QF602: Derivatives

Lecture 5:

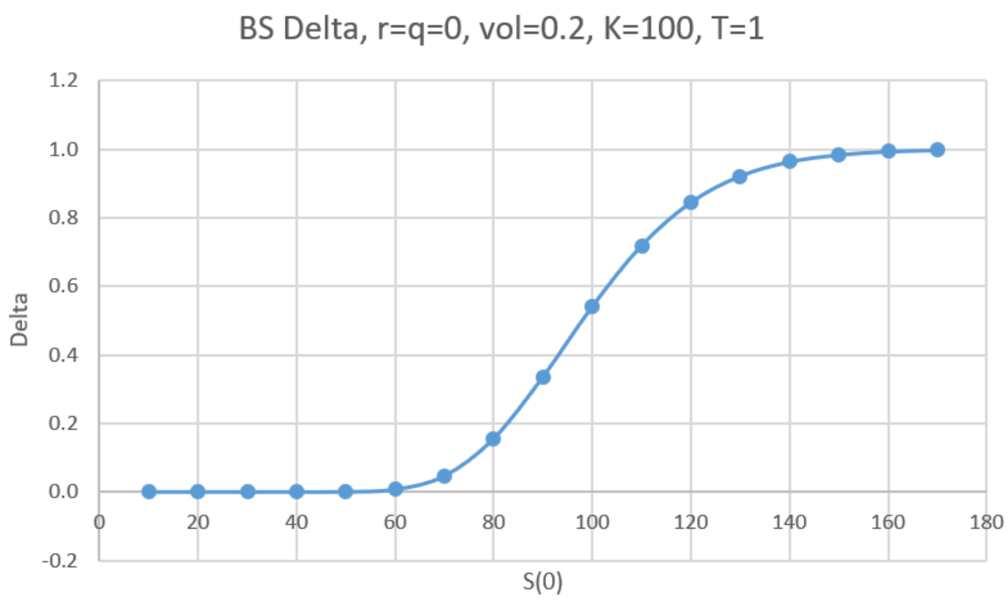
Greeks

What are Greeks

- The buying and selling a derivative creates a position with various sources of risk, some of which may be unwanted.
- Hedging is the act of reducing these risks by engaging in financial transactions that counterbalance these risks.
- When a bank sells a derivative to a client, it should understand all the risks associated with the product and hedge its accordingly.
- Once a sale is done, the product is added to an existing portfolio and the position must be risk managed.
- In order to see where the risks lie, the trader will need to know the sensitivity of the derivative's price to the various parameters that impact its value.
- The sensitivities of the option are often known as Greeks.

What are Greeks

- Note that Greeks are defined as the derivative of the price with respect to some model parameter. In other words, they are model dependent.
- Recall from Lecture 2. Delta is defined as $\frac{\partial Call}{\partial S(0)}$, 1st order sensitivity to spot. Note that this is Black Scholes Delta.



Models and Greeks

- In fact, computation of the Greeks are often one of the main reasons why model exists.
- For simple and liquid instruments, prices are largely determined by market forces.
- In those cases, we calibrate our model parameters to fit those prices. Then we compute the Greeks with respect to those parameters in order to risk manage the positions.
- In most cases, for complex and illiquid instruments, traders will calibrate to the **relevant,** simpler and liquid instruments and then ask the **calibrated model** to compute the price and the Greeks of the complex instruments.
- We will come back to this with more details when we talk about exotic derivatives.

Delta

- The most fundamental of all Greeks.
- Delta is the sensitivity of an option price to the underlying asset.
- The question that a trader would ask is, how much does the price of an option change if the underlying price is changed by x.
- To understand the concept of sensitivity, we need to look into Taylor series expansion for an option price $f(S)$ at the current spot price S_0

$$f(S_0 + x) = f(S_0) + f'(S_0)x + \frac{1}{2}f''x^2 + \cdots$$

Delta

- The change in price is given by

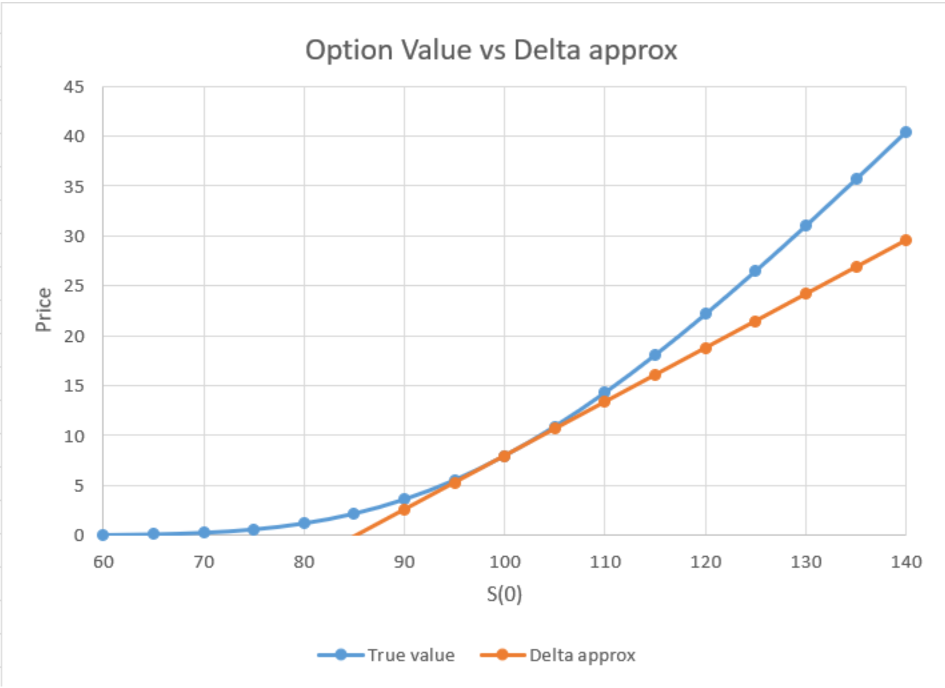
$$f(S_0 + x) - f(S_0) = f'(S_0)x + \frac{1}{2}f''x^2 + \cdots$$

- The first derivative w.r.t S_0 on the right-hand side is Delta.
- If x is small, meaning there is only a small movement in S, the price of the derivative will move by Delta times x

$$f(S_0 + x) - f(S_0) \approx f'(S_0)x$$

Delta

- $S_0 = 100, r = q = 0, K = 100, T = 1, \sigma = 0.2.$
- Option price is 7.966, Delta is 0.54.
- The curve line is the price of a call option against different value of S_0.
- The straight line is the option price compute using delta approximation. The slope of the line is 0.54. This means if S_0 is moved by \$10, the option will move by 0.54*\$10 = \$5.4.



