# The Square Tile Problem

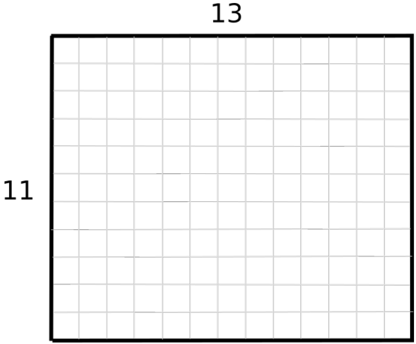
Harry Richman

University of Michigan

15 November 2018

# The Problem

Floor:



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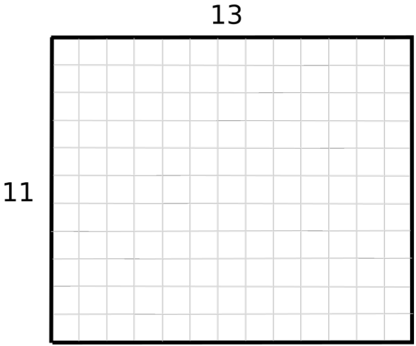


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- square tiles of any size, \$1 each
- what is minimal cost $C(11, 13)$?

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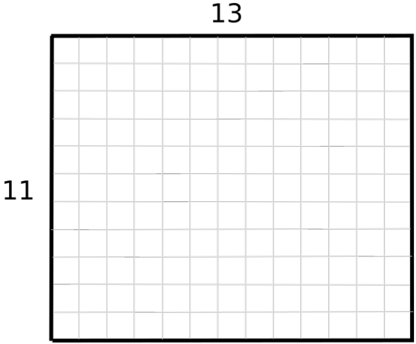


Theorem (Dehn):
A square tiling of a $\mathbb{Q}$-rectangle must have $\mathbb{Q}$-side lengths.

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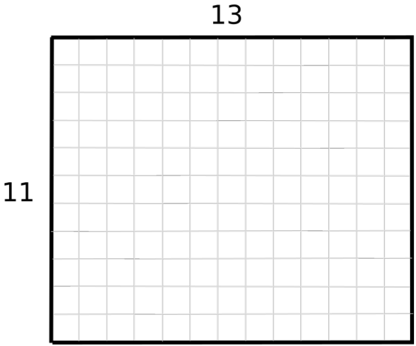


Problem:

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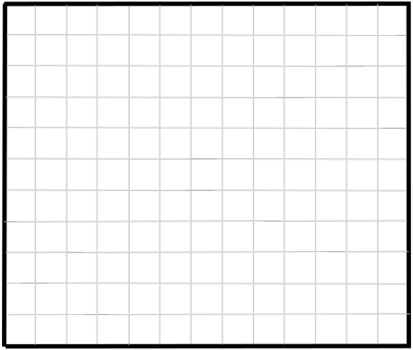
What tilings work?



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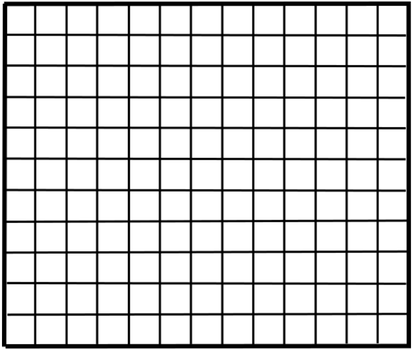

"dumb" tiling: $C(11, 13) \leq 11 \cdot 13 = 143$

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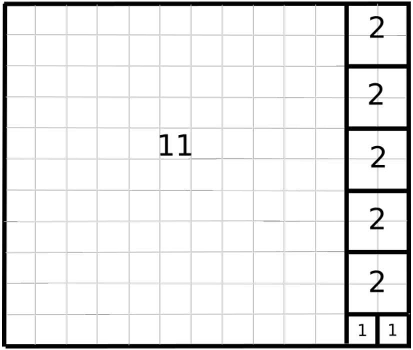


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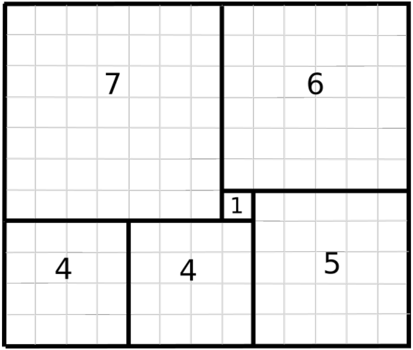


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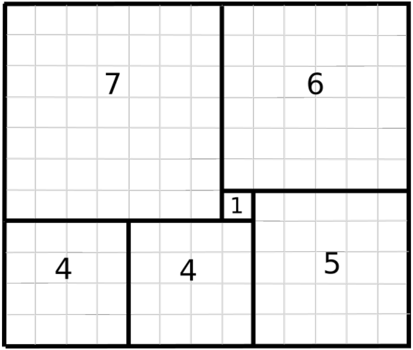


