# RESEARCH STATEMENT

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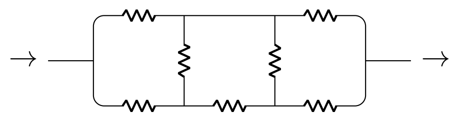
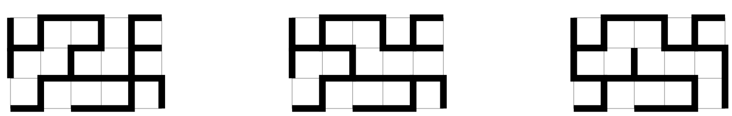
My research is in *tropical geometry*, *random graph theory*, *phylogenetics*, and *number theory*. In these areas, a central role is played by the notion of *effective resistance* of an electrical network. Early work on effective resistance was pioneered by Kirchhoff [Kir47] at the time when electricity was a new technology. As a motivating example, consider the following resistor network, where each resistor has unit resistance.



For electricity passing through this resistor network, how much effective resistance will be encountered? What fraction of the total current will pass through each individual wire in the network? Despite these old beginnings, effective resistance has found applications in cutting-edge research in arithmetic geometry [Zha93; CR93; KRZB16], probability theory [LP14; KW15] and theoretical computer science [SS11; Asa+17].

**Random graph theory.** From the beginning, the practical desire to answer these questions resulted in a surprising and beautiful connection of algebra and combinatorics, known as Kirchhoff's *matrix tree theorem*. Kirchhoff found that effective resistances and current flows could be expressed through counting *spanning trees* of the underlying network [Kir47]. These spanning tree counts, in turn, could be found by taking determinants of certain matrices. Three spanning trees in a grid graph are shown below.



Effective resistance is fundamentally connected to another natural random process, the *simple random walk* on a graph [NW59; Tet91]. Aldous [Ald90] and Wilson [Wil96] showed how to generate a UST directly from a random walk. Wilson's method for generating a UST achieved improved algorithmic efficiency, by utilizing *loop erased random walks* (LERW) [Law91].

The study of electricity was also the initial inspiration for the notion of *capacity*. When electrically charged particles are introduced to a conductive material, say a metal plate, then where do they go? Kakutani [Kak49] showed that random walks solve the graph-theoretic analogue of this problem. These statistics on spanning trees and spanning forests are also essential to calculations in string theory, where Feynman graphs describe particle interactions [Ami+16].

For my results in this area, see Section 1.

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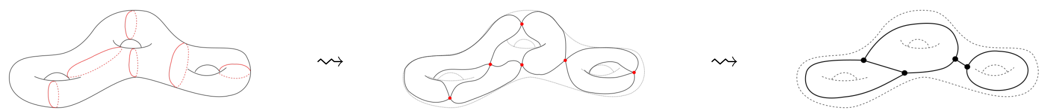
**Tropical geometry.** *Tropical geometry* forms a bridge between continuous objects of algebraic geometry and discrete objects of combinatorics. Algebraic geometry is the study of solutions to polynomial equations such as $x^3 + xy + 12y^3 = 10$. Over the complex numbers, the set of solutions is known as a Riemann surface. Tropical geometry allows us to turn a Riemann surface into a graph, as shown below.



This cartoon shows the process of *degenerating* a smooth algebraic curve to a curve with nodal singularities, then taking the dual graph of the nodal curve. This degeneration process turns meromorphic functions on the Riemann surface to piecewise linear functions on the dual graph. These tools were developed by Baker–Norine [BN07] and others [MZ08; GK08]. Tropical geometry has been used to solve difficult problems in number theory and algebraic geometry. For instance, tropical methods were used in recent work of Katz, Rabinoff, and Zureick-Brown [KRZB16] making progress toward *uniform bounds on rational points* on higher genus curves.

My research studies tropical analogues of theorems from algebraic geometry concerning special discrete subsets of algebraic curves, known as *Weierstrass points*. Effective resistance makes a surprising appearance in these results, following earlier discoveries of Zhang [Zha93] and Amini [Ami14] in arithmetic and non-Archimedean geometry. For my results in this area, see Section 2.

**Phylogenetics.** Phylogenetics is the study of evolutionary histories of living organisms. Given observed traits of modern-day organisms, how did they diverge over time from a common ancestor?

