QF602: Derivatives

Lecture 7:
More on Option Pricings

Moment Generating Function

- The moment generating function $\phi(a)$ of the random variable X is defined for all values a by
$$\phi(a) = E[e^{aX}]$$
- We call $\phi(a)$ the MGF because all of the moments of X can be obtained by successively differentiating $\phi(a)$. For example
$$\phi'(a) = \frac{d}{da}E[e^{aX}] = E\left[\frac{d}{da}e^{aX}\right] = E[Xe^{aX}]$$
- Hence, $\phi'(0) = E[X]$. Similarly,
$$\phi''(a) = \frac{d}{da}\phi'(a) = E\left[\frac{d}{da}(Xe^{aX})\right] = E[X^2e^{aX}]$$
- And so, $\phi''(0) = E[X^2]$.
- In general, $\phi^n(0) = E[X^n], n \geq 1$.

Normal Distribution

- The moment generating function $\phi(a)$ of a standard normal random variable Z is obtained as follows

$$E[e^{aZ}]$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ax}\, e^{-\frac{x^2}{2}} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2ax)}{2}} dx$$
$$= e^{\frac{a^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2}} dx$$
$$= e^{\frac{a^2}{2}}$$

Normal Distribution

- If Z is a standard normal, then $X = \sigma Z + \mu$ is normal with parameters μ and σ, MGF of Z is given as

$$E[e^{aX}] = E\left[e^{a(\sigma Z+\mu)}\right] = e^{a\mu}E[e^{a\sigma Z}]$$
$$= \exp\left(\frac{\sigma^2 a^2}{2} + a\mu\right)$$

- This is one of the most important calculations that you need to remember in derivative pricing since Gaussian distribution is used everywhere in mathematical finance.

Martingale pricing

- Let $V(t)$ be a **tradable** asset price and $N(t)$ be **a strictly positive** asset, for $t<T$, we have
$$\frac{V(t)}{N(t)} = E_t^N\left[\frac{V(T)}{N(T)}\right]$$
- The subscript t denotes the expectation is taken at time t.
- The superscript N denotes the expectation is taken under the measure induced by the numeraire asset N.
- The above formula says **any tradeable numeraire rebased asset is a martingale under the numeraire induced probability measure.**

Why the risk neutral drift is r?

- In Black Scholes, the stock price is assumed to be lognormally distributed:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)$$

- How do we show that the drift $\mu = r$ if the numeraire asset is the money market account, $\beta(t) = e^{rt}$?
- The key is to identify what a tradeable asset.

Why the risk neutral drift is r?

- Domestic investors see the stock S as a risky asset, in the BS world, it has the distribution

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

- Comparing this with the money market account, a truly risk-neutral investor must expect the two assets to have the same expected returns. The ratio of these should therefore be a martingale, we have

$$\frac{S(t)}{\beta(t)} = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)} e^{-rt}$$

$$= S(0)e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)} e^{(\mu - r)t}$$

Why the risk neutral drift is r?

- The expectation of the ratio is equal to

$$E_0^{\beta}\left[\frac{S(t)}{\beta(t)}\right] = S(0)e^{(\mu-r)t}E[e^{-\frac{1}{2}\sigma^2 t+\sigma W(t)}] = S(0)e^{(\mu-r)t}$$

- In order to be a martingale, μ must equal to r.
- The idea is to construct some quantity you know that is a martingale and then solve for the unknown drift.

FX – domestic risk neutral measure

- Assume the FX rate, X, is lognormally distributed:
$$\frac{dX(t)}{X(t)} = \mu dt + \sigma dW(\mathrm{t})$$
- How do we show that the drift $\mu = r^d - r^f$ if the numeraire asset is the domestic money market account, $\beta^d(t) = e^{r^d t}$?
- Note that FX rate itself is not a tradeable asset but a foreign money market account denominated in domestic currency is one, $X(t)\beta^f(t)$.
- The following ratio is a martingale under the domestic risk neutral measure:
$$\frac{X(t)\beta^f(t)}{\beta^d(t)} = X(0)e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)} e^{(\mu + r^f - r^d)t}$$
- In order to be a martingale, $\mu = r^d - r^f$.

