COMMUTATOR INEQUALITIES FOR DILATED FLOOR FUNCTIONS

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1. Introduction

The *floor function* $\lfloor x \rfloor$ rounds a real number down to the nearest integer. Given a real parameter α, we may define the *dilated* floor function $f_\alpha(x) = \lfloor \alpha x \rfloor$. The behavior of these functions is easy to understand, i.e. f_α is the sum of a linear function and a piecewise-linear periodic function, but the composition of two such functions $f_{\alpha,\beta}(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor$ exhibits more subtle behavior. In general they do not commute under composition. A number of identities relating dilated floor functions with different dilation factors are given in Graham, Knuth and Patashnik [11, Chap. 3]. They raised problems [11, Research problem 50, p.101] concerning compositions of dilated floor functions, some later addressed in Graham and O'Bryant [12].

Recently the authors together with T. Murayama [14] gave necessary and sufficient conditions on (α, β) for these functions to commute under composition, i.e. determining when $\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ holds for all $x \in \mathbb{R}$. Examples of commuting compositions of dilated floor functions were noted in 2010 by Cardinal [4, Lemma 6] in a number-theoretic context.

The commutator

$$[f_\alpha, f_\beta](x) := f_{\alpha,\beta}(x) - f_{\beta,\alpha}(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor$$

is a bounded generalized polynomial in the sense of Bergelson and Leibman [3]. We address the problem of determining the parameters (α, β) for which the one-sided inequality

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor \geq 0 \quad \text{for all} \quad x \in \mathbb{R} \tag{1.1}$$

holds.

1.1. Main Result.

Our main result gives a classification of the parameters (α, β) satisfying the inequality (1.1).

Theorem 1.1. *The parameters $(\alpha, \beta) \in \mathbb{R}^2$ that satisfy the inequality*

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \textit{for all} \quad x \in \mathbb{R}$$

consist of the two coordinate axes $\{(\alpha, 0) : \alpha \in \mathbb{R}\}$ and $\{(0, \beta) : \beta \in \mathbb{R}\}$ together with the following points.

- *(i)* (First Quadrant) *These points with $\alpha > 0$ and $\beta > 0$ that satisfy the inequality fall into three collections of one-parameter continuous families.*
  - *(i-a) For each integer $m_1 \geq 1$ all points with $\alpha > 0$ that lie on the line $\beta = m_1 \alpha$ of slope m_1 through the origin, i.e. $\{(\alpha, m_1 \alpha) : \alpha > 0\}$.*
  - *(i-b) For each integer $m_2 \geq 1$ all points with $\beta > 0$ that lie on the vertical line $\alpha = \frac{1}{m_2}$ i.e. $\{(\frac{1}{m_2}, \beta) : \beta > 0\}$.*

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(i-c) *For each pair of integers $m_1 \geq 1$ and $m_2 \geq 1$, all points with $\beta > 0$ that lie on the rectangular hyperbola*

$$m_1\alpha\beta + m_2\alpha - \beta = 0.$$

(ii) (Second quadrant) *All points with $\alpha < 0$ and $\beta > 0$ satisfy the inequality.*

(iii) (Third quadrant) *All points with $\alpha < 0$ and $\beta < 0$ that satisfy the inequality have $|\alpha| \geq |\beta|$. They fall into three collections of one parameter continuous families, plus additional sporadic rational solutions.*

(iii-a) *For each integer $m_1 \geq 1$ all points with $\alpha < 0$ that lie on the line $\alpha = m_1\beta$ of slope $\frac{1}{m_1}$ through the origin, i.e. $\{(\alpha, \frac{1}{m_1}\alpha) : \alpha < 0\}$.*

(iii-b) *For each positive rational $\frac{m_1}{m_2}$ given in lowest terms, all points $(-\frac{m_1}{m_2}, -\beta)$ that lie on the vertical line segment $0 < \beta \leq \frac{1}{m_2}$.*

(iii-b') *Sporadic rational solutions $(\alpha, \beta) \in \mathbb{Q}^2$. These include an infinite set $(\alpha, \beta) = (-m_1, -\frac{m_1 r}{m_1 r - j})$, parametrized by integers (m_1, j, r) with $m_1 \geq 2$, $1 \leq j \leq m_1 - 1$, and $r \geq 1$. which comprise all sporadic solutions having $\beta < -1$.*

(iii-c) *For each pair of integers $m_1 \geq 1$ and $m_2 \geq 1$, all points having $\alpha < 0$ that lie on the rectangular hyperbola*

$$m_1\alpha\beta + \alpha - m_2\beta = 0.$$

(iv) (Fourth quadrant) *No point with $\alpha > 0$ and $\beta < 0$ satisfies the inequality.*

A *sporadic rational solution* is a rational solution that does not lie on any of the one-parameter families of solutions given above. These are solutions $(\alpha, \beta) \in \mathbb{Q}^2$ of (1.1) which are not limit points of other solutions that have at least one irrational parameter. An interesting part of the proof of Theorem 1.1 is that of ruling out any sporadic rational solutions in the first quadrant case. This question is related to the Diophantine Frobenius problem in two dimensions. There exist infinitely many sporadic rational solutions in the third quadrant, having $|\beta| > 1$. We have not (yet) classified the sporadic rational solutions in the third quadrant having $-1 \leq \beta < 0$ but there are many in this region. In Proposition 9.1 we give a testable criterion for such solutions, showing they have a restricted form.

An interesting feature of the answer is the internal structures of the parametric families of solutions in the first quadrant and the third quadrant. These consist of straight lines and arcs of rectangular hyperbolas. Straight lines occur in the equality case, but a new feature here is the occurrence of families of arcs of rectangular hyperbolas in case *(i-c)* and case *(iii-c)*, respectively. These arcs of hyperbolas all pass through the origin. Some of the hyperbolas in case *(iii-c)* are continuations of the arcs of hyperbolas in case *(i-c)*; these occur exactly when the parameter $m_2 = 1$ in the two cases, as is apparent from their given equations. Figure 1.1 gives a plot of these solutions in the first and third quadrants.

1.2. **Equality case.** We recall the classification of the case of equality, which has a simple direct proof given in [14].

Theorem 1.2. *The complete set of all $(\alpha, \beta) \in \mathbb{R}^2$ such that*

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor$$

holds for all $x \in \mathbb{R}$ consists of:

(i) *Three one-parameter continuous families (α, α), $(\alpha, 0)$, $(0, \alpha)$ for all $\alpha \in \mathbb{R}$.*

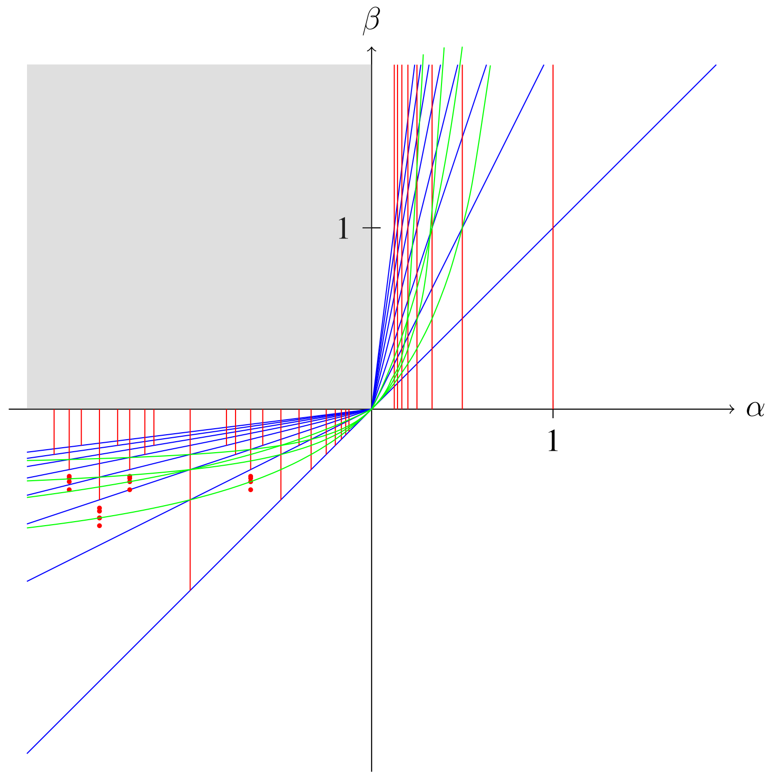

FIGURE 1.1. Solutions to (1.1) first and third quadrants of the (α, β)-plane

(ii) The infinite discrete family

$$\left\{ (\alpha, \beta) = \left(\frac{1}{m}, \frac{1}{n} \right) : m, n \geq 1 \right\},$$

where m, n are positive integers. (The continuous and discrete families overlap when $m = n$.)

To see that Theorem 1.2 follows from Theorem 1.1 we intersect the solution set of Theorem 1.1 with a copy of it obtained by reflecting around the line $\alpha = \beta$, i.e. exchanging α and β. The coordinate axes $\alpha = 0$ and $\beta = 0$ and the line $\alpha = \beta$ are clearly invariant under reflection, so it remains to study this map on the four open quadrants. This map interchanges the second and fourth quadrant and takes the first and third quadrants into themselves, reflected around the line $\alpha = \beta$.

(1) The intersection of the solutions in Theorem 1.1 in the open second quadrant and open fourth quadrant is clearly the empty set.
(2) The intersection of solutions in Theorem 1.1 in the third quadrant under the reflection includes the line $\alpha = \beta$. This intersection is exactly this line, because Theorem 1.1 states there are no solutions having $|\beta| > |\alpha|$.
(3) Finally the intersections in the first quadrant certainly includes the invariant line $\alpha = \beta$, which is $m_1 = 1$ in case *(i-a)*. We next note that there are no solutions in this quadrant having $\alpha > \beta$ aside from those coming from the case *(i-b)* vertical line solutions $\alpha = \frac{1}{m_1}$ for $m_1 \geq 1$. Since the solution set must be invariant under exchange of

variables, the second coordinate must be of the form $\beta = \frac{1}{m_2}$ for some integer $m_2 \geq 1$ as well. But all such points $(\frac{1}{m_1}, \frac{1}{m_2})$ lie in the solution set and its reflection (which have horizontal lines $\beta = \frac{1}{m_2}$.)

A proof of Theorem 1.2 via Theorem 1.1 is roundabout, in that the latter proof requires the exclusion of sporadic rational solutions in the first quadrant, which is complicated in the proof given here.

1.3. **Coordinate Changes and Symmetries.** We use two birational coordinate changes that simplify the analysis and also reveal symmetries of the solution set.

1.3.1. *Birational Coordinate Changes to (u,v)-coordinates and (u',v')-coordinates.* For the first quadrant case of (α,β) in Theorem 1.1, the set of solutions to (1.1) is simpler to analyze after making the coordinate change from (α,β) to new coordinate variables (u,v) via

$$u := \frac{1}{\alpha}, \quad v := \frac{\beta}{\alpha}.$$

This is a birational transformation that maps the open first quadrant onto itself, having inverse map $\alpha = \frac{1}{u}, \quad \beta = \frac{v}{u}$. Viewed in the first quadrant of the (u,v)-plane the classification of Theorem 1.1 becomes:

- *(i-a)* For each integer $m_1 \geq 1$ all points (u,v) that lie on the vertical line $u = m_1$, i.e. $\{(m_1, v) :\, v > 0\}$.
- *(i-b)* For each integer $m_2 \geq 1$ all points (u,v) that lie on the horizontal line $v = m_2$, i.e. $\{(u, m_2) :\, u > 0\}$.
- *(i-c)* For each pair of integers $m_1 \geq 1$ and $m_2 \geq 1$, all points (u,v) with $u, v > 0$ that lie on the rectangular hyperbola
$$\frac{m_1}{u} + \frac{m_2}{v} = 1.$$

In the (u,v)-coordinate system there is a visible connection of the hyperbola solutions in the case *(i-c)* to (generalizations of) Beatty sequences. The (u,v)-coordinate system also reveals a hidden symmetry of first quadrant solutions: it is symmetrical around the line $u = v$.

The third quadrant case of (α,β) can be analyzed similarly using a related birational coordinate change, taking (α,β) to (u',v') with

$$u' := -\frac{1}{\beta}, \quad v' := \frac{\alpha}{\beta}.$$

This map sends the (open) third quadrant onto the (open) first quadrant, and has inverse map $\alpha = -\frac{v'}{u'}, \quad \beta = -\frac{1}{u'}$. The classification of Theorem 1.1 viewed in the first quadrant of the (u',v')-plane becomes:

- *(iii-a)* For each integer $m_1 \geq 1$ all points (u',v') that lie on the horizontal line $v' = m_1$, i.e. $\{(u', m_1) :\, u' > 0\}$.
- *(iii-b)* For all integers $m_2, m_3 \geq 1$ all points (u',v') that lie on the diagonal line $v' = \frac{m_2}{m_3}u'$ with $u' \geq m_3$, i.e. $\{(u', \frac{m_2}{m_3}u') :\, u' \geq m_3\}$.
- *(iii-b')* For each integer $m_2 \geq 2$ the sporadic solutions in the parametric family $(u',v') := (1 - \frac{j}{m_2 r}, m_1 - \frac{j}{r})$ with parameters $1 \leq j \leq m_2 - 1$ and $r \geq 2$. These solutions have $0 < u' < 1$ and $v'/u' = m_2$.

(iii-c) For each pair of integers $m_1 \geq 1$ and $m_2 \geq 1$, all points (u', v') with $u', v' > 0$ that lie on the rectangular hyperbola

$$\frac{m_1}{u'} + \frac{m_2}{v'} = 1.$$

The solutions to (1.1) in these coordinates are pictured in Figures 1.2 and 1.3. These various coordinates may be explained as follows: if we consider our initial coordinates (α, β) as defining an affine open chart $i_0 : \mathbb{A}^2 \to \mathbb{P}^2$ sending $(\alpha, \beta) \mapsto [\pm 1 : \alpha : \beta] = [x_0 : x_1 : x_2]$ whose image is $\{x_0 \neq 0\}$, then (u, v) are coordinates for a second standard affine open $\{x_1 \neq 0\}$, while (u', v') are coordinates for a third affine open $\{x_2 \neq 0\}$.

The details of the analysis in the third quadrant case parallel that in the first quadrant case, but become more complicated. In the end we will also relate case *(iii-c)* to generalizations of Beatty sequences.

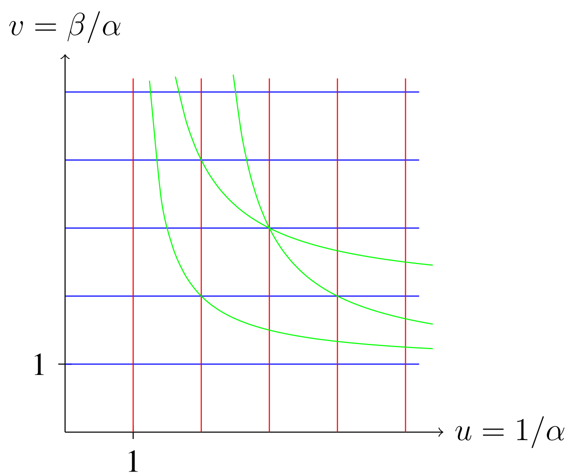

FIGURE 1.2. First quadrant solutions to (1.1) viewed in (u, v)-coordinates, with $u = 1/\alpha$ and $v = \frac{\beta}{\alpha}$

1.3.2. *Birational Coordinate Changes to (X, Y)-coordinates and (X', Y') coordinates.* There is a second interesting birational coordinate change of the first quadrant, sends (α, β) to (X, Y) with

$$X := \frac{1}{u} = \alpha, \quad Y := \frac{1}{v} = \frac{\alpha}{\beta}.$$

It is pictured in Figure 1.4. The point of this coordinate change is that under it the rectangular hyperbolas in case *(i-c)* become $m_1 X + m_2 Y = 1$. In this coordinate system all continuous families becomes linear, consisting of families of lines and line segments.

There is an analogous coordinate change in the third quadrant, sending (α, β) to coordinates $(X^{'}, Y^{'})$, given by

$$X^{'} := \frac{1}{u'} = -\beta, \quad Y^{'} := \frac{1}{v'} = \frac{\beta}{\alpha}.$$

which maps the third quadrant into the first quadrant. In this coordinate system the rectangular hyperbolas in case *(iii-c)* become $m_1 X^{'} + m_2 Y^{'} = 1$. This change of variable linearizes all continuous families of third quadrant solutions, as pictured in Figure 1.5. There is a reflection

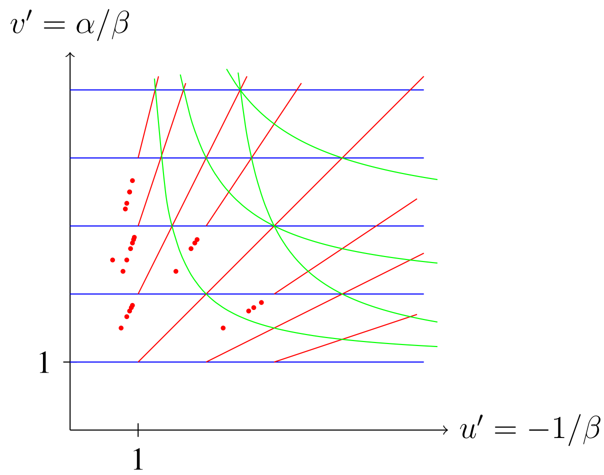

FIGURE 1.3. Third quadrant solutions to (1.1) viewed in (u', v')-coordinates, with $u' = -1/\beta$ and $v' = \frac{\alpha}{\beta}$.

symmetry around the line $X^{'} = Y^{'}$ valid for the hyperbola solutions. [CHECK THIS, FOR WHICH OTHER SOLNS DOES IT HOLD?]

The linearizations of the solution set obtainable using (X, Y)-coordinates (resp. $(X^{'}, Y^{'})$-coordinates) play an essential role in the proofs in the first and third quadrant cases. However most of the proofs in the first and third quadrants take place using the (u, v)-coordinate system which displays the hyperbolas and the connection to Beatty sequences.

1.3.3. *Broken Reflection Symmetry of Solution Set.* Another interesting feature of the solution set is that a great extent the solution set is reflection-symmetric about the line $x = -y$ of slope -1 through the origin. This map is the reflection map sending $(\alpha, \beta) \mapsto (-\beta, -\alpha)$. It preserves the second and fourth quadrant and exchanges the first and third quadrants.

(a) This symmetry is exact when restricted to the coordinate axes.
(b) The symmetry is exact for all points in the second and fourth quadrants.
(c) The reflection symmetry matches all half-lines of families *(i-a)* in the first quadrant to corresponding half-lines of the families *(iii-a)* in the third quadrant, and vice-versa.
(d) The reflection symmetry matches all hyperbolas in family *(i-c)* in the first quadrant to corresponding hyperbolas in family *(iii-c)* in the third quadrant, and vice versa.

On the other hand the reflection symmetry is broken for the straight lines in the continuous families in case *(i-b)* versus the line segments in case *(iii-b)*. The symmetry is also broken for the sporadic rational solutions in the third quadrant, which have no counterpart in the first quadrant.

1.4. **Roadmap.** Theorem 1.1 will be proved by following a path of conditions all equivalent to the one given at the beginning.

The main steps in the proof: the following conditions on (α, β) are all equivalent

(1) original inequality $\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ for all $x \in \mathbb{R}$
(2) upper level sets $S_{\alpha,\beta} \supseteq S_{\beta,\alpha}(n)$ for all $n \in \mathbb{Z}$ (Section 3)
(3) rounding function inequality $r_\alpha(n) \leq r_\beta(n)$ for all $n \in \mathbb{Z}$ (Section 4)
(4) coordinate change $r_1(x) \leq r_v(x)$ for all $x \in u\mathbb{Z}$. (Sections 5, 6)

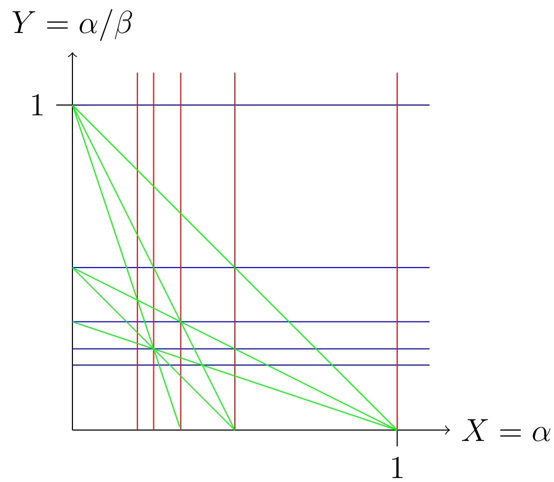

FIGURE 1.4. First quadrant solutions to (1.1) viewed in (X, Y)-coordinates, with $X = \alpha$ and $Y = \frac{\alpha}{\beta}$

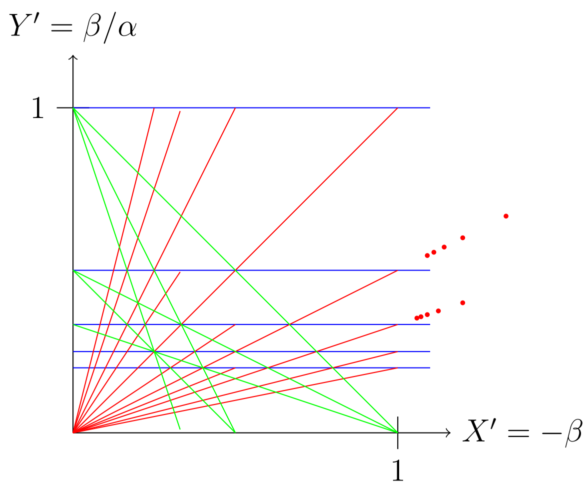

FIGURE 1.5. Third quadrant solutions to (1.1) viewed in (X', Y')-coordinates, with $X' = -\beta$ and $Y' = \frac{\beta}{\alpha}$.