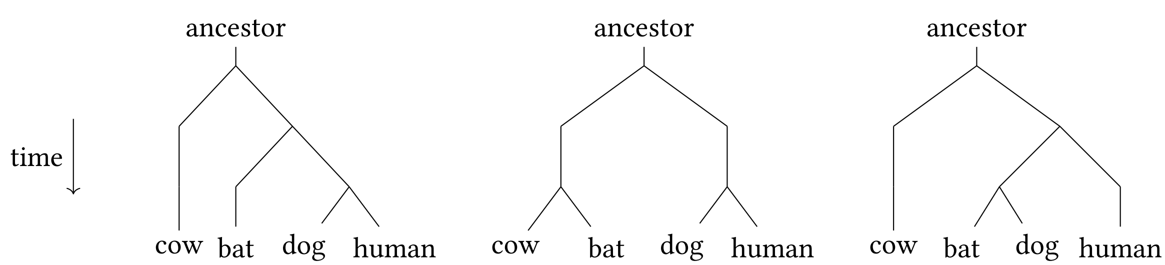



FIGURE 1. Possible phylogenetic trees on four species.

In its current form, these evolutionary histories are inferred from DNA or protein sequences. "Solving" phylogenetic inference is computationally hard. Phylogenetics research often focuses on heuristic for "guessing" better evolutionary trees. To perform such computations efficiently, it is necessary to use encodings of trees which may take advantage of non-obvious structure. For my results in this area, see Section 3.

**Number theory.** A central problem in number theory is to understand the distribution of prime numbers. The *Riemann hypothesis* can be viewed as a way of quantifying the "randomness" of the primes. This is done via bounding the growth of the partial sums of the Möbius function defined on the positive integers under divisibility. A common approach to the Riemann hypothesis is to try "deforming" the underlying structure, and testing to what extent other properties are preserved or broken. I take this approach of "deforming" the multiplicative structure of positive integers by allowing division with rounding, and study the resulting Möbius function. For my results in this area, see Section 4.

# 1. Graph theory

1.1. **Random two-forests.** In joint work with F. Shokrieh and C. Wu [RSW p; RSW24], we prove results in graph theory via use of effective resistance and potential theory on graphs. The following bound on the number of two-component spanning forests generalizes an earlier result on regular lattices motivated by loop erased random walks (LERW) [Law91].

**Theorem 1** ([RSW p]). *For any finite graph $G = (V, E)$,*

$$\frac{\kappa_2(G)}{\kappa_1(G)} \geq \frac{(|V|-1)^2}{4|E|} \tag{1}$$

*where $\kappa_1$ denotes the number of spanning trees and $\kappa_2$ the number of two-forests.*

The looping constant of the LERW can be linked to the behavior of a uniformly random two-forest. This motivates the study of random two-forests on planar lattices [KKW15; KW16; KW15; LP14], in particular, on quantifying the size of the *edge boundary* $\partial F$ of a random two-forest. The following result gives a generalization to an arbitrary finite graph.

**Theorem 2** ([RSW p]). *If $F$ is a uniformly-random two-forest on $G = (V, E)$, then*

$$\mathbb{E}(|\partial F|) \leq 2(\text{avg.deg}) \left(1 + \frac{1}{|V|-1}\right).$$

Here "avg. deg" is the average degree of the vertices, $2|E|/|V|$. In the case when $G$ is taken to be a "large" subset of the lattice $\mathbb{Z}^d$, with nearest-neighbor edges, the bound in Theorem 2 is asymptotically sharp.

Theorem 1 is related to *Mason's conjecture*, on the log concavity of matroid independence numbers. Mason's conjecture was solved by [Ana+24; BH20] in independent work, after several decades of research activity. The bound (1), in matroid theoretic language, involves independence numbers $I_{r-1}/I_r$ and $I_0/I_1$ for graphic matroids; this bound is stricter than the one implied by Mason's conjecture. We plan to study whether stronger bounds can be given on $I_{k-1}/I_k$ for graphic matroids, for other $k$.

1.2. **Distance minors of trees.** In other joint work [RSW24], we study a special case of effective resistance: on a tree, the resistance is equal to the usual shortest-path distance. Graham and Pollak [GP71] found that the determinant of the distance matrix reduces to a simple expression depending *only* on the number of vertices.

We extend their result by finding an expression for the determinant of an arbitrary principal minor of the distance matrix. Unlike in [GP71], this expression depends on combinatorics of the underlying graph, involving counts of rooted spanning forests and boundary edges.

