QF602 Derivatives

Lecture 6 - More on option pricing

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Moment generating function

- The moment generating function ϕ of the random variable x is defined for all values a by

$$\phi(a) = E\left[e^{ax}\right]$$

- We call ϕ the MGF because all of the moments of x can be obtained by successively differentiating $\phi(a)$ and set $a = 0$. For example

$$\phi^{'}(a) = \frac{\partial}{\partial a} E\left[e^{ax}\right] = E\left[xe^{ax}\right]$$

- Hence, $\phi^{'}(0) = E[x]$. Similarly,

$$\phi^{''}(a) = \frac{\partial}{\partial a} \phi^{'}(a) = E\left[x^2 e^{ax}\right]$$

- And so, $\phi^{''}(0) = E[x^2]$.
- In general, $\phi^n(0) = E[x^n], n \geq 1$.

Normal distribution

- The moment generating function $\phi(a)$ of a standard normal random variable z is obtained as follows

$$\begin{aligned} E[e^{az}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{az} e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z^2-2az)/2} dz \\ &= \frac{1}{\sqrt{2\pi}} e^{a^2/2} \int_{-\infty}^{\infty} e^{-(z-a)^2/2} dz \\ &= e^{a^2/2} \end{aligned}$$

Normal distribution

- If z is a standard normal, then $x = \sigma z + \mu$ is normal with parameters μ and σ, MGF of x is given as

$$\begin{aligned} E[e^{ax}] &= E[e^{a(\sigma z+\mu)}] \\ &= e^{a\mu} E[e^{a\sigma z}] \\ &= e^{\frac{\sigma^2 a^2}{2}+a\mu} \end{aligned}$$

- This is one of the most important calculations that you need to remember in derivative pricing since Gaussian distribution is used everywhere in mathematical finance.

Martingale pricing

- Let $V(t)$ be a tradeable asset price and $N(t)$ be a strictly positive asset, for $t < T$, we have

$$\frac{V(t)}{N(t)} = E_t \left[\frac{V(T)}{N(T)} \right]$$

- The subscript t denotes the expectation is taken at time t. The superscript N denotes the expectation is taken under the measure induced by the numeraire asset N.
- The above formula says any tradeable numeraire induced asset is a martingale under the numeraire induced probability measure.

Why the risk neutral drift is r?

- In Black Scholes, the stock price is assumed to be lognormally distributed:
$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)$$
- How do we show that the drift $\mu = r$ if the numeraire asset is the money market account, $\beta(t) = e^{rt}$?
- The key is to identify what is a tradeable asset.

Why the risk neutral drift is r?

- Domestic investors see the stock S as risky asset, in the BS world, it has the distribution

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

- Comparing this with the money market account, a truly risk-neutral investor must expect the two assets to have the same expected returns. The ratio of these should therefore be a martingale, we have

$$\begin{aligned} \frac{S(t)}{\beta(t)} &= S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)} e^{-rt} \\ &= S(0)e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)} e^{(\mu - r)t} \end{aligned}$$

Why the risk neutral drift is r?

- The expectation of the ratio is equal to

$$
\begin{aligned}
E_0^{\beta}\left[\frac{S(t)}{\beta(t)}\right] &= S(0)e^{(\mu-r)t}E\left[e^{-\frac{1}{2}\sigma^2 t+\sigma W(t)}\right] \\
&= S(0)e^{(\mu-r)t}
\end{aligned}
$$

- In order to be a martingale, μ must equal to r.
- The idea is to construct some quantity you know that is a martingale and then solve for the unknown drift.

FX - domestic risk neutral measure

- Assume an FX rate, X, is lognormally distributed:

$$\frac{dX(t)}{X(t)} = \mu dt + \sigma dW(t)$$

- How do we show that the drift, $\mu = r^d - r^f$, if the numeraire asset is the domestic money market account $\beta^d(t) = e^{r^d}$?
- Note that the FX rate itself is not a tradeable asset but a foreign money market account denominated in the domestic currency is, $X(t)\beta^f(t)$.
- The following ratio is a martingale under the domestic risk neutral measure:

$$\frac{X(t)\beta^f(t)}{\beta^d(t)} = X(0)e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)} e^{(\mu + r^f - r^d)t}$$

.

