# Weierstrass points on a tropical curve

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University of Washington AAG Seminar
29 October 2019

# What is a Weierstrass point?

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$\rightsquigarrow$ projective embedding $\phi : X \to \mathbb{P}^r$.

$$W(D_N) = \{x \in X : \exists H \subset \mathbb{P}^r \text{ s.t. } m_x(H \cap X) \geq r + 1\}$$
$$= \left\{ x \in X : \begin{array}{c} \text{“higher-than-expected” tangency with} \\ \text{some hyperplane } H \text{ at } x \end{array} \right\}$$

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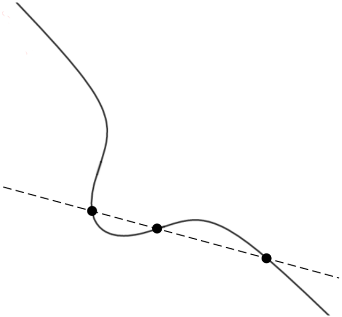

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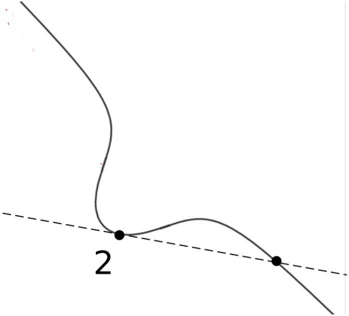


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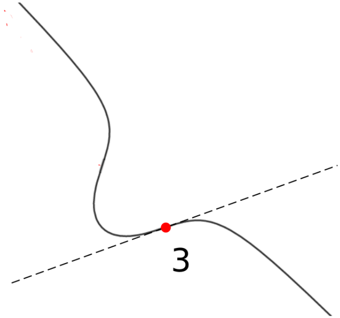


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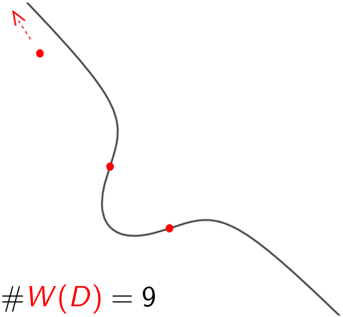

$N = 3$ , $\#W(D) = 9$

# What is a Weierstrass point?

Intution (Mumford):

| | | |
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| $N$-torsion points on an elliptic curve | $\leftrightarrow$ | Weierstrass points of $D_N$ on a higher-genus curve |

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**Problem**

How are Weierstrass points distributed on an algebraic curve?

# Weierstrass points: genus 1, complex case

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How are Weierstrass points distributed on genus 1 curve $X/\mathbb{C}$?

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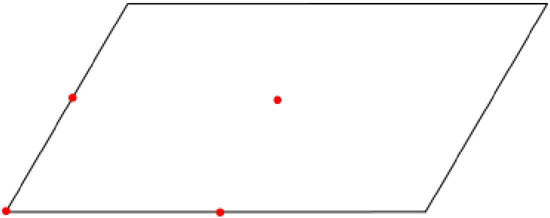

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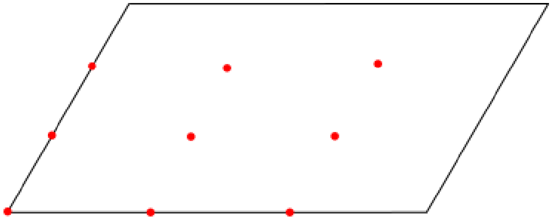

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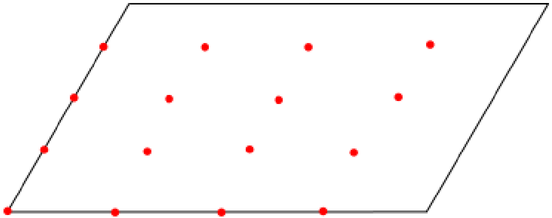

