DILATED FLOOR FUNCTIONS HAVING NONNEGATIVE COMMUTATOR II. NEGATIVE DILATIONS

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ABSTRACT. This paper completes the classification of the set S of real parameter pairs (α, β) such that the dilated floor functions $f_\alpha(x) = \lfloor \alpha x \rfloor$ and $f_\beta(x) = \lfloor \beta x \rfloor$ have a nonnegative commutator, i.e. $[f_\alpha, f_\beta](x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor \geq 0$ for all real x. This paper treats the case where both dilation parameters α, β are negative. This result is equivalent to classifying all positive α, β satisfying $\lfloor \alpha \lceil \beta x \rceil \rfloor - \lfloor \beta \lceil \alpha x \rceil \rfloor \geq 0$ for all real x. The classification analysis is connected with the theory of Beatty sequences and with the Diophantine Frobenius problem in two generators.

1. INTRODUCTION

The *floor function* $\lfloor x \rfloor$ rounds a real number down to the nearest integer. For a real parameter α, the *dilated* floor function $f_\alpha(x) = \lfloor \alpha x \rfloor$ performs discretization at the length scale α^{-1}. Besides references noted in Part I, dilated floor functions are used in describing one-dimensional quasicrystals, see de Bruijn [2, Sect. 4], [3], Brown [1] and Mingo [7].

This paper continues the study of commutators under composition of dilated floor functions

$$[f_\alpha, f_\beta](x) := f_\alpha \circ f_\beta(x) - f_\beta \circ f_\alpha(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor.$$

It completes the classification of the set S of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \text{for all } x \in \mathbb{R}, \tag{1.1}$$

begun in Part I [4]. That is, (1.1) says $[f_\alpha, f_\beta](x) \geq 0$ for all real x, which we will often abbreviate as $[f_\alpha, f_\beta] \geq 0$, omitting the variable x.

1.1. Classification theorem.

Theorem 1.1 (Negative dilations classification). *Given $\alpha < 0$, $\beta < 0$, the inequality $\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ holds for all $x \in \mathbb{R}$ if and only if one or more of the following conditions holds:*

(i) There are integers $m \geq 0$, $n \geq 1$ such that

$$m\alpha\beta - n\beta = -\alpha. \tag{1.2}$$

(ii) There are coprime integers $p, q \geq 1$ such that

$$\alpha = -\frac{q}{p}, \quad -\frac{1}{p} \leq \beta < 0. \tag{1.3}$$

(iii^*) There are coprime integers $p, q \geq 1$ and $m \geq 0, n \geq 1, r \geq 1$ such that

$$\alpha = -\frac{q}{p}, \quad \beta = -\frac{1}{p}\left(1 + \frac{1}{r}\left(\frac{m}{p} + \frac{n}{q} - 1\right)\right)^{-1}. \tag{1.4}$$

2010 *Mathematics Subject Classification.* Primary 11A25; Secondary 11B83, 11D07, 11Z05, 26D07, 52C05.

Work of the first author was partially supported by NSF grant DMS-1401224 and DMS-1701576, by MSRI as a Chern Professor and by a Simons Fellows in Mathematics grant. MSRI is partially supported by an NSF grant. Work of the second author was partially supported by NSF grant DMS-1600223 and by a Rackham Predoctoral Fellowship.

In this classification, (iii^*) is a strictly larger set than (iii) in Theorem 1.3 stated in Part I; however the two theorems are equivalent. Case (iii^*) differs from (iii) in allowing value $r = 1$ and in not imposing the side condition $0 < \frac{m}{p} + \frac{n}{q} < 1$ in (iii). However, all members of case (iii^*) coming from $r = 1$ also appear in case (i), and all members that have $\frac{m}{p} + \frac{n}{q} \geq 1$ also appear in case (ii). Condition (iii^*) parameterizes *all* negative rational solutions in S, see Theorem 6.1. The parametrization in (iii^*) is redundant: some (α, β) values arise from multiple parameter values (p, q, m, n, r).

Figure 1.1 pictures the solution set S for negative dilations viewed in the the positive (α', β')-coordinate system, letting $\alpha' = -\alpha, \beta' = -\beta$; the set pictured is $S' := -S$ in (α', β') coordinates.

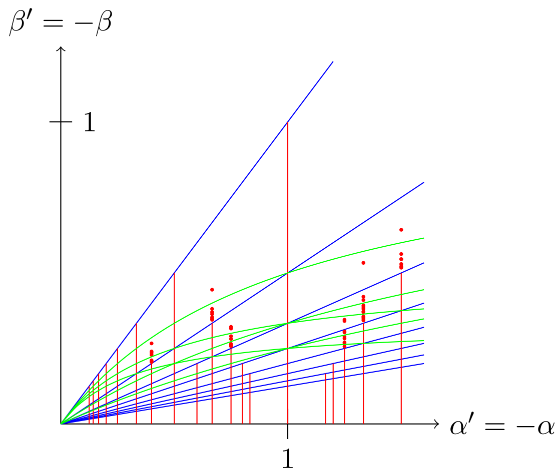

FIGURE 1.1. Negative dilation solutions $S' = -S$ viewed in (α', β')-coordinates. Sporadic solutions appear as dots.

Figure 1.1 exhibits two kinds of solutions having no analogue in the positive dilation case. The first consists of vertical line segments (of variable finite length) for all rational values of α, given by case (ii) of Theorem 1.1. The second consists of a countable set of isolated points having rational coordinates, which we term "sporadic rational solutions." They appear as dots above the vertical line segments in Figure 1.1. They form a strict subset of case (iii^*); determining this subset involves removing from case (iii^*) the overlap shared with case (i) and (ii) solutions, and removing further redundancy from the parametrization in (iii^*). This determination is related to the Diophantine Frobenius problem in the parameters (p, q), see [8]. We treat this problem elsewhere ([5]).

1.2. **The set S is closed.** In Part I we showed that the intersection of S with each of the closed first, second and fourth quadrants of the (α, β)-plane is closed. To complete the proof that S is closed, we establish in this paper that the intersection with the closed third quadrant is closed.

Theorem 1.2 (Closed set property of S)**.** *The set of all pairs of dilation factors (α, β) with $\alpha \leq 0, \beta \leq 0$ satisfying the nonnegative commutator relation $[f_\alpha, f_\beta](x) \geq 0$ is a closed subset of $\mathbb{R}^2$.*

The function $[f_\alpha, f_\beta](x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ is discontinuous in each of the variables α, β, x, so the closure property is not obvious. We establish this result in Section 7 using the classification of the set S in Theorem 1.1.

1.3. **Symmetries of parameter set S: negative dilations.** We establish several symmetries of the set S in the negative dilation case. There are a set of linear symmetries paralleling the positive dilation case.

Theorem 1.3 (Symmetries of S: negative dilations)**.** *For negative dilations $\alpha, \beta < 0$, the set S is mapped into itself under the following symmetries:*

(i) For any integer $m \geq 1$, *if* $(\alpha, \beta) \in S$, *then* $(m\alpha, \beta) \in S$.
(ii) For any integer $m \geq 1$, *if* $(\alpha, \beta) \in S$, *then* $(\frac{1}{m}\alpha, \frac{1}{m}\beta) \in S$.

An important new feature of the negative dilation case is the existence of additional symmetries having no positive dilation analogue. These symmetries act on vertical lines with fixed rational α-coordinate. When $\alpha = -\frac{q}{p}$ (in lowest terms), the symmetries act on the β-coordinate as special cases of the family of linear fractional transformations

$$\Phi_p^r(\beta) := \frac{r\beta}{(1-r)p\beta + 1}$$

defined for all real $r > 0$, which depend on p and not on q.

Theorem 1.4 (Linear fractional symmetries)**.** *Let* $\alpha = -\frac{q}{p}$ *be a fixed negative rational. For each integer* $r \geq 1$, *the linear fractional transformation*

$$\Phi_p^r(\beta) := \frac{r\beta}{(1-r)p\beta + 1}$$

maps the set of $\beta < 0$ *which satisfy* $[f_{-q/p}, f_\beta] \geq 0$ *into itself. Namely, if* $(-\frac{q}{p}, \beta) \in S$ *then* $(-\frac{q}{p}, \Phi_p^r(\beta)) \in S$.

The set of transformations Φ_p^r with p fixed and integer $r \geq 1$ forms a commutative semigroup under composition of functions:

$$\Phi_p^r \circ \Phi_p^s = \Phi_p^{rs}.$$

All elements of this semigroup have two common fixed points $z = 0$ and $z = -\frac{1}{p}$. When $r = 1$ the map is the identity.

(1) The map Φ_p^r is a homeomorphism of the open interval $(-\frac{1}{p}, 0) \to (-\frac{1}{p}, 0)$. For $\alpha = -\frac{q}{p}$ this interval is exactly the β values in Case (ii).
(2) The map Φ_p^r sends the open interval $(-\infty, -\frac{1}{p})$ to $(-\frac{r}{(r-1)p}, -\frac{1}{p})$, for $r \geq 2$. For $\alpha = -\frac{q}{p}$ the case (iii^*) solutions with $\beta \in (-\infty, -\frac{1}{p})$ are the ones with $0 < \frac{m}{p} + \frac{n}{q} < 1$, and are a discrete set. The action of Φ_p^r on these solutions is injective, but for $r \geq 2$ usually not surjective.

Theorem 1.4 is the mechanism used in this paper to show existence of all the case (iii^*) solutions. We start from the case (i) rational solutions $(-\frac{q}{p}, \beta)$ having $\beta < -\frac{1}{p}$; these solutions correspond to $r = 1$ in the parameterization in case (iii^*). From these solutions for fixed $\alpha = -\frac{q}{p}$ the full infinite set of sporadic rational solutions are obtained by applying the linear fractional maps Φ_p^r for $r \geq 2$ to these solutions. We prove Theorem 1.4 in Section 4.6.