- In order to be a martingale, the drift $\mu = r^d - r^f$.

FX - foreign risk neutral measure

- Foreign investors see the domestic money market account denominated in foreign currency is a tradeable asset, $\frac{\beta^d(t)}{X(t)}$:

$$\frac{\beta^d(t)}{X(t)} = \frac{1}{X(t)} e^{(-\mu+\frac{1}{2}\sigma^2+r^d)t-\sigma W(t)}$$

.

- The ratio between the foreign tradeable asset and foreign money market account is

$$\begin{aligned}\frac{\beta^d(t)}{X(t)\beta^f(t)} &= \frac{1}{X(t)} e^{(-\mu+\frac{1}{2}\sigma^2+r^d-r^f)t-\sigma W(t)} \\ &= \frac{1}{X(t)} e^{-\frac{1}{2}\sigma^2 t-\sigma W(t)} e^{(-\mu+\sigma^2+r^d-r^f)t}.\end{aligned}$$

- In order to be a martingale in the foreign risk neutral measure, the drift $\mu = r^d - r^f + \sigma^2$.

Margrabe Option Formula

- Assuming the dynamics of the two stock prices are

$$\begin{aligned}\frac{dS_1(t)}{S_1(t)} &= rdt + \sigma_1 dW_1(t) \\ \frac{dS_2(t)}{S_2(t)} &= rdt + \sigma_2 dW_2(t)\end{aligned}$$

- where W_1 and W_2 are correlated BMs with the correlation ρ.
- Consider the payoff

$$V(T) = (S_2(T) - S_1(T))^+.$$

- The most straightforward (or brute-force) way is to compute the expectation in the risk neutral measure:

$$V(0) = \frac{\beta(0)}{\beta(T)} E_0^{\beta}[V(T)].$$

- The expectation involves a two-dimensional integration. Is there an easier method than doing that?

Margrabe Option Formula

- One can rearrange the payoff such that

$$V(T) = (S_2(T) - S_1(T))^+ = S_1(T)\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+.$$

- Recall the martingale pricing formula with $t = 0$

$$\frac{V(0)}{N(0)} = E_0^N\left[\frac{V(T)}{N(T)}\right].$$

- If we pick the numeraire asset to be S_1, the pricing formula becomes

$$V(0) = E_0^{S_1}\left[\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+\right].$$

Margrabe Option Formula

- Now the option pricing problem reduces to computing the distribution of $Y(T) = \frac{S_2(T)}{S_1(T)}$ in the S_1 measure.
- We know that Y is a S_1 martingale (why?) and both S_1 and S_2 are lognormally distributed so we know that the ratio of them are also lognormally distributed.
- Let
$$\frac{dY(t)}{Y(t)} = \mu_Y dt + \sigma_Y dW(t).$$
- What is the value of μ_Y in the S_1 measure? It is trivial, $\mu_Y = 0$ because Y is a S_1 martingale.

Margrabe Option Formula

- Our next task is to find σ_Y.
- Recall $Y = \frac{S_2}{S_1}$, by Ito's lemma, we get

$$\frac{dY(t)}{Y(t)} = (\mu_2 - \mu_1 + \sigma_1^2 - \sigma_1\sigma_2\rho)dt + (\sigma_2 dW_2 - \sigma_1 dW_1).$$

- Let's ignore the drift since we know that it is 0. Compare the Brownian motion terms in with

$$\frac{dY(t)}{Y(t)} = \sigma_Y dW$$

and we have

$$\sigma_Y dW = \sigma_2 dW_2 - \sigma_1 dW_1$$

.

Margrabe Option Formula

- Let Z_1 and Z_2 be independent Brownian motions. Using Choleskey decomposition, we have

$$\begin{aligned} dW_1 &= dZ_1 \\ dW_2 &= \rho dZ_1 + \sqrt{1-\rho^2}\, dZ_2. \end{aligned}$$

- The effective volatility can be computed as

$$\sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}.$$

Margrabe Option Formula

- The Margrabe option can be coputed as
$$V(0) = S_1(0)E_0^{S_1}[(Y(T)-1)^+].$$
- Using the standard Black formula we have
$$\begin{aligned} V(0) &= S_1(0)[Y(0)N(d_1) - N(d_2)] \\ &= S_2(0)N(d_1) - S_1(0)N(d_2), \end{aligned}$$
where
$$d_1 = \frac{\ln Y(0) + \frac{1}{2}\sigma_Y^2 T}{\sigma_Y\sqrt{T}}, d_2 = d_1 - \sigma_Y\sqrt{T}.$$
- and $\sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$.