(5) disjoint residual sets $u\mathbb{Z} \cap R_v^{\pm} = \emptyset \quad \Leftrightarrow \quad R_u^{\pm} \cap R_v^{\pm} = \emptyset$ (Sections 5, 6)
(6) reduced Beatty sequences $\mathcal{B}_0(u) \cap \mathcal{B}_0(v) = \emptyset$ (Section 7)
(7) torus subgroup $\langle (X, Y) \rangle \subset \mathbb{R}^2/\mathbb{Z}^2$ has "small" generator (Section 7)

2. PRELIMINARIES

In this section we give some basic results used in the proof of the main result.

2.1. **Rounding functions.** The floor function and ceiling function lie naturally in the one-parameter family of *rounding functions*

$$r_\alpha(x) := \alpha \left\lceil \frac{1}{\alpha} x \right\rceil, \tag{2.1}$$

with parameter $\alpha \in \mathbb{R} - \{0\}$. Just as the floor function $\lfloor x \rfloor = r_{-1}(x)$ rounds x down to the nearest integer $\mathbb{Z} \subset \mathbb{R}$, $r_\alpha(x)$ rounds x to the nearest element of the dilated lattice $\alpha\mathbb{Z} \subset \mathbb{R}$ in a certain direction—if $\alpha < 0$ then r_α rounds down, while if $\alpha > 0$ then it rounds up. Here $r_1(x) = \lceil x \rceil$ is the ceiling function and $r_{-1}(x) = \lfloor x \rfloor$ is the floor function. (More generally

$$r_{-\alpha}(x) = -\alpha\lceil -\frac{1}{\alpha}x\rceil = \alpha\lfloor \frac{1}{\alpha}x\rfloor.)$$

From the definition of r_α it is clear that

$$x \leq r_\alpha(x) < x + \alpha \quad \text{for all } x \in \mathbb{R}$$

if α is positive, and if α is negative

$$x + \alpha < r_\alpha(x) \leq x \quad \text{for all } x \in \mathbb{R}.$$

By taking limits as $\alpha \to 0$ and $\alpha \to \pm\infty$, it is natural to extend the family of rounding functions to include $r_0(x) := x$ and

$$r_{-\infty}(x) := \begin{cases} -\infty & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad r_{\infty}(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ \infty & \text{if } x > 0. \end{cases}$$

Rounding functions satisfy the following inequalities.

Lemma 2.1. (Rounding Function Inequalities)

(i) If both $\alpha, \beta > 0$, then

$$r_\alpha(x) \geq r_\beta(x) \quad \textit{for all } x \in \mathbb{R}$$

if and only if $\frac{\alpha}{\beta}$ is a (positive) integer.

(ii) If both $\alpha, \beta < 0$, then

$$r_\alpha(x) \geq r_\beta(x) \quad \textit{for all } x \in \mathbb{R}$$

if and only if $\frac{\beta}{\alpha}$ is a (positive) integer.

(iii) If $\alpha\beta < 0$, then

$$r_\alpha(x) \geq r_\beta(x) \quad \textit{for all } x \in \mathbb{R}$$

if and only if $\alpha > 0$ and $\beta < 0$.

Proof. (i) Let both $\alpha, \beta > 0$. We claim that $r_\alpha(x) \geq r_\beta(x)$ holds for all $x \in \mathbb{R}$ if and only if $r_{\frac{\alpha}{\beta}}(x) \geq r_1(x)$ for all $x \in \mathbb{R}$. To see this, given $\alpha\lceil \frac{1}{\alpha}(x)\rceil \geq \beta\lceil \frac{1}{\beta}(x)\rceil$ let $x = \beta y$. and we obtain

$$\alpha\left\lceil \frac{\beta}{\alpha}y \right\rceil \geq \beta\lceil y \rceil \quad \text{for all } y \in \mathbb{R}.$$

Dividing both sides by $\beta > 0$ preserves the inequality and verifies the claim. Now we may assume that $\beta = 1$. If $\alpha = n$ is a positive integer, then $n\mathbb{Z} \subset \mathbb{Z}$ so rounding up to the nearest element $r_n(x)$ in $n\mathbb{Z}$ will go at least as far as $r_1(x) = \lceil x \rceil \in \mathbb{Z}$. Thus $r_n(x) \geq r_1(x)$.

If $\alpha > 0$ is not an integer, then the desired inequality fails at $x = \alpha$:

$$r_\alpha(\alpha) = \alpha < \lceil \alpha \rceil = r_1(\alpha).$$

(ii) Let $\alpha, \beta < 0$. We claim $r_\alpha(x) \geq r_\beta(x)$ for all x if and only if $r_{\frac{\beta}{\alpha}}(x) \geq r_1(x)$ for all x. Given $\alpha\lceil \frac{1}{\alpha}(x)\rceil \geq \beta\lceil \frac{1}{\beta}(x)\rceil$, let $x = \alpha y$ and we obtain

$$\beta\left\lceil \frac{\alpha}{\beta}y \right\rceil \leq \alpha\lceil y \rceil \quad \text{for all } y \in \mathbb{R}.$$

Dividing by $\alpha < 0$ reverses the inequality and gives the claim. Since β/α is positive, the result follows from part (i).

(iii) When $\alpha > 0$ and $\beta < 0$ we have $r_\alpha(x) \geq x \geq r_\beta(x)$ for all $x \in \mathbb{R}$. Otherwise if $\alpha < 0$ and $\beta > 0$, $r_\beta(x) \geq x \geq r_\alpha(x)$ and there are some x for which the inequalities are strict. □

We shall also encounter a second one-parameter family of *strict* rounding functions

$$\vec{r}_\alpha(x) := \alpha\lfloor \frac{1}{\alpha}x \rfloor + \alpha. \tag{2.2}$$

which are based on the *next integer function*

$$\vec{r}_1(x) := \lfloor x \rfloor + 1.$$

The behavior of the next integer function agrees with the ceiling function except on integers, where it picks out the least integer *strictly* greater than the argument x. From the definition it is clear that for all $x \in \mathbb{R}$,

$$x < \vec{r}_\alpha(x) \leq x + \alpha \quad \text{if } \alpha > 0, \text{ and}$$

$$x + \alpha \leq \vec{r}_\alpha(x) < x \quad \text{if } \alpha < 0.$$

This one-parameter family of rounding functions has similar properties to the one above.

Lemma 2.2. (Strict Rounding Function Inequalities)

(i) If both $\alpha, \beta > 0$, then

$$\vec{r}_\alpha(x) \geq \vec{r}_\beta(x) \quad \textit{for all } x \in \mathbb{R}$$

if and only if $\frac{\alpha}{\beta}$ is a (positive) integer.

(ii) If both $\alpha, \beta < 0$, then

$$\vec{r}_\alpha(x) \geq \vec{r}_\beta(x) \quad \textit{for all } x \in \mathbb{R}$$

if and only if $\frac{\beta}{\alpha}$ is a (positive) integer.

(iii) If $\alpha > 0$ and $\beta < 0$, then

$$\vec{r}_\alpha(x) > \vec{r}_\beta(x) \quad \textit{for all } x \in \mathbb{R}.$$

We omit proofs, since they are similar to those in Lemma 2.1.

2.2. **"Baby" case.** We observe in this section that a sufficient condition for (1.1) to hold is that

$$\alpha\lfloor \beta x \rfloor - \beta\lfloor \alpha x \rfloor \geq 0 \quad \text{for all } x \in \mathbb{R}. \tag{2.3}$$

Namely, we can apply the floor function to both sides of $\alpha\lfloor \beta x \rfloor \geq \beta\lfloor \alpha x \rfloor$, and since $x \mapsto \lfloor x \rfloor$ is (weakly) order-preserving we deduce (1.1). The classification of solutions (α, β) to this simplified condition is answered completely using results the previous sections on rounding functions. The solutions found are those in cases (i-a), (ii), (iii-a), and (iv) of Theorem 1.1.

Proposition 2.3. *The parameters $(\alpha, \beta) \in \mathbb{R}^2$ that satisfy the inequality*

$$\alpha\lfloor \beta x \rfloor \geq \beta\lfloor \alpha x \rfloor \quad \textit{for all } x \in \mathbb{R}$$

consist of the two coordinate axes $\{(\alpha, 0) : \alpha \in \mathbb{R}\}$ and $\{(0, \beta) : \beta \in \mathbb{R}\}$ together with the following points.

(i) (First Quadrant) *For each integer $m_1 \geq 1$ all points with $\alpha > 0$ that lie on the line $\beta = m_1\alpha$ of slope m_1 through the origin, i.e. $\{(\alpha, m_1\alpha) : \alpha > 0\}$.*

(ii) (Second quadrant) *All points with $\alpha < 0$ and $\beta > 0$ satisfy the inequality.*

(iii) (Third quadrant) *For each integer $m_1 \geq 1$ all points with $\alpha < 0$ that lie on the line $\alpha = m_1\beta$ of slope $\frac{1}{m_1}$ through the origin, i.e. $\{(\alpha, \frac{1}{m_1}\alpha) : \alpha < 0\}$.*

(iv) (Fourth quadrant) *No point with $\alpha > 0$ and $\beta < 0$ satisfies the inequality.*

Proof. If $\alpha = 0$ or $\beta = 0$, the inequality (2.3) holds since both sides are identically zero.

If $\alpha\beta > 0$, then (2.3) is equivalent after dividing by $\alpha\beta$ to

$$r_{1/\beta}(x) \geq r_{1/\alpha}(x) \quad \text{for all } x \in \mathbb{R}.$$

If Lemma 2.1 (i) and (ii) gives necessary and sufficient conditions for this, proving the first and third quadrant cases.

If $\alpha\beta < 0$, then (2.3) is equivalent to

$$r_{1/\beta}(x) \leq r_{1/\alpha}(x) \quad \text{for all } x \in \mathbb{R}$$

and Lemma 2.1 (iii) gives necessary and sufficient conditions for this, proving the second and fourth quadrant cases. □

The solutions to (2.3) are pictured in Figure 2.1.

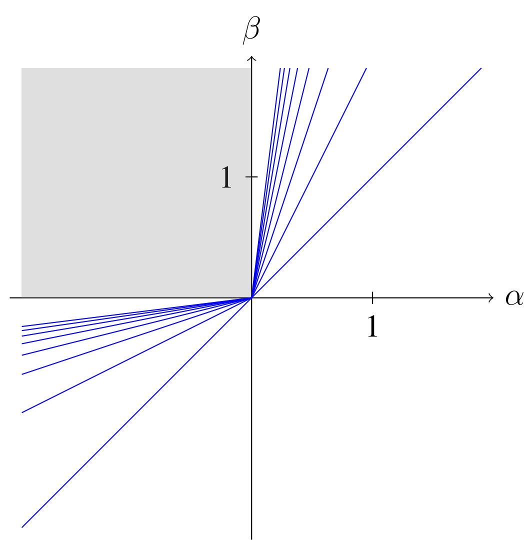

FIGURE 2.1. Solutions to (2.3) in the (α, β)-plane

2.3. **Self-similar structure of the solution set.** Here we make some general remarks about the set of all (α, β) that satisfy Theorem 1.1. Rather than being an "arbitrary" set of points, the solution locus has a self-similar structure in the following sense.

Proposition 2.4. *Let S denote the set of solutions.*

(1) If $(\alpha, \beta) \in S$ and $\alpha, \beta > 0$, then for all positive integers m_1, m_2

$$(\frac{1}{m_1}\alpha, \beta) \in S \quad \text{and} \quad (\alpha, m_2\beta) \in S.$$

(2) If $(\alpha, \beta) \in S$ and $\alpha, \beta < 0$, then for all positive integers m_1, m_2

$$(m_1\alpha,\ \beta) \in S \quad \text{and} \quad (\alpha, \frac{1}{m_2}\beta) \in S.$$

(3) *Moreover if* $(\alpha, \beta) \in S$ *then for any positive integer* m

$$(\frac{1}{m}\alpha, \frac{1}{m}\beta) \in S.$$

Proof. By assumption, $\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ for all $x \in \mathbb{R}$.

(1) If m is a positive integer, then $\lfloor mx \rfloor \geq m \lfloor x \rfloor$ and $\frac{1}{m}\lfloor x \rfloor \geq \lfloor \frac{1}{m} x \rfloor$ by Rounding Lemma 2.1. Hence we deduce that

$$\lfloor \alpha \lfloor (m_2\beta) x \rfloor \rfloor = \lfloor \alpha \lfloor \beta (m_2 x) \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha (m_2 x) \rfloor \rfloor = \lfloor \beta \lfloor m_2 (\alpha x) \rfloor \rfloor \geq \lfloor m_2 \beta \lfloor (\alpha x) \rfloor \rfloor,$$

where the last inequality uses that $x \mapsto \lfloor \beta x \rfloor$ is order-preserving. This shows $(\alpha, m_2\beta) \in S$. Since $x \mapsto \lfloor \alpha x \rfloor$ is also order-preserving, we also have

$$\lfloor \frac{1}{m_1} \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \alpha \lfloor \frac{1}{m_1} (\beta x) \rfloor \rfloor = \lfloor \alpha \lfloor \beta (\frac{1}{m_1} x) \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha (\frac{1}{m_1} x) \rfloor \rfloor = \lfloor \beta \lfloor (\frac{1}{m_1} \alpha) x) \rfloor \rfloor.$$

which shows $(\frac{1}{m_1}\alpha, \beta) \in S$.

(2) When $\alpha, \beta < 0$, multiplication by these negative constants reverses the above inequalities, i.e. $\alpha \lfloor mx \rfloor \leq \alpha m \lfloor x \rfloor$ and $\beta \frac{1}{m} \lfloor x \rfloor \leq \beta \lfloor \frac{1}{m} x \rfloor$. Thus

$$\lfloor m_1 \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \alpha \lfloor m_1 (\beta x) \rfloor \rfloor = \lfloor \alpha \lfloor \beta (m_1 x) \rfloor \rfloor \geq \lfloor \beta \lfloor (m_1 \alpha) x \rfloor \rfloor,$$

which shows $(m_1\alpha, \beta) \in S$. Next

$$\lfloor \alpha \lfloor (\frac{1}{m_2} \beta) x \rfloor \rfloor = \lfloor \alpha \lfloor \beta (\frac{1}{m_2} x) \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha (\frac{1}{m_2} x) \rfloor \rfloor = \lfloor \beta \lfloor \frac{1}{m_2} (\alpha x) \rfloor \rfloor \geq \lfloor \frac{1}{m_2} \beta \lfloor \alpha x \rfloor \rfloor$$

which shows $(\alpha, \frac{1}{m_2}\beta) \in S$.

(3) As noted above, we have $\frac{1}{m}\lfloor x \rfloor \geq \lfloor \frac{1}{m} x \rfloor$ for any positive integer m, but the difference of the two sides is at most $\frac{m-1}{m}$. So applying the floor function once more turns this into an equality: $\lfloor \frac{1}{m} \lfloor x \rfloor \rfloor = \lfloor \lfloor \frac{1}{m} x \rfloor \rfloor = \lfloor \frac{1}{m} x \rfloor$ for all real x. Thus

$$\begin{aligned}
\lfloor \frac{1}{m} \alpha \lfloor \frac{1}{m} \beta x \rfloor \rfloor &= \lfloor \frac{1}{m} \lfloor \alpha \lfloor \beta (\frac{1}{m} x) \rfloor \rfloor \rfloor \\
&\geq \lfloor \frac{1}{m} \lfloor \beta \lfloor \alpha (\frac{1}{m} x) \rfloor \rfloor \rfloor \qquad (y \mapsto \lfloor \frac{1}{m} y \rfloor \text{ is order-preserving}) \\
&= \lfloor \frac{1}{m} \beta \lfloor \frac{1}{m} \alpha x \rfloor \rfloor.
\end{aligned}$$

□

[It is worth remarking that

- part (3) does *not* distinguish whether parameters α, β are positive or negative
- in contrast with parts (1) and (2), the argument makes vital use of *both* sets of floor functions in the inequality

In other words, parts (1) and (2) describes just as well the symmetries of the set of (α, β) such that $\alpha \lfloor \beta x \rfloor \geq \beta \lfloor \alpha x \rfloor$ for all x, discussed in section 2.2. The symmetries identified here, in (α, β) coordinates, are different in quadrants I and III. Part (3), in contrast, identifies a symmetry that only holds when both sets of floor functions are present.]

Given these symmetries of S, to show one direction of Theorem 1.1 in the first quadrant, i.e. that the claimed points are in fact solutions, it reduces to showing that

(i-a0) $(\alpha, \alpha) \in S$
(i-b0) $(1, \beta) \in S$
(i-c0) $(\alpha, \beta) \in S$ if $\alpha\beta + \alpha - \beta = 0$.

To show the other direction, i.e. that this exhibits all solutions to (1.1), requires further argument.

3. Upper level sets

For any α, β, let $S_{\alpha,\beta}(y)$ denote the *upper level set* of $f_{\alpha,\beta}$ at range value $y \in \mathbb{R}$, meaning

$$S_{\alpha,\beta}(y) := \{x : \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq y\} = {f_{\alpha,\beta}}^{-1}([y, \infty)).$$

Since $\lfloor x \rfloor$ takes values in $\mathbb{Z}$, it suffices to consider these values $y = n \in \mathbb{Z}$. If $\alpha = 0$ or $\beta = 0$, the composed floor functions $f_{\alpha,\beta}$, $f_{\beta,\alpha}$ are both identically zero so the sets $S_{\alpha,\beta}(n)$ are uninteresting. For the rest of this section, we assume $\alpha\beta \neq 0$.

Lemma 3.1. *The inequality (1.1) is equivalent to*

$$S_{\alpha,\beta}(n) \supseteq S_{\beta,\alpha}(n) \quad \text{for all } n \in \mathbb{Z}. \tag{3.4}$$

Proof. We have

$$\begin{aligned} \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \forall x \in \mathbb{R} &\Leftrightarrow \lfloor \beta \lfloor \alpha x \rfloor \rfloor \geq n \Rightarrow \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq n \\ &\qquad \forall x \in \mathbb{R}, \forall n \in \mathbb{Z} \\ &\Leftrightarrow \{x : \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq n\} \supseteq \{x : \lfloor \beta \lfloor \alpha x \rfloor \rfloor \geq n\} \\ &\qquad \forall n \in \mathbb{Z} \\ &\Leftrightarrow S_{\alpha,\beta}(n) \supseteq S_{\beta,\alpha}(n) \quad \forall n \in \mathbb{Z}, \end{aligned}$$

as asserted. ☐

The functions $f_{\alpha,\beta}$ are monotonic, so the sets $S_{\alpha,\beta}(n)$ always take a particularly simple form: they are half-infinite intervals of one of the forms

$$(-\infty, x_{max}) \quad \text{or} \quad (-\infty, x_{max}] \quad \text{or} \quad (x_{min}, \infty) \quad \text{or} \quad [x_{min}, \infty).$$

This simple form makes the set inclusion (3.4) easy to check. The finite endpoints x_{max} or x_{min} are the points in the domain where the value of $f_{\alpha,\beta}$ "jumps" between two integers.

We will use precise formulas for the upper level sets at integer values, following [14], given in the next three lemmas.

Lemma 3.2. *For $\alpha > 0$ and $\beta > 0$ we have for $n \in \mathbb{Z}$ that the upper level set is the half-open interval*

$$S_{\alpha,\beta}(n) = \left[\frac{1}{\beta}\lceil \frac{1}{\alpha} n \rceil, \infty\right).$$

Proof. This result is given as [14, Lemma 1]. We use here $S_{\alpha,\beta}$ rather than $S_{1/\alpha,1/\beta}$ treated in [14], and give a proof in the new variables for the reader's convenience. We have

$$\begin{aligned}
x \in S_{\alpha,\beta}(n) \quad &\Leftrightarrow \quad \lfloor \alpha \lfloor \beta(x) \rfloor \rfloor \geq n \quad \text{(definition)} \\
&\Leftrightarrow \quad \alpha \lfloor \beta x \rfloor \geq n \quad \text{(the right side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow \quad \lfloor \beta x \rfloor \geq \frac{1}{\alpha} n \quad \text{(since } \alpha > 0\text{)} \\
&\Leftrightarrow \quad \lfloor \beta x \rfloor \geq \lceil \frac{1}{\alpha} n \rceil \quad \text{(the left side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow \quad \beta x \geq \lceil \frac{1}{\alpha} n \rceil \quad \text{(the right side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow \quad x \geq \frac{1}{\beta} \lceil \frac{1}{\alpha} n \rceil \quad \text{(since } \beta > 0\text{).}
\end{aligned}$$

□

Lemma 3.3. *For $\alpha < 0$ and $\beta < 0$ we have we have for $n \in \mathbb{Z}$ that the upper level set is the open interval*

$$S_{\alpha,\beta}(n) = \Big(\frac{1}{\beta} \lfloor \frac{1}{\alpha} n \rfloor + \frac{1}{\beta}, \infty \Big).$$

Proof. These results are obtained in [14, Lemma 4]. For $n \in \mathbb{Z}$ we have for $\alpha, \beta < 0$,

$$\begin{aligned}
x \in S_{\alpha,\beta}(n) &\Leftrightarrow \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq n && \text{(definition)} \\
&\Leftrightarrow \alpha \lfloor \beta x \rfloor \geq n && \text{(the right side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow \lfloor \beta x \rfloor \leq \frac{n}{\alpha} && \text{(since } \alpha < 0\text{)} \\
&\Leftrightarrow \lfloor \beta x \rfloor \leq \lfloor \frac{n}{\alpha} \rfloor && \text{(the left side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow \beta x < \lfloor \frac{n}{\alpha} \rfloor + 1 && \text{(the right side is in } \mathbb{Z}\text{)} \\
&\Leftrightarrow x > \frac{1}{\beta} \lfloor \frac{n}{\alpha} \rfloor + \frac{1}{\beta} && \text{(since } \beta < 0\text{),}
\end{aligned}$$

as asserted. □

Lemma 3.4. *For $\alpha < 0$ and $\beta > 0$ we have for $n \in \mathbb{Z}$ that the upper level set is an open interval involving the floor function, given by*

$$S_{\alpha,\beta}(n) = \Big(-\infty, \frac{1}{\beta} \lfloor \frac{1}{\alpha} n \rfloor + \frac{1}{\beta} \Big).$$

For the reversed variables it is a half-open interval using the ceiling function, given by

$$S_{\beta,\alpha}(n) = \Big(-\infty, \frac{1}{\alpha} \lceil \frac{1}{\beta} n \rceil \Big].$$

Proof. These results are obtained in [14, Lemma 7]. □

4. Proof of Main Result

In this section we prove Theorem 1.1, modulo two propositions whose proofs are deferred to later sections. These two propositions are the heart of the proof and in particular they locate the set of solutions forming hyperbolas, which occur in subcases (i-c) and (iii-c). Here we will reinterpret the boundary points of the upper level sets calculated in Section 3 in terms of rounding functions. This reinterpretation proves certain cases of Theorem 1.1 and motivates a change of coordinates to be used in finishing the proof.