FX – foreign risk neutral measure

- Foreign investors see the domestic money market account denominated in foreign currency is a tradeable asset, $\frac{\beta^d(t)}{X(t)}$.

$$\frac{\beta^d(t)}{X(t)} = \frac{1}{X(0)} e^{\left(-\mu+\frac{1}{2}\sigma^2+r^d\right)t-\sigma W(t)}$$

- The ratio between the foreign tradeable asset and foreign money market account is

$$\frac{\beta^d(t)}{X(t)\beta^f(t)} = \frac{1}{X(0)} e^{\left(-\mu+\frac{1}{2}\sigma^2+r^d-r^f\right)t-\sigma W(t)}$$

$$= \frac{1}{X(0)} e^{-\frac{1}{2}\sigma^2 t-\sigma W(t)} e^{\left(-\mu+\sigma^2+r^d-r^f\right)t}$$

- In order to be a martingale in the foreign risk neutral measure, $\mu = r^d - r^f + \sigma^2$.

Margrabe Option Formula

- Assuming the dynamics of the two stock prices are

$$\frac{dS_1(t)}{S_1(t)} = rdt + \sigma_1 dW_1(t)$$
$$\frac{dS_2(t)}{S_2(t)} = rdt + \sigma_2 dW_2(t)$$

- Where W_1 and W_2 are correlated BMs with the correlation ρ.
- Consider the payoff

$$V(T) = (S_2(T) - S_1(T))^+$$

- The most straightforward way is to compute

$$V(0) = \frac{\beta(0)}{\beta(T)} E_0^\beta [V(T)]$$

- The expectation involves a two-dimensional integration. Is there any easier method?

Margrabe Option Formula

- One can rearrange the payoff such that

$$V(T) = (S_2(T) - S_1(T))^+ = S_1(T)\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+$$

- Recall the martingale pricing formula with t=0

$$\frac{V(0)}{N(0)} = E_0^N\left[\frac{V(T)}{N(T)}\right]$$

- We can see that if we pick the numeraire asset to be S_1, the pricing formula becomes

$$V(0) = S_1(0)E_0^{S_1}\left[\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+\right]$$

Margrabe Option Formula

- Now the option pricing problem reduces to computing the distribution of $Y(T) = \frac{S_2(T)}{S_1(T)}$ in S_1 measure.
- We know that Y is a S_1 martingale and both S_1 and S_2 are lognormally distributed so we know that the ratio of them are also lognormally distributed.
- Let

$$\frac{dY(t)}{Y(t)} = \mu_Y dt + \sigma_Y dW(t)$$

- What is the value of μ_Y?
- We first find the drifts of S_1 in the S_1 measure.

Margrabe Option Formula

- Recall $\beta(t) = e^{rt}$, in other words, $d\beta(t) = r\beta(t)dt$.
- Let

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dW_1.$$

- By Ito's lemma

$$d\left(\frac{\beta}{S_1}\right) = \frac{1}{S_1}d\beta - \frac{\beta}{S_1^2}dS_1 + \frac{1}{2}\frac{\beta}{S_1^3}2 < dS_1 >$$

$$= \frac{r\beta dt}{S_1} - \frac{\beta}{S_1}(\mu_1 dt + \sigma_1 dW_1) + \frac{\beta}{S_1}\sigma_1^2 dt$$

$$\frac{d\left(\frac{\beta}{S_1}\right)}{\left(\frac{\beta}{S_1}\right)} = rdt - (\mu_1 dt + \sigma_1 dW_1) + \sigma_1^2 dt$$

- $\mu_1 = r + \sigma_1^2$ in order to be a martingale.

Margrabe Option Formula

- We are now in the position to compute σ_Y.
- Let

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_2.$$

- By Ito's lemma

$$d\left(\frac{S_2}{S_1}\right) = \frac{1}{S_1} dS_2 - \frac{S_2}{S_1^2} dS_1 + \frac{1}{2}\frac{S_2}{S_1^3} 2 < dS_1 > - \frac{1}{S_1^2} < dS_1 dS_2 >$$

$$= \frac{S_2}{S_1}(\mu_2 dt + \sigma_2 dW_2) - \frac{S_2}{S_1}(\mu_1 dt + \sigma_1 dW_1) + \frac{S_2}{S_1}\sigma_1^2 dt - \frac{S_2}{S_1}\sigma_1\sigma_2\rho dt$$

$$\frac{d\left(\frac{S_2}{S_1}\right)}{\left(\frac{S_2}{S_1}\right)} = \mu_2 dt + \sigma_2 dW_2 - (\mu_1 dt + \sigma_1 dW_1) + \sigma_1^2 dt - \sigma_1\sigma_2\rho dt$$

- $\mu_2 = r + \sigma_1\sigma_2\rho$ in order to be a martingale (we don't really need this but it is nice to show).