How to find the drift in the stock measure?

- Recall $\beta(t) = e^{rt}$, in other words, $d\beta(t) = r\beta(t)dt$.
- Let
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$
- Let $X = \frac{\beta}{S}$, by Ito's lemma
$$\begin{aligned} dX &= \frac{1}{S}d\beta - \frac{\beta}{S^2}dS + \frac{1}{2}\frac{\beta}{S^3}2 < dS > \\ &= \frac{r\beta}{S}dt - \frac{\beta}{S}(\mu dt + \sigma dW) + \frac{\beta}{S}\sigma^2 dt \\ \frac{dX}{X} &= (r - \mu + \sigma^2)dt - \sigma dW \end{aligned}$$
- In order for X to be martingale in the S measure, μ must equal to $r + \sigma^2$.

Quanto Option

- A quanto option is, roughly, an option that pays off in the wrong currency. The FX rate is fixed at the inception.
- Google stock is a USD tradable asset but it is perfectly legit to write an option on Google but the notional is specified in SGD.
- A non-quanto equity call is mostly defined as:

$$N_{USD}\left(\frac{S(T)}{S(0)} - K\right)^{+}.$$

- N_{USD} is the notional in the same currency as S. K is the strike factor, e.g. 1 if at-the-money. This form is normally called the "fixed notional".
- Another less common way, "fixed units" is defined as

$$n(S(T) - KS(0))^{+}.$$

- n is the number of units and the two forms are equivalent.

Quanto Option

- Using the fixed notional form, it is easier to see what is a quanto option.
- A quanto equity call can be defined as:

$$N_{SGD}\left(\frac{S(T)}{S(0)}-K\right)^{+}.$$

- N_{SGD} is the notional in SGD.
- Note that this is equivalent to

$$N_{USD}X(0)\left(\frac{S(T)}{S(0)}-K\right)^{+}.$$

- where $X(0)$ is the FX rate at inception.

Quanto Option

- The key to understand quanto option pricing is to keep a firm grasp on what the tradeable quantities are. Suppose we are a SGD investor, our unit of account is SGD money market account and we have a quanto option on a US stock.
- Note that Google stock, , is a USD tradable but not a SGD tradable. However, we can convert it into a SGD tradable by multiplying by the exchange rate to give it a price in SGD instead of USD.
- To price this option, we first identify what processes are involved.

Quanto Option

- $X(t)$ denotes teh value of one USD in SGD at time t and is assumed to be lognormally distributed.
- $S(t)$ denotes the value of the a US stock at time t and is assumed ot lognormally distributed.
- $\beta^d(t)$ denotes the SGD money market account which grows at a continuous rate r^d.
- $\beta^f(t)$ denotes the USD money market account which grows at a continuous rate r^f.

The processes that we have are

$$\begin{aligned} d\beta^d &= r^d \beta^d dt \\ d\beta^f &= r^f \beta^f dt \\ dS &= \mu_S S dt + \sigma_S S dW_S \\ dX &= \mu_X X dt + \sigma_X X dW_X \end{aligned}$$

where $E[dW_S dW_X] = \rho dt$.

Quanto Option

- We pick the numeraire to be β^d, this means that the drift of the FX process is $\mu_X = r^d - r^f$.
- The remaining quantity to be found is the drift of the US stock in teh domestic risk neutral measure.
- consider the domestically tradable asset, $X(t)S(t)$. By Ito's lemma, we have

$$\frac{d(XS)}{(XS)} = (\mu_X dt + \sigma_X dW_X) + (\mu_S dt + \sigma_S dW_S) + \sigma_X \sigma_S \rho dt$$

- We know that the drift of any domestically tradeable asset is r^d.
- In other words, $\mu_X + \mu_S + \sigma_X \sigma_S \rho = r^d$.
- We have $\mu_S = r^f - \sigma_X \sigma_S \rho$.