4.1. **Second and Fourth Quadrant Cases:** α, β **have opposite signs.** First, suppose $\alpha < 0$ and $\beta > 0$. The upper level sets $S_{\alpha,\beta}$ are calculated in Lemma 3.4:

$$S_{\alpha,\beta}(n) = \Big(-\infty,\, \frac{1}{\beta}\lfloor n\frac{1}{\alpha}\rfloor + \frac{1}{\beta}\Big) \quad \text{and} \quad S_{\beta,\alpha}(n) = \Big(-\infty,\, \frac{1}{\alpha}\lceil n\frac{1}{\beta}\rceil\Big].$$

We claim that the strict inclusion $S_{\alpha,\beta}(n) \supsetneq S_{\beta,\alpha}(n)$ holds for all n. In terms of endpoints this claim asserts

$$\frac{1}{\alpha}\lceil\frac{1}{\beta}n\rceil < \frac{1}{\beta}\lfloor\frac{1}{\alpha}n\rfloor + \frac{1}{\beta}.$$

After multiplying by $\alpha\beta < 0$, this is equivalent to

$$r_\beta(n) = \beta\lceil\frac{1}{\beta}n\rceil > \alpha\lceil\frac{1}{\alpha}n\rceil = r_\alpha(n)$$

for all $n \in \mathbb{Z}$. This follows from our assumption that $\beta > 0$ and $\alpha < 0$, since

$$r_\beta(n) \geq n > \vec{r}_\alpha(n).$$

The claim follows, which proves Theorem 1.1 (ii).

Second, suppose that the signs are switched so that $\alpha > 0$ and $\beta < 0$. The same reasoning shows the reversed strict inclusion $S_{\alpha,\beta}(n) \subsetneq S_{\beta,\alpha}(n)$ for all n. This proves Theorem 1.1 (iv).

4.2. **First Quadrant Case: Both** α, β **positive.** The upper level sets $S_{\alpha,\beta}$ are calculated in Lemma 3.2:

$$S_{\alpha,\beta}(n) = \Big[\frac{1}{\beta}\lceil\frac{1}{\alpha}n\rceil,\, \infty\Big) \quad \text{and} \quad S_{\beta,\alpha}(n) = \Big[\frac{1}{\alpha}\lceil\frac{1}{\beta}n\rceil,\, \infty\Big).$$

In view of Lemma 3.1, the inequality (1.1) is equivalent to

$$\frac{1}{\beta}\lceil\frac{1}{\alpha}n\rceil \leq \frac{1}{\alpha}\lceil\frac{1}{\beta}n\rceil \quad \text{for all } n \in \mathbb{Z}. \tag{4.1}$$

We can express condition (4.1) in terms of rounding functions, by multiplying both sides by $\alpha\beta > 0$: $r_\alpha(n) = \alpha\lceil\frac{1}{\alpha}n\rceil \leq \beta\lceil\frac{1}{\beta}n\rceil = r_\beta(n)$. We thus have the further equivalent condition

$$r_\alpha(n) \leq r_\beta(n) \quad \text{for all } n \in \mathbb{Z}. \tag{4.2}$$

There are two "obvious" cases when (4.2) holds, giving continous linear families of solutions, and one not-so-obvious case which is deferred to a later section.

Case (i-a). $\beta/\alpha = m_1$ *is a (positive) integer.* By Rounding Lemma 2.1, the inequality (4.2) holds *a fortiori* for all $n \in \mathbb{R}$ if β/α is a positive integer. Thus (1.1) holds in this case, completing case *(i-a)*.

Case (i-b). $1/\alpha = m_2$ *is a (positive) integer.*

By Rounding Lemma 2.1, we have the comparison $r_\alpha(x) = r_{1/m_2}(x) \le r_1(x) = \lceil x \rceil$ for any real x. Thus in particular, for any integer n

$$r_\alpha(n) \le \lceil n \rceil = n \le r_\beta(n)$$

as $\beta > 0$. Thus (1.1) holds, completing case *(i-b)*.

The form of these solutions to cases *(i-a)* and *(i-b)* motivates the introduction of new variables $u = \frac{1}{\alpha}$ and $v = \frac{\beta}{\alpha}$. In the new coordinates (u, v) cases *(i-a)* and *(i-b)* amount to saying that (1.1) is satisfied whenever u or v is a positive integer.

It remains to characterize those positive values (α, β) satisfying (1.1) for which both variables u, v are not integers. This case will be treated in the following proposition, which identifies the solutions arising in case *(i-c)*.

Proposition 4.1. *For $\alpha, \beta > 0$, the inequality* (1.1) *holds if and only if the associated point $(u, v) = (\frac{1}{\alpha}, \frac{\beta}{\alpha})$ lies on any one of the two-parameter family of curves*

$$\frac{m_1}{u} + \frac{m_2}{v} = 1$$

for integers $m_1 \ge 0, m_2 \ge 0$.

These curves are rectangular hyperbolas when both $m_1 \ge 1$ and $m_2 \ge 1$. If $m_1 = 0$ these are horizontal half-lines in the (u, v)-coordinates and correspond to case *(i-b)* proved above, and if $m_2 = 0$ these are vertical half-lines corresponding to case *(i-a)*. Thus we need only consider the rectangular hyperbola cases where both $m_1, m_2 \ge 1$. These rectangular hyperbolas in (α, β)-coordinates become $m_1\alpha\beta + m_2\alpha = \beta$, so Proposition 4.1 completes the proof of *(i)* in Theorem 1.1.

The proof of Proposition 4.1 is deferred to Section 5. The argument is done in the (u, v) coordinate system.

4.3. **Third quadrant: Both α, β negative.** The sets $S_{\alpha,\beta}$ are calculated in Lemma 3.3:

$$S_{\alpha,\beta}(n) = \left(\frac{1}{\beta}\lfloor\frac{1}{\alpha}n\rfloor + \frac{1}{\beta}, \infty\right) \quad \text{and} \quad S_{\beta,\alpha}(n) = \left(\frac{1}{\alpha}\lfloor\frac{1}{\beta}n\rfloor + \frac{1}{\alpha}, \infty\right).$$

Thus the equivalent inequalities (1.1) and (3.4) are now each equivalent to

$$\frac{1}{\beta}\lfloor\frac{1}{\alpha}n\rfloor + \frac{1}{\beta} \le \frac{1}{\alpha}\lfloor\frac{1}{\beta}n\rfloor + \frac{1}{\alpha} \quad \text{for all } n \in \mathbb{Z}. \tag{4.3}$$

The special case $n = 0$ implies that $\alpha \le \beta$, which establishes that $|\alpha| \ge |\beta|$ is a necessary condition for (1.1) to hold. We can rewrite (4.3) by multiplying by $\alpha\beta > 0$,

$$\alpha\lfloor\frac{1}{\alpha}n\rfloor + \alpha \le \beta\lfloor\frac{1}{\beta}n\rfloor + \beta, \tag{4.4}$$

so in terms of strict rounding functions the equivalent condition is

$$\vec{r}_\alpha(n) \le \vec{r}_\beta(n) \quad \text{for all } n \in \mathbb{Z}. \tag{4.5}$$

We now observe the inequality (1.1) holds in the first of three cases listed in Theorem 1.1.

Case (iii-a). $\frac{\alpha}{\beta} = m_1$ is a positive integer

For such α, β, Strict Rounding Lemma 2.2 implies that (4.5) holds *a fortiori* for all $n \in \mathbb{R}$. Thus (1.1) holds, completing case *(iii-a)*.

Case (iii-b). $\alpha = -\frac{m_2}{m_3}$ is rational, and $-\frac{1}{m_3} \le \beta < 0$

We first treat the case $m_2 = m_3 = 1$, so $\alpha = -1$. Then for each $n \in \mathbb{Z}$,

$$\vec{r}_\alpha(n) = \vec{r}_{-1}(n) = n - 1.$$

Consequently for any $0 > \beta \geq -1 = -\frac{1}{m_3}$,

$$\vec{r}_\alpha(n) = n - 1 \leq n + \beta \leq \vec{r}_\beta(n),$$

where the last inequality follows from the definition of the strict rounding functions in section 2.1. Thus condition (4.5) holds, which is equivalent to (1.1).

Now the general case $\alpha = -\frac{m_2}{m_3}$ rational follows from the symmetries of S discussed in section 2.3, i.e. Propositions 2.4(2) and (3). This completes case *(iii-b)*.

The remaining part of the third quadrant case will be handled after making a birational change of variables (α, β) to (u', v') where $u' = -\frac{1}{\beta}$ and $v' = \frac{\alpha}{\beta}$. (This change of variables differs from that used in the first quadrant cases.) Under this change to (u', v')-coordinates solutions in case *(iii-a)* are mapped to horizontal half-lines with integer v'-coordinate. Case *(iii-b)* solutions are mapped to half-lines of positive rational slope, extending out from the origin but only starting at the first integer lattice point after the origin. [EXPLAIN BETTER?]

The necessary condition $|\alpha| \geq |\beta|$ implied by (4.3) (when $n = 0$) translates to

$$v' = \frac{\alpha}{\beta} \geq 1 \tag{4.6}$$

in the new coordinates.

It remains to characterize the full set of allowable positive values (u', v') for which the associated (α, β) satisfy (1.1), for which v' is not an integer. This case will be covered by the following proposition. It specifies the case *(iii-c)* solutions, state some facts about (iii-d) solutions, and its main content will be the exclusion of all other possibilities.

Proposition 4.2. *Let $(u', v') = (-\frac{1}{\beta}, \frac{\alpha}{\beta})$ with both $\alpha, \beta < 0$.*

(1) All points (u', v') with $u', v' > 0$ that lie on any of the two-parameter family of rectangular hyperbolas, which for integers $m_1 \geq 1, m_2 \geq 1$ are given by

$$\frac{m_1}{u'} + \frac{m_2}{v'} = 1,$$

necessarily give rise to parameters (α, β) that satisfy the inequality (1.1).

(2) All points in the (u', v') -plane with $u', v' > 0$ that have at least one irrational coordinate u' or v', excluding those of the form:

- *(a) $v' = m_1$ for an integer $m_1 \geq 1$, with any $u' > 0$ (which is case (iii-a)),*
- *(b) $\frac{v'}{u'} = \frac{m_2}{m_3}$ for integers $m_2, m_3 \geq 1$, with $v' \geq m_2$ (which is case (iii-b)),*
- *(c) (u', v') on any rectangular hyperbola in (1) above (which is case (iii-c)),*

necessarily give rise to parameters (α, β) that do not satisfy (1.1).

(3) Infinitely many sporadic rational solutions exist in the parameter range $0 < u' < 1$. These are parametrized by integers (m_2, r, j) with $m_2 \geq 2, r \geq 2$ and $1 \leq j \leq m_2 - 1$. These are $(u', v') = (1 - \frac{j}{m_2 r}, m_2 - \frac{j}{r})$. (These "extend" the half-line solutions in case (iii-b).) All other sporadic rational solutions necessarily have $u' \geq 1$.

We defer the proof of Proposition 4.2 to Section 7. The argument is done in the new (u', v') coordinate system. The proof parallels the first quadrant case but has additional complications. The condition that at least one of u', v' is irrational is equivalent to the condition that at least one of α, β is irrational. The rectangular hyperbolas in (1) convert in (α, β)-coordinates to

$m_1\alpha\beta - m_2\beta = -\alpha$, so the first part of Proposition 4.2 specifies case *(iii-c)* in Theorem 1.1. The excluded cases (a), (b), (c) in part (2) correspond in (α, β)-coordinates to cases *(iii-a)*, *(iii-b)*, and *(iii-c)*, of Theorem 1.1, respectively. The condition $u' \geq 1$ in converts to $-1 \leq \beta < 0$.

4.4. **Proof of Theorem 1.1.** The proof of Theorem 1.1 is completed by the case analysis above, modulo giving proofs of Propositions 4.1 and 4.2.

5. Residual set and Beatty sequence criteria: First Quadrant

In this section assume $\alpha > 0$ and $\beta > 0$ are in the open first quadrant. We work using (u, v)-coordinates, taking $u = \frac{1}{\alpha}, v = \frac{\beta}{\alpha}$. These new coordinates will reveal a hidden symmetry of the solution set. The main problem (1.1) will be translated to a statement regarding Beatty sequences. We will be able to prove one direction of Theorem 1.1 in the first quadrant, that all stated parameters are solutions of (1.1). The reverse implication will be deferred to section **??**.

The effect of the birational variable change from (α, β) to (u, v)-coordinates on the solution set is pictured in Figure 5.1 below.

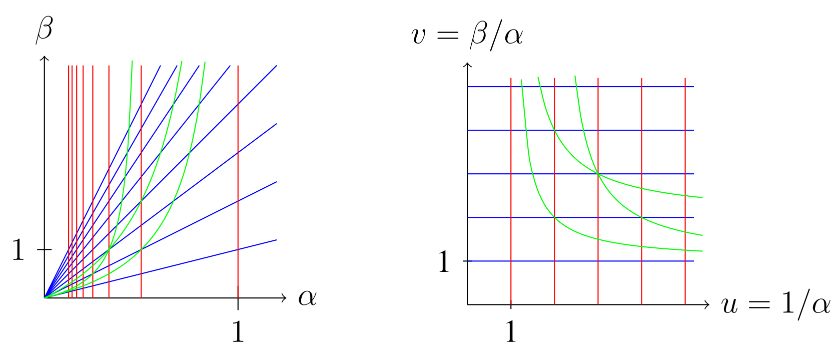

FIGURE 5.1. First quadrant solutions to (1.1) pictured in (α, β)- and (u, v)-coordinates, where $u = \frac{1}{\alpha}$, $v = \frac{\beta}{\alpha}$.

5.1. **First Quadrant Residual Sets.** In terms of the new parameters u and v, condition (4.2) reads

$$r_{1/u}(n) = \frac{1}{u}\lfloor un \rfloor \leq \frac{v}{u}\lfloor \frac{1}{v} un \rfloor = r_{v/u}(n) \quad \text{for all } n \in \mathbb{Z}.$$

Multiplying both sides by u, this condition becomes

$$r_1(un) = \lceil un \rceil \leq v \lceil \frac{1}{v} un \rceil = r_v(un).$$

Thus condition (4.2) is equivalent to

$$r_1(x) \leq r_v(x) \quad \text{for all } x \in u\mathbb{Z} \tag{5.1}$$

where $u\mathbb{Z} = \{un : n \in \mathbb{Z}\}$ is the scaled $\mathbb{R}$-lattice generated by u. We now introduce a “residual set" R_v where the rounding condition above fails, defined by

$$R_v := \{x \in \mathbb{R} : r_1(x) > r_v(x)\},$$

noting that it depends only on the variable v. The preceding argument shows that in terms of the residual set a necessary and sufficient condition for (1.1) to hold is

$$u\mathbb{Z} \cap R_v = \emptyset, \tag{5.2}$$

where $u = 1/\alpha$ and $v = \beta/\alpha$.

The residual set R_v has a nice characterization as the union of countably many half-open intervals.

Lemma 5.1. *(1) For any $v > 0$,*

$$R_v = \bigcup_{n \in \mathbb{Z}} \big(\lfloor vn \rfloor, vn\big].$$

(2) For any $v > 0$ the intersection $R_v \cap \mathbb{Z} = \emptyset$.

(3) The set $R_v = \emptyset$ if and only if v is a positive integer.

Proof. (1) We first show the $\supseteq$ direction. Suppose $\lfloor vn \rfloor < x \le vn$ for some integer n; by definition of R_v we need to show $\lceil x \rceil > v\lceil \frac{1}{v}x \rceil$. The existence of x implies $vn \notin \mathbb{Z}$, so we have strict inequalities $\lfloor vn \rfloor < vn < \lceil vn \rceil$ and x satisfies the same bounds $\lfloor vn \rfloor < x < \lceil vn \rceil$ which differ by a unit amout $\lceil vn \rceil - \lfloor vn \rfloor = 1$. Thus

$$\lfloor x \rfloor = \lfloor vn \rfloor \quad \text{and} \quad \lceil x \rceil = \lceil vn \rceil.$$

The assumption $x \le vn$ implies

$$v\lceil \tfrac{1}{v}x \rceil \le v\lceil n \rceil = vn.$$

Thus

$$\lceil x \rceil = \lceil vn \rceil > vn \ge v\lceil \tfrac{1}{v}x \rceil,$$

so $x \in R_v$ as claimed.

Next, $\subseteq$: suppose $\lceil x \rceil > v\lceil \frac{1}{v}x \rceil$ and set $n = \lceil \frac{1}{v}x \rceil$. The inequality $\lceil x \rceil > vn$ implies

$$\lceil x \rceil > \lfloor vn \rfloor \quad \Leftrightarrow \quad x > \lfloor vn \rfloor$$

while the choice of n implies

$$\frac{1}{v}x \le n \quad \Leftrightarrow \quad x \le nv.$$

Thus $x \in (\lfloor vn \rfloor, vn]$. The claim follows.

(2) Each interval $(\lfloor vn \rfloor, vn]$ contains no integer, so $R_v \cap \mathbb{Z} = \emptyset$.

(3) If $v \in \mathbb{Z}$ then $(\lfloor vn \rfloor, vn] = \emptyset$. If $v \notin \mathbb{Z}$ the interval $(\lfloor v \rfloor, v]$ is nonempty. ∎

At first glance the condition $u\mathbb{Z} \cap R_v = \emptyset$ of does not look symmetric in u, v. But this is misleading. Let $R_v^{\pm}$ denote the *symmetrized residual set*

$$R_v^{\pm} := R_v \bigcup -R_v$$

where $-S = \{-x : x \in S\}$ denotes the point-wise negation of a set. We first observe that

$$\begin{aligned}
R_v \cup (-R_v) &= \Big(\bigcup_{n \in \mathbb{Z}} \big(\lfloor nv \rfloor, nv\big] \Big) \bigcup \Big(\bigcup_{n \in \mathbb{Z}} \big[-nv, -\lfloor nv \rfloor\big) \Big) \\
&= \Big(\bigcup_{n \in \mathbb{Z}} \big(\lfloor nv \rfloor, nv\big] \Big) \bigcup \Big(\bigcup_{n' \in \mathbb{Z}} \big[n'v, -\lfloor -n'v \rfloor\big) \Big) \\
&= \Big(\bigcup_{n \in \mathbb{Z}} \big(\lfloor nv \rfloor, nv\big] \Big) \bigcup \Big(\bigcup_{n' \in \mathbb{Z}} \big[n'v, \lceil n'v \rceil\big) \Big).
\end{aligned}$$

Reordering the last union to match n with n', we obtain

$$R_v^{\pm} = \bigcup_{n\in\mathbb{Z}} \big(\lfloor nv\rfloor, \lceil nv\rceil\big) = \bigcup_{x\in v\mathbb{Z}} \big(\lfloor x\rfloor, \lceil x\rceil\big). \tag{5.3}$$

This formula shows that $R_v^{\pm}$ is a union of open unit intervals whose endpoints are lattice points.

Lemma 5.2. *For $u, v > 0$ the following conditions are equivalent.*

1. *The inequality* (1.1) *holds for parameters (α, β) with $\alpha = \frac{1}{u}, \beta = \frac{v}{u}$.*
2. *The inequality $r_1(un) \le r_v(un)$ holds for all $n \in \mathbb{Z}$.*
3. *$u\mathbb{Z} \bigcap R_v = \emptyset$.*
4. *$u\mathbb{Z} \bigcap R_v^{\pm} = \emptyset$.*
5. *$R_u^{\pm} \bigcap R_v^{\pm} = \emptyset$.*

Proof. We showed above that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). (This is the discussion around (5.1) and (5.2).)