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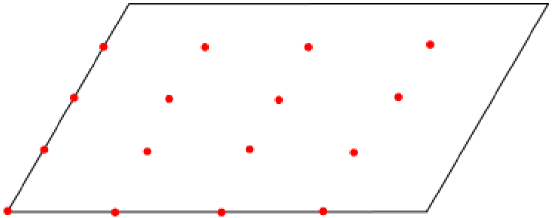

$\rightsquigarrow$ Weierstrass points distribute **uniformly**, w.r.t. $\mathbb{C} \to \mathbb{C}/\Lambda$

# Weierstrass points: genus $\geq 2$, complex case

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How are Weierstrass points distributed on higher genus curve $X/\mathbb{C}$?

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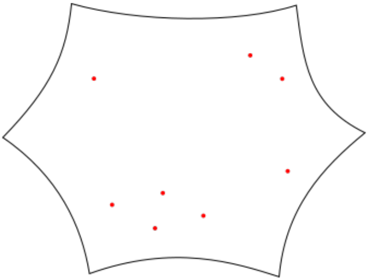

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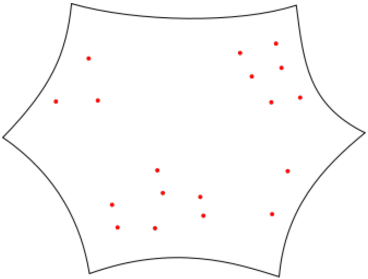

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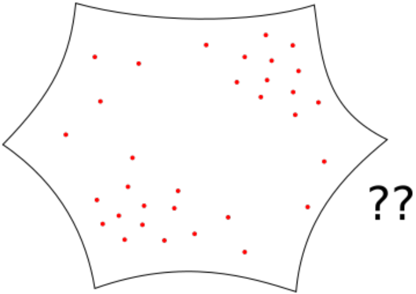

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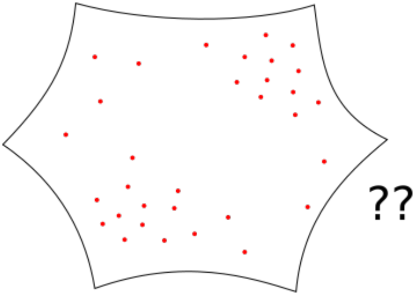


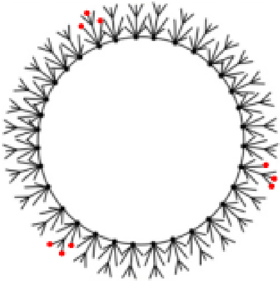

## Theorem (Neeman, 1984)

*Suppose $X$ is a complex algebraic curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Bergman measure as $N \to \infty$.*

# Weierstrass points: non-Archimedean case

## Problem

How are Weierstrass points distributed on $X/\mathbb{K}$, $val : \mathbb{K}^\times \to \mathbb{R}$?

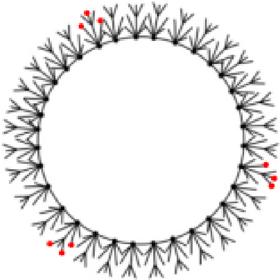


Source: Matt Baker's math blog

# Weierstrass points: non-Archimedean case

## Problem

How are Weierstrass points distributed on ~~$X/\mathbb{K}$~~ $X^{an}$?



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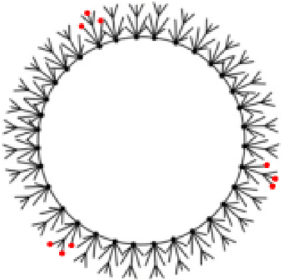

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## Theorem (Amini, 2014)

*Suppose $X^{an}$ is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Zhang measure as $N \to \infty$.*

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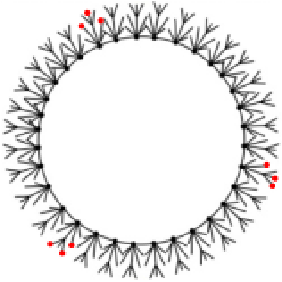

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Does the distribution follow from considering only the skeleton $\Gamma \subset X^{an}$?