Delta Hedging using Spot

- To hedge against movements in the underlying, a seller of a call option needs to buy delta units of the underlying.
- In our previous example, the seller needs to buy 0.54 unit of stocks.
- Now, the portfolio consists of a short 1 option and long 0.54 units of stock and is delta neutral.
- As delta changes with movements in the underlying, the amount of the underlying that needs to be held to remain delta neutral will need to be adjusted. This is the so-called dynamics hedging.
- Delta of a call option is non-negative.
- As the option goes deep into the money, i.e. $S_t \gg K$, the delta tends to 1.
- As the option goes deep out-of-they-money, the delta tends to 0.

Delta Hedging using Forward

- Other than trading the underlying to delta hedge, it is also possible to use forwards.
- Assume no dividend, recall the forward price is given by
$$F(t) = S(t)\exp(r(T-t))$$
- The delta of the forward price w.r.t the Spot price is
$$\frac{\partial F(t)}{\partial S(t)} = \exp\big(r(T-t)\big)$$
- One would need to buy $\frac{\partial P(t)}{\partial S(t)}\exp(-r(T-t))$ amount of forward contracts.

Delta Hedging using other asset

- One can further exploit correlations between assets to delta hedge.
- It is useful if you sell an option P on an underlying S_1 that you cannot easily buy/sell but you know there is a highly correlated asset S_2 which you can access to.
- This is an application of the chain rule:

$$\frac{\partial P}{\partial S_1} = \frac{\partial P}{\partial S_2} \frac{\partial S_2}{\partial S_1}$$

- If we assume both S_1 and S_2 are lognormally distributed and denote ρ to be the correlation between their log returns then we can compute the change in S_2 for a change in S_1 as

$$\frac{\partial S_2}{\partial S_1} = \rho \frac{\sigma_2 S_2}{\sigma_1 S_1}$$

A Proof

- Show

$$\frac{\partial S_2}{\partial S_1} = \rho \frac{\sigma_2 S_2}{\sigma_1 S_1}$$

- Assume

$$\frac{dS_1}{S_1} = \sigma_1 dW_1(t), \frac{dS_2}{S_2} = \sigma_2 dW_2(t), E[dW_1 dW_2] = \rho dt$$

- The solution of the SDEs are
- $S_1(t) = S_1(0) \exp\left(-\frac{1}{2}\sigma_1^2 t + \sigma_1 \sqrt{t} x_1\right)$
- $S_2(t) = S_2(0) \exp\left(-\frac{1}{2}\sigma_2^2 t + \sigma_2 \sqrt{t} (\rho x_1 + \sqrt{1-\rho^2} x_2)\right)$
- x_1 and x_2 are independent standard normal random variables.
- The above is an example of Cholesky decomposition.

A Proof

- Apply the chain rule and we can show that , $\frac{\partial S_2}{\partial S_1} = \frac{\partial S_2}{\partial x_1}\frac{\partial x_1}{\partial S_1}$

$$\frac{\partial S_1}{\partial x_1} = \sigma_1\sqrt{t}S_1$$

$$\frac{\partial S_2}{\partial x_1} = \rho\sigma_2\sqrt{t}S_2$$

- Substitute in the terms and we get

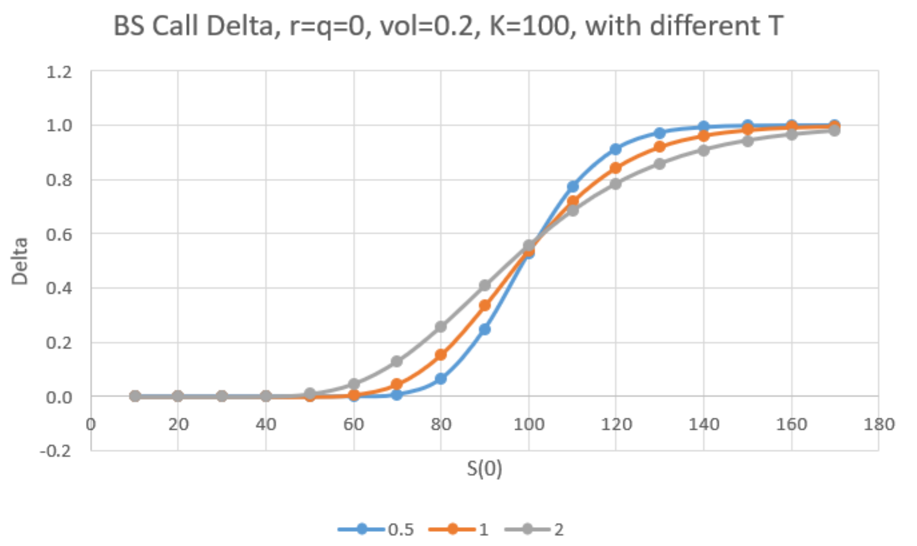
$$\frac{\partial S_2}{\partial S_1} = \rho\,\frac{\sigma_2 S_2}{\sigma_1 S_1}$$

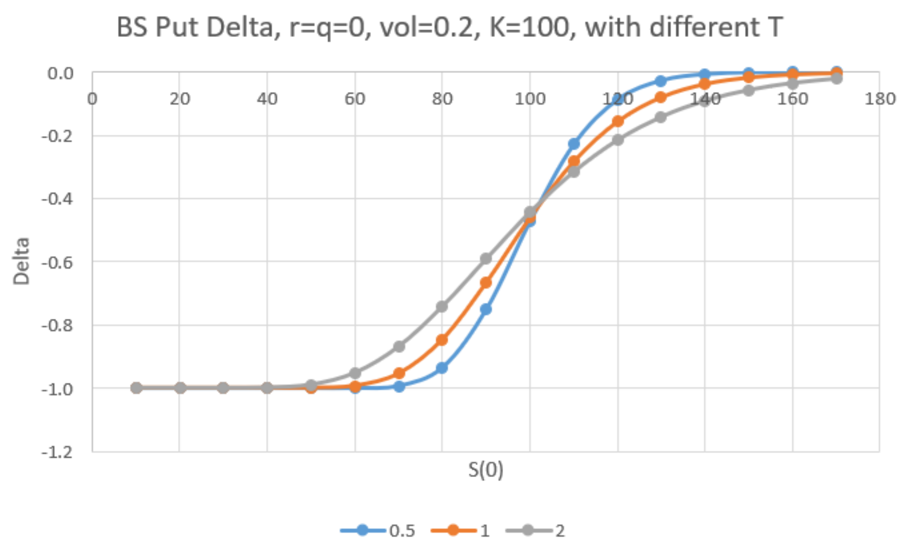
Black Scholes Delta

- Assuming no dividend yield, the Black Scholes Deltas are given as

$$\Delta_{call} = N\left(\frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right), \Delta_{put} = \Delta_{call} - 1$$