Clearly (4) $\Rightarrow$ (3). For the direction (3) $\Rightarrow$ (4), if we negate both sets pointwise (i.e. apply $x \mapsto -x$), then the lattice $u\mathbb{Z}$ remains fixed but R_v does not. In particular (3) implies $u\mathbb{Z} \bigcup (-R_v) = \emptyset$, whence (4) holds.

To show (4) and (5) are equivalent we will show the contrapositive. The set decomposition for $R_v^{\pm}$ shows that it is a union of all open unit intervals containing a non-integer point of the lattice $v\mathbb{Z}$. In particular $v\mathbb{Z} \setminus \mathbb{Z} \subset R_v^{\pm}$. By Lemma 5.1(2) if $u\mathbb{Z} \bigcap R_v^{\pm} \ne \emptyset$ then the intersection consists exclusively of non-integer members of $u\mathbb{Z}$. All such members belong to $R_u^{\pm}$, whence $R_u^{\pm} \bigcap R_v^{\pm} \ne \emptyset$. Conversely, if $R_u^{\pm} \bigcap R_v^{\pm} \ne \emptyset$, then since both sets are unions of open unit intervals, one full open interval overlaps and that interval contains a non-integer member of $u\mathbb{Z}$, whence $u\mathbb{Z} \bigcap R_v^{\pm} \ne \emptyset$. ☐

The condition (5) is manifestly symmetric in u and v, whence we have

$$u\mathbb{Z} \cap R_v^{\pm} = \emptyset \quad \Leftrightarrow \quad R_u^{\pm} \cap R_v^{\pm} = \emptyset \quad \Leftrightarrow \quad R_u^{\pm} \cap v\mathbb{Z} = \emptyset. \tag{5.4}$$

so condition (5.2) *is* in fact symmetric in terms of the parameters u, v.

In what follows we will study in detail the symmetric condition (5), given as (5.4). We already know that (1.1) holds when either u or v is a positive integer (cases (i-a) and (i-b) of Theorem 1.1).

Lemma 5.3. *Suppose that neither u or v is a positive integer. Then $u > 1$ and $v > 1$ are necessary conditions for $R_u^{\pm} \bigcap R_v^{\pm} = \emptyset$ to hold.*

Proof. By the symmetry of exchanging u and v, it suffices to show that $R_u^{\pm} \bigcap R_v^{\pm} \ne \emptyset$ when $0 < u < 1$ and v is not an integer. Now $0 < u < 1$ yields $R_u^{\pm} = \mathbb{R} \setminus \mathbb{Z}$. The requirement that v is not an integer implies that $R_v^{\pm}$ contains $v \in (\lfloor v\rfloor, \lceil v\rceil)$, and so does $R_u^{\pm}$. ☐

5.2. **Beatty sequences.** To study the problem when $R_u^{\pm} \cap R_v^{\pm} = \emptyset$, we convert it to a problem concerning Beatty sequences.

Definition 5.4. Given a non-zero real number u, the *Beatty sequence* $\mathcal{B}^+(u)$ is

$$\mathcal{B}^+(u) := \{\lfloor nu\rfloor \,:\, n \ge 1\}$$

We recall the following well-known result, which was raised as a problem by Sam Beatty in 1926, with solutions given in 1927 by L. Ostrowski and J. Hyslop, and by A. C. Aitken, see [2].

Theorem 5.5. ("Beatty's Theorem") *Let u, v be positive irrational numbers with*

$$\frac{1}{u} + \frac{1}{v} = 1,$$

a condition which requires $u, v > 1$. Then the Beatty sequences $\mathcal{B}^+(u)$ and $\mathcal{B}^+(v)$ partition the positive integers $\mathbb{N}$, namely

$$\mathcal{B}^+(u) \cap \mathcal{B}^+(v) = \emptyset \quad \text{and} \quad \mathcal{B}^+(u) \cup \mathcal{B}^+(v) = \mathbb{N}.$$

The converse also holds: If u and v do not satisfy the above conditions (either u, v are rational or the equality $\frac{1}{u} + \frac{1}{v} = 1$ fails) then the sets $\mathcal{B}^+(u)$ and $\mathcal{B}^+(v)$ do not partition the positive integers $\mathbb{N}$.

The result was not proved by Beatty, who only posed the problem in the irrational case. Besides the solvers above, this result was independently found by Uspensky [25] in 1927, in the process of solving Wythoff's game, see also Graham [9]. More general results, together with a detailed history can be found in O'Bryant [18].

We will connect the residual sets $R_u^{\pm}$ to Beatty sequences extended to $\mathbb{Z}$.

Definition 5.6. (1) Given a non-zero real number u, the *full Beatty sequence* $\mathcal{B}(u)$ is:

$$\mathcal{B}(u) := \{\lfloor nu \rfloor : n \in \mathbb{Z}\} = \{\lfloor x \rfloor : x \in u\mathbb{Z}\}.$$

(2) The *reduced Beatty sequence* $\mathcal{B}_0(u)$ is:

$$\mathcal{B}_0(u) := \{\lfloor x \rfloor : x \in u\mathbb{Z} \smallsetminus \mathbb{Z}\}.$$

All full Beatty sequences contain $0 = \lfloor 0 \rfloor$. For $0 < u < 1$ one has $\mathcal{B}_0(u) = \mathcal{B}(u) = \mathbb{Z}$. If $u > 1$ is irrational then the reduced Beatty sequence $\mathcal{B}_0(u) = \mathcal{B}(u) \smallsetminus \{0\}$. If $u = r/s > 1$ is rational, given in lowest terms, then the reduced Beatty sequence $\mathcal{B}_0(u) = \mathcal{B}(u) \smallsetminus r\mathbb{Z}$. If u is an integer then $\mathcal{B}_0(u) = \emptyset$, while if u is not an integer, then $\mathcal{B}_0(u)$ is an infinite set.

Lemma 5.7. *(1) For any $u > 0$,*

$$R_u^{\pm} = \bigcup_{m \in \mathcal{B}_0(u)} (m, m+1).$$

(2) For $u, v > 0$,

$$R_u^{\pm} \bigcap R_v^{\pm} = \emptyset \quad \Leftrightarrow \quad \mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset. \tag{5.5}$$

Proof. (1) This is a reinterpretation of the equality $R_u^{\pm} = \bigcup_{x \in u\mathbb{Z}} \big(\lfloor x \rfloor, \lceil x \rceil\big)$. When $x \in u\mathbb{Z} \cap \mathbb{Z}$, the open interval $(\lfloor x \rfloor, \lceil x \rceil)$ is empty since both endpoints coincide. So the union may be equivalently taken over $u\mathbb{Z} \smallsetminus \mathbb{Z}$, namely $R_u^{\pm} = \bigcup_{x \in u\mathbb{Z} \smallsetminus \mathbb{Z}} \big(\lfloor x \rfloor, \lceil x \rceil\big)$. The upper endpoint $\lceil x \rceil$ is always one greater than the lower endpoint $\lfloor x \rfloor$, and the lower endpoints are precisely the points in the reduced Beatty sequence $\mathcal{B}_0(u)$.

(2) This equivalence follows directly from (1). $\square$

5.3. **Intersections of Beatty sequences: Torus subgroup criterion.** The criterion of Lemma 5.7 (2) reduces the analysis of (1.1) in the first quadrant to a disjointness property of reduced Beatty sequences. The next lemma gives formula describing the intersection of two given reduced Beatty sequences. It also gives a parallel formula for the intersection of a full Beatty sequence with a reduced Beatty sequence, the latter case being needed for the third quadrant case, to be treated in Section 7.

Lemma 5.8. *Let $u, v > 1$ be given.*
(1) There holds

$$\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \{ -n \in \mathbb{Z} : 0 < \{\frac{n}{u}\} < \frac{1}{u} \quad \textit{and} \quad 0 < \{\frac{n}{v}\} < \frac{1}{v} \}.$$

(2) In addition there holds

$$\mathcal{B}(u) \bigcap \mathcal{B}_0(v) = \{ -n \in \mathbb{Z} : 0 \le \{\frac{n}{u}\} < \frac{1}{u} \quad \textit{and} \quad 0 < \{\frac{n}{v}\} < \frac{1}{v} \}.$$

Proof. Suppose that the integer $-n \in \mathcal{B}_0(u) \bigcap \mathcal{B}_0(v)$. That is, there exist integers m_1, m_2 with $m_1 u = -n + x_1$ with $0 < x_1 < 1$ and $m_2 v = -n + x_2$ with $0 < x_2 < 1$. Dividing the first equality by u yields

$$-\frac{n}{u} + \frac{x_1}{u} \equiv 0 \, (\text{mod } 1).$$

Since $0 < \frac{x_1}{u} < \frac{1}{u} < 1$, the above equation implies $\{\frac{n}{u}\} = \frac{x_1}{u}$ so we obtain $0 < \{\frac{n}{u}\} < \frac{1}{u}$. In the same fashion we obtain $0 < \{\frac{n}{v}\} < \frac{1}{v}$ from the second equality. This gives the equivalence in one direction. All the steps are reversible, so the converse implication follows.

The argument for $\mathcal{B}(u) \bigcap \mathcal{B}_0(v)$ is similar, except we start with a non-strict inequality $0 \le x_1 < 1$ in $m_1 u = -n + x_1$. ☐

6. First Quadrant Analysis: Proof of Proposition 4.1

We now complete the analysis of the first quadrant case, and prove Proposition 4.1 in Section 6.2.

Our first object in this section is to a disjointness criterion for reduced Beatty sequences. The final section 6.5 presents a criterion for sporadic rational solutions. It uses (X, Y)-coordinates, with $(X, Y) := (\frac{1}{u}, \frac{1}{v}) = (\alpha, \frac{\alpha}{\beta})$, and makes a connection with the two-dimensional Diophantine Frobenius Problem.

6.1. Intersections of Beatty Sequences.

A disjointness criterion for Beatty sequences was found in 1957 by Th. Skolem [21, Theorem 8], which is stated below. Skolem's work was discussed further in section 3 of a supplementary note by T. Bang [1]. Later work, starting with Skolem [22], studied disjointness of shifted Beatty sequences $\lfloor \alpha n + \gamma \rfloor$. We mention[1] Fraenkel [6], [7] , Graham [10], Morikawa [17], with further work surveyed in [18].

Theorem 6.1. (Skolem) *The Beatty sequences*

$$\mathcal{B}^+(u) \bigcap \mathcal{B}^+(v) = \emptyset$$

are disjoint if and only if u, v are irrational and satisfy

$$\frac{m_1}{u} + \frac{m_2}{v} = 1$$

for some integers $m_1, m_2 \ge 1$.

Skolem's basic arguments, supplemented by Bang [1, Sect. 5], apply to our case, where we replace $\mathcal{B}^+(u)$ with the reduced Beatty sequence $\mathcal{B}_0(u)$. The following result classifies when the intersection of two reduced Beatty sequences is empty.

[1]Fraenkel [6] noted that a subsequent work of Skolem [22] on intersective properties of shifted Beatty sequences has incorrectly stated theorems with defective proofs.

Theorem 6.2. *(1) For rational or irrational numbers $u, v > 1$, if there are integers $m_1, m_2 \geq 0$ such that*

$$\frac{m_1}{u} + \frac{m_2}{v} = 1,$$

then the intersection of reduced Beatty sequences

$$\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset.$$

(2) Conversely, if $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$ with $u, v > 1$ then (u, v) must satisfy an equation of the form $\frac{m_1}{u} + \frac{m_2}{v} = 1$, for some integers $m_1, m_2 \geq 0$ (not both zero).

Our proof uses ideas from Skolem's arguments which apply in the case where at least one of u or v is irrational. However when both u and v are rational then the conclusion of Theorem 6.2 differs from that of Theorem 6.1. This case requires a completely new argument; in Section 6.3 we relate this case to the 2-dimensional Diophantine problem of Frobenius.

6.2. **Orbits on a torus.** The proof of Theorem 6.2 is based on a a study the $\mathbb{Z}$-orbit $\mathcal{O}(\mathbf{v})$ of the vector $\mathbf{v} := (\frac{1}{u}, \frac{1}{v})$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$. This torus is defined using the (X, Y)-coordinate system, since $X = \alpha = \frac{1}{u}$ and $Y = \frac{\alpha}{\beta} = \frac{1}{v}$. This two-sided orbit is

$$\mathcal{O}(\mathbf{v}) := \{n\mathbf{v} : n \in \mathbb{Z}\} \subset \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2.$$

Its structure depends the dimension of the $\mathbb{Q}$-vector space spanned by $[1, \frac{1}{u}, \frac{1}{v}]$, which can be $1, 2$ or 3. We treat these cases separately. The first two cases follow approach of Skolem, while the third case (dimension 1) requires new arguments which make a connection with the 2-dimensional Frobenius problem.

6.2.1. *Case 1. Dimension 3 vector space.* In the dimension 3 case both coordinates u, v are irrational.

Lemma 6.3. *For $u, v > 1$ suppose that $[1, \frac{1}{u}, \frac{1}{v}]$ in $\mathbb{R}$ are linearly independent over $\mathbb{Q}$, so span a vector space of dimension 3 over $\mathbb{Q}$. In this case the orbit $\mathcal{O}(\mathbf{v})$ is dense on $\mathbb{T}^2$, and*

$$\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) \neq \emptyset.$$

Proof. It is well known that in this case the $\mathbb{Z}$-orbit of $(\frac{1}{u}, \frac{1}{v})$ on the torus $\mathbb{R}/\mathbb{Z}$ is dense. In consequence there are infinitely many orbit points in the open rectangular region $(0, \frac{1}{u}) \times (0, \frac{1}{v})$. Now Lemma 5.8 applies to show nonempty intersection. □

6.2.2. *Case 2. Dimension 2 vector space.* In the dimension 2 case at least one coordinate u or v must be irrational.

Lemma 6.4. *For $u, v > 1$ suppose that the numbers $[1, \frac{1}{u}, \frac{1}{v}]$ in $\mathbb{R}$ have dimension 2 as a vector space over $\mathbb{Q}$. In this case the orbit is dense on a finite set of lines all having the same rational slope on $\mathbb{R}^2/\mathbb{Z}^2$. The condition $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$ holds if and only if there are nonnegative integers m_1, m_2, with at least one positive, such that*

$$\frac{m_1}{u} + \frac{m_2}{v} = 1.$$

Proof. By hypothesis there is a linear dependence relation over $\mathbb{Q}$, which can be written

$$\frac{m_1}{u} + \frac{m_2}{v} = m_3,$$

for integers m_1, m_2, m_3. Without loss of generality we may suppose at least one of m_1, m_2 is positive. The lemma asserts there is a restriction on the form of this linear relation: both m_1, m_2 are nonnegative, with at least one of them positive, and $m_3 = 1$.

In this case the orbit of $\mathbf{v} = (\frac{1}{u}, \frac{1}{v})$ lies on the set

$$S_{m_1,m_2} := \{(x,y) \in \mathbb{R}^2/\mathbb{Z}^2 : m_1 x + m_2 y \equiv 0 \,(\mathrm{mod}\, 1).\}$$

This set viewed in $[0,1]^2 \subset \mathbb{R}^2$ is a finite collection of line segments having the same rational slope. Since at least one of u, v is irrational, the orbit $\mathcal{O}(\mathbf{v}) = \{n\mathbf{v} : n \in \mathbb{Z}\}$ can be shown to be dense in S_{m_1,m_2}, the projection to $\mathbb{T}^2$ of the line segments above. [ADD MORE DETAILS?]

Using the criterion of Lemma 5.8 (1) the disjointness condition is equivalent to none of the lines $m_1 x + m_2 y = n$, for all $n \in \mathbb{Z}$, crossing the open set

$$\{(x,y) : 0 < x < \frac{1}{u}, 0 < y < \frac{1}{v}\}. \tag{6.1}$$

If we had integers m_1, m_2 of strictly opposite sign (meaning $m_1 m_2 < 0$), then the line $m_1 x + m_2 y = 0$ will intersect the interior of this region, since it exits from the point $(0,0)$ into the first quadrant. We then obtain points where the criterion (i) of Lemma 5.8 implies $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) \neq \emptyset$, whence (1.1) cannot be satisfied. See Figure 6.2.2. Thus necessarily both $m_1 \geq 0$ and $m_2 \geq 0$.

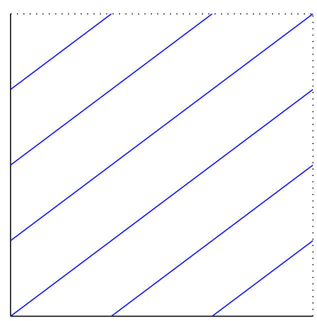

FIGURE 6.1. Solutions to $3x - 4y \in \mathbb{Z}$ in the torus $\mathbb{R}^2/\mathbb{Z}^2$

We first treat the the cases where both $m_1, m_2 \geq 1$. We need to determine the conditions on m_3 such that that no line segments of the configuration go inside the region (6.1). See Figure 6.2.2. We must have $m_3 \geq 1$ to have strictly positive solutions. If $m_3 \geq 2$ we find that some line segment of S_{m_1,m_2} extends strictly inside the region (6.1). Now Lemma 5.8 (ii) implies that for $m_3 \geq 2$ we have $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) \neq \emptyset$, ruling this possibility out. In the remaining case $m_3 = 1$, all line segments outside the open region (6.1), hence $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$ by Lemma 5.8(1) again.[ADD MORE DETAILS?]

It remains to treat cases where one of $m_1, m_2 = 0$. These cases are symmetric, so it suffices to give the argument for $m_1 = 0$. (See Figure 6.2.2 for an example.) In that case $m_2 > 0$ and $\frac{1}{v} = \frac{m_3}{m_2}$ is rational, so the theorem hypotheses require that u be irrational. The condition $v > 1$ implies that necessary $1 \leq m_3 < m_2$. We may suppose without loss of generality $\gcd(m_2, m_3) = 1$. In this case the locus S_{0,m_2} in $\mathbb{R}^2/\mathbb{Z}^2$ is a collection of horizontal line segments $y \equiv \frac{j}{m_2} \,(\mathrm{mod}\, 1)$ where $0 < j < m_2$ with $\gcd(j, m_2) = 1$, and $0 \leq x \leq 1$. The orbit is dense on these horizontal lines. We aim to apply the criterion of Lemma 5.8 (ii), so need to know when any of these lines enter the region (6.1), $0 \leq x \leq \frac{1}{u}$, $0 \leq y \leq \frac{1}{v}$. This

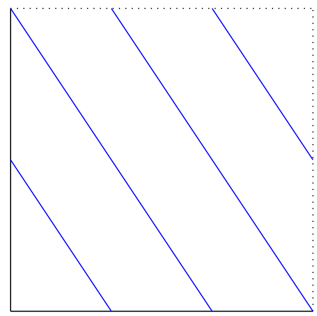

FIGURE 6.2. Solutions to $3x + 2y \in \mathbb{Z}$ in the torus $\mathbb{R}^2/\mathbb{Z}^2$

condition requires that $\frac{m_3}{m_2}$ be minimal among all the $\frac{j}{m_2}$ with $0 < j < m_2$ and $\gcd(j, m_2) = 1$. This requirement forces $m_3 = 1$. Indeed if $m_3 = 1$ then $v = \frac{m_2}{m_3} = m_2 \geq 1$ is an integer, so $\mathcal{B}_0(v) = \emptyset$ and $\mathcal{B}_0(u) \cap \mathcal{B}_0(v) = \emptyset$. (This case corresponds to case (i-a) of Theorem 1.1.) In the case $m_2 = 0$ a similar argument shows $m_3 = 1$ and $u > 1$ is an integer. (This case corresponds to case (i-b) of Theorem 1.1.) □

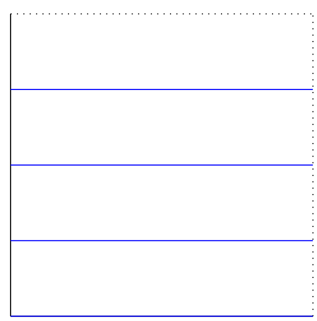

FIGURE 6.3. Solutions to $4y \in \mathbb{Z}$ in the torus $\mathbb{R}^2/\mathbb{Z}^2$

6.2.3. *Case 3: Dimension 1 vector space: rational solutions.* We turn to the case where $\dim_{\mathbb{Q}}[1, u, v] = 1$, which is the case where u, v are both rational. We start with some solutions that exist.

Lemma 6.5. *For $m_1, m_2 \geq 1$ all rational points with $u, v > 0$ on the rectangular hyperbola*

$$\frac{m_1}{u} + \frac{m_2}{v} = 1$$

satisfy $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$; such points necessarily have $u, v > 1$. All points (α, β) in the first quadrant such that $(u, v) = (\frac{1}{\alpha}, \frac{\beta}{\alpha})$ falls on this hyperbola satisfy (1.1).

Proof. In this case the orbit $\mathcal{O}(\mathbf{v})$ is a finite set of points, and all these points lie on the set S_{m_1,m_2} inside $\mathbb{R}/\mathbb{Z}$. The argument of Lemma 6.4 showed that S_{m_1,m_2} contains no points inside the open box (6.1). Since the finite set lies inside S_{m_1,m_2}, we conclude from the criterion of Lemma 5.8 that $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$. Now the criteria of Lemma 5.7 and Lemma 5.2 combine to show that property (1.1) holds for (α, β). □

Lemma 6.5 shows that in Theorem 6.2 there is no restriction to irrational points on the hyperbola, such as occurs in “Beatty’s Theorem" (Theorem 5.5).

It remains to analyze the possibility that there are “sporadic" rational points u, v satisfying $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$ that do not lie on any hyperbola $\frac{m_1}{u} + \frac{m_2}{v} = 1$ or any half-line $\frac{m_1}{u} = 1$ or $\frac{m_2}{v} = 1$.