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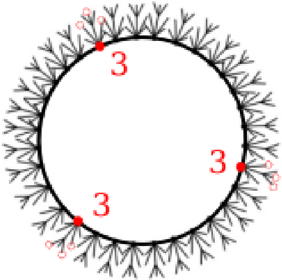


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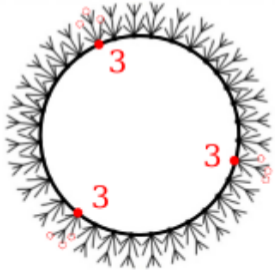
**Problem (Amini, 2014)**

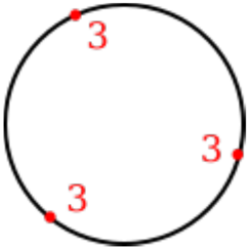
Does the distribution follow from considering <u>only</u> the skeleton $\Gamma \subset X^{\mathrm{an}}$?

# What is a tropical curve?

Tropical curve ( $=$ a skeleton of $X^{\mathrm{an}}$)




$\rightsquigarrow$



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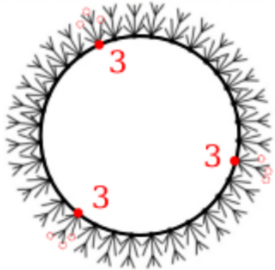
Tropical curve $( =$ a skeleton of $X^{an})$

(combinatorics) $=$ finite graph with edge lengths

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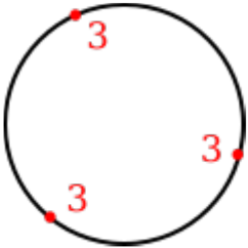

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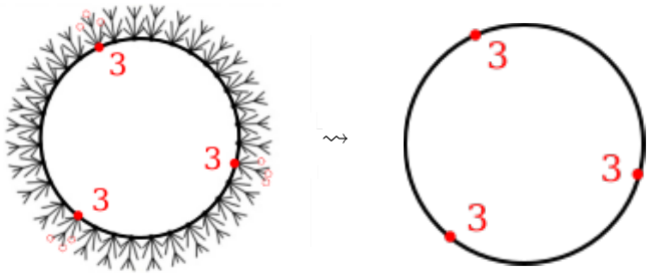

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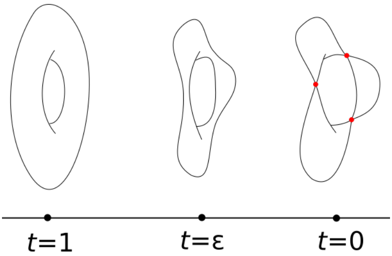

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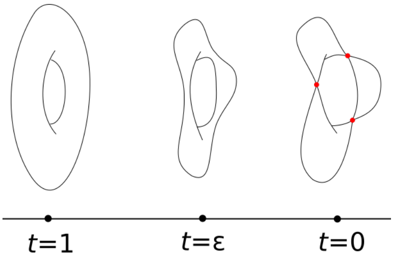
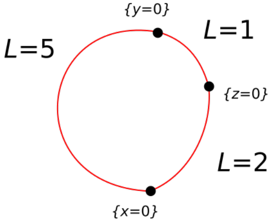



$\rightsquigarrow$
dual graph of $X_0$



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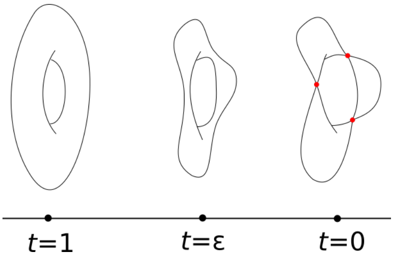


