# Notes on mode summation

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The displacement is given by

$$\mathbf{s}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(\frac{1}{{}_n\omega_\ell^2 + {}_n\gamma_\ell^2}\right){}_n\mathbf{A}_\ell(\mathbf{x}) \\ \times \Big( \left(\frac{{}_n\omega_\ell^2 - {}_n\gamma_\ell^2}{{}_n\omega_\ell^2 + {}_n\gamma_\ell^2}\right)[1 - \cos({}_n\omega_\ell t)\exp(-{}_n\gamma_\ell t)] \\ - \left(\frac{2{}_n\omega_{\ell n}\gamma_\ell}{{}_n\omega_\ell^2 + {}_n\gamma_\ell^2}\right)\sin({}_n\omega_\ell t)\exp(-{}_n\gamma_\ell t)\Big) \quad \text{(D\&T 10.51)}$$

or, in the limit of small attenuation

$$\mathbf{s}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} {}_n\omega_\ell^{-2}\,{}_n\mathbf{A}_\ell(\mathbf{x})\,[1 - \cos({}_n\omega_\ell t)\exp(-{}_n\gamma_\ell t)] \quad \text{(D\&T 10.61)}$$

or zero attenuation

$$\mathbf{s}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} {}_n\omega_\ell^{-2}\,{}_n\mathbf{A}_\ell(\mathbf{x})\,[1 - \cos({}_n\omega_\ell t)] \quad \text{(from D\&T 10.61)}$$

where (dropping the labels for each mode)

$$\mathbf{A}(\mathbf{x}) = \frac{2\ell+1}{4\pi}\mathbf{D}(r,\Theta,\Phi)A(\Theta,\Phi) \quad \text{(D\&T 10.52)}$$

The velocity can be found by differentiating the displacement

$$\mathbf{v}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}({}_n\omega_\ell^2 + {}_n\gamma_\ell^2)^{-1}\,{}_n\mathbf{A}_\ell(\mathbf{x})({}_n\omega_\ell \sin {}_n\omega_\ell t - {}_n\gamma_\ell \cos {}_n\omega_\ell t)\exp(-{}_n\omega_\ell t) \quad (1)$$

which simplifies only slightly in the low-attenuation approximation to

$$\mathbf{v}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} {}_n\omega_\ell^{-2}\,{}_n\mathbf{A}_\ell(\mathbf{x})({}_n\omega_\ell \sin {}_n\omega_\ell t - {}_n\gamma_\ell \cos {}_n\omega_\ell t)\exp(-{}_n\omega_\ell t) \quad (2)$$

and in the case of zero attenuation the velocity becomes

$$\mathbf{v}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} {}_n\omega_\ell^{-1} \, {}_n\mathbf{A}_\ell(\mathbf{x}) \sin {}_n\omega_\ell t \tag{3}$$

The acceleration can be found by differentiating a second time. It has a simple form, even without assuming small attenuation:

$$\mathbf{a}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} {}_n\mathbf{A}_\ell(\mathbf{x}) \cos({}_n\omega_\ell t) \exp(-{}_n\gamma_\ell t) \qquad \text{(D\&T 10.63)}$$

This simplifies further in the case of zero attenuation:

$$\mathbf{a}(\mathbf{x},t) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} {}_n\mathbf{A}_\ell(\mathbf{x}) \cos({}_n\omega_\ell t) \tag{4}$$

The displacement operator $\mathbf{D}$ for spheroidal modes is

$$\mathbf{D} = \hat{\mathbf{r}} U_r + \hat{\mathbf{\Theta}} k^{-1} V_r \partial_\Theta + \hat{\mathbf{\Phi}} k^{-1} V_r (\sin\Theta)^{-1} \partial_\Phi \qquad \text{(from D\&T 10.60)}$$

which in the case of radial modes simplifies to

$$\mathbf{D} = \hat{\mathbf{r}} U_r \qquad \text{(from D\&T 10.60)}$$

whilst for toroidal modes it is

$$\begin{aligned}\mathbf{D} &= \hat{\mathbf{\Theta}} k^{-1} W_r (\sin\Theta)^{-1} \partial_\Phi - \hat{\mathbf{\Phi}} k^{-1} W_r \partial_\Theta && \text{(from D\&T 10.60)} \\ &= k^{-1} W_r \left( \hat{\mathbf{\Theta}} (\sin\Theta)^{-1} \partial_\Phi - \hat{\mathbf{\Phi}} \partial_\Theta \right)\end{aligned}$$