Lemma 6.6. *There are no “sporadic" rational points in the first quadrant $u, v > 0$ whose associated points $(\alpha, \beta) := (\frac{1}{u}, \frac{v}{u})$ in the first quadrant satisfy* (1.1).

In the next subsection we will show there are no “sporadic" rational solutions in the first quadrant.

6.3. **Sporadic Rational Solutions and the Diophantine Frobenius Problem.** We now switch to the (X, Y)-coordinates given by $X := \frac{1}{u} = \alpha$ and $Y := \frac{1}{v} = \frac{\alpha}{\beta}$. This is a birational change of variable $\alpha = X$ and $\beta = \frac{X}{Y}$. The condition $(X, Y) \in \mathbb{Q}^2$ is equivalent to $(\alpha, \beta) \in \mathbb{Q}^2$. We write $(X, Y) = (\frac{a}{t}, \frac{b}{t})$ where t is a common denominator of such a rational. That is $\gcd(a, b, t) = 1$, however we may have $\gcd(a, b) > 1$. The corresponding $(\alpha, \beta) = (\frac{a}{t}, \frac{a}{b})$, where the second fraction may not be in lowest terms. The corresponding $(u, v) = (\frac{t}{a}, \frac{t}{b})$.

Proposition 6.7. (Sporadic Rational Solution Criterion-First Quadrant) *Set $(X, Y) = (\frac{a}{t}, \frac{b}{t})$ with integers $a, b, t > 0$ and $\gcd(a, b, t) = 1$. Then $(\alpha, \beta) = (\frac{a}{t}, \frac{a}{b})$ is a sporadic rational solution to* (1.1) *if and only if the denominator $t > \max\{a, b\}$ and:*

(i) There is no integer $0 \le n < t$ such that

$$0 < \{\frac{na}{t}\} < \{\frac{a}{t}\} \quad \textit{and} \quad 0 < \{\frac{nb}{t}\} < \{\frac{b}{t}\}.$$

(ii) The linear Diophantine equation

$$aX_1 + bX_2 = t \tag{6.2}$$

is unsolvable in non-negative integers (X_1, X_2).

Proof. We first show that criterion (i) is equivalent to (1.1) holding. We have $X = \frac{1}{u}, Y = \frac{v}{u}$ with $u = \frac{1}{\alpha} = \frac{t}{a}$ and $v = \frac{\beta}{\alpha} = \frac{b}{a}$. Lemma 5.3 shows that $u > 1$ and $v > 1$ are necessary conditions for (1.1) to hold. They require $t > a$ and $t > b$, whence $t > \max\{a, b\}$. The condition (1.1) holds if and only if $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$, combining Lemma 5.3 and Lemma 5.7(2). The criterion of Lemma 5.8(1) shows that this intersection is empty if and only if the criterion given in (i) above holds for all $n \in \mathbb{Z}$. The fractional parts $\{\frac{na}{t}\}$ and $\{\frac{nb}{t}\}$ are periodic with period t so it suffices to check there is no solution over one period. This establishes the equivalence.

We next show criterion (ii) is equivalent to (u, v) not falling on any one-parameter solution to (1.1). By Proposition 4.1 and the discussion after its statement, the one-parameter solutions are given by $\frac{m_1}{u} + \frac{m_2}{v} = 1$ for any integer $m_1, m_2 \ge 0$ with at least one m_i nonzero. The condition for $(u, v) = (\frac{t}{a}, \frac{t}{b})$ falling on such a curve is $m_1 \frac{a}{t} + m_2 \frac{b}{t} = 1$, which on clearing denominators becomes $m_1 a + m_2 b = t$. This condition is easily seen to be equivalent to the equation

$$aX_1 + bX_2 = t,$$

having no solution in non-negative integers (X_1, X_2). $\square$

We use Proposition 6.7 to exclude sporadic rational solutions having $\gcd(a, b) > 1$.

Lemma 6.8. *Suppose we are given positive integers* (a, b, t) *with* $\gcd(a, b, t) = 1$. *Set* $\frac{1}{u} = \frac{a}{t}, \frac{1}{v} = \frac{b}{t}$, *with* $\gcd(a, b, t) = 1$, *and suppose that* $0 < \frac{1}{u} < 1$ *and* $0 < \frac{1}{v} < 1$, *i.e.* $t > \max\{a, b\}$. *Then the condition* $\gcd(a, b) > 1$ *implies that there exists a positive integer* $1 \leq n < t$ *such that*

$$0 < \{\frac{n}{u}\} < \frac{1}{u} \quad \text{and} \quad 0 < \{\frac{n}{v}\} < \frac{1}{v}.$$

Thus $(\alpha, \beta) = (\frac{a}{t}, \frac{a}{b})$ *is not a sporadic rational solution of* (1.1).

Proof. We look at the orbit of $\mathbf{v} = (X, Y) := (\frac{a}{t}, \frac{b}{t}) = (\frac{1}{u}, \frac{1}{v})$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$. We consider the line segment $\mathcal{L} := \{(\frac{a}{t}\lambda, \frac{b}{t}\lambda) : 0 \leq \lambda \leq t\}$ in $\mathbb{R}^2$, which connects the lattice point $(0, 0)$ to (a, b). If $\gcd(a, b) = r > 1$ then the line segment hits $r - 1$ interior lattice points $\lambda = \frac{j}{r}$, $1 \leq j \leq r-1$. By hypothesis $\gcd(r, t) = 1$, and the condition $\gcd(a, b) \geq 2$ and $t > \max(a, b)$ implies $t \geq 3$. We study the projection modulo one of this line segment $\mathcal{L}$ into the unit square, which will be a set of line segments, and we note that $\mathcal{L}$ traces each of these line segments r times. We choose $n := \lceil \frac{t}{r} \rceil$. Then $2 \leq n < t$ since $r \geq 2$ and $t \geq 3$, and $\gcd(t, r) = 1$ gives $0 < n - \frac{t}{r} < 1$. Then we have

$$n(\frac{a}{t}) = \frac{a}{r} + (n - \frac{t}{r})\frac{a}{t} \equiv (n - \frac{t}{r})\frac{a}{t} \pmod{1}.$$

Since $t > a$ it follows that $0 < \{\frac{na}{t}\} < \frac{a}{t}$. By similar reasoning we obtain $0 < \{\frac{nb}{t}\} < \frac{b}{t}$. Now (α, β) is not a sporadic rational solution to (1.1) since Proposition 6.7 (i) does not hold. $\square$

The remaining problem relates to the linear Diophantine equation of Frobenius, for which see the book of Ramírez Alfonsín [20]. This general problem is concerned with bounding above the largest positive integer t for which

$$a_1X_1 + a_2X_2 + \cdots + a_nX_n = t \tag{6.3}$$

has no solution in nonnegative integers $(X_1, X_2, ..., X_n)$ as a function of positive integers $a_1, a_2, ..., a_n$ having $\gcd(a_1, a_2, ..., a_n) = 1$, We encounter the $n = 2$ case here. It is well known that when $\gcd(a, b) = 1$ there are only finitely many nonnegative integers t for which the linear Diophantine equation (6.2) is unsolvable in nonnegative integers (X, Y), and that the largest such value is $t = ab - a - b$; this result goes back to J. J. Sylvester in 1882. Proposition 6.7 (ii) specifies that t belongs to the *unsolvable Frobenius set* for the pair (a, b), which we define as

$N^\star(a, b) := \{t \geq 0 :\ t$ is not representable as a nonnegative integer combination of a and $b.\}$.

Sylvester ([23] [24]) showed that for each $0 \leq t \leq ab - a - b$ exactly one member t^* of the pair $\{t, ab - a - b - t\}$ has $aX_1 + bX_2 = t^*$ solvable in nonnegative integers. Thus the unsolvable Frobenius set has cardinality $\frac{1}{2}(a - 1)(b - 1)$ ([20, Theorem 5.1.1]).

We can now complete the proof that there are no sporadic rational solutions.

Lemma 6.9. *Let* $a, b \geq 1$ *be positive integers with* $(a, b) = 1$. *Suppose that* $t \in N^*(a, b)$ *with* $t > \max(a, b)$. *Then* $(\alpha, \beta) = (\frac{t}{a}, \frac{b}{a})$ *is not a sporadic rational solution to* (1.1).

Proof. We will exclude (α, β) using the criterion of Proposition 6.7. It is a classical result that:

Claim. $t \in N^*(a, b)$ if and only if

$$ab - t = na + mb \quad \text{for some} \quad n, m \geq 1. \tag{6.4}$$

To verify the claim, a well known solution to the problem of Sylvester, see [20, pp. 103–105] shows that for $0 \le t \le ab - a - b$ one has

$$t \in N^*(a,b) \quad \text{if and only if} \quad ab - a - b - t \notin N^*(a,b).$$

The latter condition says that

$$ab - a - b - t = na + mb \quad \text{with} \quad m, n \ge 0,$$

and it is clearly equivalent to

$$ab - t = na + mb \quad \text{with} \quad m, n \ge 1.$$

The claim follows.

Given such t, the choice of $n, m \ge 1$ is in fact unique, since $t > 0$ implies

$$na < na + mb < ab \quad \Rightarrow \quad 0 < n < b,$$

and a similar argument shows that $0 < m < a$. By assumption that a, b are coprime, the only way for $na + mb = n'a + m'b$ is to have $n - n' \in b\mathbb{Z}$ and $m - m' \in a\mathbb{Z}$, so the given inequalities uniquely determine n and m as claimed.

Reducing (6.4) modulo t, we have $ab \equiv na + mb \pmod{t}$ whence $(b-n)a \equiv mb \pmod{t}$ and we deduce that

$$N_1 := \frac{b-n}{b} \equiv \frac{m}{a} \pmod{t},$$

where the fractions are taken as multiplicative inverses in $(\mathbb{Z}/t\mathbb{Z})^\times$. Thus for the vector $\mathbf{v} = (\frac{a}{t}, \frac{b}{t})$ viewed in $\mathbb{R}^2/\mathbb{Z}^2$,

$$N_1 \mathbf{v} = N_1(\frac{a}{t}, \frac{b}{t}) = (\frac{N_1 a}{t}, \frac{N_1 b}{t}) \equiv (\frac{m}{t}, \frac{b-n}{t}) \pmod{\mathbb{R}^2/\mathbb{Z}^2}.$$

Now we have, using $t > \max\{a, b\}$,

$$0 < \{\frac{N_1 a}{t}\} = \frac{m}{t} < \frac{a}{t} < 1 \quad \text{and} \quad 0 < \{\frac{N_1 b}{t}\} = \frac{b-n}{t} < \frac{b}{t} < 1,$$

violates the fractional-part condition (i) in Proposition 6.7, so $(\frac{a}{t}, \frac{a}{b})$ is not a sporadic rational solution. ☐

6.4. **Proof of Disjoint Intersection Theorem 6.2.**

Proof. (1) The already proved cases *(i-a)* and *(i-b)* show that all rational and irrational points on any of the curves $\frac{m_1}{u} + \frac{m_2}{v} = 1$ with $m_1 = 0$ or $m_2 = 0$ yield solutions to the inequality (1.1). Lemma 6.4 shows that for all $m_1, m_2 \ge 1$ those points on for the hyperbolas having irrational coordinates yield solutions to (1.1). Finally Lemma 6.5 shows that for $m_1, m_2 \ge 1$ all points with rational coordinates on these hyperbolas yield solutions to (1.1).

(2) Lemma 6.3 rules out all points in the plane having $\dim_{\mathbb{Q}}[1, u, v] = 3$ from giving any solutions $(\alpha, \beta) = (\frac{1}{u}, \frac{v}{u})$ to (1.1). Lemma 6.4 shows that all points having $\dim_{\mathbb{Q}}[1, u, v] = 2$ that satisfy (1.1) must lie on a hyperbola or straight line solution for some pair $m_1, m_2 \ge 0$. There remain the case of rational points, where $\dim_{\mathbb{Q}}[1, u, v] = 1$. Lemma 6.9 rules out all possible rational solutions not on any of these curves. ☐

6.5. Completion of first quadrant case of Theorem 1.1.

Proof of Proposition 4.1. (1) Lemma 5.2 establishes that (1.1) will hold in the first quadrant case if and only if $R_u^{\pm} \cap R_v^{\pm} = \emptyset$. Lemma 5.7 shows that this condition is equivalent to $\mathcal{B}_0(u) \bigcap \mathcal{B}_0(v) = \emptyset$. In consequence Theorem 6.2 (1) establishes (1) of Proposition 4.1.

(2) Lemma 5.3 shows that if neither of u, v is a positive integer, then $u > 1$ and $v > 1$ are necessary conditions for (1.1) to hold. Now Theorem 6.2 (2) applies to prove (2) of Proposition 4.1. $\square$

7. Residual set and modified Beatty sequence criteria: Third Quadrant

We now begin the analysis of the third quadrant case, which parallels that for the first quadrant but is more complicated, having sporadic rational solutions.

In this section (α, β) falls in the open third quadrant $\alpha < 0$ and $\beta < 0$. We will work using the coordinates $u' = -\frac{1}{\beta}, v' = \frac{\alpha}{\beta}$, which differs from the coordinate change (u, v) used in Section 5. This variable change maps the open third quadrant (α, β) birationally and homeomorphically to the open first quadrant (u', v'), with inverse $\alpha = -\frac{v'}{u'}$, $\beta = -\frac{1}{u'}$. Our object is to prove Proposition 4.2, to classify an infinite subclass of sporadic rational solutions in the third quadrant, and to restrict the form of possible additional sporadic rational solutions.

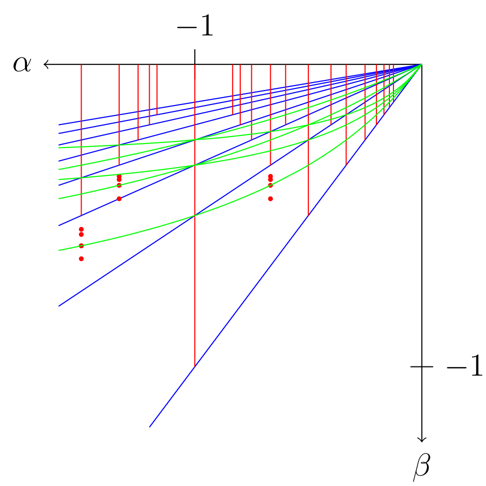

FIGURE 7.1. Third quadrant solutions in the (α, β) coordinates

7.1. Third Quadrant Residual Sets.

In terms of the new parameters $u' = -\frac{1}{\beta}$ and $v' = \frac{\alpha}{\beta}$, condition (4.3), which is equivalent to (1.1), becomes

$$-u'\left\lfloor -\frac{u'}{v'}n \right\rfloor - u' \leq -\frac{u'}{v'}\lfloor -u'n \rfloor - \frac{u'}{v'}, \tag{7.1}$$

The condition (7.1) becomes, after multiplying both sides by $-\frac{v'}{u'} = \alpha < 0$,

$$\vec{r}_{v'}(-u'n) = v'\left\lfloor -\frac{u'}{v'}n \right\rfloor + v' \geq \lfloor -u'n \rfloor + 1 = \vec{r}_1(-u'n),$$

which after setting $x = -u'n \in u'\mathbb{Z}$ becomes

$$\vec{r}_{v'}(x) \geq \vec{r}_1(x) \quad \text{for all} \quad x \in u'\mathbb{Z}. \tag{7.2}$$

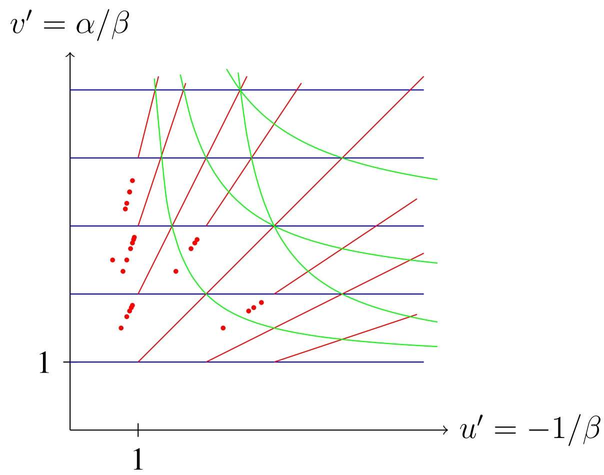

FIGURE 7.2. Third quadrant solutions in the (u', v') coordinates

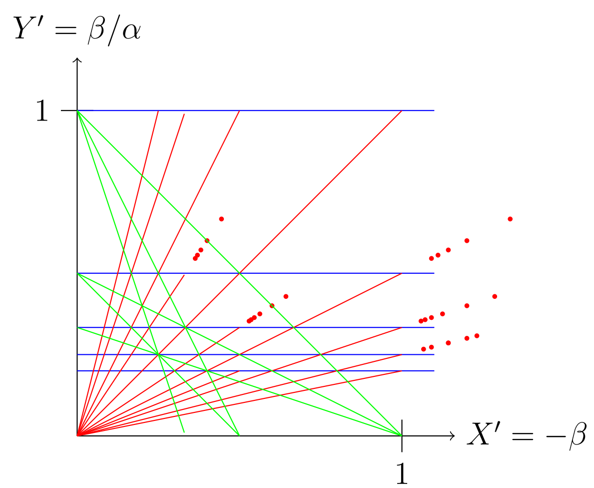

FIGURE 7.3. Third quadrant solutions in the (X', Y') coordinates

Proceeding analogously to the first quadrant case, we define for $v' > 0$ the *(third quadrant) residual set* $\widetilde{R}_{v'}$ by

$$\widetilde{R}_{v'} := \{x \in \mathbb{R} : \vec{r}_{v'}(x) < \vec{r}_1(x)\}.$$

The preceding argument shows that a necessary and sufficient condition for (1.1) to hold is that

$$u'\mathbb{Z} \cap \widetilde{R}_{v'} = \emptyset, \tag{7.3}$$

with $u' = -\frac{1}{\beta}$ and $v' = \frac{\alpha}{\beta}$. The residual set $\widetilde{R}_{v'}$ has a nice union-of-intervals characterization.

Lemma 7.1. *For any $v' > 0$, the (third quadrant) residual set*

$$\widetilde{R}_{v'} = \bigcup_{n \in \mathbb{Z}} \big[\lfloor v'n \rfloor, v'n\big).$$

Proof. We first show the $\supseteq$ direction. Suppose $\lfloor v'n \rfloor \le x < v'n$ for some integer n; by definition of $\widetilde{R}_{v'}$ we need to show $\vec{r}_{v'}(x) < \vec{r}_1(x)$. Now $x < v'n$ implies $\vec{r}_{v'}(x) \le v'n$, and $\lfloor v'n \rfloor \le x$ implies $\lfloor v'n \rfloor \le \lfloor x \rfloor$. We deduce

$$\vec{r}_{v'}(x) \le v'n < \vec{r}_1(v'n) = \lfloor v'n \rfloor + 1 \le \lfloor x \rfloor + 1 = \vec{r}_1(x).$$

as required.

For the other direction ($\subseteq$): suppose x satisfies $\vec{r}_{v'}(x) = v'\lfloor \frac{1}{v'}x \rfloor + v' < \lfloor x \rfloor + 1 = \vec{r}_1(x)$, we wish to deduce $\lfloor v'n \rfloor \le x < v'n$ for some integer n. Set $n = \lfloor \frac{1}{v'}x \rfloor + 1$; then we have $x < \vec{r}_{v'}(x) = v'\lfloor \frac{1}{v'}x \rfloor + v' = v'n$. For the other inequality $v'n = \vec{r}_{v'}(x) < \vec{r}_1(x)$ implies $\lfloor v'n \rfloor < \vec{r}_1(x) = \lfloor x \rfloor + 1$, whence we obtain

$$\lfloor v'n \rfloor \le \lfloor x \rfloor \le x < v'n$$

as required. ☐

We next introduce the *symmetrized (third quadrant) residual set*

$$\widetilde{R}^{\pm}_{v'} := \widetilde{R}_{v'} \cup (-\widetilde{R}_{v'}).$$

Lemma 7.2. *For $u', v' > 0$ the following conditions are equivalent.*

(1) The inequality (1.1) *holds for negative parameters* (α, β) *given by* $\alpha = -\frac{v'}{u'}, \beta = -\frac{1}{u'}$.
(2) The inequality $\vec{r}_{v'}(x) = v'\lfloor \frac{1}{v'}x \rfloor + v' \ge \lfloor x \rfloor + 1 = \vec{r}_1(x)$ *holds for all* $x \in u'\mathbb{Z}$.
(3) $u'\mathbb{Z} \bigcap \widetilde{R}_{v'} = \emptyset$.
(4) $u'\mathbb{Z} \bigcap \widetilde{R}^{\pm}_{v'} = \emptyset$.

Proof. We showed above that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). (It follows from the discussion made around (7.1) and (7.2).)

The equivalence (3) $\Leftrightarrow$ (4) follows because the lattice $u'\mathbb{Z}$ is invariant under the reflection $u' \mapsto -u'$. ☐

Remark 7.3. Note that the equivalent conditions given in the previous lemma do not include one that in symmetric under interchanging the coordinates u' and v'. In fact this interchange does not preserve third-quadrant solutions in general (although it does preserve many of them), in contrast to what we saw in the first quadrant. This indicates an extra subtlety of the problem of classifying third quadrant solutions.