2. Outline of proofs

The negative dilation case analysis is similar to the positive dilation case, but is more complicated. Part I noted an analogy with the classification of disjoint Beatty sequences; a similar connection appears here in Lemma 4.10.

Section 3 introduces the family of *strict rounding functions*, whose definition parallels that of rounding functions in Part I, replacing the rescaled floor function with rescaled versions of the *strict floor function*. Proposition 3.3 gives a criterion for (1.1) to hold in terms of strict rounding functions.

Section 4 establishes that S is closed under the linear fractional transformations given in Theorem 1.4 (for rational α). These symmetries provide a mechanism to construct all type (iii^*) solutions starting from type (i) rational solutions. This proof proceeds in a new coordinate system, the (μ', ν')-coordinates, given by $\mu' = -\frac{1}{\beta}$, $\nu' = \frac{\alpha}{\beta}$, which take positive values.

Section 5 establishes the sufficiency direction of Theorem 1.1. It shows all parameters given in cases $(i), (ii), (iii^*)$ of Theorem 1.1 are solutions in S. It uses (μ', ν')-coordinates.

Section 6 establishes the necessity direction of Theorem 1.1. Its main result partitions all negative dilation solutions into two irrational cases (i^*) and (ii^*) and the rational case (iii^*) which it shows enumerates all the rational points in S. It uses (σ', τ')-coordinates, with $\sigma' = \frac{1}{\mu'}$, $\tau' = \frac{1}{\nu'}$.

Section 7 proves that the part of S restricted to the region of nonpositive dilation factors is a closed set. This result completes the proof that S is closed in $\mathbb{R}^2$, stated in Part I [4, Theorem 1.4].

2.1. **Notational conventions.** (1) The moduli space parametrizing dilations has coordinates (α, β). The proofs use different coordinate systems for this moduli space, starting with the negation (α', β') of the coordinate system (α, β). Thus $S' = -S$ denotes the solution set S in (α', β')-coordinates. In Section 5 we use the (μ', ν')-coordinate systems and in Section 6 the (σ', τ')-coordinate system. These coordinate systems take positive real values.

(2) Variables x and y are never moduli space coordinates. They are real-valued, and are used in dilated floor functions, in commutators $[f_\alpha, f_\beta](x)$, and in lattices and quotient tori.

3. STRICT ROUNDING FUNCTIONS

In Part I, rounding functions $\lfloor x \rfloor_\alpha := \alpha \left\lfloor \frac{1}{\alpha} x \right\rfloor$ were used for analyzing the relation $[f_\alpha, f_\beta] \geq 0$ in the positive dilation case. Here we use modified rounding functions, obtained by replacing the floor function with the *strict floor function* $\lfloor x \rfloor'$, which returns the largest integer strictly smaller than x. Thus $\lfloor x \rfloor' := \max\{n \in \mathbb{Z} : n < x\}$, which also has $\lfloor x \rfloor' = \lceil x \rceil - 1$.

Definition 3.1. Given nonzero α, let $\lfloor x \rfloor'_\alpha$ denote the *(lower) strict rounding function* defined by

$$\lfloor x \rfloor'_\alpha := \alpha \left\lfloor \frac{1}{\alpha} x \right\rfloor' = \alpha \left(\left\lceil \frac{1}{\alpha} x \right\rceil - 1 \right),$$

We additionally set $\lfloor x \rfloor'_0 = x$.

We may also define the *strict ceiling function* $\lceil x \rceil' = -\lfloor -x \rfloor' = \lfloor x \rfloor + 1$ and the *(upper) strict rounding function* defined by $\lceil x \rceil'_\alpha := \alpha \left\lceil \frac{1}{\alpha} x \right\rceil'$. These definitions yield a single family of strict rounding functions, because

$$\lceil x \rceil'_\alpha = \lfloor x \rfloor'_{-\alpha} \tag{3.1}$$

for all $\alpha \neq 0$. The continuous function $\lfloor x \rfloor'_0 = \lceil x \rceil'_0 = x$ arises as the pointwise limit of the functions $\lfloor x \rfloor'_\alpha$ as $\alpha \to 0$.

3.1. **Strict rounding functions: ordering inequalities.** We classify when the graph of one strict rounding function lies (weakly) below the other.

Proposition 3.2 (Strict Rounding Function: Ordering Inequalities)**.** *Given positive α, β, the strict rounding functions satisfy the inequalities*

(a) $\lfloor x \rfloor'_\alpha \leq \lfloor x \rfloor'_\beta$ holds for all $x \in \mathbb{R}$ if and only if $\alpha = m\beta$ for some integer $m \geq 1$.

(b) $\lceil x \rceil'_\alpha \leq \lceil x \rceil'_\beta$ holds for all $x \in \mathbb{R}$ if and only if $\beta = m\alpha$ for some integer $m \geq 1$.

Proof. This result parallels [4, Prop. 4.2], making use of the following general observation: if functions $f(x)$ and $g(x)$ are both piecewise continuous and left-continuous (resp. both right-continuous), then $f \leq g$ holds identically if and only if it holds on the interior of the domain of continuity of f and g. Here the strict floor function $\lfloor x \rfloor'_\alpha$ (resp. $\lceil x \rceil'_\alpha$) agrees with $\lfloor x \rfloor_\alpha$ (resp. $\lceil x \rceil_\alpha$) on the interior of its domain of continuity. $\square$

3.2. **Strict rounding function criterion: negative dilations.** We give a (separated variable) criterion in terms of strict rounding functions equivalent to the nonnegative commutator condition $[f_\alpha, f_\beta] \ge 0$ on dilated floor functions with negative dilation factors. We state it in $(\alpha', \beta') := (-\alpha, -\beta)$ coordinates.

Proposition 3.3 (Nonnegative commutator relation: strict rounding function criterion)**.** *Given* $\alpha', \beta' > 0$, *the following properties are equivalent.*

(R1') *The nonnegative commutator relation* $[f_{-\alpha'}, f_{-\beta'}] \ge 0$ *holds.*
(R2') *(Lower strict rounding function) There holds*

$$\lfloor n \rceil'_{\alpha'} \le \lfloor n \rceil'_{\beta'} \quad \textit{for all } n \in \mathbb{Z}, \tag{3.2}$$

where $\lfloor x \rceil'_{\alpha'}$ *is the strict rounding function with parameter* α'.

Proof. The proof parallels that of Proposition 4.3 in [4, p. 284]. The fundamental observation is that

$$\{x : f_{-\alpha'} \circ f_{-\beta'}(x) \ge n\} = \{x : x > \frac{1}{\alpha'\beta'} \lfloor n \rceil'_{\alpha'}\}. \tag{3.3}$$

We leave the details to the interested reader. □

3.3. **Symmetries of S for negative dilations: Proof of Theorem 1.3.**

Proof of Theorem 1.3. We suppose $\alpha', \beta' > 0$ and are to show:

(i) for any integer $m \ge 1$, if $(-\alpha', -\beta') \in S$ then $(-\alpha', -m\beta') \in S$.
(ii) for any integer $m \ge 1$, if $(-\alpha', -\beta') \in S$ then $(-\frac{1}{m}\alpha', -\frac{1}{m}\beta') \in S$.

These follow from Proposition 3.2 and Proposition 3.3; the proof parallels that of Theorem 2.1 in [4, p. 285]. □

4. Negative dilations classification: linear fractional symmetries

This section establishes the linear fractional symmetries of S for negative dilations. We will use these symmetries to generate all the rational solutions in case (iii^*), in particular to produce the sporadic rational solutions.

4.1. **Birational coordinate change: (μ', ν')-coordinates.** The proofs use a birational change of coordinates of the parameter space describing the two negative dilations. We map (α', β')-coordinates of the parameter space to (μ', ν')-coordinates, given by

$$(\mu', \nu') := \left(\frac{1}{\beta'}, \frac{\alpha'}{\beta'}\right) = \left(-\frac{1}{\beta}, \frac{\alpha}{\beta}\right). \tag{4.1}$$

Its inverse map is $(\alpha', \beta') = \left(\frac{\nu'}{\mu'}, \frac{1}{\mu'}\right)$. The negative dilation part of the solution set S is pictured in the new coordinates in Figure 4.1.

In (μ', ν') coordinates, Theorem 1.3 says the following.

Theorem 4.1 (Linear Symmetries)**.** *For any integer* $m \ge 1$, *solutions to* $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \ge 0$ *are preserved under the maps*

$$(\mu', \nu') \mapsto (m\mu', \nu') \quad \textit{and} \quad (\mu', \nu') \mapsto (\mu', m\nu')$$

The linear symmetries of the set S are visually apparent in Figure 4.1 after this coordinate change.

4.2. **Lattice disjointness criterion: negative dilations.** We reformulate the nonnegative commutator relation in terms of the new parameters (μ', ν') as follows. The criterion involves a disjointness property of a rectangular lattice $\Lambda_{\mu',\nu'}$ from an "enlarged diagonal set" $\mathcal{D}'$; see (P2') of Proposition 4.4 below.

