\sqrt{T_{1,2}}$.

Forward Starting Option and Forward Implied Volatility

- We can further simplify as

$$S(T_1)e^{rT_{1,2}}N(d_1) - \alpha S(T_1)N(d_2) = S(T_1)\left(e^{rT_{1,2}}N(d_1) - \alpha N(d_2)\right)$$

  and note that all the terms inside the bracket are non-random.
- Now go back to Equation (1) and we have

$$\begin{aligned} V(0) &= e^{-rT_2}E_0\left[S(T_1)\left(e^{rT_{1,2}}N(d_1) - \alpha N(d_2)\right)\right] \\ &= e^{-rT_2}S(0)e^{rT_1}\left(e^{rT_{1,2}}N(d_1) - \alpha N(d_2)\right) \\ &= S(0)\left(N(d_1) - \alpha e^{-rT_{1,2}}N(d_2)\right) \end{aligned}$$

- The notion of forward implied volatility is that we are given a price of the forward starting option and then we find σ_{fs} such that it produces the same price by plugging in the Black Scholes forward starting option formula.

Local Volatility Model

- All the models that we talked about in the lecture can only produce limited types of implied volatility shapes.
- An obvious question to ask is that: what is the simplest model that can calibrate to the whole implied volatility smile?
- The answer is local volatility (LV) model. Assuming no dividend, a local volatility model can be specified as

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma(S(t), t)dW(t).$$

- The state and time dependent diffusion coefficient $\sigma(S(t), t)$ known as the local volatility function.
- It is due to the seminal paper by Bruno Dupire called "Pricing with a smile" at 1994.

Local Volatility Model

- It possess many desire properties in terms of option pricing:
  - It is a single factor diffusion model.
  - It is easy to simulate and apply numerical PDE methods to price exotic options.
  - It has closed form solution for calibration. In other words, there is a closed form formula such that we can compute the diffusion coefficient $\sigma(S(t), t)$ using the implied volatility smile directly.
- There is a main drawback with the local volatility model:
  - It is well known that the forward smile (reads the implied volatility smile in the future time) generated from the model is not consistent with implied volatility surfaces observed in the real world.

Local Volatility Model

- For FX derivatives, banks would use it to price relatively short dated exotic options (we will come back to this later).
- For longer dated exotics, they are normally priced using local stochastic volatility (LSV) model.
- The LSV model is an upgrade on the LV model such that the volatility is allowed to be a random process.
- This allows the user to have more control on the smile dynamics.
- The calibration routine is more involved and requires more than just European options.
- A very good book about local volatility model is "The volatility surface" by Jim Gatheral.
- Another book which contains many details on LV and LSV is "Foreign Exchange Option Pricing A Practitioners Guide" by Iain J. Clark.