Margrabe Option Formula

- Recall
$$\frac{dY(t)}{Y(t)} = \sigma_Y dW(t).$$
- Compare with the previous slide we have
$$\sigma_Y dW(t) = \sigma_2 dW_2 - \sigma_1 dW_1$$
- Let Z_1 and Z_2 be independent Brownian motions. Using Choleskey decomposition, we have
$$dW_1 = dZ_1$$
$$dW_2 = \rho dZ_1 + \sqrt{1-\rho^2} dZ_2$$
- The effective volatility can be computed as $\sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$

Margrabe Option Formula

- The Margrabe option can be computed as

$$V(0) = S_1(0)E_0^{S_1}[(Y(T) - 1)^+]$$

- Using the standard Black formula we have

$$\begin{aligned} V(0) &= S_1(0)[Y(0)N(d_1) - N(d_2)], \\ &= S_2(0)N(d_1) - S_1(0)N(d_2) \end{aligned}$$

$$d_1 = \frac{\ln\big(Y(0)\big) + \frac{1}{2}\sigma_Y^2 T}{\sigma_Y\sqrt{T}}, d_2 = d_1 - \sigma_Y\sqrt{T}$$

- Where $\sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$

Quanto Option

- A quanto option is, roughly, an option that pays off in the wrong currency. The FX rate is fixed at the inception.
- Google stock is a USD tradable asset but it is perfectly legit to write an option on Google but the notional is specified in SGD.
- A non-quanto equity call is mostly defined as:

$$N_{USD}\left(\frac{S(T)}{S(0)} - K\right)^+$$

- N_{USD} is the notional in the same currency as S. K is the strike factor, e.g. 1 if at-the-money. This form is normally called the "fixed notional".
- Another less common way, "fixed units" is defined as:

$$n\big(S(T) - KS(0)\big)^+$$

- n is the number of units and the two forms are equivalent.

Quanto Option

- Using the fixed notional form, it is easier to see what is a quanto option.
- A quanto equity call can be defined as:

$$N_{SGD}\left(\frac{S(T)}{S(0)} - K\right)^{+}$$

- N_{SGD} is the notional in SGD.
- Note that this is equivalent to

$$N_{USD}X(0)\left(\frac{S(T)}{S(0)} - K\right)^{+}$$

- Where X(0) is the FX rate at inception.

Quanto Option

- The key to understand quanto option pricing is to keep a firm grasp on what the tradeable quantities are. Suppose we are a SGD investor, our unit of account is SGD money market account and we have a quanto option on a US stock.
- Note that Google stock, S, is a USD tradable but not a SGD tradable. However, we can convert it into a SGD tradable by multiplying by the exchange rate to give it a price in SGD instead of USD.
- To price this option, we first identify what processes are involved.

Quanto Option

- Let
  - $X(t)$ denotes the value of one USD in SGD at time t and is assumed to be lognormally distributed.
  - S(t) denotes the value of a US stock time t and is assumed to be lognormally distributed.
  - $\beta^d(t)$ denotes the SGD money market account which grows at continuous rate r^d.
  - $\beta^f(t)$ denotes the USD money market account which grows at continuous rate r^f.
- The processes that we have are

$$d\beta^d = r^d \beta^d dt$$
$$d\beta^f = r^f \beta^f dt$$
$$dS = \mu_S S dt + \sigma_S S dW_S.$$
$$dX = \mu_X X dt + \sigma_X X dW_X.$$

- W_S and W_X are Brownian motions correlated with coefficient ρ.

Quanto Option

- We pick the numeraire to be β^d, this means that the drift of the FX process is $\mu_X = r^d - r^f$.
- The remaining quantity to be found is the drift of the US stock in the domestic risk neutral measure.
- Consider the domestically tradable asset, $X(t)S(t)$. By Ito's lemma, we have

$$d(XS) = SdX + XdS + < dXdS >$$

$$\frac{d(XS)}{(XS)} = (\mu_X dt + \sigma_X dW_X) + (\mu_S dt + \sigma_S dW_S) + \sigma_X \sigma_S \rho dt$$

- We know that the drift of any domestically tradable asset is r^d.
- In other words, $\mu_X + \mu_S + \sigma_X \sigma_S \rho = r^d$.
- We have μ_S = $r^d - (r^d - r^f) - \sigma_X \sigma_S \rho = r^f - \sigma_X \sigma_S \rho$