Lemma 7.1 shows that

$$\begin{aligned}\widetilde{R}^{\pm}_{v'} &= \Big(\bigcup_{n\in\mathbb{Z}} \big[\lfloor v'n \rfloor, v'n\big)\Big) \bigcup \Big(\bigcup_{n\in\mathbb{Z}} \big(-v'n, -\lfloor v'n \rfloor\big]\Big) \\ &= \bigcup_{n\in\mathbb{Z}} \Big(\big[\lfloor v'n \rfloor, v'n\big) \cup \big(v'n, \lceil v'n \rceil\big]\Big) = \bigcup_{n\in\mathbb{Z}} \Big(\big[\lfloor v'n \rfloor, \lceil v'n \rceil\big] \smallsetminus \{v'n\}\Big),\end{aligned}$$

where we exchanged n with $-n$ in the second union. Each of these intervals is punctured in its middle to remove a (non-integral) lattice point, but in general these intervals are *not* disjoint, e.g. when $v' < \frac{1}{2}$, each interval will overlap with an adjacent interval. This residual set may be simplified to

$$\widetilde{R}^{\pm}_{v'} = \bigcup_{x\in v'\mathbb{Z}\smallsetminus\mathbb{Z}} \Big(\big[\lfloor x \rfloor, x\big) \cup \big(x, \lceil x \rceil\big]\Big) = \bigcup_{x\in v'\mathbb{Z}\smallsetminus\mathbb{Z}} \Big(\big[\lfloor x \rfloor, \lceil x \rceil\big] \smallsetminus \{x\}\Big). \tag{7.4}$$

If $v' \geq 1$, then the adjacent intervals in the union are disjoint and

$$\widetilde{R}_{v'}^{\pm} = \left(\bigcup_{x \in v'\mathbb{Z}} \big[\lfloor x \rfloor, \lceil x \rceil \big] \right) \smallsetminus v'\mathbb{Z}. \tag{7.5}$$

7.2. **Comparison of first quadrant and third quadrant residual sets.** We compare the third-quadrant residual set criterion (7.5) in Lemma 7.2 with that for first quadrant solutions, (5.3), which states

$$R_v^{\pm} = \bigcup_{x \in v\mathbb{Z} \smallsetminus \mathbb{Z}} \big(\lfloor x \rfloor, \lceil x \rceil \big).$$

They are quite similar; there are only two differences, namely that the intervals in $\widetilde{R}_{v'}^{\pm}$ now include both endpoints, but are punctured in the middle by the removal of a point in $v'\mathbb{Z}$.

This similarity motivates introducing the *closed symmetrized residual set*

$$\bar{R}_v := \bigcup_{x \in v\mathbb{Z} \smallsetminus \mathbb{Z}} \big[\lfloor x \rfloor, \lceil x \rceil \big] = \bigcup_{m \in \mathcal{B}_0(v)} [m, m+1]$$

that is the common closure of the residual sets in quadrants one and three. That is, we have the following lemma.

Lemma 7.4. *(1) The closure of the first-quadrant symmetrized residual set $R_v^{\pm}$ is $\bar{R}_v$, and the difference is a discrete set of points*

$$\bar{R}_v \smallsetminus R_v^{\pm} \subset \mathbb{Z}.$$

(2) The closure of the third-quadrant symmetrized residual set $\widetilde{R}_{v'}^{\pm}$ is $\bar{R}_{v'}$, and the set difference is a discrete set of points

$$\bar{R}_{v'} \smallsetminus \widetilde{R}_{v'}^{\pm} \subset v'\mathbb{Z}.$$

Proof. (1) To go from $\bar{R}_v$ to $R_v^{\pm}$, we simply remove the endpoints of all closed intervals to make them open intervals. But the endpoints all have the form $\lfloor x \rfloor$ or $\lceil x \rceil$, which are clearly in $\mathbb{Z}$.

(2) The punctures in the intervals making up $\widetilde{R}_{v'}^{\pm}$ are located at $x \in v'\mathbb{Z} \smallsetminus \mathbb{Z}$, so the set difference is contained in $v'\mathbb{Z}$.

□

Lemma 7.2 (5) shows that the inequality (1.1) with (α, β) in the third quadrant is equivalent to the empty intersection condition

$$u'\mathbb{Z} \bigcap \widetilde{R}_{v'}^{\pm} = \emptyset \tag{7.6}$$

in the new (u', v') coordinates. For (α, β) in the first quadrant, we had the equivalence of (1.1) with the empty intersection condition $u\mathbb{Z} \cap R_v^{\pm} = \emptyset$ stated in (5.4). We now relate the various empty-intersection conditions among residual sets that pertain to the main problem.

Lemma 7.5. *(1) For any parameters $u, v > 0$, if $u\mathbb{Z} \cap \bar{R}_v = \emptyset$ then both $u\mathbb{Z} \cap R_v^{\pm} = \emptyset$ and $u\mathbb{Z} \cap \widetilde{R}_v^{\pm} = \emptyset$.*

(2) If u is irrational, then $u\mathbb{Z} \cap R_v^{\pm} = \emptyset$ if and only if $u\mathbb{Z} \cap \bar{R}_v = \emptyset$.

(3) If u/v is irrational, then $u\mathbb{Z} \cap \widetilde{R}_v^{\pm} = \emptyset$ if and only if $u\mathbb{Z} \cap \bar{R}_v = \emptyset$.

Proof. (1) The closed residual set $\bar{R}_v$ contains both $R_v^{\pm}$ and $\widetilde{R}_v^{\pm}$.

(2) It suffices to show the ($\Rightarrow$) direction, so assume $u\mathbb{Z} \cap R_v^{\pm} = \emptyset$. If u is irrational then $u\mathbb{Z} \cap \mathbb{Z} = \{0\}$. By the previous lemma, $\bar{R}_v \subset R_v^{\pm} \cup \mathbb{Z}$ so
$$u\mathbb{Z} \cap \bar{R}_v \subset u\mathbb{Z} \cap (R_v^{\pm} \cup \mathbb{Z}) = (u\mathbb{Z} \cap R_v^{\pm}) \cup (u\mathbb{Z} \cap \mathbb{Z}) = \emptyset \cup \{0\} = \{0\}.$$
If $v \geq 1$ then $\bar{R}_v$ does not contain 0 (*a fortiori*, it does not intersect the open interval $(-1, 1)$). So the intersection is empty, as desired. If $v < 1$, then $R_v^{\pm} = \mathbb{R} - \mathbb{Z}$. But since u is assumed irrational, $u \in \mathbb{R} - \mathbb{Z}$ and this contradicts our assumption that $u\mathbb{Z} \cap R_v^{\pm} = \emptyset$.

(3) Again here it suffices to show the ($\Rightarrow$) direction, so assume $u\mathbb{Z} \cap \widetilde{R}_v^{\pm} = \emptyset$. If u/v is irrational then $u\mathbb{Z} \cap v\mathbb{Z} = \{0\}$. By the previous lemma, $\bar{R}_v \subset \widetilde{R}_v^{\pm} \cup v\mathbb{Z}$ so
$$u\mathbb{Z} \cap \bar{R}_v \subset u\mathbb{Z} \cap (\widetilde{R}_v^{\pm} \cup v\mathbb{Z}) = (u\mathbb{Z} \cap \widetilde{R}_v^{\pm}) \cup (u\mathbb{Z} \cap v\mathbb{Z}) = \{0\}.$$
If $v \geq 1$ then the intersection $u\mathbb{Z} \cap \bar{R}_v$ is empty because $\bar{R}_v$ does not contain 0. If $0 < v < 1$, then $\widetilde{R}_v^{\pm} \subset \mathbb{R} - v\mathbb{Z}$. Since u/v is assumed irrational, $u \in \mathbb{R} - v\mathbb{Z}$ so this contradicts our initial empty-intersection hypothesis.

□

7.3. **Connection with Beatty sequences: Third Quadrant Case.** By Lemma 7.2 (1.1) is equivalent to $u'\mathbb{Z} \bigcap \widetilde{R}_{v'}^{\pm} = \emptyset$. When $u' \geq 1$ we can obtain a third quadrant criterion in terms of the intersection of a full Beatty sequences $\mathcal{B}(u')$ with the reduced Beatty sequences $\mathcal{B}_0(v')$.

Lemma 7.6. *(1) For $u' \geq 1$ and $v' > 1$, the condition $u'\mathbb{Z} \bigcap \widetilde{R}_{v'}^{\pm} = \emptyset$ is equivalent to the condition*
$$\left(\mathcal{B}(u') \bigcap \mathcal{B}_0(v')\right) \smallsetminus X_{u',v'} = \emptyset, \tag{7.7}$$
in which the exceptional set $X_{u',v'}$ is defined by
$$X_{u',v'} := \{\lfloor w \rfloor : w \in u'\mathbb{Z} \cap v'\mathbb{Z}\}.$$
(2) If in addition $\frac{u'}{v'}$ is irrational then (7.7) *simplifies to*
$$\mathcal{B}(u') \bigcap \mathcal{B}_0(v') = \emptyset. \tag{7.8}$$
(3) If in addition both $\frac{u'}{v'}$ and u' are irrational then (7.7) *simplifies to*
$$\mathcal{B}_0(u') \bigcap \mathcal{B}_0(v') = \emptyset. \tag{7.9}$$

Proof. (1) Suppose $u'\mathbb{Z} \cap \widetilde{R}_{v'}^{\pm}$ is nonempty and contains some point n_1u'. Since $v' > 1$ the residual set is a disjoint union of punctured intervals
$$\widetilde{R}_{v'}^{\pm} = \bigcup_{x \in v'\mathbb{Z} \smallsetminus \mathbb{Z}} \Big([\lfloor x \rfloor, \lceil x \rceil] \smallsetminus \{x\} \Big) = \Big(\bigcup_{n \in \mathbb{Z}} [\lfloor nv' \rfloor, \lceil nv' \rceil] \Big) \smallsetminus v'\mathbb{Z}.$$
Hence $n_1u' \in \widetilde{R}_{v'}^{\pm}$ implies $n_1u' \in [\lfloor n_2v' \rfloor, \lceil n_2v' \rceil]$ for a unique integer n_2, for which
(i) $n_2v' \notin \mathbb{Z}$ so this interval has length one.
(ii) $n_1u' \neq n_2v'$ since $n_1u' \notin v'\mathbb{Z}$.
If n_1u' lies in the half-open interval $[\lfloor n_2v' \rfloor, \lceil n_2v' \rceil)$, then we claim taking the floor $k := \lfloor n_1u' \rfloor$ produces an element of $\big(\mathcal{B}(u') \cap \mathcal{B}_0(v')\big) \smallsetminus X_{u',v'}$. Indeed it is clear that
$$k = \lfloor n_1u' \rfloor = \lfloor n_2v' \rfloor \in \mathcal{B}(u') \bigcap \mathcal{B}(v')$$

and since n_2v' is not integral, k lies in the reduced Beatty set $\mathcal{B}_0(v')$. Moreover $k = \lfloor n_2v' \rfloor$ does not lie in the exceptional set $X_{u',v'}$ since if $k = \lfloor w \rfloor$ for some $w \in u'\mathbb{Z} \cap v'\mathbb{Z}$, then w would be contained in the same unit interval as both n_1u' and n_2v' (distinct points), which contradicts our assumption that $u', v' > 1$. This proves the claim.

If, on the other hand, n_1u' is equal to the upper endpoint $\lceil n_2v' \rceil$ of our chosen interval then the above argument applied to $-n_1u' \in [-\lceil n_2v' \rceil, -\lfloor n_2v' \rfloor) = [\lfloor -n_2v' \rfloor, \lceil -n_2v' \rceil)$ shows that $k' := \lfloor -n_1u' \rfloor = \lfloor -n_2v' \rfloor$ lies in $\big(\mathcal{B}(u') \cap \mathcal{B}_0(v')\big) \smallsetminus X_{u',v'}$.

Conversely, suppose $\big(\mathcal{B}(u') \cap \mathcal{B}_0(v')\big) \smallsetminus X_{u',v'}$ is nonempty and contains some integer k. Since $u' > 1$ and $k \in \mathcal{B}(u')$, there is a unique $n_1u' \in u'\mathbb{Z}$ such that $k = \lfloor n_1u' \rfloor$; in fact $n_1u' = r_{u'}(k)$. Then we claim $n_1u' \in u'\mathbb{Z} \cap \widetilde{R}^{\pm}_{v'}$. Indeed, $k \in \mathcal{B}_0(v')$ implies that $n_2v' = r_{v'}(k)$ is a non-integral point of the lattice $v'\mathbb{Z}$, such that the closed interval $[\lfloor n_2v' \rfloor, \lceil n_2v' \rceil] = [k, k+1]$. Then our choice of n_1, $n_1u' \in [k, k+1)$ lies in this interval as well. Finally n_1u' cannot be equal to the puncture $n_2v' \in [k, k+1]$, since in that case $k = \lfloor n_1u' \rfloor = \lfloor n_2v' \rfloor$ lies in the exceptional set $X_{u',v'}$ contradicting our hypothesis.

(2) If $\frac{u'}{v'}$ is irrational then the intersection $u'\mathbb{Z} \cap v'\mathbb{Z} = \{0\}$ is trivial so the exceptional set $X_{u',v'} = \{0\}$. However 0 is omitted from $\mathcal{B}_0(v')$ by our assumption $v' > 1$, so we obtain (7.8).

(3) If u' is irrational then $\mathcal{B}_0(u') = \mathcal{B}(u') \smallsetminus \{0\}$, and we may replace $\mathcal{B}(u')$ with $\mathcal{B}_0(u')$ in (7.7) in the intersection. Thus if both irrationality conditions hold we may additionally remove $X_{u',v'}$ from the intersection, obtaining (7.9). ☐

7.4. **Intersections of Beatty sequences: Torus subgroup criterion.** The criterion of Lemma 7.6 requires an analysis of (1.1) in the third quadrant of the structure of the intersection of a full Beatty sequence $\mathcal{B}(u')$ with a reduced Beatty sequence $\mathcal{B}_0(v')$. The next lemma gives a formula for such an intersection. This parallels Lemma 5.8.

Lemma 7.7. (Intersection of full and reduced Beatty sequences) *Let $u', v' > 1$ be given. There holds*

$$\mathcal{B}(u') \bigcap \mathcal{B}_0(v') = \Big\{ -n \in \mathbb{Z} :\ 0 \le \{\frac{n}{u'}\} < \frac{1}{u'} \quad \textit{and} \quad 0 < \{\frac{n}{v'}\} < \frac{1}{v'} \Big\}.$$

Proof. The argument for $\mathcal{B}(u') \bigcap \mathcal{B}_0(v')$ is similar to that in Lemma 5.8, except that we start with the non-strict inequality $0 \le x_1 < 1$ in $m_1u' = -n + x_1$, but we still have $m_2v' = -n + x_2$ with $0 < x_2 < 1$. ☐

Lemma 7.8. *The "exceptional set"*

$$X_{u,v} := \{ \lfloor x \rfloor : x \in (u\mathbb{Z} \cap v\mathbb{Z}) \} = \mathcal{B}(\mathrm{lcm}(u,v))$$

satisfies

$$X_{u,v} = \Big\{ -n \in \mathbb{Z} :\ (\{\frac{n}{u}\}, \{\frac{n}{v}\}) = \lambda(\frac{1}{u}, \frac{1}{v}) \quad \textit{for some } 0 \le \lambda < 1 \Big\}$$

Proof. Let $w = \mathrm{lcm}(u,v)$, so there are integers m_1, m_2 such that $w = m_1u$ and $w = m_2v$. Suppose $-n = \lfloor mw \rfloor \in X_{u,v}$ where m is an integer. Then

$$mm_1u = mw = \lfloor mw \rfloor + \{mw\} = -n + \{mw\},$$

so after dividing by u we have

$$mm_1 = -\frac{n}{u} + \frac{\{mw\}}{u} \equiv 0 \,(\mathrm{mod}\ 1).$$

Rearranging this equation gives the fractional part of $\frac{n}{u}$:

$$\{\frac{n}{u}\} = \{mw\}\frac{1}{u}$$

since $0 \le \{mw\}\frac{1}{u} < \frac{1}{u} < 1$. Similarly, we have

$$\{\frac{n}{v}\} = \{mw\}\frac{1}{v},$$

so taking $\lambda = \{mw\} \in [0, 1)$ gives one direction of the claim, that is

$$X_{u,v} \subseteq \big\{ -n \in \mathbb{Z} :\ (\{\frac{n}{u}\}, \{\frac{n}{v}\}) = \lambda(\frac{1}{u}, \frac{1}{v}) \quad \text{for some } 0 \le \lambda < 1 \big\}.$$

The other direction follows from taking the same steps in reverse: given n and λ satisfying the specified conditions, $\frac{n}{u} = \lfloor \frac{n}{u} \rfloor + \{\frac{n}{u}\} = m_1 + \frac{\lambda}{u}$ so multiplying by u gives $n = m_1 u + \lambda$. This implies $\lambda - n = -m_1 u \in u\mathbb{Z}$, and the same reasoning gives $\lambda - n \in v\mathbb{Z}$. Thus $\lambda - n \in u\mathbb{Z} \cap v\mathbb{Z} = w\mathbb{Z}$. Since $0 \le \lambda < 1$, taking the floor gives

$$-n = \lfloor \lambda - n \rfloor \in X_{u,v}$$

as desired. ☐

The preceeding two lemmas may be combined to give a convenient criterion for when (1.1) is satisfied in the third quadrant. This criterion uses the new "torus coordinates" $X' = \frac{1}{u'}$, $Y' = \frac{1}{v'}$ analogous to the first quadrant.

Proposition 7.9. (Torus subgroup criterion) *For $u' \ge 1$ and $v' \ge 1$, the set*

$$\Big(\mathcal{B}(u') \bigcap \mathcal{B}_0(v')\Big) \smallsetminus X_{u',v'} = \emptyset$$

if and only if all integers n such that

$$0 \le \{nX'\} < X' \quad \textit{and} \quad 0 < \{nY'\} < Y'. \tag{7.10}$$

satisfy

$$(\{nX'\}, \{nY'\}) = \lambda(X', Y') \quad \textit{for some } 0 \le \lambda < 1. \tag{7.11}$$

Proof. This follows directly from Lemmas 7.7 and 7.8. ☐

As a consequence of Lemma 7.6 and Lemma 7.2, the conditions of the previous proposition are equivalent to the main inequality (1.1) for α and β, where $(u', v') = (-\frac{1}{\beta}, \frac{\alpha}{\beta})$ and $(X', Y') = (-\beta, \frac{\beta}{\alpha})$,

8. Third Quadrant Analysis: Proposition 4.2

We now complete the analysis of the third quadrant case in Theorem 1.1, proving Proposition 4.2 in Section 8.5. In this case sporadic rational solutions do exist.

8.1. **Case analysis for third quadrant solutions.** We recall that $v' \geq 1$ is a necessary condition for (1.1) to hold in the third quadrant case (see (4.6) in Section 4.3); furthermore (1.1) always holds for $v' = 1$ by case *(iii-a)* with $m_1 = 1$. It therefore suffices to treat the case $v' > 1$ in this section.

We split the analysis of criterion (7.6) when $v' > 1$ splits into cases, depending on whether $0 < u' < 1$ or $u' \geq 1$. This distinction is made because solutions on hyperbolas are confined to the second case $u' > 1$, The case division we use is:

(1) Case 1. $0 < u' < 1$; line segment and sporadic rational solutions (Sect. 8.2)
(2) Case 2. $u' \geq 1$ (Sect. 8.3 and 8.4)
  (a) $\frac{u'}{v'}$ is irrational. (Sect. 8.3)
    (i) u' is irrational: hyperbola solutions (Sect. 8.3.1)
    (ii) u' is rational: no solutions (Sect. 8.3.2)
  (b) $\frac{u'}{v'}$ is rational. (Sect. 8.4)
    (i) one-parameter solutions, straight line segments (Sect. 8.4.1)
    (ii) rational solutions on hyperbolas (Sect. 8.4.2)
    (iii) sporadic rational solutions (Sect. 8.4.3)

The case (2)(b)(iii) is not covered in Proposition 4.2 and we treat it further in Sect. 9. We consider these possibilities in turn, in the next three subsections.

8.2. **Case 1.** $0 < u' < 1$**.** We assume $v' > 1$, and treat the range $0 < u' < 1$.

Lemma 8.1. *For* $0 < u' < 1 < v'$*, the condition*

$$u'\mathbb{Z} \bigcap \widetilde{R}^{\pm}_{v'} = \emptyset$$

holds if and only if one of the following holds.

(a) (Continuous family) For integer $m_1 \geq 1$ *all values* (u', v') *with* $u' > 0$ *and* $v' = m_1$*.*
(b) (Sporadic solutions) For integers $m_2 \geq 2$*,* $r \geq 2$ *and* j *with* $1 \leq j \leq m_2 - 1$ *and* j *not a multiple of* r*, all points* (u', v') *with* $u' = 1 - \frac{j}{m_2 r}$ *and* $v' = m_2 - \frac{j}{r}$

Proof. (a) If $v' = m_1$ for $m_1 \geq 1$ an integer, then $\mathcal{B}_0(v') = \mathcal{B}_0(m_1) = \emptyset$, whence $\widetilde{R}^{\pm}_{v'} = \emptyset$.

(b) Suppose next that v' is not an integer. Then the residual set $\widetilde{R}^{\pm}_{v'}$ contains the punctured interval $\big[\lfloor v' \rfloor, \lceil v' \rceil\big] \smallsetminus v'$ of unit length. The condition $0 < u' < 1$ implies that $u'\mathbb{Z}$ contains at least one element in every open unit interval $(n, n+1)$. In particular when $n = \lfloor v' \rfloor$, the disjointness of the intersection implies that some element of $u'\mathbb{Z}$ must fall at the puncture v'. Thus we obtain the constraint $v' = m_2 u'$ for some integer m_2, whence $v'/u' = m_2$; necessarily $m_2 \geq 2$ because $u' < 1 < v'$.

