Notes on mode summation

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$$
\begin{aligned}
\mathbf{s}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} & \left(\frac{1}{{}_n\omega_\ell^2 + {}_n\gamma_\ell^2} \right) {}_n\mathbf{A}_\ell(\mathbf{x}) \\
& \times \Big(\left(\frac{{}_n\omega_\ell^2 - {}_n\gamma_\ell^2}{{}_n\omega_\ell^2 + {}_n\gamma_\ell^2} \right) [1 - \cos({}_n\omega_\ell t) \exp(-{}_n\gamma_\ell t)] \\
& \qquad - \left(\frac{2{}_n\omega_\ell {}_n\gamma_\ell}{{}_n\omega_\ell^2 + {}_n\gamma_\ell^2} \right) \sin({}_n\omega_\ell t) \exp(-{}_n\gamma_\ell t) \Big) \qquad \text{(D\&T 10.51)}
\end{aligned}
$$

where (dropping the labels for each mode)

$$
\mathbf{A}(\mathbf{x}) = \frac{2\ell + 1}{4\pi} \mathbf{D}(r, \Theta, \Phi) A(\Theta, \Phi) \qquad \text{(D\&T 10.52)}
$$

The displacement operator $\mathbf{D}$ for spheroidal modes is

$$
\mathbf{D} = \hat{\mathbf{r}} U_r + \hat{\boldsymbol{\Theta}} k^{-1} V_r \partial_\Theta + \hat{\boldsymbol{\Phi}} k^{-1} V_r (\sin\Theta)^{-1} \partial_\Phi \qquad \text{(from D\&T 10.60)}
$$

and for toroidal modes it is

$$
\begin{aligned}
\mathbf{D} &= \hat{\boldsymbol{\Theta}} k^{-1} W_r (\sin\Theta)^{-1} \partial_\Phi - \hat{\boldsymbol{\Phi}} k^{-1} W_r \partial_\Theta \qquad \text{(from D\&T 10.60)} \\
&= k^{-1} W_r \left(\hat{\boldsymbol{\Theta}} (\sin\Theta)^{-1} \partial_\Phi - \hat{\boldsymbol{\Phi}} \partial_\Theta \right)
\end{aligned}
$$

The excitation function A can be written as

$$
A(\Theta, \Phi) = \sum_{m=0}^{2} P_{\ell m}(\cos\Theta)(A_m \cos m\Phi + B_m \sin m\Phi) \qquad \text{(D\&T 10.53)}
$$

where the coefficients are written in terms of the radial eigenfunctions evaluated at the source depth and the moment-tensor components; for the spheroidal modes these are

$$
\begin{aligned}
&A_0 = M_{rr}\dot{U}_s + (M_{\theta\theta} + M_{\phi\phi})\left(U_s - \tfrac{1}{2}kV_s\right) r_s^{-1} && \text{(D\&T 10.54)} \\
&B_0 = 0 && \text{(D\&T 10.55)} \\
&A_1 = k^{-1}M_{r\theta}\left(\dot{V}_s - r_s^{-1}V_s + kr_s^{-1}U_s\right) && \text{(from D\&T 10.56)} \\
&B_1 = k^{-1}M_{r\phi}\left(\dot{V}_s - r_s^{-1}V_s + kr_s^{-1}U_s\right) && \text{(from D\&T 10.57)} \\
&A_2 = \tfrac{1}{2}k^{-1}r_s^{-1}\left(M_{\theta\theta} - M_{\phi\phi}\right)V_s && \text{(from D\&T 10.58)} \\
&B_2 = k^{-1}r_s^{-1}M_{\theta\phi}V_s && \text{(from D\&T 10.59)}
\end{aligned}
$$

and for the toroidal modes these are

$$
\begin{aligned}
&A_0 = 0 && \text{(from D\&T 10.54)} \\
&B_0 = 0 && \text{(D\&T 10.55)} \\
&A_1 = -k^{-1}M_{r\phi}\left(\dot{W}_s - r_s^{-1}W_s\right) && \text{(from D\&T 10.56)} \\
&B_1 = k^{-1}M_{r\theta}\left(\dot{W}_s - r_s^{-1}W_s\right) && \text{(from D\&T 10.57)} \\
&A_2 = -k^{-1}r_s^{-1}M_{\theta\phi}W_s && \text{(from D\&T 10.58)} \\
&B_2 = \tfrac{1}{2}k^{-1}r_s^{-1}(M_{\theta\theta} - M_{\phi\phi})W_s && \text{(from D\&T 10.59)}
\end{aligned}
$$

Evaluating the differential operators in 10.60, the components of displacement in the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\Theta}}$ and $\hat{\boldsymbol{\Phi}}$ directions for spheroidal modes are:

$$A_r = U_r A(\Theta, \Phi) \tag{1}$$

$$A_\Theta = -k^{-1}V_r \sin\Theta \left(\sum_{m=0}^{2} P'_{\ell m}(\cos\Theta)(A_m \cos m\Phi + B_m \sin m\Phi)\right) \tag{2}$$

$$
\begin{aligned}
A_\Phi = k^{-1}V_r\,(\sin\Theta)^{-1}\Big(& P_{\ell 1}(\cos\Theta)\,(B_1\sin\Phi - A_1\cos\Phi) + \\
& 2P_{\ell 2}(\cos\Theta)\,(B_2\sin 2\Phi - A_2\cos 2\Phi)\Big)
\end{aligned} \tag{3}
$$

while for the toroidal modes they are

$$A_r = 0 \tag{4}$$

$$
\begin{aligned}
A_\Theta = k^{-1}W_r\,(\sin\Theta)^{-1}\Big(& P_{\ell 1}(\cos\Theta)\,(B_1\sin\Phi - A_1\cos\Phi) + \\
& 2P_{\ell 2}(\cos\Theta)\,(B_2\sin 2\Phi - A_2\cos 2\Phi)\Big)
\end{aligned} \tag{5}
$$

$$A_\Phi = k^{-1}W_r \sin\Theta \left(\sum_{m=0}^{2} P'_{\ell m}(\cos\Theta)(A_m \cos m\Phi + B_m \sin m\Phi)\right) \tag{6}$$

where

$$P'_{\ell m} \equiv \frac{\mathrm{d}}{\mathrm{d}\cos\Theta} P_{\ell m}(\cos\Theta)$$