Quanto Option

- Therefore, the US stock in SGD risk neutral measure has the following dynamics:

$$\frac{dS(t)}{S(t)} = (r^f - \sigma_X \sigma_S \rho)dt + \sigma_S dW_S(t).$$

- The value of the payoff, set $N_{SGD} = 1$,

$$V(T) = \left(\frac{S(T)}{S(0)} - K \right)^+$$

- can be computed as

$$V(0) = \frac{\beta^d(0)}{\beta^d(T)} E_0^d \left[\left(\frac{S(T)}{S(0)} - K \right)^+ \right]$$

- which is a standard Black formula.

LIBOR in arrears

- Let the zero coupon bond at time t with maturity T be Z(t,T).
- The LIBOR rate can be defined as

$$L(t, T_1, T_2) = \frac{1}{\delta}\left(\frac{Z(t,T_1)}{Z(t,T_2)} - 1\right)$$

- L(t,T1,T2) is a martingale under the T2 forward measure. The corresponding numeraire asset is Z(t,T2).
- Note that the LIBOR rate is fixed at time T1. In other words, after T1, the LIBOR is not a random variable anymore.
- The payoff of an FRA at time t is

$$V(t) = \delta(L(t, T_1, T_2) - K)Z(t, T_2)$$

- In FRA, the corresponding LIBOR rate is fixed at T1 and the payment is at T2 but settled at T1.

LIBOR in arrears

- Using the martingale pricing formula:

$$V(0) = N(0)E\left[\frac{V(T_1)}{N(T_2)}\right]$$

- Pick N(t) = Z(t,T2) and we have

$$V(0) = Z(0,T_2)E\left[\frac{\delta(L(T_1,T_1,T_2) - K)Z(T_1,T_2)}{Z(T_1,T_2)}\right]$$

$$V(0) = Z(0,T_2)\delta E[L(T_1,T_1,T_2) - K]$$

- Recall the LIBOR is a martingale under T2 forward measure, we have

$$V(0) = Z(0,T_2)\delta(L(0,T_1,T_2) - K)$$

LIBOR in arrears

- What about if the LIBOR is fixed at T1 but the payment is also at T1?

- This is called the LIBOR in arrears and it has the following payoff

$$V(t) = \delta(L(t,T_1,T_2) - K)Z(t,T_1)$$

- Using the martingale pricing formula and pick N(t) = Z(t,T2)

$$V(0) = Z(0,T_2)E\left[\frac{\delta(L(T_1,T_1,T_2) - K)Z(T_1,T_1)}{Z(T_1,T_2)}\right]$$

- Note that the ZCB in the expectation don't cancel out in this case.

LIBOR in arrears

- Recall the definition of LIBOR:

$$L(t) = L(t, T_1, T_2) = \frac{1}{\delta}\left(\frac{Z(t,T_1)}{Z(t,T_2)} - 1\right)$$

- Rearrange the terms and we have

$$1 + \delta L(t) = \frac{Z(t, T_1)}{Z(t, T_2)}$$

- We then substitute it into

$$V(0) = Z(0, T_2)E\left[\frac{\delta(L(T_1) - K)Z(T_1, T_1)}{Z(T_1, T_2)}\right]$$

- To simplify the notation we assume δ=1, we have

$$V(0) = Z(0, T_2)E[\,(L(T_1) - K)(1 + L(T_1))]$$

LIBOR in arrears

- Rearrange the terms, we have

$$V(0) = Z(0,T_2)E[(L(T_1) - K)] + Z(0,T_2)E[L(T_1)(L(T_1) - K)]$$

- The green term is the FRA payoff and the red term is the convexity adjustment.

- We now concentrate on the convexity adjustment term. It contains a LIBOR squared term:

$$Z(0,T_2)E[L(T_1)^2 - KL(T_1)] = Z(0,T_2)E[L(T_1)^2] - Z(0,T_2)KL(0)$$

LIBOR in arrears

- There are a few ways to compute the term $E[L(T_1)^2]$
- Assume a dynamics for the LIBOR rate, say, it follows a lognormal process in the T2 forward measure

$$dL(T_1) = \sigma L(T_1) dW(t).$$

- Or using Breeden-Litzenberger formula to replicate the square payoff using a collection of caplets and floorlets.