**Theorem 3** ([RSW24]). *Suppose $G = (V, E)$ is a tree with distance matrix $D$, and let $S \subset V$ be a nonempty subset of vertices. Then the principal submatrix $D[S]$ has determinant*

$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left( |E| \cdot \kappa_1(G; S) - \sum_{\mathcal{F}_2(G;S)} (\deg^o(F, *) - 2)^2 \right).$$

This result is proved using methods from potential theory—we associate to the data $(G, S)$ a certain distribution which is "extremal" for the energy functional $\mathbf{u} \mapsto -\frac{1}{2}\mathbf{u}^T D[S]\mathbf{u}$, and then manipulate algebraically. Theorem 3 also has a generalization to graph with edges lengths.

Bapat, Lal, and Pati [BLP06] found that the Graham–Pollak determinant formula generalizes to $q$-distance matrices (in two ways). In the future, we hope to investigate how Theorem 3 generalizes to $q$-analogues of distance. We also hope that the expression in Theorem 3 may be used to make progress toward the *lower bound conjecture* for the tau constant on metric graphs [CR93].

1.3. **Curvature on graphs.** Curvature is a fundamental notion in the study of smooth manifolds. Curvature-based tools such as Ricci flow have been used to solve long-standing conjectures in differential geometry [Per02]. This has inspired researchers in graph theory to define curvature on graphs in various ways [LLY11; Oll09] and study the resulting Ricci flow.

A recent approach by Devriendt–Lambiotte [DL22] uses effective resistance to define a new curvature on the vertices of a graph. Inspired by this, in [Daw+25a], [Daw+25b] we adapt their definition to study a notion of curvature on graph edges. We define the *Ricci–Foster* curvature of a graph $(G, \ell)$ by

$$K_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{1}{2}\frac{\omega_e}{\ell_e}, \tag{2}$$

where $\omega_e$ is the effective resistance across an edge, and $\ell_e$ is the resistance of the edge itself. Under this notion of curvature, we consider the Ricci flow $\left\{ \frac{d}{dt}\ell_e(t) = -K_e(t) : e \in E(G) \right\}$. In the first of these papers, we show that Ricci flow is well-defined on metric graphs. We also prove the following analogue of classical (geometric) Ricci flow.

**Theorem 4** ([Daw+25a])**.** *Ricci flow preserves the class of edge-weighted graphs with positive Ricci–Foster curvature* (2)*.*

In [Daw+25b], we study the behavior of Ricci–Foster curvature under the operation of graph Cartesian product, in particular for the product of two path graphs.

## 2. Tropical geometry

2.1. **Tropical Weierstrass points.** Weierstrass points are a higher genus analogue of torsion points on an elliptic curve [Mum75]. Namely, for a given embedding in projective space, the *Weierstrass points* (of the corresponding divisor) are the points where the curve intersects some hyperplane with "higher-than-expected" multiplicity.

In [Ric24b], I study a natural analogue of Weierstrass points for a tropical curve. In particular, the number of Weierstrass points for a generic divisor is determined as a function of the degree and genus, and a limiting distribution is proved as the degree grows to infinity.

**Theorem 5** ([Ric24b])**.** *On a metric graph of genus $g$, a generic divisor of degree $n \geq g$ has $g(n-g+1)$ distinct Weierstrass points.*

**Theorem 6** ([Ric24b])**.** *Let $\Gamma$ be a metric graph of genus $g \geq 2$, and let $\delta_n$ be the unit discrete measure supported on the Weierstrass locus of a generic divisor of degree $n$. Then the sequence of normalized measures $\frac{1}{gn}\delta_n$ converges weakly to Zhang's canonical measure on $\Gamma$.*

The distribution result mirrors parallel results of Neeman [Nee84] and Amini [Ami14] for algebraic curves over the complex numbers $\mathbb{C}$ and over a field with non-Archimedean valuation, respectively. The value of Theorem 6 over this previous work is that we can bypass much of the technical machinery and instead directly use properties of effective resistance.

2.2. **Weierstrass weights.** Although [Ric24b] gives a fairly complete description of the tropical Weierstrass locus for a generic divisor, many divisors of particular interest are not generic. Most prominently, it does not address the tropical Weierstrass locus of the *canonical divisor* $K$. In joint work with O. Amini and L. Gierczak [AGR23], we develop techniques for addressing the tropical

Weierstrass locus on an arbitrary divisor, including $K$. We consider how Weierstrass points on an algebraic curve, over a non-Archimedean field, tropicalize to its "skeleton" tropical curve [Bak08; Ber90]. We give a combinatorial formula for *weights* that match the number of algebraic Weierstrass points under inverse-tropicalization.