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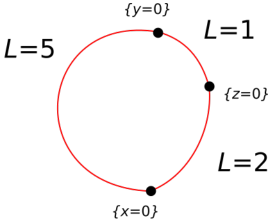

# Tropical curves: divisor theory

Tropical curve = metric graph

| alg. curve $X$ | | tropical curve $\Gamma$ |
|---|---|---|
| divisors $\mathrm{Div}(X)$ | $\rightsquigarrow$ | divisors $\mathrm{Div}(\Gamma)$ |
| meromorphic functions | $\rightsquigarrow$ | piecewise $\mathbb{Z}$-linear functions |
| linear system $\lvert D\rvert$ | $\rightsquigarrow$ | linear system $\lvert D\rvert$ |
| $= \mathbb{P}^r$ | | $=$ polyhedral complex of dim $\geq r$ |
| rank $r = \dim \lvert D\rvert$ | $\rightsquigarrow$ | rank $r =$ Baker-Norine rank |

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Intuition: linear equivalence on $\Gamma$ = "discrete current flow"

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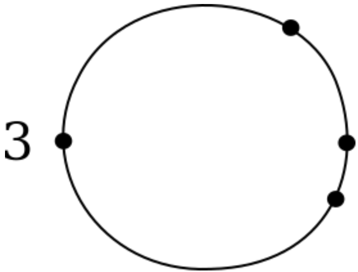

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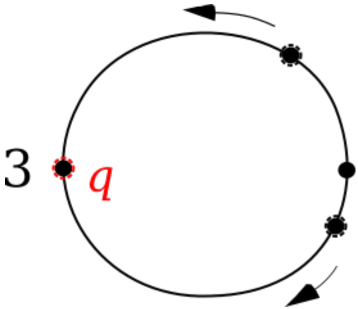

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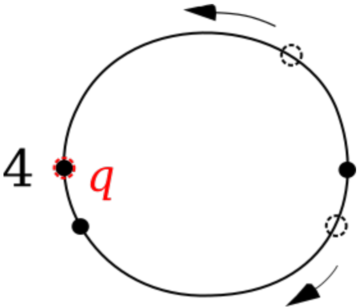

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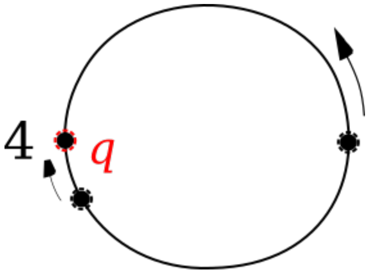

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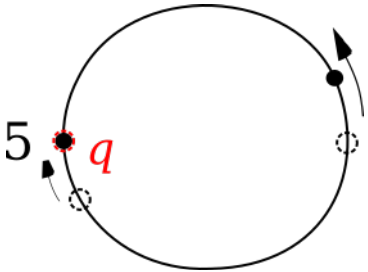

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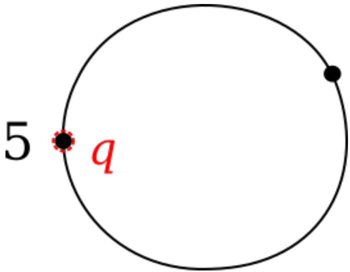

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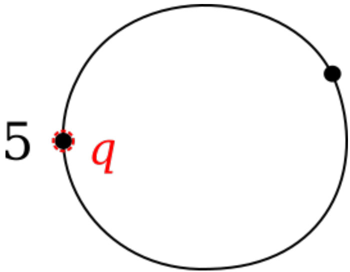


What happens as $q$ varies?