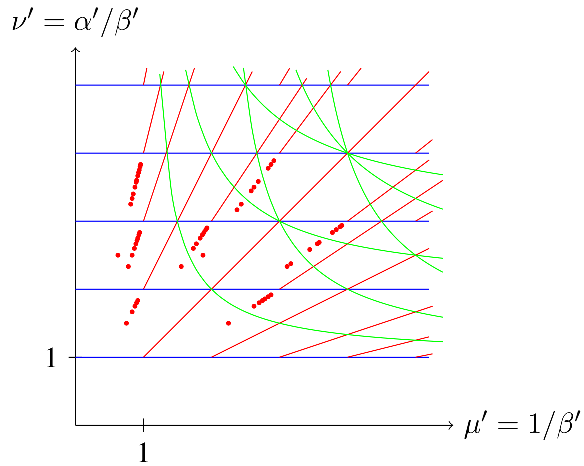

FIGURE 4.1. Negative dilation part of S viewed in (μ', ν')-coordinates

4.2.1. *Enlarged diagonal set* $\mathcal{D}'$. The positive dilation case in Part I used an "approximate diagonal" set of the plane $\mathbb{R}^2$ given by

$$\mathcal{D} := \bigcup_{n \in \mathbb{Z}} \{(x, y) : n < x < n+1 \text{ and } n < y < n+1\}, \tag{4.2}$$

see [4, Definition 5.1]. The set $\mathcal{D}$ is invariant under the negation map $(x, y) \mapsto (-x, -y)$ and reflection map $(x, y) \mapsto (y, x)$. It is pictured in Figure 4.2(a). The negative dilation case will use instead the following set.

Definition 4.2. The *punctured enlarged diagonal set* $\mathcal{D}'$ in $\mathbb{R}^2$ is the region

$$\mathcal{D}' := \bigcup_{n \in \mathbb{Z}} \{(x, y) : n \le x \le n+1,\ n < y < n+1,\ \text{ and } x \ne y\}.$$

Here $\mathcal{D}'$ differs from $\mathcal{D}$ in being a "punctured approximate diagonal" since it omits the exact diagonal line; it is pictured in Figure 4.2(b). It is characterized in terms of rounding functions by

$$\mathcal{D}' = \{(x, y) : \text{either} \quad \lfloor y \rfloor \le x < y \quad \text{or} \quad y < x \le \lceil y \rceil\}, \tag{4.3}$$

and in terms of strict rounding functions by

$$\mathcal{D}' = \{(x, y) : \text{either} \quad \lfloor x \rfloor' < y < x \quad \text{or} \quad x < y < \lceil x \rceil'\}. \tag{4.4}$$

The set $\mathcal{D} \cup \mathcal{D}'$ will also be important in the arguments in Sections 5 and 6; it is pictured in Figure 4.2(c).

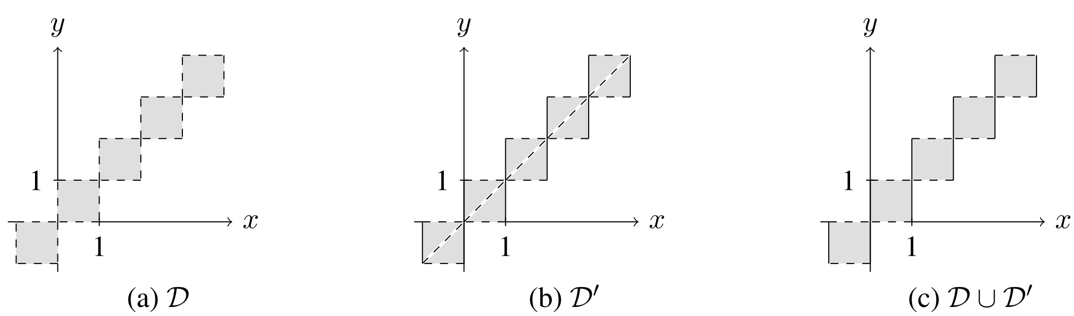

FIGURE 4.2. Regions $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D} \cup \mathcal{D}'$ in $\mathbb{R}^2$, in gray.

Remark 4.3. $\mathcal{D}'$ is invariant under the negation map $(x, y) \mapsto (-x, -y)$. However, it is not invariant under the reflection map $(x, y) \mapsto (y, x)$.

Proposition 4.4 (Lattice disjointness criterion: negative dilations)**.** *For $\mu', \nu' > 0$, the following three properties are equivalent.*

(P1') *The nonnegative commutator relation holds:*

$$[f_{-\nu'/\mu'}, f_{-1/\mu'}](x) \geq 0 \quad \textit{for all } x \in \mathbb{R}.$$

(P2') *The two-dimensional rectangular lattice $\Lambda_{\mu',\nu'} = \mu'\mathbb{Z} \times \nu'\mathbb{Z}$ is disjoint from the region $\mathcal{D}' = \{(x, y) : \lfloor y \rfloor \leq x \leq \lceil y \rceil,\, x \neq y\}$. That is,*

$$\Lambda_{\mu',\nu'} \bigcap \mathcal{D}' = \emptyset.$$

(P3') *The set $\Lambda^{+}_{\mu',\nu'}$ is disjoint from $\mathcal{D}'$, where $\Lambda^{+}_{\mu',\nu'} = \bigcup_{k\in\mathbb{Z}} \big(\Lambda_{\mu',\nu'} + (k, k)\big)$ is the union of all translates of $\Lambda_{\mu',\nu'}$ by integer diagonal vectors (k, k).*

Proof. (P1') $\Leftrightarrow$ (P2') The argument parallels the proof in [4] of (P1) $\Leftrightarrow$ (P2) in Proposition 5.2. We omit the details. The main step is that

$$\lfloor n \rfloor'_{\nu'/\mu'} > \lfloor n \rfloor'_{1/\mu'} \quad \text{for some } n \in \mathbb{Z}, \tag{4.5}$$

from Proposition 3.3, is equivalent to the condition

$$\text{there exist } m, n \in \mathbb{Z} \text{ such that } \lfloor n\mu' \rfloor' < m\nu' < n\mu'. \tag{4.6}$$

(P2') $\Leftrightarrow$ (P3') Since $\Lambda_{\mu',\nu'} \subseteq \Lambda^{+}_{\mu',\nu'}$, the implication (P3') $\Rightarrow$ (P2') holds. In the other direction, suppose $\Lambda_{\mu',\nu'}$ is disjoint from $\mathcal{D}'$. The enlarged diagonal set $\mathcal{D}'$ is sent to itself by translation by an integer diagonal vector (k, k), so the translated lattice $\Lambda_{\mu',\nu'} + (k, k)$ is disjoint from $\mathcal{D}'$. This holds for all integers k, so $\Lambda^{+}_{\mu',\nu'}$ is disjoint from $\mathcal{D}'$. Consequently (P2') $\Rightarrow$ (P3'). ☐

4.3. **Extension of GCD and LCM to positive real numbers.** We formulate an extension of the notion of $\gcd$ and lcm to pairs of positive real numbers, for use in arguments to prove the linear fractional rescaling symmetries. This extension is based on commensurability of lattices.

Definition 4.5. Given real numbers $u, v > 0$, we define their *extended greatest common divisor* by

$$\mathrm{egcd}(u, v) := \begin{cases} w & \text{if } u\mathbb{Z} + v\mathbb{Z} = w\mathbb{Z},\ w > 0 \\ 0 & \text{if } u\mathbb{Z} + v\mathbb{Z} \text{ is dense in } \mathbb{R}. \end{cases}$$

We also define their *extended least common multiple*

$$\mathrm{elcm}(u, v) := \begin{cases} w & \text{if } u\mathbb{Z} \cap v\mathbb{Z} = w\mathbb{Z},\ w > 0 \\ +\infty & \text{if } u\mathbb{Z} \cap v\mathbb{Z} = \{0\}. \end{cases}$$

These definitions match the usual notions when u, v are positive integers. We will frequently use the following proposition, whose proof we omit.

Proposition 4.6. *Suppose u, v are positive real numbers.*

(i) We have $d = \mathrm{egcd}(u, v) > 0$ if and only if there are coprime positive integers p, q such that $u = pd \quad$ and $\quad v = qd$.

(ii) We have $m = \mathrm{elcm}(u, v) < +\infty$ if and only if there are coprime positive integers p, q such that $m = qu \quad$ and $\quad m = pv$.

4.4. **Disjointness of lattices and regions: part 1.** We prove several lemmas characterizing when various lattices are disjoint from $\mathcal{D}'$, resp. $\mathcal{D} \cup \mathcal{D}'$, to facilitate use of Proposition 4.4. The disjointness criteria involving $\mathcal{D} \cup \mathcal{D}'$ are needed for the necessity proofs in Section 6.

The first two lemmas will be used to show existence of the type (iii^*) rational solutions in the sufficiency proof in Section 5.

Lemma 4.7 (Diagonal expansion-contraction)**.** *Let $r \geq 2$ be an integer, and suppose u, v are real parameters satisfying $1 + u > v$. Then the following conditions are equivalent.*

(S1') *The lattice* $\Lambda = \mathrm{row.span}_{\mathbb{Z}} \begin{pmatrix} 1+u & v \\ 1 & 1 \end{pmatrix}$ *is disjoint from* $\mathcal{D} \cup \mathcal{D}'$.

(S2') *The lattice* $\Lambda = \mathrm{row.span}_{\mathbb{Z}} \begin{pmatrix} 1+u & v \\ 1 & 1 \end{pmatrix}$ *is disjoint from* $\mathcal{D}'$.

(S3') *The lattice* $\Lambda^{(r)} = \mathrm{row.span}_{\mathbb{Z}} \begin{pmatrix} 1+\frac{1}{r}u & \frac{1}{r}v \\ \frac{1}{r} & \frac{1}{r} \end{pmatrix}$ *is disjoint from* $\mathcal{D}'$.

Proof. To see that (S1') $\Leftrightarrow$ (S2'), observe that the lattice Λ only intersects the diagonal Δ of $\mathbb{R}^2$ at integer points due to the assumption that $1 + u > v$. Thus Λ must be disjoint from $\mathcal{D} \setminus \mathcal{D}' = \Delta \setminus \mathbb{Z}^2$.