**Theorem 7** ([AGR23]). *Suppose $A \subset \Gamma$ is a closed, connected subset which is $W(K)$-measurable. Then, the total weight of Weierstrass points of $\mathcal{W}(\mathcal{K})$ tropicalizing to points in $A$ is precisely*

$$\deg\left(\mathcal{W}(\mathcal{K})|_{\tau^{-1}(A)}\right) = g\bigg((g+1)\left(g(A)-1\right) - \sum_{\nu \in \partial^{\text{out}} A} \left(s_0^{\nu}(K) - 1\right)\bigg).$$

This result significantly generalizes work of Eisenbud and Harris [EH87] on Weierstrass points on stable nodal curves. For a generic genus $g$ algebraic curve, the number of Weierstrass points is $g^3 - g$ (assuming the base field is algebraically closed) [Hur92]. In the tropical case, studying low-genus examples indicate that there are at most $g^2 - 1$ Weierstrass points. These expressions differ by a factor of $g$, and strongly suggest that the tropicalization map on Weierstrass points is generically $g$-to-1. Theorem 7 confirms that this is indeed the case.

We also show that there are strong local topological constraints on the position of the tropical Weierstrass locus.

**Corollary 8** ([AGR23]). *Suppose $\Gamma$ is a tropical curve of genus $g \geq 2$. Then every cycle in $\Gamma$ contains a Weierstrass point in $W(K)$.*

2.3. **Tropical Manin–Mumford conjecture.** By analogy with Mordell's conjecture on finiteness of rational points, Manin and Mumford conjectured that a higher genus algebraic curve has finitely many torsion points in its Abel–Jacobi embedding. This conjecture was proved by Raynaud [Ray83].

For a metric graph $\Gamma$, there is an analogous Jacobian [MZ08]. In [Ric23], I study the tropical version of the Manin–Mumford conjecture, which asks: when a metric graph is embedded in its Jacobian, how often does it pass through torsion points of the Jacobian? We give a conditionally uniform bound in the tropical case.

**Theorem 9** ([Ric23]). *Let $\Gamma$ be a metric graph of genus $g \geq 2$. If $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}})$ is finite, then $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq 3g - 3$.*

However, the tropical Manin–Mumford conjecture fails in general: for any metric graph with integer edge lengths, there are infinitely many torsion points with respect to any basepoint, no matter how large the genus is. On the other hand, I show that a metric graph does satisfy the Manin–Mumford condition if we impose certain additional constraints.

**Theorem 10** ([Ric23]). *Let $G$ be a biconnected graph of genus $g \geq 2$. For a very general choice of edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, the metric graph $\Gamma = (G, \ell)$ has $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq g + 1$.*

There are natural higher-dimensional analogues of the Manin–Mumford condition, where we embed the $d$-th symmetric power of a curve, or metric graph, into its Jacobian. I found that a modification of the *girth* of a graph gives an upper bound on the dimension $d$ for which the higher Manin–Mumford condition is satisfied.

**Theorem 11** ([Ric23]). *Suppose $\Gamma = (G, \ell)$ is a graph with very general edge lengths. Then $\mathrm{Sym}^d(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}$ is finite if and only if*

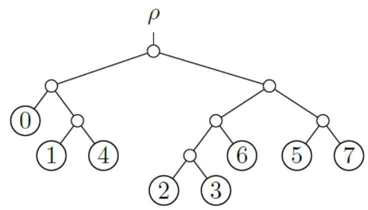
$$d \leq \min_{C \subset E(G)} \{\mathrm{rank}_{M^{\perp}(G)}(C)\}, \tag{3}$$

*where the minimum is taken over all cycles of $G$ and $M^{\perp}(G)$ denotes the cographic matroid of $G$.*

We call this bound in (3) the *independent girth* of $G$; note that the girth is $\min_{C \subset E(G)}\{\#C\}$.

## 3. Phylogenetics

In order to solve a phylogenetics problem with a computer, it is necessary to translate a tree into a computer-readable format. Along with coauthors, I introduced a new format to encode rooted, binary phylogenetic trees, called the *ordered leaf attachment (OLA) code* [RZM25].




$\leftrightarrow$ `(0,-1,2,1,-3,-3,5)`

The OLA code provides a bijection from the set of $n$-leaf trees with the integer vectors

$$\mathcal{C}_{n-1} = \{(a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1} : -i < a_i < i\}.$$

Note that $\mathcal{C}_{n-1}$ is the intersection of $\mathbb{Z}^{n-1}$ with a convex region of $\mathbb{R}^{n-1}$. We investigated how the OLA-code-induced distance is connected to others commonly-used in phylogenetics, such as the *subtree-prune-regraft* (SPR) distance. We also prove that the OLA encoding is computational efficient:

**Theorem 12** ([RZM25])**.** *The OLA encoding and decoding algorithms are linear-time in the number of leaves.*

The OLA encoding has the potential to improve the efficiency of many phylogenetic computations. It was used in [Xie+25] for building a machine learning model for rapidly generating phylogenetic trees. In another ongoing project, we evaluate different tree encoding formats, along with OLA, using a *contrastive learning* framework. The goal is to find which format causes groups of plausible trees to be "clumped together." The "groups of plausible trees" comes from datasets collected by other researchers, using *Bayesian phylogenetics* which is typically computationally slow.