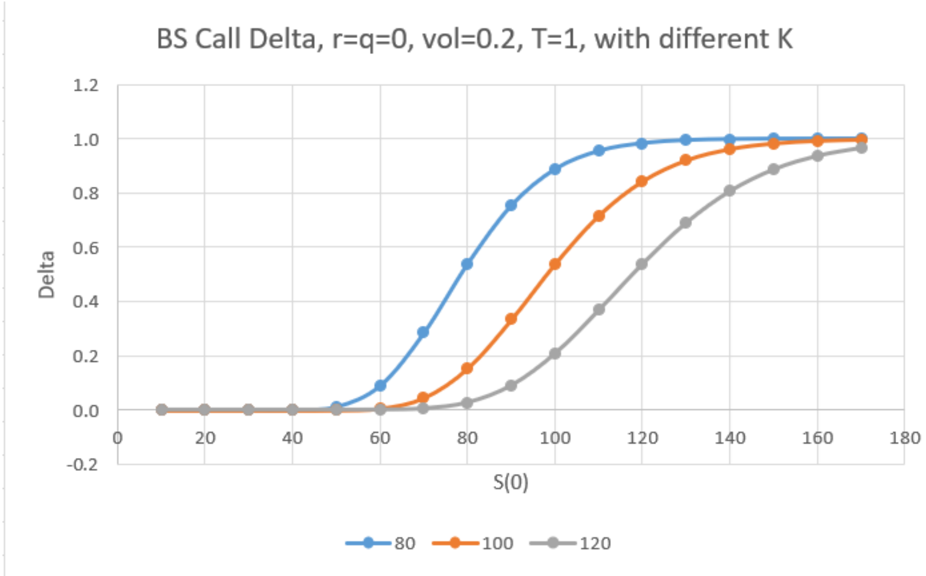
- The graphs below show the delta profile for options with different maturities.





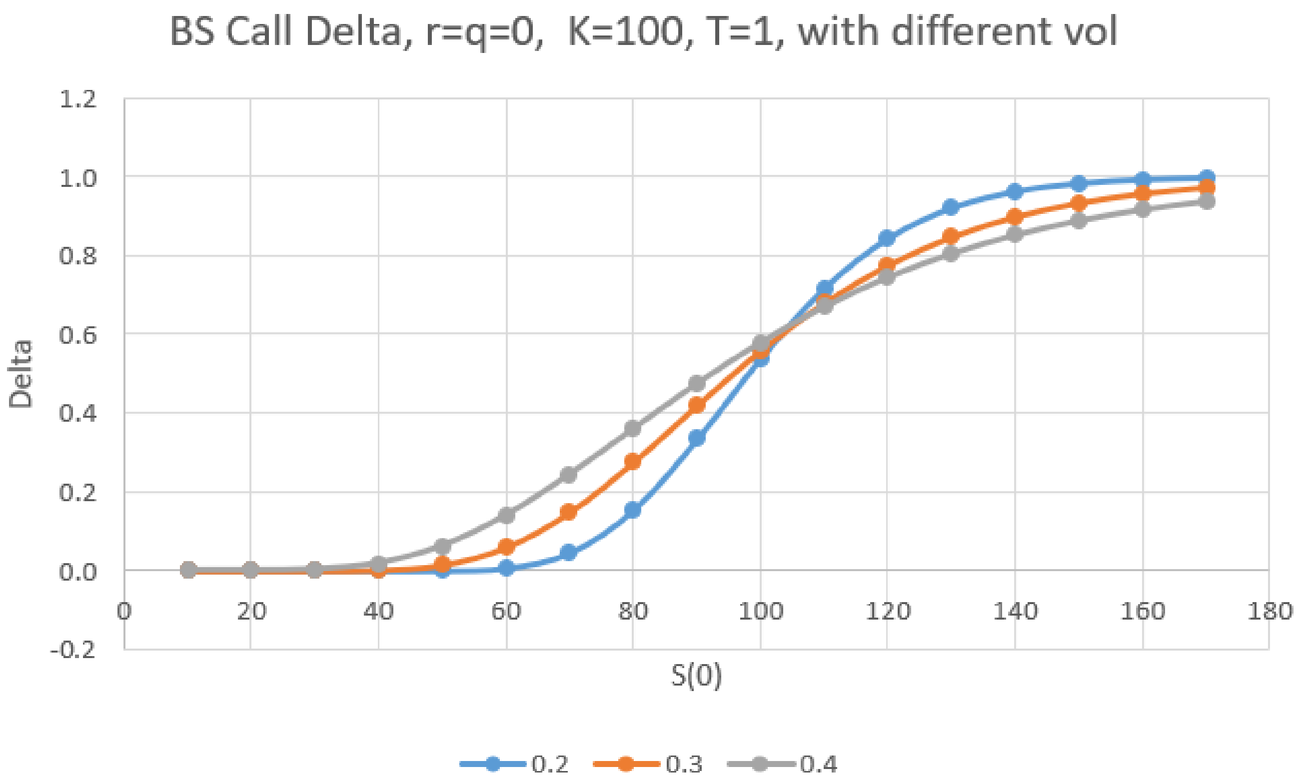
Black Scholes Delta

- Delta of a European option is sensitive to the maturity, the volatility of the underlying and the difference between the strike and the spot prices (aka moneyness)
- The graphs below show the delta profile for call options with different moneyness.



Black Scholes Delta

- The graphs below show the delta profile for a call options with different volatilities.



Gamma

- Gamma represents the second order sensitivity of the option to a movement in the underlying.
- It is the second term in the Taylor series expansion for an option price $f(S)$ at the current spot price S_0

$$f(S_0 + x) = f(S_0) + f'(S_0)x + \frac{1}{2}f''x^2 + \cdots$$

- For a non-small move x, the second order effect is not negligible. This is often called the convexity.
- Gamma is the first order sensitivity of Delta to a movement in the underlying. It tells us how much Delta will move if the underlying moves.