The excitation function $A$ can be written as

$$A(\Theta,\Phi) = \sum_{m=0}^{2} P_{\ell m}(\cos\Theta)(A_m \cos m\Phi + B_m \sin m\Phi) \qquad \text{(D\&T 10.53)}$$

where the coefficients are written in terms of the radial eigenfunctions evaluated at the source depth and the moment-tensor components; for the spheroidal modes these are

$$\begin{aligned}
A_0 &= M_{rr} \dot{U}_s + (M_{\theta\theta} + M_{\phi\phi}) \left( U_s - \tfrac{1}{2} k V_s \right) r_s^{-1} && \text{(D\&T 10.54)} \\
B_0 &= 0 && \text{(D\&T 10.55)} \\
A_1 &= k^{-1} M_{r\theta} \left( \dot{V}_s - r_s^{-1} V_s + k r_s^{-1} U_s \right) && \text{(from D\&T 10.56)} \\
B_1 &= k^{-1} M_{r\phi} \left( \dot{V}_s - r_s^{-1} V_s + k r_s^{-1} U_s \right) && \text{(from D\&T 10.57)} \\
A_2 &= \tfrac{1}{2} k^{-1} r_s^{-1} (M_{\theta\theta} - M_{\phi\phi}) V_s && \text{(from D\&T 10.58)} \\
B_2 &= k^{-1} r_s^{-1} M_{\theta\phi} V_s && \text{(from D\&T 10.59)}
\end{aligned}$$

It is not obvious from the textbook what the correct expressions are for radial modes, in particular, which terms are zero, but, given the well-known property that the radial modes have only radial nodes, it seems that the correct forms are

$$
\begin{aligned}
A_0 &= M_{rr}\dot{U}_s + (M_{\theta\theta} + M_{\phi\phi})\, U_s r_s^{-1} && \text{(from D\&T 10.54)}\\
B_0 &= 0 && \text{(D\&T 10.55)}\\
A_1 &= 0\\
B_1 &= 0\\
A_2 &= 0\\
B_2 &= 0
\end{aligned}
$$

For the toroidal modes, the coefficients are

$$
\begin{aligned}
A_0 &= 0 && \text{(from D\&T 10.54)}\\
B_0 &= 0 && \text{(D\&T 10.55)}\\
A_1 &= -k^{-1} M_{r\phi} \left( \dot{W}_s - r_s^{-1} W_s \right) && \text{(from D\&T 10.56)}\\
B_1 &= k^{-1} M_{r\theta} \left( \dot{W}_s - r_s^{-1} W_s \right) && \text{(from D\&T 10.57)}\\
A_2 &= -k^{-1} r_s^{-1} M_{\theta\phi} W_s && \text{(from D\&T 10.58)}\\
B_2 &= \tfrac{1}{2} k^{-1} r_s^{-1} (M_{\theta\theta} - M_{\phi\phi}) W_s && \text{(from D\&T 10.59)}
\end{aligned}
$$

Evaluating the differential operators in 10.60, the components of displacement in the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\Theta}}$ and $\hat{\boldsymbol{\Phi}}$ directions for spheroidal modes are:

$$
A_r = L U_r A(\Theta, \Phi) \tag{5}
$$

$$
A_\Theta = -L k^{-1} V_r \sin\Theta \left( \sum_{m=0}^{2} P_{\ell m}^{'}(\cos\Theta)(A_m \cos m\Phi + B_m \sin m\Phi) \right) \tag{6}
$$

$$
\begin{aligned}
A_\Phi = L k^{-1} V_r \left(\sin\Theta\right)^{-1} \Big( & P_{\ell 1}(\cos\Theta) \left(B_1 \sin\Phi - A_1 \cos\Phi\right) + \\
& 2P_{\ell 2}(\cos\Theta) \left(B_2 \sin 2\Phi - A_2 \cos 2\Phi\right) \Big)
\end{aligned} \tag{7}
$$

For radial modes, these reduce to

$$
\begin{aligned}
A_r &= U_r A_0 / 4\pi\\
A_\Theta &= 0\\
A_\Phi &= 0
\end{aligned}
$$

and for toroidal modes they are

$$A_r = 0 \tag{8}$$

$$A_\Theta = Lk^{-1}W_r\,(\sin\Theta)^{-1}\Bigg( P_{\ell 1}(\cos\Theta)\,(B_1\sin\Phi - A_1\cos\Phi) + 2P_{\ell 2}(\cos\Theta)\,(B_2\sin 2\Phi - A_2\cos 2\Phi)\Bigg) \tag{9}$$

$$A_\Phi = Lk^{-1}W_r\sin\Theta\left(\sum_{m=0}^{2} P^{'}_{\ell m}(\cos\Theta)(A_m\cos m\Phi + B_m\sin m\Phi)\right) \tag{10}$$

where

$$P^{'}_{\ell m} \equiv \frac{\mathrm{d}}{\mathrm{d}\cos\Theta}P_{\ell m}(\cos\Theta)$$