- Therefore, the US stock in SGD risk neutral measure has the following dynamics:

$$\frac{dS(t)}{S(t)} = (r^f - \sigma_X \sigma_S \rho)dt + \sigma_S dW_S.$$

- The value of the payoff, set $N_{SGD} = 1$,

$$V(T) = \left(\frac{S(T)}{S(0)} - 1\right)^+$$

can be computed as

$$V(0) = \frac{\beta^d(0)}{\beta^d(T)} E\left[\left(\frac{S(T)}{S(0)} - 1\right)^+\right]$$

which is a standard Black formula.

LIBOR in arrears

- Let the zero coupon bond at time t with maturity T be $Z(t, T)$.
- The LIBOR rate can be defined as

$$L(t, T_1, T_2) = \frac{1}{\delta}\left(\frac{Z(t, T_1)}{Z(t, T_2)} - 1\right).$$

- $L(t, T_1, T_2)$ is a martingale under the T_2 forward measure. The corresponding numeraire asset is $Z(t, T_2)$.
- Note that the LIBOR rate is fixed at time T_1. In other words, after T_1, the LIBOR is not a random variable anymore.
- The payoff of an FRA at time T_1 is

$$V(T_1) = \delta(L(T_1, T_1, T_2) - K)Z(T_1, T_2).$$

- In FRA, the corresponding LIBOR rate is fixed at T_1 and the payment is at T_2 but settled at T_1.

LIBOR in arrears

- Pick $N(t) = Z(t, T_2)$, using the martingale pricing formula and we have

$$\begin{aligned} V(0) &= Z(0, T_2) E_0 \left[\frac{\delta(L(T_1, T_1, T_2) - K) Z(T_1, T_2)}{Z(T_1, T_2)} \right] \\ &= Z(0, T_2) \delta E_0 \left[L(T_1, T_1, T_2) - K \right]. \end{aligned}$$

- Recall the LIBOR is a T_2 martingale, we have

$$V(0) = Z(0, T_2) \delta (L(0, T_1, T_2) - K).$$

LIBOR in arrears

- What about if the LIBOR is fixed at T_1 but the payment is also at T_1?
- This is called the LIBOR in arrears and it has the following payoff at T_1

$$V(T_1) = \delta(L(T_1, T_1, T_2) - K)Z(T_1, T_1).$$

- Using the martingale pricing formula and pick $N(t) = Z(t, T_2)$

$$V(0) = Z(0, T_2)E_0\left[\frac{\delta(L(T_1, T_1, T_2) - K)Z(T_1, T_1)}{Z(T_1, T_2)}\right] \quad (1)$$

- Note that the ZCB in the expectation don't cancel out in this case.

LIBOR in arrears

- Recall the definition of LIBOR:

$$L(t) = L(t, T_1, T_2) = \frac{1}{\delta}\left(\frac{Z(t, T_1)}{Z(t, T_2)} - 1\right).$$

- Rearrange the terms and we have

$$1 + \delta L(t) = \frac{Z(t, T_1)}{Z(t, T_2)}.$$

- We then substitute it into Equation (1) and get

$$V(0) = Z(0, T_2) E_0 \left[\delta(L(T_1) - K)(1 + \delta L(T_1))\right]$$

LIBOR in arrears

- For ease of notation, let $\delta = 1$.
- Rearrange the terms and we have

$$
\begin{aligned}
V(0) &= Z(0, T_2)E_0\left[(L(T_1) - K)\right] \\
&+ Z(0, T_2)E_0\left[L(T_1)^2\right] \\
&- Z(0, T_2)KE_0\left[L(T_1)\right]
\end{aligned}
$$

- The first term is the FRA and rest are the convexity adjustments.
- All the terms can be computed without further modelling assumptions except for the second term which involves a non-linear function of $L(T_1)$

$$E_0\left[L(T_1)^2\right]$$

LIBOR in arrears

- There are a few ways to compute the

$$E_0\left[L(T_1)^2\right]$$

- One way is to assume a dynamics for the LIBOR rate, say, it follows a lognormal process in the T_2 forward measure

$$dL(t) = \sigma L(t) dW(t).$$

- Or using Breeden-Litzenberger formula to replicate the square payoff using a collection of caplets and floorlets.