Next we show that (S2') $\Leftrightarrow$ (S3'). Observe that Λ is disjoint from $\mathcal{D}'$ if and only if the point $(1+u, v)$ lies below (or to the right of) the diagonal region $\mathcal{D}'$, which holds if and only if $v \leq \lceil u \rceil$. See the left side of Figure 4.3.

Since the lattice $\Lambda^{(r)}$ has $(\frac{1}{r}, \frac{1}{r})$ as a generator, $\Lambda^{(r)}$ is invariant under translation by $\frac{j}{r}(1,1)$ for any integer j. Hence in order for $\Lambda^{(r)}$ to be disjoint from $\mathcal{D}'$, it must also be disjoint with each translate $\frac{j}{r}(1,1) + \mathcal{D}'$, for integers $j = 1, \ldots, r-1$. The latter fact holds if and only if the other generator $(1 + \frac{1}{r}u, \frac{1}{r}v)$ of $\Lambda^{(r)}$ lies below the union of all translates $\frac{j}{r}(1,1) + \mathcal{D}'$, which is equivalent to $\frac{1}{r}v \leq \lceil \frac{1}{r}u \rceil_{1/r}$. See the right side of Figure 4.3.

Finally, we observe that for any fixed integer $r \geq 1$, the condition on (u, v) that $\frac{1}{r}v \leq \lceil \frac{1}{r}u \rceil_{1/r}$ is equivalent to $v \leq \lceil u \rceil$. This equivalence implies the equivalence of the conditions (S2') and (S3'). $\square$

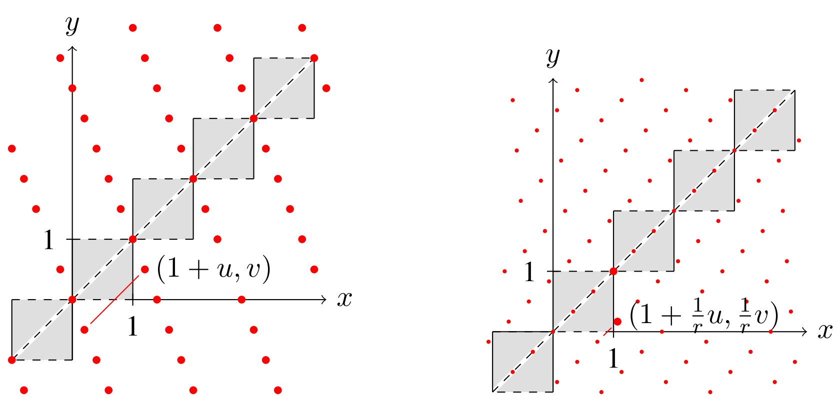

FIGURE 4.3. Lattices Λ and $\Lambda^{(r)}$ with $r = 3$ in red, shown with punctured approximate diagonal region $\mathcal{D}'$ in gray.

The condition (a) of the next lemma provides a recipe for producing an infinite number of new rational solutions in S, starting from an initial rational solution in S associated to $\Lambda_{\ell p, \ell q}$.

Lemma 4.8. *Suppose $p, q \geq 1$ are coprime integers and $\ell > 0$ is a real number such that the lattice $\Lambda_{\ell p, \ell q}$ is disjoint from the set*

$$\mathcal{D}' = \bigcup_{n \in \mathbb{Z}} \{(x, y) : n \leq x \leq n+1,\ n < y < n+1,\ \text{ and }\ x \neq y\}.$$

(a) If $\ell > 0$ is rational, then for any integer $r \geq 1$, the lattice $\Lambda_{\lambda p, \lambda q}$ with

$$\lambda = \lambda_{r,\ell} = 1 + \frac{1}{r}(\ell - 1)$$

is disjoint from $\mathcal{D}'$. (Note that when $r = 1$, $\lambda_{1,\ell} = \ell$.)

(b) If ℓ is irrational, then necessarily $\ell > 1$. Conversely, if $\ell > 1$ is irrational then the lattice $\Lambda_{\ell p, \ell q}$ is disjoint from the set $\mathcal{D}'$.

Proof. Set $\mu' = \ell p$, $\nu' = \ell q$. Note that $\ell = \operatorname{egcd}(\mu', \nu')$. Recall that $\Lambda_{\ell p, \ell q} = \Lambda_{\mu', \nu'}$ denotes the image of $(m, n) \mapsto (m\mu', n\nu')$ for $(m, n) \in \mathbb{Z}^2$. Equivalently, $\Lambda_{\mu', \nu'}$ is the integer row span of the diagonal matrix

$$\Lambda_{\mu', \nu'} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} \mu' & 0 \\ 0 & \nu' \end{pmatrix}.$$

We first find a second basis for $\Lambda_{\mu', \nu'}$ which is convenient for comparison with $\mathcal{D}'$. Namely, we choose one basis vector to lie on the main diagonal. Let $\Delta = \{(x, x) : x \in \mathbb{R}\}$ denote the diagonal of $\mathbb{R}^2$.

Let $m_0, n_0 \geq 1$ be integers such that $m_0 p - n_0 q = 1$.

Claim. *The lattice $\Lambda_{\mu', \nu'}$ is equal to* $\operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} m_0\mu' & n_0\nu' \\ q\mu' & p\nu' \end{pmatrix}$.

Proof. Note that

$$\begin{pmatrix} m_0\mu' & n_0\nu' \\ q\mu' & p\nu' \end{pmatrix} = \begin{pmatrix} m_0 & n_0 \\ q & p \end{pmatrix} \begin{pmatrix} \mu' & 0 \\ 0 & \nu' \end{pmatrix}.$$

The integer change of basis matrix $\begin{pmatrix} m_0 & n_0 \\ q & p \end{pmatrix}$ has determinant 1. This proves the claim.

Now $\Lambda_{\mu', \nu'}$ is disjoint from $\mathcal{D}'$ by hypothesis. Hence the implication (P2') $\Leftrightarrow$ (P3') in Proposition 4.4 implies that $\Lambda^+_{\mu', \nu'}$ is disjoint from $\mathcal{D}'$ where

$$\Lambda^+_{\mu', \nu'} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} \mu' & 0 \\ 0 & \nu' \\ 1 & 1 \end{pmatrix}.$$

The above claim implies that $\Lambda^+_{\mu', \nu'}$ is equal to the $\mathbb{Z}$-row span of vectors

$$\Lambda^+_{\mu', \nu'} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} m_0\mu' & n_0\nu' \\ q\mu' & p\nu' \\ 1 & 1 \end{pmatrix} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} m_0\mu' & n_0\nu' \\ \ell pq & \ell pq \\ 1 & 1 \end{pmatrix}.$$

(a) Suppose first that ℓ is rational. Since the second and third rows are on the diagonal Δ while the first row is off the diagonal, we have

$$\Lambda^+_{\mu', \nu'} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} m_0\mu' & n_0\nu' \\ \operatorname{egcd}(1, \ell pq) & \operatorname{egcd}(1, \ell pq) \end{pmatrix}$$

by definition of the extended greatest common divisor, and the assumption that ℓ is rational. Let s be the positive integer such that $\frac{1}{s} = \operatorname{egcd}(1, \ell pq)$.

Next we apply Lemma 4.7 with $u = s(m_0\mu' - 1)$ and $v = sn_0\nu'$, so

$$\Lambda^{(s)} := \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} 1 + (m_0\mu' - 1) & n_0\nu' \\ \frac{1}{s} & \frac{1}{s} \end{pmatrix} = \Lambda^+_{\mu', \nu'}.$$

Since $\Lambda^{(s)}$ is disjoint from $\mathcal{D}'$, the implication (S3') $\Rightarrow$ (S2') of Lemma 4.7 guarantees that

$$\Lambda := \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} 1 + s(m_0\mu' - 1) & sn_0\nu' \\ 1 & 1 \end{pmatrix}$$

is disjoint from $\mathcal{D}'$. But then (S2') $\Rightarrow$ (S3') implies that

$$\Lambda^{(rs)} := \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} 1 - \frac{1}{r} + \frac{1}{r}m_0\mu' & \frac{1}{r}n_0\nu' \\ \frac{1}{rs} & \frac{1}{rs} \end{pmatrix}.$$

is disjoint from $\mathcal{D}'$ for any integer $r \geq 1$.

Finally, to see that the lattice $\Lambda_{\lambda p,\lambda q}$ is disjoint from $\mathcal{D}'$ it suffices to show that $\Lambda_{\lambda p,\lambda q}$ is contained in $\Lambda^{(rs)}$, i.e. it suffices to verify that

$$\Lambda^{(rs)} \supset \Lambda_{\lambda p,\lambda q} := \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} \lambda p & 0 \\ 0 & \lambda q \end{pmatrix}$$

Recall that $\lambda = 1 + \frac{1}{r}(\ell - 1)$; this containment follows from the matrix relation

$$\begin{pmatrix} \lambda p & 0 \\ 0 & \lambda q \end{pmatrix} = \begin{pmatrix} p & -n_0 s\ell pq \\ -q & (r-1)sq + m_0 s\ell pq \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{r} + \frac{1}{r}m_0\ell p & \frac{1}{r}n_0\ell q \\ \frac{1}{rs} & \frac{1}{rs} \end{pmatrix}. \tag{4.7}$$

Here we recall that $m_0 p - n_0 q = 1$, and that $s\ell pq$ is an integer by the choice $\frac{1}{s} = \operatorname{egcd}(1, \ell pq)$. Thus the second matrix in (4.7) has integer entries. This shows that $\Lambda_{\lambda p,\lambda q} \subset \Lambda^{(rs)}$ so $\Lambda_{\lambda p,\lambda q}$ is disjoint from $\mathcal{D}'$ as desired.