and

$$L = \frac{2\ell+1}{4\pi}$$

## Case of zero epicentral distance

In the limit $\Theta \to 0$, some expressions have to be modified. In the expressions for $A_\Theta$ (equations 6 and 9), the term with $m = 1$ can be ignored because

$$\lim_{\Theta\to 0}\frac{\sin\Theta}{P^{'}_{\ell 1}(\cos\Theta)} = 0 \tag{11}$$

In the expressions for $A_\Phi$, (equations 7 and 10), the term with $m = 1$ can be simplified by noting that

$$\lim_{\Theta\to 0^+}\frac{P_{\ell 1}(\cos\Theta)}{\sin\Theta} = \ell(\ell+1)/2 \tag{12}$$

and the term with $m = 2$ can be ignored because

$$\lim_{\Theta\to 0}\frac{P_{\ell 2}(\cos\Theta)}{\sin\Theta} = 0 \tag{13}$$

## Surface-averaged value of excitation

In the source receiver geometry (D&T section 10.1), an area element is given by

$$\mathrm{d}A = r^2\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\Phi \tag{14}$$

which gives an expected total area of

$$A = \int \mathrm{d}A \tag{15}$$

$$= \int_{\Theta=0}^{\pi}\int_{\Phi=0}^{2\pi} r^2\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\Phi \tag{16}$$

$$= 4\pi r^2 \tag{17}$$

We can define the RMS excitation for an excitation coefficient (for example, the vertical component $A_r$) as:

$$E = \left( \int A_r^2 \,\mathrm{d}A / 4\pi r^2 \right)^{1/2} \tag{18}$$

Substituting equation 5, this becomes

$$\begin{aligned} E &= \frac{(2\ell+1)}{8\pi^{3/2}} |U_r| \left( \int_{\Theta=0}^{\pi} \int_{\Phi=0}^{2\pi} A^2(\Theta, \Phi) \sin\Theta \,\mathrm{d}\Theta \,\mathrm{d}\Phi \right)^{1/2} &(19) \\ &= \frac{(2\ell+1)}{8\pi^{3/2}} |U_r| I^{1/2} &(20) \end{aligned}$$

Given that $A(\Theta, \Phi)$ has five non-zero terms (see equation D&T 10.53, above), the squared term appears to be very complicated. However, the integral over azimuth eliminates all of the cross-terms, so that the overall integral reduces to

$$\begin{aligned} I = A_0^2 &\int_{\Theta=0}^{\pi} P_{\ell 0}^2 \sin\Theta \,\mathrm{d}\Theta \int_{\Phi=0}^{2\pi} \mathrm{d}\Phi \\ + A_1^2 &\int_{\Theta=0}^{\pi} P_{\ell 1}^2 \sin\Theta \,\mathrm{d}\Theta \int_{\Phi=0}^{2\pi} \cos^2\Phi \,\mathrm{d}\Phi \\ + B_1^2 &\int_{\Theta=0}^{\pi} P_{\ell 1}^2 \sin\Theta \,\mathrm{d}\Theta \int_{\Phi=0}^{2\pi} \sin^2\Phi \,\mathrm{d}\Phi \\ + A_2^2 &\int_{\Theta=0}^{\pi} P_{\ell 2}^2 \sin\Theta \,\mathrm{d}\Theta \int_{\Phi=0}^{2\pi} \cos^2 2\Phi \,\mathrm{d}\Phi \\ + B_2^2 &\int_{\Theta=0}^{\pi} P_{\ell 2}^2 \sin\Theta \,\mathrm{d}\Theta \int_{\Phi=0}^{2\pi} \sin^2 2\Phi \,\mathrm{d}\Phi \end{aligned} \tag{21}$$

The geometric integrals then simplify to give

$$\begin{aligned} I = A_0^2 &\cdot 2(2\ell+1)^{-1} \cdot 2\pi \\ + A_1^2 &\cdot 2\ell(\ell+1)(2\ell+1)^{-1} \cdot \pi \\ + B_1^2 &\cdot 2\ell(\ell+1)(2\ell+1)^{-1} \cdot \pi \\ + A_2^2 &\cdot 2(\ell-1)\ell(\ell+1)(\ell+2)(2\ell+1)^{-1} \cdot \pi \\ + B_2^2 &\cdot 2(\ell-1)\ell(\ell+1)(\ell+2)(2\ell+1)^{-1} \cdot \pi \end{aligned} \tag{22}$$

which simplifies further to

$$\begin{aligned} I = 2\pi(2\ell+1)^{-1} \Big( &2A_0^2 + \ell(\ell+1)(A_1^2 + B_1^2) \\ &+ (\ell-1)\ell(\ell+1)(\ell+2)(A_2^2 + B_2^2) \Big) \end{aligned} \tag{23}$$