The condition that for each $k \in \mathbb{Z}$ the interval $[\lfloor kv' \rfloor, \lceil kv' \rceil]$ not contain two points in $u'\mathbb{Z}$ is equivalent to the condition that the two neighboring points $kv' \pm u'$ lie in different unit intervals from kv'; that is,

$$kv' - u' < \lfloor kv' \rfloor \quad \text{and} \quad kv' + u' > \lceil kv' \rceil.$$

Equivalently

$$\{kv'\} < u' \quad \text{and} \quad 1 - \{kv'\} < u'$$

which requires that

$$\frac{1}{2} \leq \max\Big(\{kv'\}, 1 - \{kv'\}\Big) < u' < 1$$

holds for all $k \in \mathbb{Z}$ such that $kv' \notin \mathbb{Z}$. If v' is irrational then the fractional parts $\{kv'\}$ are dense in $(0,1)$ so this case is ruled out. It remains to consider rational $v' = \frac{s}{r}$, with the fraction given in lowest terms. Then the nonzero values of fractional parts $\{kv'\}$ fall in the interval $[\frac{1}{r}, 1-\frac{1}{r}]$, and attain both endpoints. It follows that $u' = \frac{s}{m_2 r}$ must satisfy

$$1 - \frac{1}{r} < u' = \frac{s}{m_2 r} < 1,$$

and this is sufficient to force the "different neighboring intervals" condition to hold for all $k \in \mathbb{Z}$ (with $kv' \notin \mathbb{Z}$). After multiplying by $m_2 r$, this condition becomes

$$m_2 r - m_2 < s < m_2 r,$$

whence we have $s = m_2 r - j$ for some j with $1 \le j \le m_2 - 1$, (with fraction $\frac{s}{m_2 r}$ not necessarily in lowest terms). We conclude that the variable v' must be a rational number satisfying the conditions specified in case (b). Conversely if a rational $v' = m_2 - \frac{j}{r}$ does satisfy the case (b) conditions then we obtain a disjoint intersection, by reversing the argument. □

Remark 8.2. We express the two cases in Lemma 8.1 in terms of the (α, β) parameters.

(1) Case (a) gives continuous families $\beta = \frac{1}{m_1}\alpha$, with $m_1 \ge 1$ which belong to the already-proved case *(iii-a)* in Theorem 1.1, but allow only a restricted range of α parameters.
(2) Case (b) corresponds to $\alpha = -m_2$ for integers $m_2 \ge 2$ but has a discrete set of solutions $(\alpha, \beta) = (-m_2, -(\frac{m_2 r}{m_2 r - j}))$ with $r \ge 1$ and $1 \le j \le m_2 - 1$, whence $|\beta| > 1$. The case (b) solutions are all sporadic rational solutions in the third quadrant. That is, they do not lie on any line segment specified in cases *(iii-a)* and *(iii-b)* or on any hyperbola given in case *(iii-c)* of Theorem 1.1. Choosing $m_2 = r = 2, j = 1$ yields the explicit example $(u, v) = (\frac{1}{2}, \frac{2}{3})$ with associated $(\alpha, \beta) = (-2, -\frac{4}{3})$.

8.3. **Case 2 (a).** $u' \ge 1$, $\frac{u'}{v'}$ **irrational.** In this subsection we treat the case where the parameter ratio $\frac{u'}{v'}$ is irrational.

8.3.1. *Subcase 2 (a)(i).* $\frac{u'}{v'}$ *irrational,* u' *irrational.* We treat the subcase when $u' \ge 1$ is irrational (which forces $u' > 1$.)

Lemma 8.3. *(1) For* u', v' *satisfying* $u', v' \ge 1$*, such that* u' *and* $\frac{u'}{v'}$ *are both irrational, the condition* $u'\mathbb{Z} \cap \widetilde{R}^{\pm}_{v'} = \emptyset$ *holds if and only if there are integers* $m_1, m_2 \ge 1$ *such that*

$$\frac{m_1}{u'} + \frac{m_2}{v'} = 1. \tag{8.12}$$

(2) All solutions to equation (8.12) *such that* u' *is irrational, also have* $\frac{u'}{v'}$ *irrational, so have* $u'\mathbb{Z} \cap \widetilde{R}^{\pm}_{v'} = \emptyset$.

Proof. (1) Under these hypotheses Lemma 7.6 (3) yields an equivalence of $u'\mathbb{Z} \cap \widetilde{R}^{\pm}_{v'} = \emptyset$ to the condition

$$\mathcal{B}_0(u') \bigcap \mathcal{B}_0(v') = \emptyset.$$

We now may apply the criterion of Theorem 6.2 to u', which is irrational, to obtain the criterion that for such points there must exist integers $m_1, m_2 \ge 0$ such that

$$\frac{m_1}{u'} + \frac{m_2}{v'} = 1.$$

The case $m_1 = 0$ cannot occur because in that case $v' = m_2$ would be rational, and $m_2 = 0$ cannot occur because $u' = \frac{1}{m_1}$ would be rational. Thus $m_1, m_2 \ge 1$.

(2) Conversely, if for $m_1, m_2 \geq 1$ the equation has a solution with u' irrational, then we claim $\mathcal{B}_0(u') \bigcap \mathcal{B}_0(v') = \emptyset$ will hold. To see this, for $m_1, m_2 \geq 1$ the equation can be rewritten as

$$m_1 + m_2 \frac{u'}{v'} = u',$$

which implies that $\frac{u'}{v'}$ is irrational since u' is irrational. Now that both u' and $\frac{u'}{v'}$ are irrational, we have $\mathcal{B}_0(u') \bigcap \mathcal{B}_0(v') = \emptyset$ equivalent to $u'\mathbb{Z} \cap \widetilde{R}^{\pm}_{v'} = \emptyset$ by Lemma 7.6 (3). ☐

Remark 8.4. Changing variables using $u' = -\frac{1}{\beta}$ and $v' = \frac{\alpha}{\beta}$ and simplifying yields the rectangular hyperbola

$$m_1 \alpha \beta + \alpha = m_2 \beta.$$

Thus Lemma 8.3 establishes the irrational part of case *(iii-c)* of Theorem 1.1.

8.3.2. *Subcase 2 (a) (ii).* $\frac{u'}{v'}$ *irrational,* u' *rational.* We show there are no solutions in this subcase.

Lemma 8.5. *If* u', v' *satisfy* $v' > 1, u' \geq 1$*, such that* $\frac{u'}{v'}$ *is irrational and* u' *is rational, then* $u'\mathbb{Z} \bigcap \widetilde{R}^{\pm}_{v'} \neq \emptyset$.

Proof. Since $\frac{u'}{v'}$ is irrational, Lemma 7.6 (2) applies to show that empty intersection is equivalent to $\mathcal{B}(u') \bigcap \mathcal{B}_0(v') = \emptyset$. Now $u' = \frac{r}{s}$ is rational, hence $\mathcal{B}(r) \subseteq \mathcal{B}(u')$ and $\mathcal{B}(r) = r\mathbb{Z}$ is an infinite arithmetic progression. To show $\mathcal{B}(u') \bigcap \mathcal{B}_0(v') \neq \emptyset$ it suffices to show $\mathcal{B}(r) \bigcap \mathcal{B}_0(v') \neq \emptyset$, i.e. that $\mathcal{B}_0(v')$ contains a nonzero element $\lfloor kv' \rfloor$ divisible by r. The hypothesis that $\frac{u'}{v'}$ is irrational and u' is rational implies that $\theta := v'$ is irrational. More generally, for any irrational $\theta > 1$, there is some $\lfloor n\theta \rfloor$ that is divisible by any fixed integer r. For any Diophantine approximation $\frac{p}{q}$ with $0 < \theta - \frac{p}{q} \leq \frac{1}{q^2}$ we have $jp \leq \lfloor jq\theta \rfloor \leq jp + \lfloor \frac{j}{q} \rfloor$ so $\lfloor jq\theta \rfloor = jp$ for $1 \leq j \leq q-1$. Finding a large enough approximation, with $q > r+1$ we may choose $j = r$ and get the result. It follows that the intersection is nonempty; in fact it is infinite. ☐

8.4. **Case 2 (b):** $u' \geq 1$**:** $\frac{u'}{v'}$ **rational.** We suppose $v' > 1$ and $u' \geq 1$, and treat cases where the parameter ratio $\frac{u'}{v'}$ is rational. (Equivalently, α is rational.) We use the criterion in Lemma 7.6 (1), noting that the exceptional set $X_{u',v'}$ is an infinite set whenever $\frac{u'}{v'}$ is rational.

8.4.1. *Subcase 2 (b)(i).* $\frac{u'}{v'}$ *rational, continuous parameter cases.* Certain one-dimensional families for $\alpha = -\frac{m_1}{m_2}$ a fixed rational were found in the proof of case (iii-b) of Theorem 1.1, which was given in Section 4.3. For the reader's convenience we restate that result in the (u', v')-coordinates, as follows, noting that $\frac{u'}{v'} = -\frac{1}{\alpha} = \frac{m_2}{m_1}$ and $u' = -\frac{1}{\beta} \geq m_2$.

Lemma 8.6. *For each rational* $\frac{u'}{v'} = \frac{m_2}{m_1}$ *given in lowest terms, with* $m_1, m_2 \geq 1$*, one has* $u'\mathbb{Z} \bigcap \widetilde{R}^{\pm}_{v'} = \emptyset$ *for all points on the rational-slope line segment* $(u', \frac{m_1}{m_2}u')$ *having* $u' \geq m_2$. *Equivalently,* $v' \geq m_1$ *and* $u' = \frac{m_2}{m_1}v'$.

For rational $\frac{u'}{v'} = \frac{m_2}{m_1}$ the next lemma rules out any further irrational points (u', v') not covered by Lemma 8.6.

Lemma 8.7. *Suppose that* $\frac{u'}{v'} = \frac{m_2}{m_1}$ *is rational, given in lowest terms with* $m_1, m_2 \geq 1$. *Then each irrational* v' *with* $1 < v' < m_1$ *has* $u'\mathbb{Z} \bigcap \widetilde{R}^{\pm}_{v'} \neq \emptyset$.

Proof. In torus coordinates $X' = \frac{1}{u'}$ and $Y' = \frac{1}{v'}$, we have $\frac{X'}{Y'} = \frac{m_1}{m_2}$ is rational and Y' is irrational with $\frac{1}{m_1} < Y' < 1$. Let $\mathcal{O}(\mathbf{v}) = \{n\mathbf{v} : n \in \mathbb{Z}\}$ denote the subgroup generated by the point $\mathbf{v} = (X', Y') \in \mathbb{R}^2/\mathbb{Z}^2$. We define the $\mathbb{R}$-orbit of $\mathbf{v}$ by

$$\mathcal{O}_{\mathbb{R}}(\mathbf{v}) = \{\pi(\lambda\mathbf{v}) : \lambda \in \mathbb{R}\}$$

where the multiplication $\lambda\mathbf{v}$ is done in $\mathbb{R}^2$ and $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is the canonical projection map. Since $\mathbf{v}$ lies at rational slope $\frac{m_1}{m_2}$ from the origin, the $\mathbb{R}$-orbit $\mathcal{O}_{\mathbb{R}}(\mathbf{v})$ will be a closed one-dimensional submanifold of the torus, and since Y' is irrational the subgroup $\mathcal{O}(\mathbf{v})$ is dense inside of $\mathcal{O}_{\mathbb{R}}(\mathbf{v})$.

The generator $\mathbf{v}$ lies on the "main diagonal" of $\mathcal{O}(\mathbf{v})$ coming out of the origin. The assumption that $Y' > \frac{1}{m_1}$ means the rectangular region C (C for "corner") to the lower-left of $\mathbf{v}$ will contain at least two addition line segments, one on each side of the main diagonal (see figure below). Thus C contains points of $\mathcal{O}(\mathbf{v})$ not on main diagonal. By the criterion in Proposition 7.9, this implies $u'\mathbb{Z} \cap \widetilde{R}^{\pm}_{v'} \neq \emptyset$ as claimed.

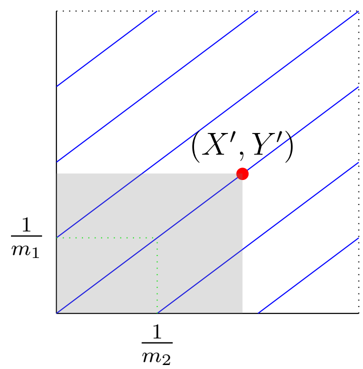

FIGURE 8.1. Real orbit $\mathcal{O}_{\mathbb{R}}(\mathbf{v})$

□

8.4.2. *Sub-case 2 (b)(ii). $\frac{u'}{v'}$ rational, non-sporadic solutions on hyperbolas.* We now consider points with rational $\frac{u'}{v'}$ that "glue into" the hyperbola solutions given in Lemma 8.3 and show that they are themselves solutions to (1.1).

Lemma 8.8. *Suppose that $u', v' > 0$ with $\frac{u'}{v'}$ rational, and that there are integers $m_1, m_2 \geq 1$ such that*

$$\frac{m_1}{u'} + \frac{m_2}{v'} = 1.$$

then $u'\mathbb{Z} \cap \widetilde{R}^{\pm}_{v'} = \emptyset$ holds. In addition v' is rational, with $v' > 1$ and $u' > 1$.

Proof. The inequalities $v' > m_1 \geq 1$ and $u' > m_2 \geq 1$ are immediate from the hyperbola equation, since both values must be positive by assumption. Lemma 7.6(1) now applies to shows the empty intersection is equivalent to the condition

$$\left(\mathcal{B}(u') \bigcap \mathcal{B}_0(v')\right) \smallsetminus X_{u',v'} = \emptyset,$$

in which

$$X_{u',v'} := \{\lfloor w \rfloor : w \in u'\mathbb{Z} \bigcap v'\mathbb{Z}\}.$$

We again study the orbit $\mathcal{O}(\mathbf{v}') = \{n\mathbf{v}' : n \in \mathbb{Z}\}$ of the vector $\mathbf{v}' := (\frac{1}{u'}, \frac{1}{v'}) = (X', Y')$ viewed on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. All members (x, y) of this orbit lie on the locus

$$S_{m_1,m_2} := \{(x, y) \in \mathbb{T}^2 : m_1 x + m_2 y \equiv 0 \,(\mathrm{mod}\, 1)\}.$$

Since $m_1, m_2 \geq 1$ this locus intersects the closure of the rectangular region

$$C_{X',Y'} := \{(x, y) : 0 \leq x < X', 0 < y < Y'\}$$

on $\mathbb{T}^2$ only in the points $(0, 0)$ and $(\frac{v'}{u'}, v')$, and neither of these points lies in $C_{X',Y'}$ itself since each y-coordinate is disallowed. By Lemma 7.7 this gives the desired result. ☐

8.4.3. *Characterizing empty intersections: Case 2(b)(ii).* $u' \geq 1$. The range $u' \geq 1$ corresponds to $\beta = -\frac{1}{u'}$ having $|\beta| \leq 1$. Here $v' = \frac{\alpha}{\beta} > 1$. We already know that for $\alpha = -\frac{m_1}{m_2}$ we have solutions for all $|\beta| \leq \frac{1}{m_2}$ so it remains to consider rational solutions with $\frac{1}{m_2} < |\beta| \leq 1$, which are sporadic if they do not fall on any hyperbola. Such sporadic solutions do exist for every such α by applying suitable self-similar symmetries in Section 2.3. We defer further study of them to Section 9.

8.5. **Completion of third quadrant case of Theorem 1.1.** Now we can complete the proof of Theorem 1.1 in the third quadrant case, which we reduced in Section 4.3 to showing Proposition 4.2. That proposition classifies some, but not all, sporadic rational solutions.

Proof of Proposition 4.2. (1) Lemma 8.3 shows that irrational solutions on a rectangular hyperbola (8.12) with $m_1, m_2 \geq 1$ will satisfy $u'\mathbb{Z} \bigcap \widetilde{R}^{\pm}_{v'} = \emptyset$, whence Lemma 7.2 implies that (1.1) holds for the associated (α, β). Lemma 8.8 shows that all rational solutions on these hyperbolas satisfy $u'\mathbb{Z} \bigcap \widetilde{R}^{\pm}_{v'} = \emptyset$, whence (1.1) holds for the associated rational (α, β).

(2) For the parameter range $0 < u' < 1$, Lemma 8.1 excludes all (u', v') having at least one irrational coordinate, in that the solutions listed in (a) are removed by exclusion (a), while those listed in (b) have no irrational coordinate. For the parameter value $u' = 1$, u'/v' and v' are both irrational or both rational, so in the irrational case, u'/v' is irrational, whence Lemma 8.6 excludes all solutions. For the remaining parameter range $u' > 1$, suppose first u'/v' is rational. Writing $\frac{u'}{v'} = \frac{m_2}{m_1}$, Lemma 8.6 excludes all irrational v having $1 < v' < m_1$, which covers the complete range of v' not excluded by exclusion (b). (Recall that $v' \leq 1$, see (3) below.) Finally suppose $u' > 1$ and u'/v' is irrational. In this case Lemma 8.3 and Lemma **??** together exclude all solutions (u', v') not on one of the hyperbolas excluded by exclusion (c).

(3) Lemma 8.1 (b) classified all the sporadic rational solutions having $0 < u' < 1 < v'$, which are all such solutions having $u' < 1$. These consist of and infinite set (u', v') with $u' = 1 - \frac{j}{m_1 r}$ and $v' = m_1 - \frac{j}{r}$ for some integers $m_1 \geq 2$, $r \geq 2$ and $1 \leq j \leq m_1 - 1$. All other sporadic rational solutions necessarily have $u' > 1$. ☐

9. Third Quadrant Sporadic Rational Solutions with $u', v' \geq 1$

We follow the analysis of sporadic rational solutions in the first quadrant case, in giving a necessary and sufficient condition for their existence.

9.1. **Sporadic rational solutions in Third Quadrant.** We assume $u' \geq 1$. This case considers points with rational u' do not belong to any parametric continuous families, those given in Lemmas 8.3 and 8.6.

We have classified the sporadic rational solutions (α, β) in the third quadrant case corresponding to (u', v')-values having $0 < u' < 1$, where $(u', v') = (-\frac{1}{\beta}, \frac{\alpha}{\beta})$. It remains to deal with possible sporadic rational solutions having $u' \geq 1$. These are all rational points (u', v') with associated (α, β) satisfying (1.1) excluding any of the following types:

(iii-a) For integer $m_1 \geq 1$, any $v' = m_1$ with $u' \geq 1$.
(iii-b) For integers $m_1, m_2 \geq 1$ any $u' \geq m_1$ with $v' = \frac{m_2}{m_1} u' \geq m_2$.
(iii-c) For integers $m_1, m_2 \geq 1$ any rational point with $u', v' > 0$ on the hyperbola

$$\frac{m_1}{u'} + \frac{m_2}{v'} = 1.$$

These are the three parametric continuous families (a)-(c) excluded in Proposition 4.2 (2).

We recall the (X', Y')-coordinates defined by

$$X' := \frac{1}{u'} = -\beta, \quad Y' := \frac{1}{v'} = \frac{\beta}{\alpha}.$$

These coordinates are a birational change of variable from (α, β) in the open third quadrant, which range over the open first quadrant $X' > 0$, $Y' > 0$. We coordinatize a rational point in these coordinates as

$$(X', Y') = (\frac{c}{t}, \frac{d}{t}),$$

with least common denominator t, requiring $\gcd(c, d, t) = 1$.

(Explicitly, t is chosen to be the unique positive integer such that

$$\frac{1}{t}\mathbb{Z} = \mathbb{Z} + X'\mathbb{Z} + Y'\mathbb{Z},$$

i.e. $\frac{1}{t} =$ " gcd "$(1, X', Y')$, and then c and d are determined by $c = tX'$, $d = tY'$.)

The following result gives a criterion such sporadic rational solutions must satisfy.

Proposition 9.1. (Sporadic Rational Solution Criterion-Third Quadrant) *Let* $(u', v') = (-\frac{1}{\beta}, \frac{\alpha}{\beta})$ *and suppose that* $u' \geq 1$ *is rational. Set* $(X', Y') = (\frac{1}{u'}, \frac{1}{v'}) := (\frac{c}{t}, \frac{d}{t})$, *with integers* $c, d, t > 0$ *with* $\gcd(c, d, t) = 1$.

(i) The point $(\alpha, \beta) = (-\frac{c}{d}, -\frac{c}{t})$ *is a rational solution to* (1.1) *if and only if all integers* $0 \leq n < t$ *such that*

$$0 \leq \{nX'\} < X' \quad \textit{and} \quad 0 < \{nY'\} < Y'.$$

satisfy

$$(\{nX'\}, \{nY'\}) = \lambda(X', Y') \quad \textit{for some } 0 \leq \lambda < 1. \tag{9.13}$$

(ii) The point $(\alpha, \beta) = (-\frac{c}{d}, -\frac{c}{t})$ *is a sporadic rational solution if, in addition, the denominator* t *has the property that* $t \leq \frac{cd}{(c,d)} = [c, d]$ *and the linear Diophantine equation*

$$cn_1 + dn_2 = t \tag{9.14}$$

is unsolvable in non-negative integers (n_1, n_2) *having* $n_2 \geq 1$.

Proof. In parallel to Proposition 6.7, the role of condition (i) is to make (1.1) hold, and the role of condition (ii) is to exclude membership in the three one-parameter families of solutions.