(b) Now suppose ℓ is irrational. As above, we consider when $\mathcal{D}'$ is disjoint from

$$\Lambda^{+}_{\mu',\nu'} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} \ell m_0 p & \ell n_0 q \\ \ell pq & \ell pq \\ 1 & 1 \end{pmatrix}.$$

When ℓ is irrational, the set $\ell pq\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$. Hence the topological closure of $\Lambda^{+}_{\ell p,\ell q}$ inside $\mathbb{R}^2$ contains the diagonal line Δ of $\mathbb{R}^2$. Using this observation it is straightforward to verify that the closure of $\Lambda^{+}_{\ell p,\ell q}$ is

$$\overline{\Lambda^{+}_{\ell p,\ell q}} = \{(x, y) \in \mathbb{R}^2 : x - y \in \ell\mathbb{Z}\}.$$

If $0 < \ell < 1$, then the closure $\overline{\Lambda^{+}_{\ell p,\ell q}}$ intersects the interior of $\mathcal{D}'$ (at a point with $x - y = \pm\ell$) so $\Lambda^{+}_{\ell p,\ell q}$ must also intersect $\mathcal{D}'$. On the other hand if $\ell > 1$, then $\overline{\Lambda^{+}_{\ell p,\ell q}}$ is disjoint from $\mathcal{D}'$ so $\Lambda^{+}_{\ell p,\ell q}$ is also disjoint from $\mathcal{D}'$. □

4.5. **Disjointness of lattices and regions: part 2.** We prove two further lemmas needed for the necessity proof in Section 6.4, which are not needed in the sufficiency proof in Section 5. The first shows that every rational lattice $\Lambda_{\lambda p,\lambda q}$ which is associated to a solution in S is associated to some rational lattice $\Lambda_{\ell p,\ell q}$ associated to a case (i) solution in S which is disjoint from $\mathcal{D} \cup \mathcal{D}'$. Lemma 4.10 characterizes all such lattices.

Lemma 4.9. *Suppose that p, q are coprime positive integers and $\lambda > 0$ is rational such that the lattice $\Lambda_{\lambda p,\lambda q}$ is disjoint from $\mathcal{D}'$. Then there necessarily exists some positive integer r such that for*

$$\ell = \ell_{r,\lambda} = 1 + r(\lambda - 1),$$

the lattice $\Lambda_{\ell p,\ell q}$ is disjoint from $\mathcal{D} \cup \mathcal{D}'$.

Proof. Assume the same notation as in the proof of Lemma 4.8. By the argument there, $\mathcal{D}'$ is disjoint from

$$\Lambda^{+}_{\lambda p,\lambda q} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} \lambda m_0 p & \lambda n_0 q \\ \operatorname{egcd}(1, \lambda pq) & \operatorname{egcd}(1, \lambda pq) \end{pmatrix}.$$

Let r denote the positive integer which satisfies $\frac{1}{r} = \mathrm{egcd}(1, \lambda pq)$. (Recall that λ is rational.)

Now we apply Lemma 4.7: let $\Lambda^{(r)} = \Lambda^{+}_{\lambda p, \lambda q}$ so that $1 + \frac{1}{r}u = m_0 \lambda p$ and $\frac{1}{r}v = n_0 \lambda q$ and (S3') holds, i.e. $\Lambda^{(r)}$ is disjoint from $\mathcal{D}'$. Then (S1') holds, namely

$$\Lambda := \mathrm{row.\,span}_{\mathbb{Z}} \begin{pmatrix} 1 - r + \lambda m_0 pr & \lambda n_0 qr \\ 1 & 1 \end{pmatrix}$$

is disjoint from $\mathcal{D} \cup \mathcal{D}'$. It remains to check that $\Lambda \supset \Lambda_{\ell p, \ell q}$ where $\ell = 1 + r(\lambda - 1)$, i.e. that

$$\Lambda \supset \mathrm{row.\,span}_{\mathbb{Z}} \begin{pmatrix} (1 - r + \lambda r)p & 0 \\ 0 & (1 - r + \lambda r)q \end{pmatrix}.$$

This amounts to the calculation

$$\begin{pmatrix} (1 - r + \lambda r)p & 0 \\ 0 & (1 - r + \lambda r)q \end{pmatrix} = \begin{pmatrix} p & -\lambda n_0 pqr \\ -q & q - rq + \lambda m_0 pqr \end{pmatrix} \begin{pmatrix} 1 - r + m_0 \lambda pr & n_0 \lambda qr \\ 1 & 1 \end{pmatrix},$$

where we note that λpqr is an integer by the choice $\frac{1}{r} = \mathrm{egcd}(1, \lambda pq)$. This proves the disjointness of $\Lambda_{\ell p, \ell q}$ and $\mathcal{D} \cup \mathcal{D}'$ as desired. $\square$

The next lemma shows that disjointness from $\mathcal{D} \cup \mathcal{D}'$ requires that (μ', ν') must lie on a rectangular hyperbola or on a horizontal line.

Lemma 4.10 (Combined lattice/diagonal disjointness classification)**.** *For rational parameters $\mu', \nu' > 0$ the following conditions are equivalent.*

(M1') *The lattice $\Lambda_{\mu', \nu'} = \mathrm{row.\,span}_{\mathbb{Z}} \begin{pmatrix} \mu' & 0 \\ 0 & \nu' \end{pmatrix}$ is disjoint from $\mathcal{D} \cup \mathcal{D}'$.*

(M2') *There exist integers $m \geq 0$, $n \geq 1$ such that*

$$\frac{m}{\mu'} + \frac{n}{\nu'} = 1.$$

Proof. (M2')⇒(M1') Suppose there exist integers $m \geq 0, n \geq 1$ such that $\frac{m}{\mu'} + \frac{n}{\nu'} = 1$. By [4, Theorem 5.4] we have the relation $[f_{1/\mu'}, f_{\nu'/\mu'}] \geq 0$, and then the equivalence [4, Proposition 5.2] implies $\Lambda_{\mu', \nu'}$ is disjoint from $\mathcal{D}$. The arguments used in proving [4, Theorem 5.4 and Lemma 5.5] also imply that $\Lambda_{\mu', \nu'}$ is disjoint from $\mathcal{D}'$. Thus (M1') holds. In this direction, it is not necessary to assume that μ', ν' are rational.

(M1')⇒(M2') Now suppose that $\Lambda_{\mu', \nu'}$ is disjoint from $\mathcal{D} \cup \mathcal{D}'$.

Case 1. Suppose that μ' is not an integer. By hypothesis $\Lambda_{\mu', \nu'}$ is disjoint from $\mathcal{D}$, so by [4, Proposition 5.2] there exist integers $m, n \geq 0$ such that $\frac{m}{\mu'} + \frac{n}{\nu'} = 1$. Since μ' is not an integer we must have $n \geq 1$ in any such relation so (M2') holds.

Case 2. Suppose μ' is an integer. Then set $\lambda := \mathrm{egcd}(\mu', \nu')$. Since μ', ν' are both rational, λ is a positive rational number. We claim that $\lambda \geq 1$.

By definition of extended greatest common divisor, there exist integers m_0, n_0 such that $m_0 \mu' - n_0 \nu' = \lambda$. Consider the lattice point

$$(m_0 \mu', n_0 \nu') = (m_0 \mu', m_0 \mu' - \lambda) \in \Lambda_{\mu', \nu'}.$$

Since $m_0 \mu'$ is an integer, the region $\mathcal{D}'$ contains the open segment

$$\{(m_0 \mu', y) : m_0 \mu' - 1 < y < m_0 \mu'\} \subset \mathcal{D}'.$$

Thus the hypothesis that $\Lambda_{\mu', \nu'}$ is disjoint from $\mathcal{D}'$ implies that $\lambda \geq 1$ as claimed.

Let p, q be the coprime integers such that $(\mu', \nu') = (\lambda p, \lambda q)$, and let $m = \mu' - p = \lambda p - p$ and $n = q \geq 1$. Here m is an integer since μ' is an integer, and $m \geq 0$ since $\lambda \geq 1$. Then

$$\frac{m}{\mu'} + \frac{n}{\nu'} = \frac{\lambda p - p}{\lambda p} + \frac{q}{\lambda q} = 1 - \frac{1}{\lambda} + \frac{1}{\lambda} = 1.$$

Both m and n are integers, so condition (M2') holds as desired. $\square$

4.6. **Proof of linear fractional symmetries: Theorem 1.4.** We view Theorem 1.4 in (μ', ν') coordinates. Recall that $\alpha = -\frac{\nu'}{\mu'}$, $\beta = -\frac{1}{\mu'}$. The map $\Phi_p^r(\beta)$ is conjugate under $\mathbf{J}(\mu') = -\frac{1}{\mu'} := \beta$ to $\Psi_p^r(\mu') = \mathbf{J}^{-1} \circ \Phi_p^r \circ \mathbf{J}(\mu')$, which acts linearly in the μ'-coordinate. Theorem 1.4 becomes:

Theorem 4.11 (Linear fractional symmetries in (μ', ν')-coordinates)**.** *If $\frac{\nu'}{\mu'} = \frac{q}{p}$ is a fixed positive rational given in lowest terms, then for any integer $r \geq 1$ the set of μ' satisfying $[f_{-q/p}, f_{-1/\mu'}] \geq 0$ is mapped to itself under*

$$\mu' \mapsto \Psi_p^r(\mu') := \frac{1}{r}\left(\mu' - p\right) + p.$$

This map sends $(\mu', \frac{q}{p}\mu') \in S$ to $(\Psi_p^r(\mu'), \frac{q}{p}\Psi_p^r(\mu')) \in S$ in (μ', ν') coordinates.