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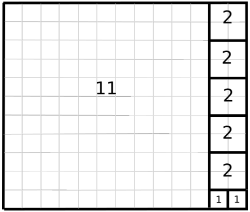

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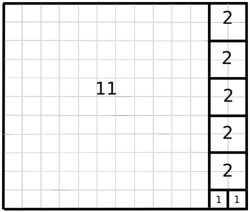

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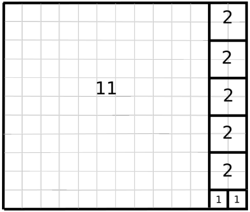


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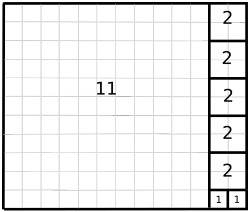


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$$\Rightarrow C(m,n) \leq \text{(sum of continued-fraction terms)}$$

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If "aspect ratio" $n/m$ is bounded, lower bound CAN be improved

# Electrical networks

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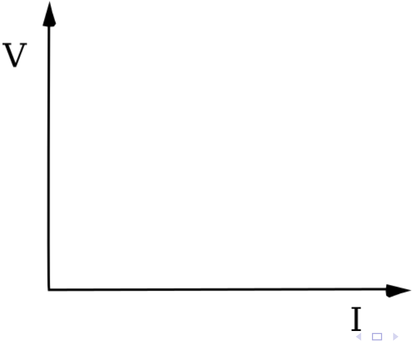

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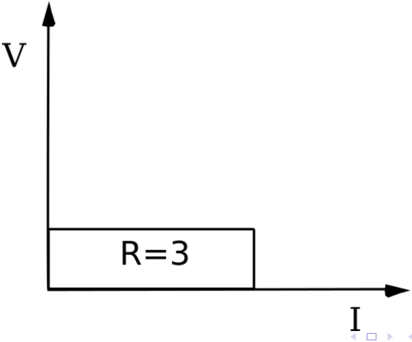

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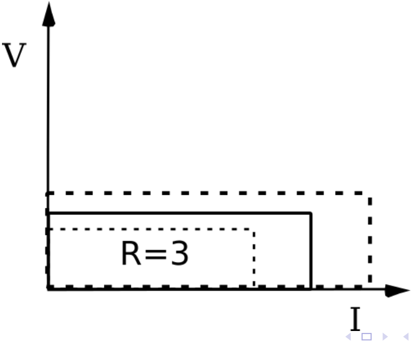

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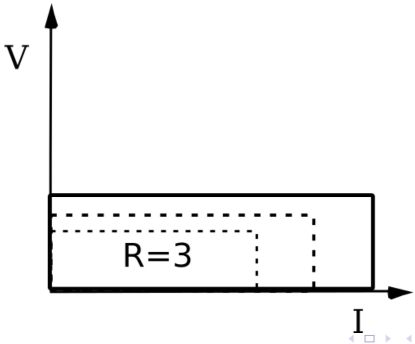

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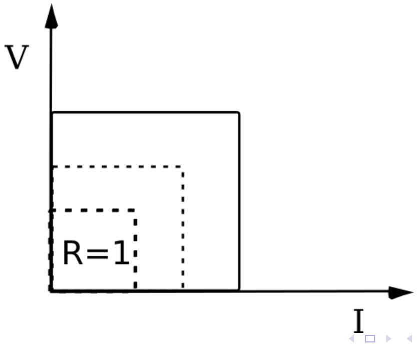

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Consider "current-voltage space":

Square tiles $\leftrightarrow$ unit resistors in IV-space!

# A Lower Bound

**Theorem (Kenyon)**

*Let $C(m, n)$ be the minimal cost of square-tiling a floor of size $m$ by $n$, where $m$ and $n$ have no common factor. Then*

$$C(m, n) \geq \log_2(n)$$

# References

R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte (1940)
The Dissection of Rectangles into Squraes
*Duke Math. J.*, **7**, no. 1, pp. 312–340.

Richard Kenyon (1996)
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*J. Combin. Theory Ser. A*, **76**, pp. 272–291.

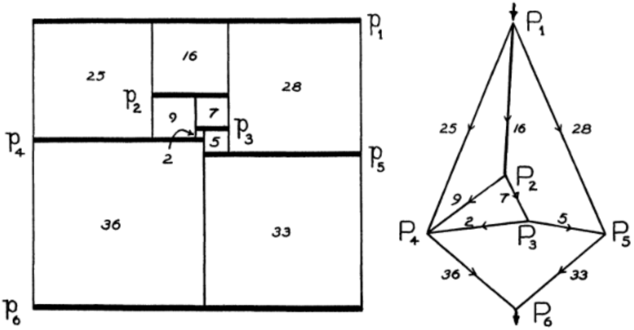


FIG. 1

# Thank you!