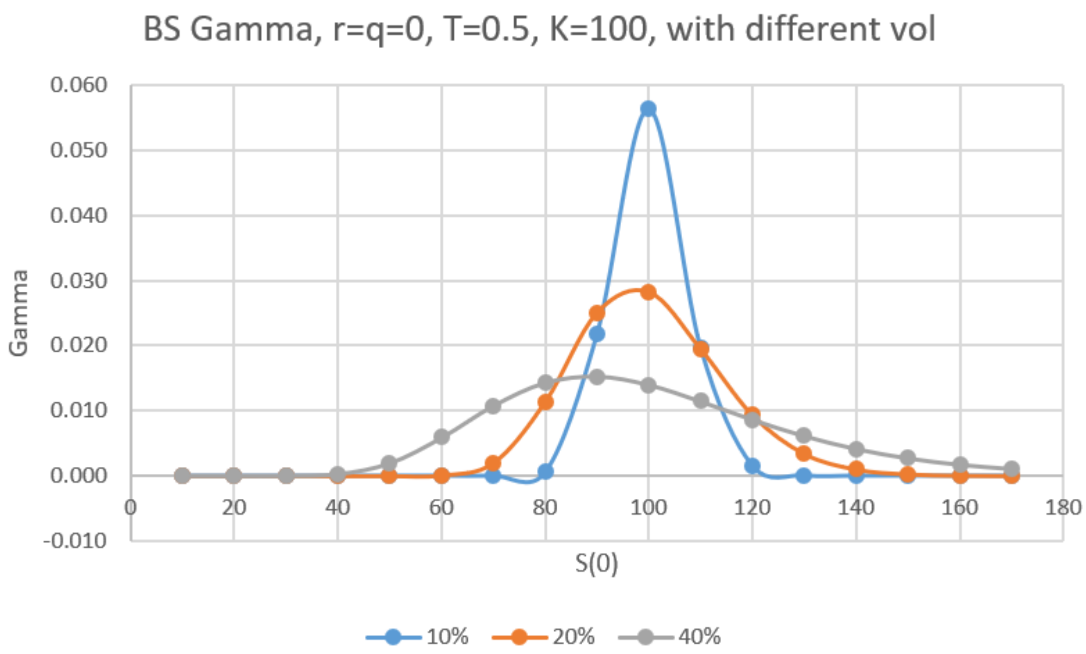
Gamma

- Assuming no dividend, the **Black Scholes Gamma** for both calls and puts is given by

$$\Gamma = \frac{\phi\left(\frac{\ln\left(\frac{F}{K}\right)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)}{S\sigma\sqrt{T}}, \phi(a) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}$$

Black Scholes Gamma

- The graph below shows the gamma profile for options with different volatilities.
- A higher volatility lowers the Gamma when underlying is near the strike but raises it when the underlying is away from the strike.



Black Scholes Gamma

- We can think of this effect in terms of time value of the option.
- Higher the gamma, higher the time value. We will come back to this later.
- For low levels of volatility, the Gamma is low for deep ITM and OTM options because these options have little time value. Vice-versa for high levels of volatility.

Gamma and Delta-Hedge

- The concept of a delta-hedged portfolio of options means that the portfolio has been hedged by trading the underlying assets against small movements in these assets.
- Gamma represents how much of rebalancing that one needs to do in order to maintain delta-hedged when there is large market movement.
- Since underlying only has delta, in order to hedge gamma, one would need to trade options.
- By lowering gamma of the portfolio, we lower the need for a large and frequent rebalancing of delta.

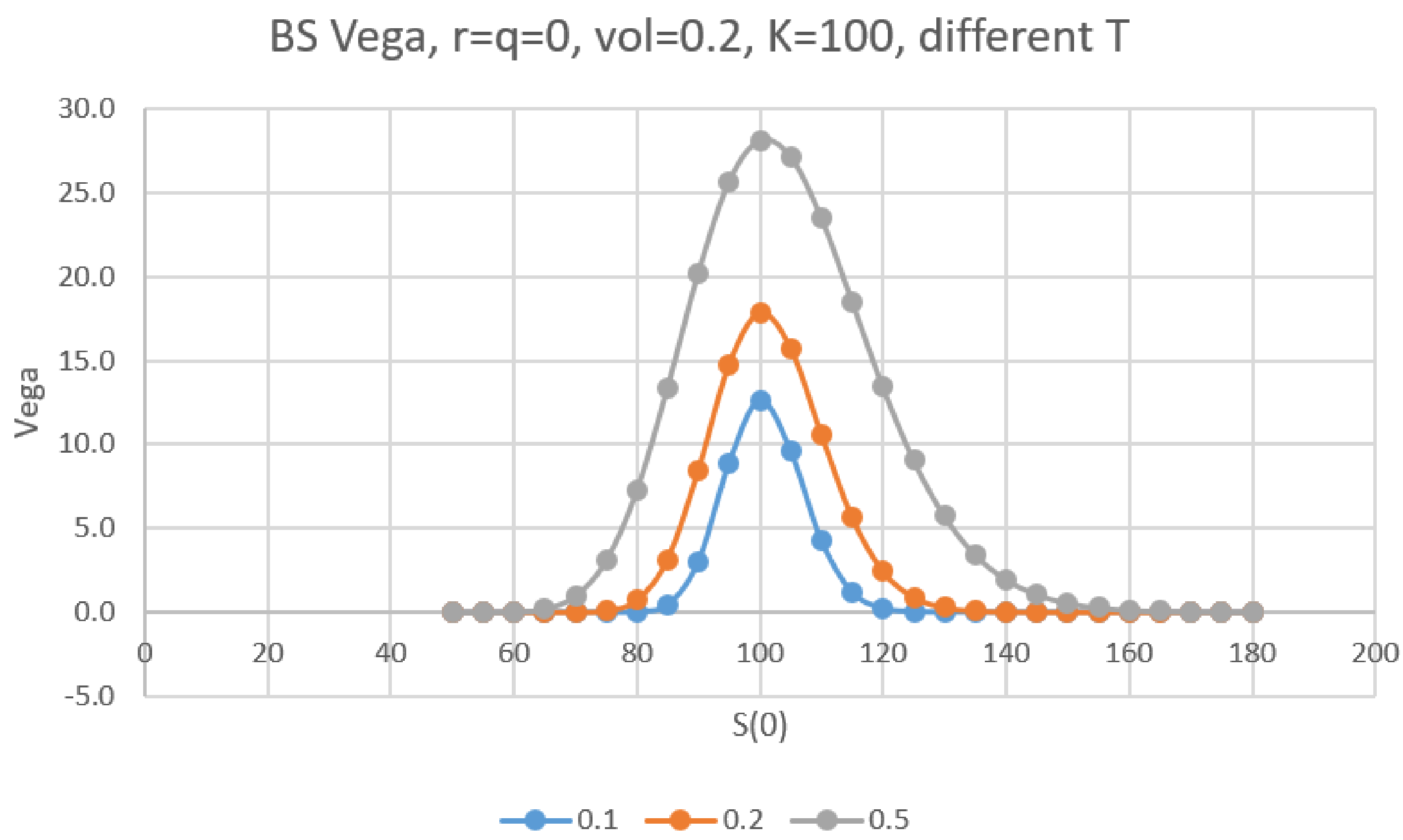
Vega

- Vega is the sensitivity of the option price to a movement in the implied volatility of the underlying asset.
- Strictly speaking, there should be no Vega in the Black Scholes world as volatility is assumed to be constant.
- However, it is very important to see how an option price changes as the result of change in the implied volatility.
- Mathematically, there is no stopping us to compute the derivative of a Black Scholes option price w.r.t the implied volatility (assume no dividend):