Criterion (i) is simply a restatement of Proposition 7.9, with the added observation that the torus orbit generated by $\mathbf{v} = (X', Y')$ is periodic modulo t, so is suffices to check a representative from each residue class. This reduces the question of checking rational solutions to a finite computation.

We show that criterion (ii) is equivalent to $(\frac{1}{u'}, \frac{1}{v'})$ not falling on any one-parameter solution to (1.1). These are classified as exclusion types *(iii-a)* through *(iii-c)* in this subcase. By Proposition 4.2 and the discussion after its statement, the one-parameter solutions of exclusion type *(iii-c)* are given by $\frac{m_1}{u'} + \frac{m_2}{v'} = 1$ for any integer $m_1, m_2 \geq 1$. The condition for $(\frac{1}{u'}, \frac{1}{v'}) = (\frac{c}{t}, \frac{d}{t})$ to fall on such a curve is $m_1\frac{c}{t} + m_2\frac{d}{t} = 1$, which on clearing denominators becomes $m_1 c + m_2 d = t$. This condition is easily seen to be equivalent to the equation

$$cX_1 + dX_2 = t,$$

having no solution in positive integers $X_1, X_2 \geq 1$.

Exclusion type *(iii-a)* removes $v' = m_1$ for all integers $m_1 \geq 1$, which corresponds to the equation $dX_2 = t$ having no positive integer solution; this excludes solutions above having $X_1 = 0$.

Exclusion *(iii-b)* removes all solutions with $\frac{u'}{v'} = \frac{m_2}{m_1}$ having also $v' \geq m_1$, if $\frac{m_2}{m_1}$ is in lowest terms. Suppose $\frac{c_0}{d_0} = \frac{c}{d} = \frac{X'}{Y'} = \frac{v'}{u'}$ expresses this ratio in lowest terms. Explicitly, $c_0 = \frac{c}{(c,d)}$ and $d_0 = \frac{d}{(c,d)}$ where (c, d) is the greatest common divisor. Then $v' = \frac{t}{d} \geq c_0$ is equivalent to

$$t \geq c_0 d = \frac{cd}{(c, d)} = [c, d]$$

as claimed, where $[c, d]$ is the least common multiple. $\square$

We now provide some intuition for how sporadic solutions may be found computationally with an extended example.

Example 9.2. Consider when (X', Y') lies on the line of slope 2/3, so $(X', Y') = (\frac{3n}{t}, \frac{2n}{t})$ for some coprime integers n, t. (i.e. $(c, d, t) = (c_0 n, d_0 n, t)$ with $(c_0, d_0) = (3, 2)$) Since we assume here $X', Y' \leq 1$, these integers must satisfy $\frac{n}{t} \leq \frac{1}{3}$. If $\frac{n}{t} \leq \frac{1}{6}$, then the necessary condition *(i)* is satisfied. These are non-sporadic rational solutions, falling in case *(iii-b)*.

If $\frac{1}{6} < \frac{n}{t} \leq \frac{1}{3}$ satisfies (i) and (ii), it is a sporadic rational solution. Since n and t are coprime, the subgroup H generated by (X', Y') will have t points in the torus. (i.e. $H = \langle (\frac{3n}{t}, \frac{2n}{t}) \rangle = \langle (\frac{3}{t}, \frac{2}{t}) \rangle$) To satisfy (i), there are essentially two obstructions: the lower-left rectangle could contain a "bad" point along the line segment $L_1 \subset H_{\mathbb{R}}$ starting at $(\frac{1}{2}, 0)$ on the bottom axis, or it could contain a "bad" point along the segment $L_2 \subset H_{\mathbb{R}}$ starting at $(0, \frac{1}{3})$ on the left axis. The smallest point in H on segment L_1 has horizontal coordinate $\frac{3}{t}(\lfloor \frac{1}{2}t \rfloor + 1) - 1$, so a necessary condition is

$$\text{(x-coord)} \quad \frac{3n}{t} \leq \frac{3}{t}(\lfloor \frac{1}{2}t \rfloor + 1) - 1.$$

The smallest point in H on segment L_2 has vertical coordinate $\frac{2}{t}\lceil \frac{2}{3}t \rceil - 1$, so a second necessary condition is

$$\text{(y-coord)} \quad \frac{2n}{t} \leq \frac{2}{t}\lceil \frac{2}{3}t \rceil - 1.$$

Solving these inequalities for n yields

$$n \le \min\left\{ \left\lfloor \frac{1}{2}t \right\rfloor + 1 - \frac{t}{3}, \quad \left\lceil \frac{2}{3}t \right\rceil - \frac{t}{2} \right\}$$

The floor and ceiling functions in the two expressions may be broken down into cases modulo 2 and 3, respectively. For example

$$\left\lfloor \frac{1}{2}t \right\rfloor + 1 - \frac{t}{3} = \begin{cases} (\frac{t}{2}) + 1 - \frac{t}{3} = \frac{t}{6} + 1 & \text{if } t \text{ even} \\ (\frac{t-1}{2}) + 1 - \frac{t}{3} = \frac{t}{6} + \frac{1}{2} & \text{if } t \text{ odd} \end{cases}$$

Considering the residue of t modulo 6, this is equivalent to

$$n \le \begin{cases} \frac{t}{6} & \text{if } t \equiv 0, 3\,(6) \\ \frac{t}{6} + \frac{1}{3} & \text{if } t \equiv 1, 4\,(6) \\ \frac{t}{6} + \frac{2}{3} & \text{if } t \equiv 2\,(6) \\ \frac{t}{6} + \frac{1}{2} & \text{if } t \equiv 5\,(6). \end{cases}$$

Since n must be integral, this may be further simplified by applying floors to the right-hand expressions. This gives

$$n \le \begin{cases} \lfloor \frac{t}{6} \rfloor & \text{if } t \equiv 0, 1, 3\,(6) \\ \lfloor \frac{t}{6} \rfloor + 1 & \text{if } t \equiv 2, 4, 5\,(6). \end{cases}$$

The first case, $n \le \lfloor \frac{t}{6} \rfloor$ is not interesting, because it produces a non-sporadic solution. The sporadic solutions come from the second case, $n > \lfloor \frac{t}{6} \rfloor$:

$$n = \begin{cases} \frac{t+4}{6} & \text{if } t \equiv 2\,(6) \\ \frac{t+2}{6} & \text{if } t \equiv 4\,(6) \\ \frac{t+1}{6} & \text{if } t \equiv 5\,(6). \end{cases}$$

The corresponding values of (X, Y) are, for $t \equiv 2\,(6)$,

$$(X, Y) = (\frac{3n}{t}, \frac{2n}{t}) = (\frac{t+4}{2t}, \frac{t+4}{3t})$$

or, taking $6r = t + 4$,

$$(X, Y) = (\frac{6r}{12r - 8}, \frac{6r}{18r - 12}) = (\frac{3r}{6r - 4}, \frac{2r}{6r - 4}).$$

It should be straightforward to carry out this same computation with (3,2) replaced with any pair of coprime positive integers (c_0, d_0). The end result will be

$$n \le \begin{cases} \lfloor \frac{t}{c_0 d_0} \rfloor & \text{if } t \equiv a_i\,(c_0 d_0) \\ \lfloor \frac{t}{c_0 d_0} \rfloor + 1 & \text{if } t \equiv b_i\,(c_0 d_0). \end{cases}$$

for some residue classes a_i, b_i modulo $c_0 d_0$. Why can we never have $n = \lfloor \frac{t}{c_0 d_0} \rfloor + 2$? If so, then $(X', Y') = (\frac{c_0 n}{t}, \frac{d_0 n}{t})$ will span a lower-left rectangle that intersects $L_1 \subset H_{\mathbb{R}}$, the segment starting at $(\frac{1}{d_0}, 0)$, at a horizontal-length strictly greater than $\frac{c_0}{t}$. Since the points in H are evenly spaced, this implies the intersection must contain at least one point in its interior, violating the condition of Proposition 9.1 (i). This has some immediate consequences.

Proposition 9.3. *Suppose*

$$(X', Y') = (\frac{c_0 n}{t}, \frac{d_0 n}{t})$$

corresponds to a sporadic rational solution to (1.1)*, where* c_0, d_0 *are coprime positive integers and* n, t *are pairwise coprime positive integers. Then*

$$n \geq 2$$

and

$$t > c_0 d_0$$

and

$$\frac{t}{c_0 d_0} < n < \frac{t}{c_0 d_0} + 1.$$

Proof. If $n = 1$, then the exceptional set only contains the 0 residue class modulo t by the criterion in (ref. diagonal line lemma). Thus the Beatty sequence disjointness criterion is equivalent to the disjointness condition in the first quadrant. These solutions are non-sporadic since they fall in case *(a)* or *(c)*. Thus $n \geq 2$ is necessary for (X', Y') to be a sporadic solution.

Assuming for now we have proved the third line of inequalities, the bounds

$$2 \leq n < \frac{t}{c_0 d_0} + 1$$

clearly imply $t > c_0 d_0$ after subtracting 1 from both sides.

We now prove the inequalities in the third line by contradiction. If $n \leq \frac{t}{c_0 d_0}$, then the corresponding point $(u', v') = (\frac{t}{c_0 n}, \frac{t}{d_0 n})$ satisfies

$$v' = \frac{t}{d_0} \cdot \frac{1}{n} \geq \frac{t}{d_0} (\frac{c_0 d_0}{t}) = c_0$$

which is the numerator of the ratio $\frac{v'}{u'} = \frac{c_0}{d_0}$ in lowest terms. Thus (X', Y') is non-sporadic since it falls in case *(b)* according to Proposition 4.2.

If $n \geq \frac{t}{c_0 d_0} + 1$, then we must consider the configuration of the torus subgroup more carefully. As before let $H \subset \mathbb{R}^2/\mathbb{Z}^2$ denote the subgroup generated by the point (X', Y'), so that H consists of all integer multiples of (X', Y'), and let $H_{\mathbb{R}}$ consist of all real multiples of (X', Y'). (Since $\mathbb{R}^2/\mathbb{Z}^2$ is not uniquely divisible, to define $H_{\mathbb{R}}$ we should first take all $\mathbb{R}$-multiples of (X', Y') in $\mathbb{R}^2$, and then let $H_{\mathbb{R}}$ be the image under the projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.) Note that $H_{\mathbb{R}}$ depends only on the parameters (c_0, d_0), and not on (n, t).

Let R denote the region to the lower-right of the point (X', Y'). Let L_1 be the first segment below the main diagonal in R, and let L_2 be the first segment in R to the left of the main diagonal. (See figure below.) Because of the Beatty sequence criterion (ref?), we consider L_1 with both endpoints open, and L_2 with its left-endpoint closed and its right-endpoint open. If L_2 contains a point of the subgroup H, then (X', Y') fails the Beatty intersection criterion (ref?).

We claim that this is the case if $n \geq \frac{t}{c_0 d_0} + 1$. If so, the height of L_2 (i.e. the length of its projection on the y-axis) is

$$\text{height}(L_2) = Y' - \frac{1}{c_0} = \frac{d_0 n}{t} - \frac{1}{c_0} \geq \frac{d_0}{t}\left(\frac{t}{c_0 d_0} + 1\right) - \frac{1}{c_0} = \frac{d_0}{t}.$$

However, since n and t are assumed coprime the subgroup H is generated by the point

$$\frac{1}{n}(X', Y') = (\frac{c_0}{t}, \frac{d_0}{t}).$$

Thus the vertical height between consecutive points of H is exactly $\frac{d_0}{t}$. Since Y' has height no smaller than this gap-distance $\frac{d_0}{t}$, it must contain a point in the subgroup H. This violates the disjointness criterion so (X', Y') does not correspond to a solution. □

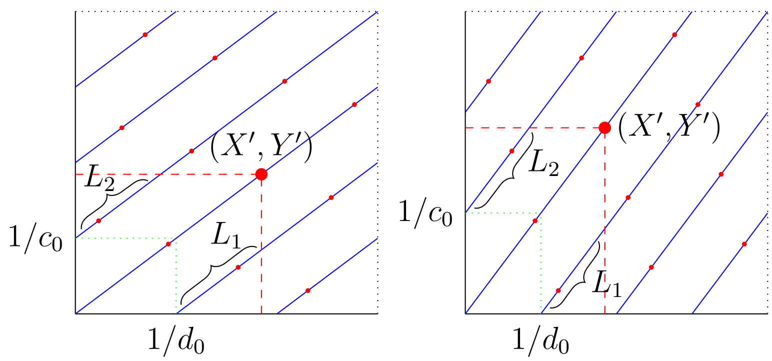

FIGURE 9.1. Torus criterion with $(c_0, d_0) = (4, 3)$. The subgroup H is indicated by red dots; $H_{\mathbb{R}}$ is indicated by blue lines.

Corollary 9.4. *If*

$$(X', Y') = (\frac{c_0 n}{t}, \frac{d_0 n}{t})$$

corresponds to a sporadic rational solution to (1.1)*, with* c_0, d_0 *coprime and* n, t *coprime, then*

$$\frac{1}{d_0} < X' < \frac{2}{d_0} \quad \textit{and} \quad \frac{1}{c_0} < Y' < \frac{2}{c_0}.$$

Proof. By the previous proposition, sporadic rational solutions satisfy $\frac{t}{c_0 d_0} < n$ which implies the lower bound on X'. The previous proposition also shows

$$n < \frac{t}{c_0 d_0} + 1 \quad \text{and} \quad t > c_0 d_0.$$

These bounds on n and t imply

$$X' = \frac{c_0}{t} n < \frac{c_0}{t}\left(\frac{t}{c_0 d_0} + 1\right) = \frac{1}{d_0} + \frac{c_0}{t} < \frac{1}{d_0} + \frac{c_0}{c_0 d_0} = \frac{2}{d_0}.$$

The same argument applies for Y'. □

Remark 9.5. This corollary should be interpreted as saying that the "sporadic range" extends at most twice as far as the "stable range" i.e. case *(iii-b)*.

Corollary 9.6. *If* $r = \frac{c_0}{d_0}$ *is a fixed positive rational in reduced terms, then the set* S_r^* *of sporadic solutions to (1) in (X',Y')-coordinates satisfying* $\frac{X'}{Y'} = r$ *is a discrete set of points in the open segment from* $(\frac{1}{d_0}, \frac{1}{c_0})$ *to* $(\frac{2}{d_0}, \frac{2}{c_0})$*, with at most a single limit point at the lower endpoint* $(\frac{1}{d_0}, \frac{1}{c_0})$*.*

Proof. We have shown that all sporadic solutions lie in the stated range. It remains to show that the topology of these points is as claimed. The bounds

$$\frac{t}{c_0 d_0} < n < \frac{t}{c_0 d_0} + 1$$

and the restriction that n is integral implies that for fixed choice of c_0, d_0, and t, there is at most one n-value that produces a sporadic solution, namely $n = \lceil \frac{t}{c_0 d_0} \rceil$. This condition is necessary but not sufficient; for example, one also needs n to be relatively prime to t. This implies the sporadic solutions are a subset $S_r^* \subset T_r$ where

$$T_r := \{(\frac{c_0}{t}\lceil \frac{t}{c_0 d_0} \rceil, \frac{d_0}{t}\lceil \frac{t}{c_0 d_0} \rceil) : t \geq c_0 d_0 + 1\} = \{(r_{c_0/t}(\frac{1}{d_0}), r_{d_0/t}(\frac{1}{c_0}) : t \geq c_0 d_0 + 1\}.$$

The set T_r is discrete and has a single limit point $(\frac{1}{d_0}, \frac{1}{c_0})$. Thus the same is true for any subset, which proves the claim. $\square$

9.2. **Closure of solution set.** We conclude here with a proof that the solutions to (1.1) form a closed subset of the plane, as an application of the partial results on sporadic solutions in the previous section.

Corollary 9.7. *The set S of solutions to* (1.1) *is a closed subset of* $\mathbb{R}^2$.

Proof. This claim clearly holds in the (closed) first, second and fourth quadrants by our classification of solutions given above. It remains to show that the third quadrant solutions form a closed subset of the plane.

Let S_1 denote the union of the coordinate axes with all non-sporadic solutions in the third quadrant, meaning those in cases *(iii-a)*, *(iii-b)* and *(iii-c)* of Theorem 1.1. Let S_2 denote the union of the coordinate axes with third quadrant solutions in case *(iii-b)* and case *(iii-b')*. It suffices to show that both S_1 and S_2 are closed. For S_1 this is clear.

To see that S_2 is closed we consider some Cauchy sequence of points $p_i = (\alpha_i, \beta_i)$ in S_2 which converge to $p = \lim p_i$ in $\mathbb{R}^2$. If the β-coordinates β_i tend to zero, then the limit p lies on the α-axis which is contained in S_2. Otherwise, if $\beta(p) < 0$ there exists some $\epsilon > 0$ such that $\beta_i < -\epsilon$ for sufficiently large i. By Corollary 9.6 (translated into (α, β) coordinates) the points in S_2 with α-coordinate of the form $\alpha = -\frac{m}{n}$ in lowest terms all have β-coordinate in the range

$$0 < |\beta| < \frac{2}{n}.$$

Therefore the bound $|\beta_i| > \epsilon$ implies the corresponding α-coordinate $\alpha_i = -\frac{m_i}{n_i}$ in reduced terms must satisfy $\epsilon < \frac{2}{n_i}$ or equivalently $n_i < \frac{2}{\epsilon}$. The set of rationals with bounded denominator forms a discrete subset of $\mathbb{R}$ so the α_i-coordinates are eventually constant. Corollary 9.6 proves that the restriction of S_2 to a fixed α-coordinate is closed, so again the limit p lies in S_2. This completes the proof. $\square$

REFERENCES

1. Th. Bang, *On the sequences* $[n\alpha], n = 1, 2, ...$ *Supplementary note to the preceding paper by Th. Skolem,* Math. Scand. **5** (1957), 69–76.
2. Samuel Beatty, *Problem 3173*, Amer. Math. Monthly **33** (1926), No. 3, 159 (Solution: Amer. Math. Monthly **34** (1927), 159–160.)
3. V. Bergelson and A. Leibman, Distribution of values of bounded generalized polynomials, *Acta Math.* **198**(2007), no.2, 155–230.

4. J.-P. Cardinal, *Symmetric matrices related to the Mertens function,* Lin. Alg. Appl. **432** (2010), no. 1, 161–172.
5. P. Erdős and R. L. Graham, *On a linear Diophantine problem of Frobenius,* Acta. Arithmetica **21** (1972), 399–408.
6. A. S. Fraenkel, *The bracket function and complementary sets of integers,* Canad. J. Math. **21** (1969), 6–27.
7. A. S. Fraenkel, *Complementing and exactly covering sequences,* J. Comb. Th. Series A **14** (1973), 8–20.
8. A. S. Fraenkel, *Iterated floor function, algebraic numbers, discrete chaos, Beatty subsequences, semigroups,* Trans. Amer. Math. Soc. **341** (1994), no. 2, 639–664.
9. R. L.Graham, *On a theorem of Uspensky,* Amer. Math. Monthly **70** (1963), 407–409.
10. R. L. Graham, *Covering the positive integers by disjoint sets of the form* $\{\lfloor n\alpha+\beta\rfloor : n = 1, 2, ...\}$, J. Comb. Theory, Series A **15** (1973), 354–358.
11. R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics. Second Edition,* Addison-Wesley: Reading, Mass. 1994.
12. R. Graham and K. O'Bryant, *Can you hear the shape of a Beatty sequence?* in: *Additive Number Theory, Festscrift in Honor of the Sixtieth Birthday of Melvyn B. Nathanson,* D. Chudnovsky and G. Chudnovsky (Eds.), pp. 39–52, Springer: New York 2010.
13. Inger Johanne Høaland and D. E. Knuth, *Polynomials involving the floor function,* Math. Scand. **76** (1995), 194–200.
14. J. C. Lagarias, T. Murayama, and D. H. Richman, *Dilated floor functions that commute,* American Math. Monthly, to appear.
15. S. Lang, *Introduction to Diophantine Approximations,* Addison-Wesley, Reading, MA 1966.
16. P. Moree, *Numerical semigroups, cyclotomic polynomials and Bernoulli numbers,* Amer. Math. Monthly **121** (2014), No 10, 890–902.
17. Ryozo Morikawa, *Disjoint sequences generated by the bracket function,* Bull. Faculty Liberal Arts, Nagasaki Univ. (Natural Science) **26**, (1985), no. 1, 1–13.
18. K. O'Bryant, *Fraenkel's partition and Brown's decomposition,* Integers **3** (2003), A11, 17pp.
19. J. L. Ramírez-Alfonsín, *Complexity of the Frobenius problem,* Combinatorica **16** (1996), no. 1, 143–147.
20. J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem,* Oxford University Press: Oxford 2005.
21. T. Skolem, *On certain distributions of integers in pairs with given differences,* Math. Scand. **5** (1957), 57–68.
22. T. Skolem, *Über einige Eigenschaften der Zahlenmengen $[\alpha n + \beta]$ bei irrationalem α mit einleitenden Bermerkungen über einige kombinatorische Probleme,* Norske Vid. Selsk. Forh., Trondheim **30** (1957), 42–49.
23. J. J. Sylvester, *On subinvariants, i.e. semi-invariants to a binary quantity of an unlimited order,* Amer. J. Math. **5** (1882), 119–136.
24. J. J. Sylvester, *Problem 7382,* Educational Times bf 37 (1884), 26; reprinted in Mathematical questions with their solution, Educational Times **41** (1884), 21.
25. J. V. Uspensky, *On a problem arising out of the theory of a certain game,* Amer. Math. Monthly **34** (1927), No. 10, 516–521.

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