Proof. For integer $r \geq 1$, the map $\Psi_p^r(\mu')$ sends rational numbers to rational numbers and irrationals to irrationals. We treat these two cases separately.

Case (i). Suppose that μ' is irrational. Lemma 4.8 (b) says $[f_{-q/p}, f_{-1/\mu'}] \geq 0$ if and only if $\mu' \geq p$. The map $\Psi_p^r(\mu')$, which is equivalent to $\mu' - p \mapsto \frac{1}{r}(\mu' - p)$, sends the set $\{\mu' : \mu' \geq p\}$ to itself.

Case (ii). Suppose that μ' is rational. Lemma 4.8 (a) and Proposition 4.4 then imply: if $[f_{-q/p}, f_{-1/\mu'}] \geq 0$ then also $[f_{-q/p}, f_{-1/\Psi_p^r(\mu')}] \geq 0$. ☐

5. NEGATIVE DILATIONS CLASSIFICATION: SUFFICIENCY

In this section we prove the sufficiency part of Theorem 1.1.

5.1. **Sufficiency condition in (μ', ν')-coordinates.** We use the (μ', ν') change of coordinates. We restate the sufficiency condition in Theorem 1.1 in terms of (μ', ν')-coordinates.

Theorem 5.1 (Sufficiency condition in (μ', ν') coordinates)**.** *Given parameters $\mu', \nu' > 0$, the nonnegative commutator relation $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \geq 0$ holds if any one of the following conditions holds.*

(i) There are integers $m \geq 0, n \geq 1$ such that

$$\frac{m}{\mu'} + \frac{n}{\nu'} = 1.$$

(ii) There are coprime integers $p, q \geq 1$ such that

$$\frac{\mu'}{p} = \frac{\nu'}{q} \geq 1.$$

(iii) There are coprime integers $p, q \geq 1$ and integers $m \geq 0, n \geq 1, r \geq 1$ such that*

$$\frac{\mu'}{p} = \frac{\nu'}{q} = 1 + \frac{1}{r}\left(\frac{m}{p} + \frac{n}{q} - 1\right).$$

Note that in terms of extended greatest common divisor (see Section 4.3), case (ii) of Theorem 5.1 is equivalent to $\mathrm{egcd}(\mu', \nu') \geq 1$. The condition in case (i) is related to disjoint Beatty sequences, see Part I [4, Prop. 2.5]. We prove Theorem 5.1 in the remainder of this section.

5.2. **Sufficiency: Hyperbola and half-line cases.** The next result shows existence of solutions for parameters corresponding to $m = n = 1$, to $m = 0$, and to $p = q = 1$. Later in the proof we will use the linear symmetries of S to construct solutions for other m, n, p, q parameters.

Lemma 5.2 (Rectangular hyperbola and half-line sufficiency)**.** *Suppose $\mu', \nu' > 0$.*

(i) If (μ', ν') lies on the hyperbola $\frac{1}{\mu'} + \frac{1}{\nu'} = 1$, then $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \geq 0$ holds.

(ii) If (μ', ν') lies on the half-line $\nu' = 1$ ($\mu' > 0$), then $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \geq 0$ holds.

(iii) If (μ', ν') lies on the half-line $\mu' = \nu' \geq 1$, then $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \geq 0$ holds.

Proof. It suffices to verify for each case that condition (P2') of Proposition 4.4 holds, whence the Proposition (P1') gives the result. To show that the lattice $\Lambda_{\mu',\nu'} = \{(m\mu', n\nu') : m, n \in \mathbb{Z}\}$ is disjoint from $\mathcal{D}'$ (resp. $\mathcal{D}$), it suffices to show they are disjoint in the closed positive quadrant since both sets are preserved by $(x, y) \mapsto (-x, -y)$ and $\mathcal{D}'$ (resp. $\mathcal{D}$) does not intersect the open second or fourth quadrants.

(i) This case parallels Lemma 5.5 in [4].

(ii) This case is clear from the definition of $\mathcal{D}'$ and $\Lambda_{\mu',\mu'}$.

(iii) Suppose that (μ', ν') lies on the half-line $\{(\mu', \nu') : \mu' = \nu' \geq 1\}$, and consider the point $(m\mu', n\mu')$ in the lattice $\Lambda_{\mu',\mu'}$. Note that $\mathcal{D}'$ is contained in the open region $\{0 < |x - y| < 1\} \subset \mathbb{R}^2$. If $m = n$ then the point $(m\mu', n\mu') = (m\mu', m\mu')$ is contained in the diagonal of $\mathbb{R}^2$ so it is not in $\mathcal{D}'$. Next suppose $m \neq n$. Then the lattice point $(m\mu', n\mu')$ has coordinates which satisfy $|m\mu' - n\mu'| = |m - n| \cdot |\mu'| \geq |m - n| \geq 1$, so again the point is not in $\mathcal{D}'$. Thus $\Lambda_{\mu',\nu'}$ is disjoint from $\mathcal{D}'$. □

5.3. **Proof of Sufficiency Theorem 5.1.**

Proof of Theorem 5.1. There are three sufficient conditions to check.

Case (i). Suppose μ', ν' satisfy $\frac{m}{\mu'} + \frac{n}{\nu'} = 1$ with $m, n \geq 1$. Let ${\mu'}_0 = \frac{\mu'}{m}$ and ${\nu'}_0 = \frac{\nu'}{n}$; these satisfy $\frac{1}{{\mu'}_0} + \frac{1}{{\nu'}_0} = 1$. Lemma 5.2 (i) implies that $({\mu'}_0, {\nu'}_0)$ satisfies $[f_{-{\nu'}_0/{\mu'}_0}, f_{-1/{\mu'}_0}] \geq 0$. It follows from Theorem 4.1 that $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \geq 0$.

If $m = 0$, then let ${\mu'}_0 = \mu'$ and ${\nu'}_0 = \frac{\nu'}{n}$. We repeat the same argument with Lemma 5.2 (ii) and Theorem 4.1.

Case (ii). Suppose μ', ν' satisfy $\frac{\mu'}{p} = \frac{\nu'}{q} \geq 1$ for coprime integers p, q, then let ${\mu'}_0 = \frac{\mu'}{p}$ and ${\nu'}_0 = \frac{\nu'}{q}$. Now Lemma 5.2 (iii) implies that $[f_{-{\nu'}_0/{\mu'}_0}, f_{-1/{\mu'}_0}] \geq 0$, so it follows from Theorem 4.1 that $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \geq 0$. (This argument works without assuming coprimality of p and q, but dropping this assumption does not yield additional solutions.)

Case (iii). Suppose that μ', ν' satisfy

$$\frac{\mu'}{p} = \frac{\nu'}{q} = 1 + \frac{1}{r}\left(\frac{m}{p} + \frac{n}{q} - 1\right) \tag{5.1}$$

for coprime integers $p, q \geq 1$ and integers $m \geq 0, n \geq 1, r \geq 1$. Let

$$\frac{{\mu'}_0}{p} = \frac{m}{p} + \frac{n}{q} \qquad \text{and} \qquad \frac{{\nu'}_0}{q} = \frac{m}{p} + \frac{n}{q}$$

so that $\frac{m}{{\mu'}_0} + \frac{n}{{\nu'}_0} = 1$. By case (i) we have $[f_{-{\nu'}_0/{\mu'}_0}, f_{-1/{\mu'}_0}] \geq 0$.

To show that $[f_{-\nu'/\mu'}, f_{-1/\mu'}] \geq 0$, we apply Theorem 4.11. Since $\mu' = \Psi_p^r({\mu'}_0)$ and $\nu' = \frac{q}{p}\Psi_p^r({\mu'}_0)$, the conclusion follows. □

6. Negative dilations classification: necessity

This section proves the necessity for membership in S of the conditions in Theorem 1.1.

6.1. **Birational Coordinate change: (σ', τ')-coordinates.**

We birationally transform the parameter space to a third set of coordinates, (σ', τ')-coordinates, given by

$$(\sigma', \tau') := \left(\frac{1}{\mu'}, \frac{1}{\nu'}\right) = \left(\beta', \frac{\beta'}{\alpha'}\right) \quad \text{where} \quad \alpha', \beta' > 0. \tag{6.1}$$

The inverse map is $(\alpha', \beta') = (\frac{\sigma'}{\tau'}, \sigma')$. The negative dilation part of the set S viewed in (σ', τ')-coordinates is pictured in Figure 6.1.

The (σ', τ') parameters take values in the positive quadrant. The negative dilation portion of S viewed in (σ', τ') coordinates consists of line segments and isolated points.

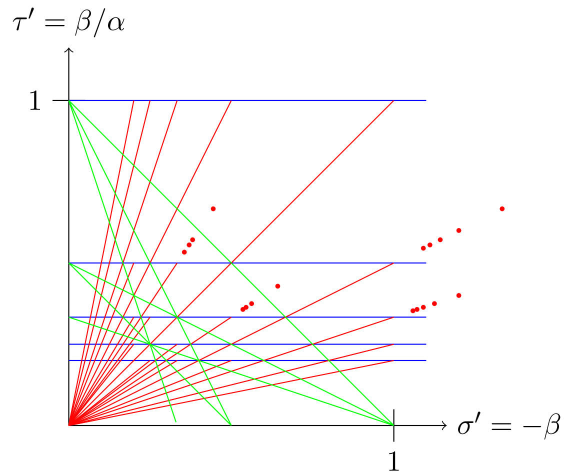

FIGURE 6.1. Negative dilation solutions of S in (σ', τ')-coordinates: $\sigma' = -\beta$, $\tau' = \alpha/\beta$

6.2. **Necessity condition in** (σ', τ')**-coordinates.** We reformulate the necessity condition in Theorem 1.1 in terms of (σ', τ')-coordinates.