$$\frac{\partial Call}{\partial \sigma} = \frac{\partial Put}{\partial \sigma} = S(0)\phi(d_1)\sqrt{T}$$

Black Scholes Vega

- The graphs below show the vega profile for call options with different maturities.



Black Scholes Vega

- Vega is the largest when K=S(0) and decays exponentially on both sides.
- When the option is ATM, a change in volatility of the underlying asset can send the option either ITM or OTM, thus the large effect on the price.
- When the option is either ITM or OTM, a change in volatility does not have as much impact as for ATM option.

Theta

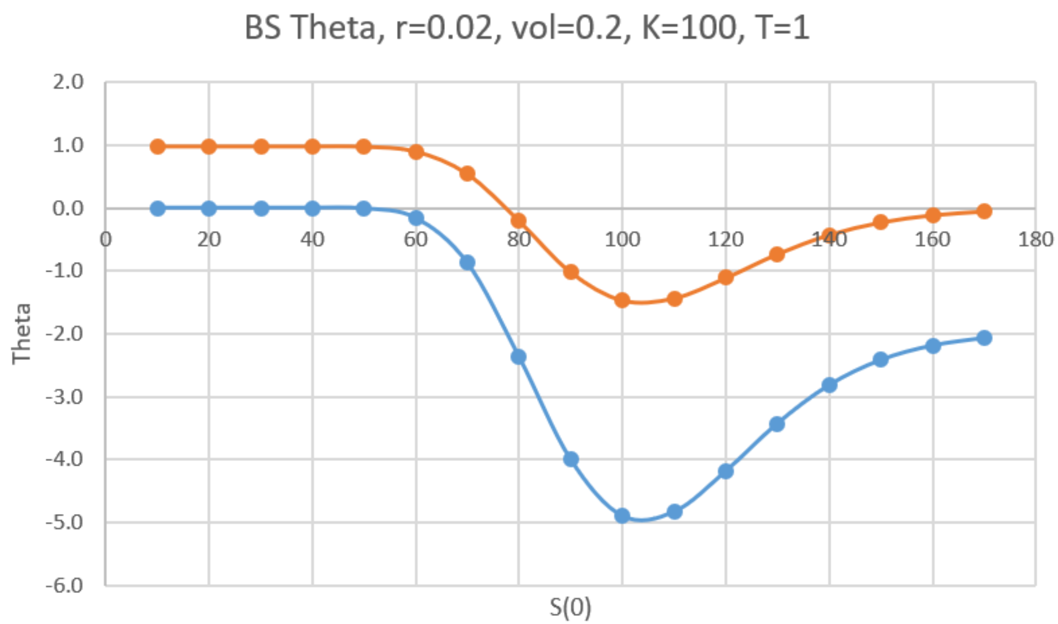
- Theta is the rate at which the option price changes as time goes by. In other words, it is asking how much does the option price change after one day, all else being equal?
- The Black Scholes call and put options thetas are given as (no dividend)

$$\frac{\partial Call}{\partial t} = -\frac{S(0)\sigma\phi(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

$$\frac{\partial Put}{\partial t} = -\frac{S(0)\sigma\phi(d_1)}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Theta

- The graph below shows the Black Scholes thetas of a call and put.
- For call option, theta is always non-positive.
- For put option, when it is deep in the money, theta becomes positive.

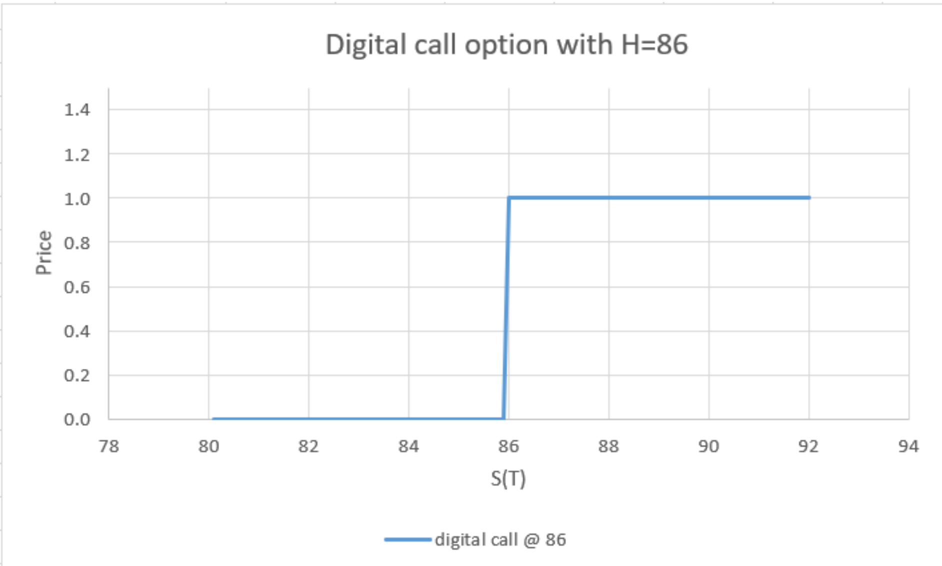


Digital Option

- So far, all the Greeks analysis are done using European call and put options.
- Let's look at the Greeks for digital call option.
- Recall the payoff of the digital call option:

$$1_{S(T)>H}$$

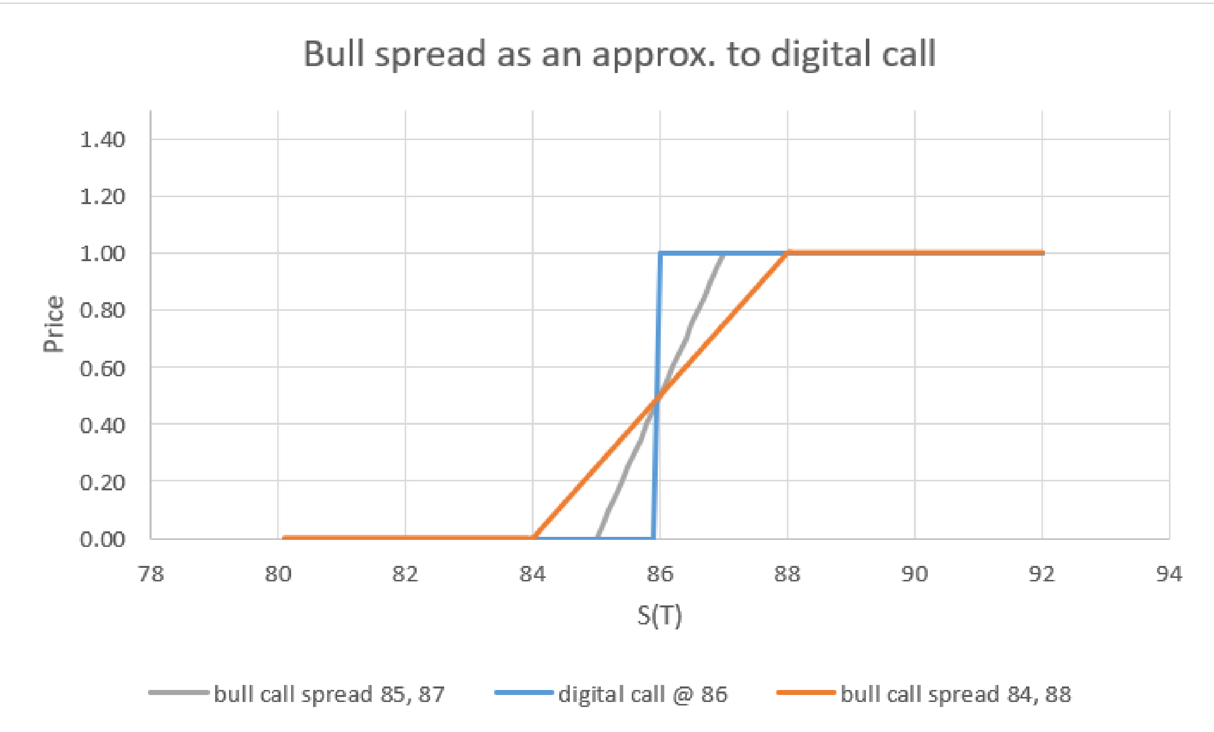
- 1 is the indicator function and H is the barrier.
- The graph below is the payoff diagram with H=86.



Digital Option

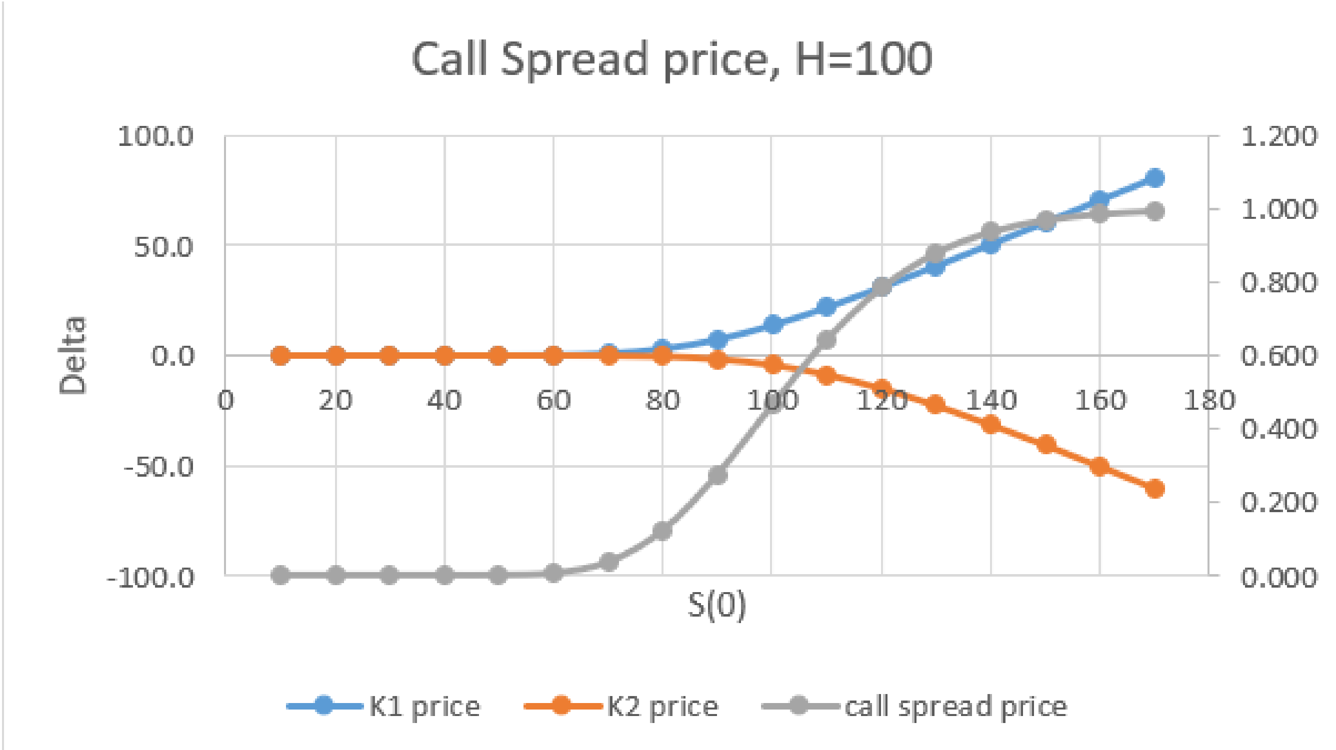
- Recall that we can use a bull (call) spread as an approx. to a digital call.
- Digital call option can be thought as a limit of a call spread:

$$DigitalCall(H) = \lim_{\epsilon \to 0} \frac{Call(H-\epsilon) - Call(H+\epsilon)}{2\epsilon} = -\frac{\partial Call(K)}{\partial K}$$