## 4. Number theory

### 4.1. Floor quotients.

In multiplicative number theory, one studies the prime factorization of numbers by considering the numbers as a partially ordered set (poset) under division. From this poset, we have the Möbius function $\mu(n)$, which depends on the prime factorization structure of $n$. It is a classical result that the Riemann Hypothesis is equivalent to the bound

$$\sum_{n \leq x} \mu(n) = O(x^{1/2+\epsilon}) \quad \text{as } x \to \infty, \text{ for any } \epsilon > 0.$$

In joint work with J. Lagarias, we attempt to gain a better understanding of the classical Möbius function by defining a deformation of the partial order of natural numbers under divisibility. Using the floor function, we define the *floor quotients* of $n$ as the numbers of the form $\{d : d = \lfloor n/k \rfloor \text{ for } k = 1, 2, \ldots, n\}$. Surprisingly, this defines a partial order relation on the positive integers. This observation was implicit in [Car10]. In Figure 2, we show the floor quotients of $n = 16$ and their poset structure.

In [LR24], we investigate the structure of the poset of floor quotients, answering questions in parallel with usual results in multiplicative number theory. We obtain polynomial bounds on the Möbius function of the floor quotient poset. (The results in [Car10] suggest that $\mu_{FQ}(n)$ should be considered an analogue of the sum $\displaystyle\sum_{n/2 < i \leq n} \mu(i)$ of the usual Möbius function, rather than an analogue of $\mu(n)$.)

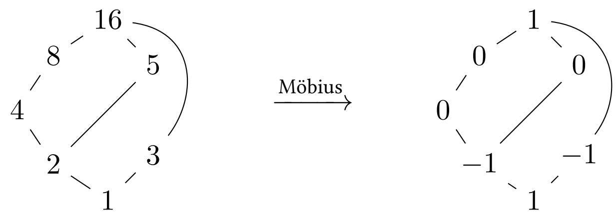



FIGURE 2. Poset of floor quotients of 16, with Möbius values $\mu_{FQ}(1, d)$ on right.

**Theorem 13** ([LR24]). *Let $\mu_{FQ}(n)$ denote the Möbius function of the floor quotient poset interval from $1$ to $n$. There is some constant $C > 0$ such that*

$$|\mu_{FQ}(n)| < Cn^{1.729} \qquad \textit{for all } n.$$

We believe the bound in Theorem 13 is not optimal. We hope that further investigation will yield improved bounds on the Möbius function. For example: for any $\epsilon > 0$, is there some constant $C(\epsilon)$ such that

$$|\mu_{FQ}(n)| < C(\epsilon)\, n^{1+\epsilon} \qquad \text{for all } n?$$

In [LR25], we generalize the floor quotient partial order to an infinite family, parametrized by a positive integer $a$, such that the limit $a \to \infty$ recovers the usual multiplicative structure of positive integers.

4.2. **Rounding functions.** Discretization is the process of sending a continuous input to a discrete output, which is fundamental in many applications such as computer imaging, digital communication, and finance. *Rounding functions* are a form of regularly-spaced discretization. These create interesting behavior in the context of elementary algebra and number theory. In [LMR16; LR19; LR20] we study commutators of dilated floor functions under different scales. [Ric24a] studies the self-similar structure of Farey staircases, which are constructed from taking cumulative averages of rescaled floor functions.

4.3. **$p$-adic continuity and combinatorial sequences.** The $p$-adic topology on the integers provides a way of viewing the usually-discrete integers, or functions on integers, in an almost-continuous way. In joint work with A. O'Desky [OR23], we generalize the observation that counting derangements gives a $p$-adically continuous function to a larger class of derangement-like counting problems. As an application, certain classes of the counts can be combined to form a $p$-adic analogue of the two-variable incomplete gamma function $\Gamma(s, x) = \displaystyle\int_x^\infty t^{s-1}e^{-t}dt$.

4.4. **Counting and geometry.** In [AAR23], joint with D. Aulicino and J. Athreya, we study a generalization of the classical problem of counting closed geodesics of bounded length in a torus.

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