Theorem 6.1 (Necessity in (σ', τ')-coordinates)**.** *For parameters $\sigma', \tau' > 0$, the nonnegative commutator relation $[f_{-\sigma'/\tau'}, f_{-\sigma'}] \geq 0$ does not hold unless one of the following conditions holds.*

(a) If at least one of σ', τ' is irrational, then exactly one of the following two conditions holds.

(i^*) *There are integers $m \geq 0, n \geq 1$ such that*

$$m\sigma' + n\tau' = 1.$$

(ii^*) *There are coprime integers p, q such that*

$$p\sigma' = q\tau' \leq 1.$$

(b) If σ', τ' are both rational, then:

(iii) There are coprime integers p, q and integers $m \geq 0, n \geq 1$, and $r \geq 1$ such that*

$$p\sigma' = q\tau' = \left(1 + \frac{1}{r}\left(\frac{m}{p} + \frac{n}{q} - 1\right)\right)^{-1}.$$

The necessary conditions in Theorem 6.1 classify solutions into three disjoint cases. This classification removes all rational solutions from cases (i) and (ii), giving cases (i^*) and (ii^*) respectively, with $(i^*) \subset (i)$ and $(ii^*) \subset (ii)$. One sees that (i^*) and (ii^*) are disjoint by observing that any common solution requires solving two linear equations in (σ', τ') with integer coefficients, which either are inconsistent or else have a unique rational solution (σ', τ').

6.3. **Torus subgroup criterion: negative dilations.** To prove necessity of the classification given in Theorem 6.1 we establish a criterion for nonnegative commutator, given in terms of (σ', τ')-coordinates. It is expressed in terms of a cyclic subgroup of the 2-dimensional torus[1]

$$\mathbf{T} := \mathbb{R}^2/\mathbb{Z}^2$$

avoiding a set $\widetilde{\mathcal{C}}'_{\sigma',\tau'}$ which also varies with the parameters. The set to be avoided, viewed in $\mathbb{R}^2$, is an open rectangular region with one corner at the origin, modified to exclude a diagonal and to include two vertical sides of its boundary. It is neither open nor closed as a subset of $\mathbb{R}^2$.

Given a real number x, we let $\widetilde{x}$ denote its image under the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

[1]In the published version of Part I ([4]), $\mathbf{T}$ was denoted $\mathbb{T}$.

Definition 6.2. We define the *modified corner rectangle* ${\mathcal{C}'}_{\sigma',\tau'}$ to be the region

$$ {\mathcal{C}'}_{\sigma',\tau'} := \{(x,y) : 0 \le x \le \sigma',\, 0 < y < \tau',\, \frac{x}{\sigma'} \neq \frac{y}{\tau'}\} \subset \mathbb{R}^2. \tag{6.2} $$

We define $\widetilde{\mathcal{C}}'_{\sigma',\tau'}$ to be the image of ${\mathcal{C}'}_{\sigma',\tau'}$ under coordinatewise projection $\mathbb{R}^2 \to \mathbf{T}$ to the torus $\mathbf{T} = \mathbb{R}^2/\mathbb{Z}^2$.

We set $\mathcal{C}' = {\mathcal{C}'}_{1,1}$, so that

$$ \mathcal{C}' := {\mathcal{C}'}_{1,1} = \{(x,y) : 0 \le x \le 1,\, 0 < y < 1,\, x \neq y\} \subset \mathbb{R}^2 \tag{6.3} $$

We call $\mathcal{C}'$ the *unit modified corner rectangle*.

The projected modified corner rectangle $\widetilde{\mathcal{C}}'_{\sigma',\tau'}$ consists of two connected components which are triangles, as long as $0 < \sigma', \tau' < 1$. If $\sigma' \ge 1$ or $\tau' > 1$ then some "wraparound" occurs in the projection to the torus $\mathbb{R}^2 \to \mathbf{T} = \mathbb{R}^2/\mathbb{Z}^2$. See Figure 6.2 for examples of the projections $\widetilde{\mathcal{C}}'_{\sigma',\tau'}$.

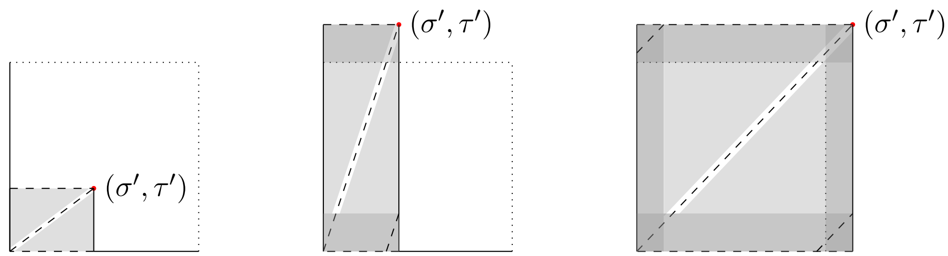

FIGURE 6.2. Modified corner rectangle $\widetilde{\mathcal{C}}'_{\sigma',\tau'}$ in the torus $\mathbf{T} = \mathbb{R}^2/\mathbb{Z}^2$; with various (σ', τ').

Proposition 6.3 (Torus subgroup criterion-negative dilations)**.** *For $\sigma', \tau' > 0$, the following conditions are equivalent.*

(Q1') *The nonnegative commutator relation holds:*

$$ [f_{-\sigma'/\tau'}, f_{-\sigma'}](x) \ge 0 \quad \textit{for all} \quad x \in \mathbb{R}. $$

(Q2') *The cyclic torus subgroup*

$$ \langle(\widetilde{\sigma'}, \widetilde{\tau'})\rangle_{\mathbf{T}} = \{(\widetilde{n\sigma'}, \widetilde{n\tau'}) : n \in \mathbb{Z}\} \subset \mathbb{R}^2/\mathbb{Z}^2 = \mathbf{T} $$

is disjoint from the modified corner rectangle

$$ \widetilde{\mathcal{C}}'_{\sigma',\tau'} = \{(\widetilde{x}, \widetilde{y}) : 0 \le x \le \sigma',\, 0 < y < \tau',\, \frac{x}{\sigma'} \neq \frac{y}{\tau'}\} \subset \mathbf{T}. $$

Proof. The proof follows the same argument as that of [4, Proposition 6.2], replacing (σ, τ) with (σ', τ') and replacing [4, Proposition 5.2] with Proposition 4.4. ☐

6.4. **Proof of Theorem 6.1.** We first recall a technical result needed in the following proofs.

Theorem 6.4 (Closed subgroup theorem)**.** *Given a Lie group G and a subgroup $H \subset G$, the topological closure $\bar{H}$ of the subspace H is a Lie subgroup $\bar{H} \subset G$.*

Proof. See Lee [6, Theorem 20.12, p. 523]. Under these hypotheses, $\bar{H}$ is a closed subgroup of G. ☐

Proof of Theorem 6.1 (a). Suppose that at least one of σ', τ' is irrational. Then the map

$$ \begin{aligned} \widetilde{\phi}_{\sigma',\tau'} &: \mathbb{Z} \to \mathbf{T} \\ k &\mapsto (\widetilde{k\sigma'}, \widetilde{k\tau'}) \end{aligned} $$

is injective and the subgroup $H = \langle(\widetilde{\sigma'}, \widetilde{\tau'})\rangle_{\mathbf{T}}$ of the torus is infinite cyclic. Since $\mathbf{T}$ is compact, H cannot be discrete. Thus the closure $\bar{H}$ of this subgroup must be a Lie subgroup $\bar{H}$ of dimension 1 or 2, by the closed subgroup theorem for Lie groups [6, Theorem 20.12, p. 523].

Case 1. If $\bar{H}$ is dimension 2 then H is dense in $\mathbf{T}$ and will intersect the non-empty region $\widetilde{\mathcal{C}}'_{\sigma',\tau'}$. So condition (Q2') of Proposition 6.3 is not satisfied, hence the nonnegative commutator relation (Q1') does not hold.

Case 2. If $\bar{H}$ is dimension 1, then the parameters σ', τ' must satisfy an integer relation of the form

$$m\sigma' + n\tau' = k, \quad m, n, k \in \mathbb{Z}, \quad m, n, k \text{ coprime}$$

and the subgroup $\bar{H}$ is the projection to $\mathbf{T}$ of the lines $\{(x, y) \in \mathbb{R}^2 : mx + ny \in \mathbb{Z}\}$. In the cases below, we will find solutions only when $k = 0$, giving case (ii^*), or $k = \pm 1$, giving case (i^*).

Case 2a-1. If the integers m, n are nonzero and have opposite sign, then the lines in $\bar{H}$ have positive slope. If in addition $k \neq 0$ then the lines in $\bar{H}$ will have different slope ($= -\frac{m}{n}$) than the punctured diagonal of $\mathcal{C}'_{\sigma',\tau'}$ (slope $= \frac{\tau'}{\sigma'}$), so the intersection $\bar{H} \cap \mathcal{C}'_{\sigma',\tau'}$ will be non-empty in a neighborhood of $(0, 0)$. See Figure 6.3. Thus the commutator relation $[f_{-\sigma'/\tau'}, f_{-\sigma'}] \geq 0$ does not hold by Proposition 6.3.

Case 2a-2. If m, n have opposite sign and $k = 0$, then the assumption $\gcd(m, n, k) = 1$ implies m, n are coprime. Set $p = |m|$, $q = -|n|$ so that p, q coprime positive integers, with $p\sigma' - q\tau' = 0$ and $\bar{H} = \{(x, y) : px - qy \in \mathbb{Z}\}$. $\bar{H}$ will not intersect $\mathcal{C}'_{\sigma',\tau'}$ in a neighborhood of $(0, 0)$, since the segment of $\bar{H}$ at $(0, 0)$ lies precisely in the diagonal of $\mathcal{C}'_{\sigma',\tau'}$. See Figure 6.3.