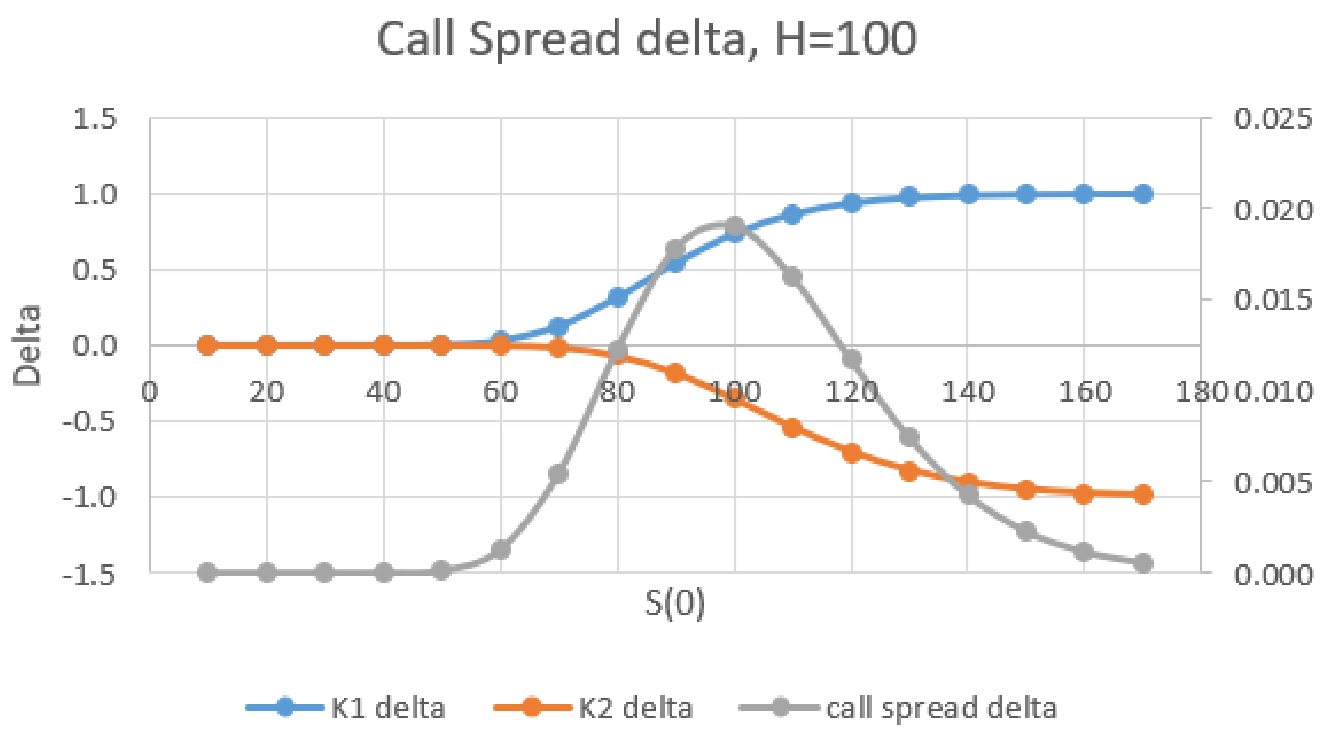
Call Spread Price

- Let $H = 100, \epsilon = 10, K_1 = H - \epsilon =, K_2 = H + \epsilon$.
- The graph below shows the price profile of a call spread and the call options with r=q=0, vol=0.2, T=1.



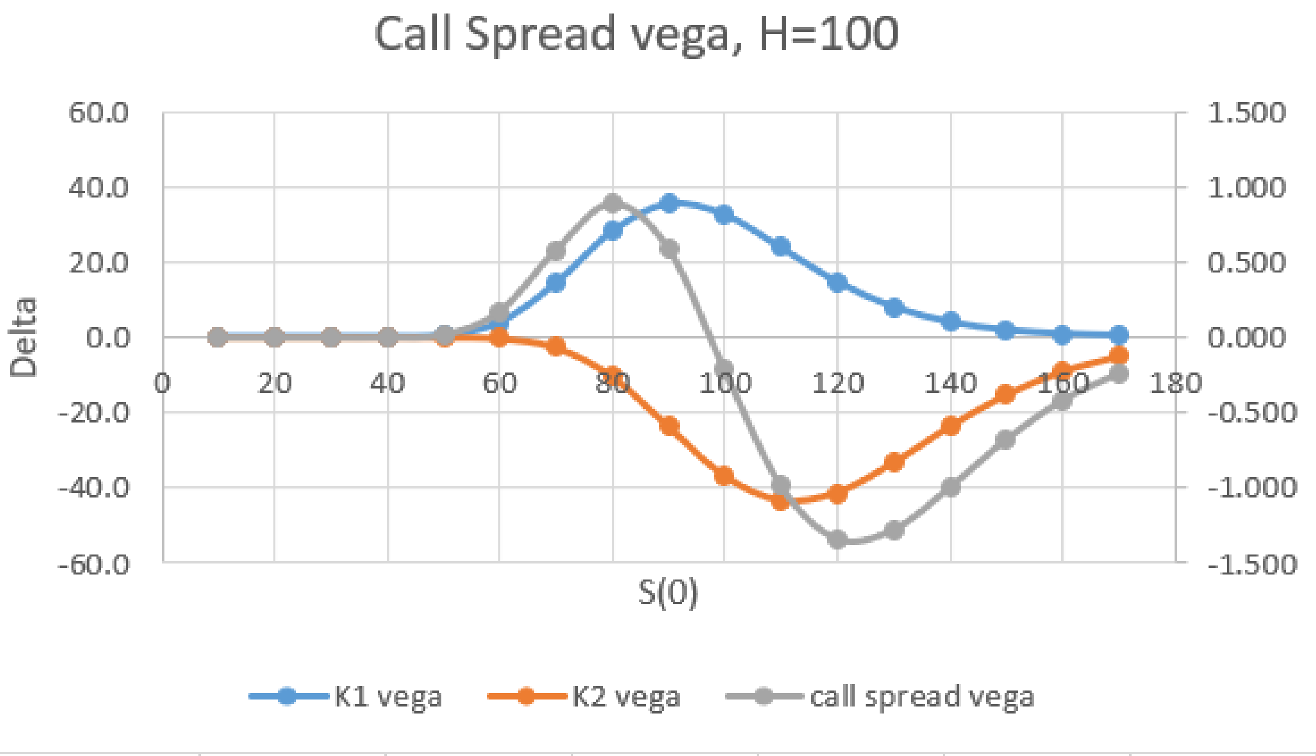
Call Spread Delta

- The graph below shows the delta profile of a call spread and the call options with r=q=0, vol=0.2, T=1.



