

Notes on mode summation

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The displacement is given by

$$\begin{aligned} \mathbf{s}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{1}{n\omega_{\ell}^2 + n\gamma_{\ell}^2} \right) n \mathbf{A}_{\ell}(\mathbf{x}) \\ \times \left(\left(\frac{n\omega_{\ell}^2 - n\gamma_{\ell}^2}{n\omega_{\ell}^2 + n\gamma_{\ell}^2} \right) [1 - \cos(n\omega_{\ell}t) \exp(-n\gamma_{\ell}t)] \right. \\ \left. - \left(\frac{2n\omega_{\ell}n\gamma_{\ell}}{n\omega_{\ell}^2 + n\gamma_{\ell}^2} \right) \sin(n\omega_{\ell}t) \exp(-n\gamma_{\ell}t) \right) \quad (\text{D\&T 10.51}) \end{aligned}$$

or, in the limit of small attenuation

$$\mathbf{s}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} n\omega_{\ell}^{-2} n \mathbf{A}_{\ell}(\mathbf{x}) [1 - \cos(n\omega_{\ell}t) \exp(-n\gamma_{\ell}t)] \quad (\text{D\&T 10.61})$$

or zero attenuation

$$\mathbf{s}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} n\omega_{\ell}^{-2} n \mathbf{A}_{\ell}(\mathbf{x}) [1 - \cos(n\omega_{\ell}t)] \quad (\text{from D\&T 10.61})$$

where (dropping the labels for each mode)

$$\mathbf{A}(\mathbf{x}) = \frac{2\ell + 1}{4\pi} \mathbf{D}(r, \Theta, \Phi) A(\Theta, \Phi) \quad (\text{D\&T 10.52})$$

The displacement operator \mathbf{D} for spheroidal modes is

$$\mathbf{D} = \hat{\mathbf{r}}U_r + \hat{\Theta}k^{-1}V_r\partial_{\Theta} + \hat{\Phi}k^{-1}V_r(\sin\Theta)^{-1}\partial_{\Phi} \quad (\text{from D\&T 10.60})$$

and for toroidal modes it is

$$\begin{aligned} \mathbf{D} &= \hat{\Theta}k^{-1}W_r(\sin\Theta)^{-1}\partial_{\Phi} - \hat{\Phi}k^{-1}W_r\partial_{\Theta} \\ &= k^{-1}W_r \left(\hat{\Theta}(\sin\Theta)^{-1}\partial_{\Phi} - \hat{\Phi}\partial_{\Theta} \right) \end{aligned} \quad (\text{from D\&T 10.60})$$

The excitation function A can be written as

$$A(\Theta, \Phi) = \sum_{m=0}^2 P_{\ell m}(\cos\Theta) (A_m \cos m\Phi + B_m \sin m\Phi) \quad (\text{D\&T 10.53})$$

where the coefficients are written in terms of the radial eigenfunctions evaluated at the source depth and the moment-tensor components; for the spheroidal modes these are

$$A_0 = M_{rr}\dot{U}_s + (M_{\theta\theta} + M_{\phi\phi}) \left(U_s - \frac{1}{2}kV_s \right) r_s^{-1} \quad (\text{D\&T 10.54})$$

$$B_0 = 0 \quad (\text{D\&T 10.55})$$

$$A_1 = k^{-1}M_{r\theta} \left(\dot{V}_s - r_s^{-1}V_s + kr_s^{-1}U_s \right) \quad (\text{from D\&T 10.56})$$

$$B_1 = k^{-1}M_{r\phi} \left(\dot{V}_s - r_s^{-1}V_s + kr_s^{-1}U_s \right) \quad (\text{from D\&T 10.57})$$

$$A_2 = \frac{1}{2}k^{-1}r_s^{-1} (M_{\theta\theta} - M_{\phi\phi}) V_s \quad (\text{from D\&T 10.58})$$

$$B_2 = k^{-1}r_s^{-1}M_{\theta\phi}V_s \quad (\text{from D\&T 10.59})$$

and for the toroidal modes these are

$$A_0 = 0 \quad (\text{from D\&T 10.54})$$

$$B_0 = 0 \quad (\text{D\&T 10.55})$$

$$A_1 = -k^{-1}M_{r\phi} \left(\dot{W}_s - r_s^{-1}W_s \right) \quad (\text{from D\&T 10.56})$$

$$B_1 = k^{-1}M_{r\theta} \left(\dot{W}_s - r_s^{-1}W_s \right) \quad (\text{from D\&T 10.57})$$

$$A_2 = -k^{-1}r_s^{-1}M_{\theta\phi}W_s \quad (\text{from D\&T 10.58})$$

$$B_2 = \frac{1}{2}k^{-1}r_s^{-1}(M_{\theta\theta} - M_{\phi\phi})W_s \quad (\text{from D\&T 10.59})$$

Evaluating the differential operators in 10.60, the components of displacement in the $\hat{\mathbf{r}}$, $\hat{\Theta}$ and $\hat{\Phi}$ directions for spheroidal modes are:

$$A_r = U_r A(\Theta, \Phi) \quad (1)$$

$$A_\Theta = -k^{-1}V_r \sin \Theta \left(\sum_{m=0}^2 P'_{\ell m}(\cos \Theta) (A_m \cos m\Phi + B_m \sin m\Phi) \right) \quad (2)$$

$$A_\Phi = k^{-1}V_r (\sin \Theta)^{-1} \left(P_{\ell 1}(\cos \Theta) (B_1 \sin \Phi - A_1 \cos \Phi) + 2P_{\ell 2}(\cos \Theta) (B_2 \sin 2\Phi - A_2 \cos 2\Phi) \right) \quad (3)$$

while for the toroidal modes they are

$$A_r = 0 \quad (4)$$

$$A_\Theta = k^{-1}W_r (\sin \Theta)^{-1} \left(P_{\ell 1}(\cos \Theta) (B_1 \sin \Phi - A_1 \cos \Phi) + 2P_{\ell 2}(\cos \Theta) (B_2 \sin 2\Phi - A_2 \cos 2\Phi) \right) \quad (5)$$

$$A_\Phi = k^{-1}W_r \sin \Theta \left(\sum_{m=0}^2 P'_{\ell m}(\cos \Theta) (A_m \cos m\Phi + B_m \sin m\Phi) \right) \quad (6)$$

where

$$P'_{\ell m} \equiv \frac{d}{d\cos\Theta} P_{\ell m}(\cos\Theta)$$