The "modified corner rectangle" $\mathcal{C}'_{\sigma',\tau'}$ will then be disjoint from the subgroup $\bar{H}$ if and only if the rectangle is contained in the open region $\{(x, y) : |px - qy| < 1\}$. This containment occurs if and only if the extremal corners $(\sigma', 0)$ and $(0, \tau')$ lie in the closed region $\{(x, y) : |px - qy| \leq 1\}$. (These corners are in the closure of $\mathcal{C}'_{\sigma',\tau'}$, but not in $\mathcal{C}'_{\sigma',\tau'}$ itself.) This means $p\sigma' = q\tau' \leq 1$, so we are in case (ii^*) of the classification.

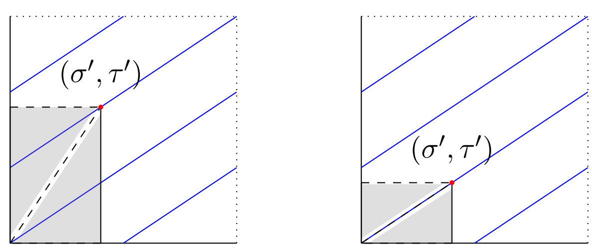

FIGURE 6.3. Solutions to $2x - 3y \in \mathbb{Z}$ compared to $\mathcal{C}'_{\sigma',\tau'}$, when $k \neq 0$ and $k = 0$.

Case 2b. Suppose m, n do not have opposite sign so we may assume that m, n are nonnegative without loss of generality. Then $k \geq 1$ since $\sigma', \tau' > 0$.

Case 2b-1. If $n = 0$, then $\bar{H}$ consists of vertical lines $\{(x, y) : mx \in \mathbb{Z}\}$ which intersect the vertical boundary segments of $\mathcal{C}'_{\sigma',\tau'}$. Thus the commutator relation $[f_{-\sigma'/\tau'}, f_{-\sigma'}] \geq 0$ does not hold by Proposition 6.3.

Case 2b-2. Suppose $n \geq 1$ and $k \geq 2$. Then the point $(\frac{1}{k}\sigma', \frac{1}{k}\tau')$ lies in the closed subgroup $\bar{H}$, and $\bar{H}$ intersects the modified corner rectangle $\mathcal{C}'_{\sigma',\tau'}$, in any open neighborhood of this point. See Figure 6.4. Thus H also intersects $\mathcal{C}'_{\sigma',\tau'}$ and the commutator relation $[f_{-\sigma'/\tau'}, f_{-\sigma'}] \geq 0$ does not hold by Proposition 6.3.

Case 2b-3. Suppose $n \geq 1$ and $k = 1$. Then the closed subgroup $\bar{H}$ does not intersect the modified corner rectangle $\mathcal{C}'_{\sigma',\tau'}$ because $\mathcal{C}'_{\sigma',\tau'}$ lies in the region $\{(x, y) : 0 < mx + ny < 1\}$; see Figure 6.4. This subcase is case (i^*) of the classification.

Theorem 6.1 (a) is established. $\square$

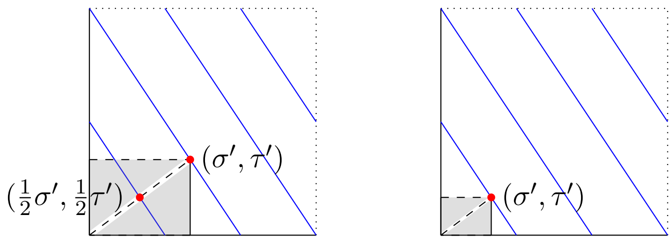

FIGURE 6.4. Solutions to $3x + 2y \in \mathbb{Z}$ in the torus $\mathbf{T}$; with $k = 2$ and $k = 1$

To prove Theorem 6.1 (b), we mainly use (μ', ν')-coordinates rather than (σ', τ')-coordinates.

Proof of Theorem 6.1 (b). If σ', τ' are both rational, then $H = \langle \widetilde{(\sigma', \tau')} \rangle_{\mathbf{T}}$ is a discrete subgroup of the torus. By Lemma 4.4 the commutator inequality $[f_{-\sigma'/\tau'}, f_{-\sigma'}] \geq 0$ means $\Lambda_{\mu',\nu'}$ is disjoint from $\mathcal{D}'$, where $\mu' = {\sigma'}^{-1}$ and $\nu' = {\tau'}^{-1}$ and $\Lambda_{\mu',\nu'} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} \mu' & 0 \\ 0 & \nu' \end{pmatrix}$.

Let $\lambda = \operatorname{egcd}(\mu', \nu')$, so that $(\mu', \nu') = (\lambda p, \lambda q)$ for coprime integers p, q. By Lemma 4.9, there exists an integer $r \geq 1$ such that the lattice

$$\Lambda_{\ell p, \ell q} = \operatorname{row.span}_{\mathbb{Z}} \begin{pmatrix} \ell p & 0 \\ 0 & \ell q \end{pmatrix} \quad \text{with} \quad \ell = 1 + r(\lambda - 1)$$

is disjoint from $\mathcal{D} \cup \mathcal{D}'$. This means $\Lambda_{\ell p, \ell q}$ must be of the form in Lemma 4.10: there exist integers $m \geq 0, n \geq 1$ such that $(\ell p, \ell q)$ lies on the hyperbola

$$\{(x, y) \in \mathbb{R}^2 : \frac{m}{x} + \frac{n}{y} = 1\}$$

so we must have $\ell = \frac{m}{p} + \frac{n}{q}$. The relation $\ell = 1 + r(\lambda - 1)$ is equivalent to $\lambda = 1 + \frac{1}{r}(\ell - 1)$. Thus we have

$$\frac{\mu'}{p} = \frac{\nu'}{q} = \lambda = 1 + \frac{1}{r}\left(\frac{m}{p} + \frac{n}{q} - 1\right).$$

Finally, we recall that $\sigma' = {\mu'}^{-1}$ and $\tau' = {\nu'}^{-1}$ so taking reciprocals of the above equalities gives

$$p\sigma' = q\tau' = \left(1 + \frac{1}{r}\left(\frac{m}{p} + \frac{n}{q}\right) - 1\right)^{-1}.$$

Theorem 6.1 (b) is established. ☐

Theorem 1.1 is now proved, combining Theorem 5.1 and Theorem 6.1.

7. PROOF OF CLOSURE THEOREM 1.2 FOR NEGATIVE DILATIONS

Proof. Let $\mathbb{R}^2_- = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha, \beta < 0\}$ denote the open third quadrant, and let $S_- = S \cap \mathbb{R}^2_- = \{(\alpha, \beta) \in \mathbb{R}^2_- : [f_\alpha, f_\beta] \geq 0\}$. Let $S_{(i)}$, $S_{(ii)}$, and $S_{(iii^*)}$ denote the subsets of S_- corresponding to cases (i), (ii), and (iii*) of Theorem 1.1. It is straightforward to check that $S_{(i)}$ and $S_{(ii)}$ are closed in $\mathbb{R}^2_-$. The set $S_{(iii^*)}$ contains the "sporadic rational solutions"; by itself, $S_{(iii^*)}$ does not form a closed set in $\mathbb{R}^2_-$.

Claim 1. *If a sequence (α_i, β_i) of points in $S_{(iii^*)} \smallsetminus S_{(i)}$ has a limit point in the open third quadrant, then the values α_i are eventually constant.*

Proof of Claim 1: Suppose (α_i, β_i) is a sequence in $S_{(iii^*)} \smallsetminus S_{(i)}$ which converges to a limit (α, β) in the open third quadrant. Since the coordinate β is strictly negative, there is some positive integer N

such that $\beta < -\frac{1}{N}$. This implies $\beta_i < -\frac{1}{N}$ for all sufficiently large i. Suppose $\alpha_i = -\frac{q_i}{p_i}$ as a reduced fraction. Case (iii^*) of Theorem 1.1 says that

$$\beta_i = -\frac{1}{p_i}\left(1 + \frac{1}{r_i}\left(\frac{m_i}{p_i} + \frac{n_i}{q_i} - 1\right)\right)^{-1} \tag{7.1}$$

for some integers $m_i \geq 0$, $n_i \geq 1$, and $r_i \geq 1$; since the point is not in case (i), we have $r_i \geq 2$. From these conditions it follows that $-\frac{2}{p_i} < \beta_i$. Since eventually $\beta_i < -\frac{1}{N}$, these bounds imply that $\frac{1}{p_i} > \frac{1}{2N}$ for sufficiently large i. This means that for sufficiently large i, j, if $\alpha_i \neq \alpha_j$ then $|\alpha_i - \alpha_j| = \left|\frac{q_i}{p_i} - \frac{q_j}{p_j}\right| > \frac{1}{4N^2}$. Thus our assumption that the sequence (α_i, β_i) converges implies that α_i must be eventually constant, which proves Claim 1.

Claim 2. *The set of points in $S_{(iii^*)} \setminus S_{(i)}$ with fixed $\alpha = -\frac{q}{p}$ has all its limit points in $S_{(ii)}$.*

This claim is straightforward to verify using (7.1), with $p_i = p$ and $q_i = q$ fixed.

The two claims above imply that all limit points of $S_{(iii^*)}$ are in $S_{(i)}$ or $S_{(ii)}$, so the union $S_- = S_{(i)} \cup S_{(ii)} \cup S_{(iii^*)}$ is a closed subset of $\mathbb{R}^2_-$. $\square$

Acknowledgements

The second author thanks David Speyer and Yilin Yang for helpful conversations.

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