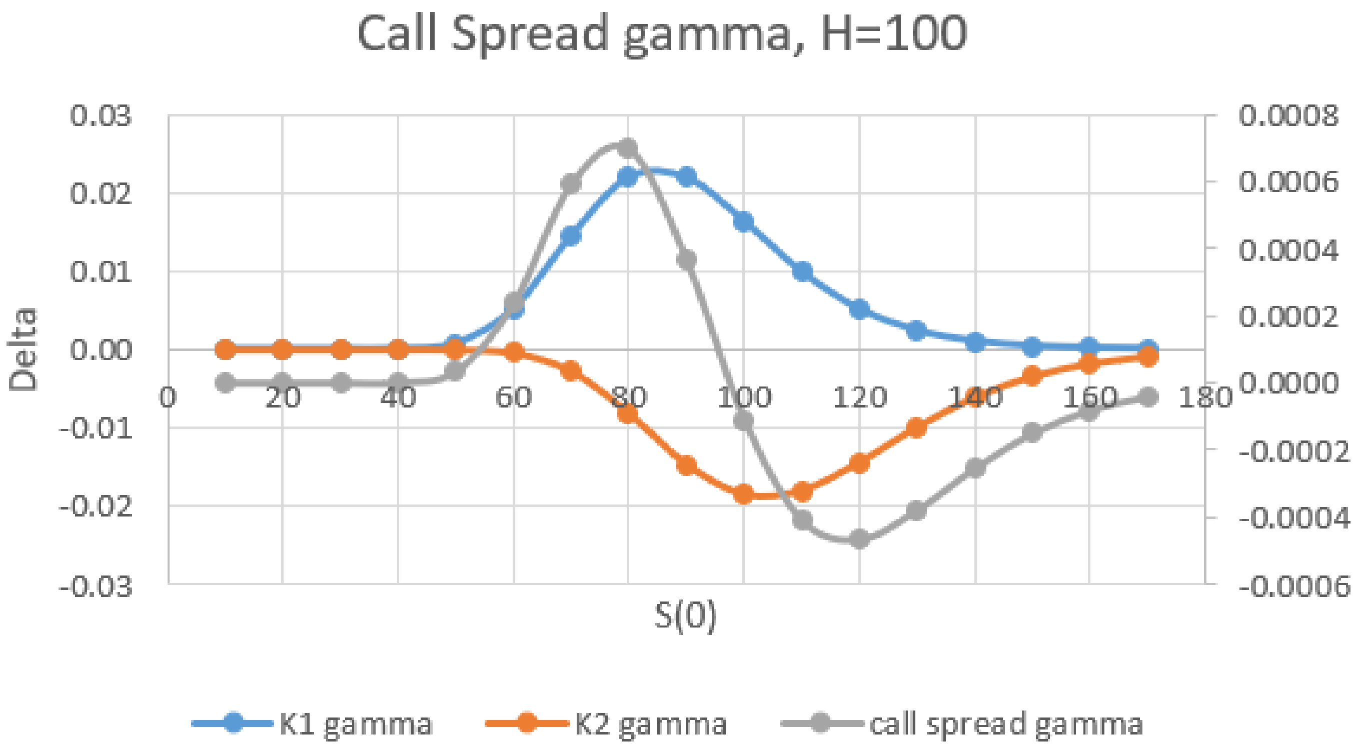
Call Spread Vega

- The graph below shows the vega profile of a call spread and the call options with r=q=0, vol=0.2, T=1.



Call Spread Gamma

- The graph below shows the gamma profile of a call spread and the call options with r=q=0, vol=0.2, T=1.



Trading the volatility

- Consider the case that one has bought a call option at strike 100, spot at 100, maturity is 1 year, rate and dividend are 0.
- The diagram below shows the the spot price movements, the corresponding option prices and deltas.
- Assume one is always delta hedged as time goes by, what is the PnL at the end of the period?
- TTM stands for time to maturity.
- Notice that spot at TTM=0.96 is the same as inception.

TTM	spot	option price	delta
1	100.0	3.99	0.52
0.99	104.6	6.77	0.69
0.98	99.0	3.45	0.48
0.97	103.6	6.03	0.66
0.96	100.0	3.91	0.52

Trading the volatility

- At inception, one would short 0.52 stocks @ 100.
- At TTM=0.99, spot moves up to 104.6, delta of the call increases to 0.69, one would need to sell more stocks, i.e. short another 0.17 stocks @ 104.6.
- At TTM=0.98, spot moves down to 99, delta drops, one needs to buy the stocks to remain delta hedged, i.e. long 0.2127 @ 99.
- Similar for TTM=0.97 and 0.96. Total PnL = 1.5567.

TTM	**spot**	**option price**	**delta**	change in delta in the hedge portfolio	PnL of the hedge portfolio	PnL of option	Total PnL
1	100.0	3.99	0.52	-0.5199			
0.99	104.6	6.77	0.69	-0.1722	-2.3932	2.7805	
0.98	99.0	3.45	0.48	0.2127	3.8746	-3.3177	
0.97	103.6	6.03	0.66	-0.1777	-2.1849	2.5775	
0.96	100.0	3.91	0.52	0.1376	2.3406	-2.1208	
					1.6372	-0.0805	**1.5567**

Trading the volatility

- Consider the similar case but the spot volatility is higher.
- In order to have a fair comparison, note that the implied volatility for the option is the same as the previous slide.
- Notice that the Total PnL is larger.

TTM	**spot**	**option price**	**delta**	change in delta in the hedge portfolio	PnL of the hedge portfolio	PnL of option	Total PnL
1	100.0	3.99	0.52	-0.5199			
0.99	108.3	9.56	0.80	-0.2834	-4.3304	5.5761	
0.98	96.1	2.22	0.36	0.4419	9.8413	-7.3426	
0.97	104.1	6.37	0.68	-0.3141	-2.8924	4.1528	
0.96	100.0	3.91	0.52	0.1560	2.7571	-2.4667	
					5.3755	-0.0805	**5.2950**

Trading the volatility

- Consider the similar case but the spot volatility is 0.
- In order to have a fair comparison, note that the implied volatility for the option is the same as the previous slide.
- Notice that the Total PnL is negative. This is due to the negative theta.

TTM	**spot**	**option price**	**delta**	change in delta in the hedge portfolio	PnL of the hedge portfolio	PnL of option	Total PnL
1	100.0	3.99	0.52	-0.5199			
0.99	100.0	3.97	0.52	0.0001	0.0000	-0.0200	
0.98	100.0	3.95	0.52	0.0001	0.0000	-0.0201	
0.97	100.0	3.93	0.52	0.0001	0.0000	-0.0202	
0.96	100.0	3.91	0.52	0.0001	0.0000	-0.0203	
					0.0000	-0.0805	**-0.0805**