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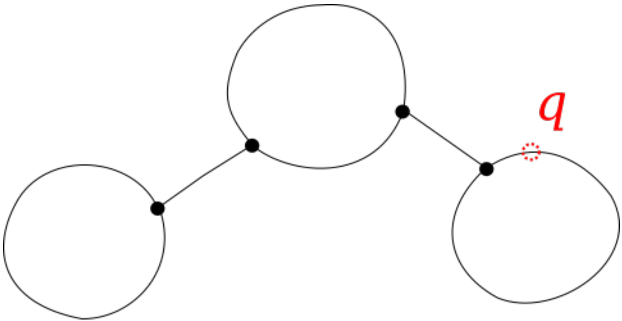

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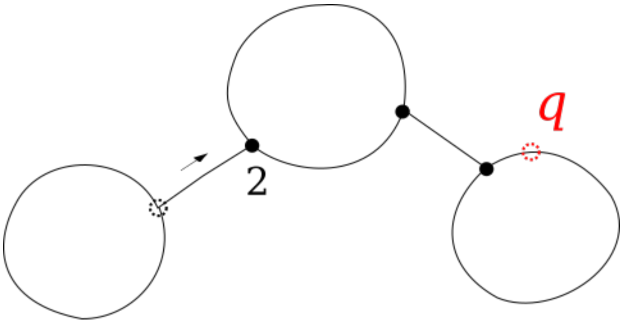

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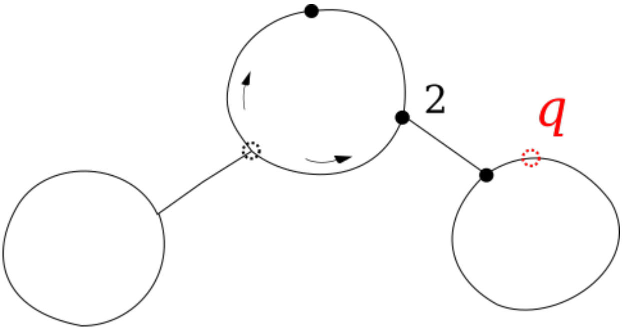

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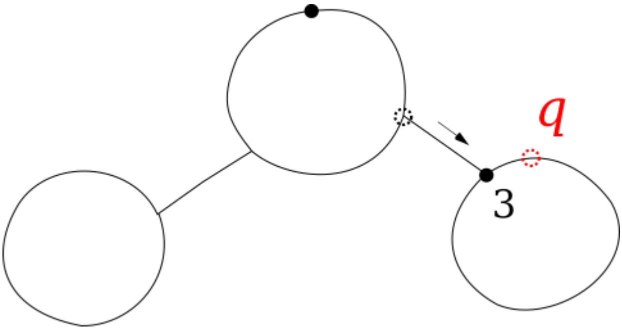

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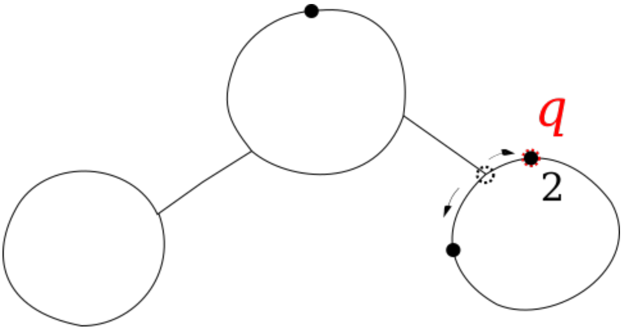

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How are Weierstrass points distributed on a tropical curve?

Definition: $\Gamma =$ metric graph, $D_N$ divisor of degree $N$
$\rightsquigarrow$ Baker–Norine rank $r = r(D_N)$

$$W(D_N) = \{x \in X : \mathrm{red}_x[D_N] \geq (r+1)x\}$$

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Definition: $\Gamma =$ metric graph, $D_N$ divisor of degree $N$

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# Tropical curves: Weierstrass points

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How are Weierstrass points distributed on a tropical curve?

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EXCEPT sometimes $\#(\text{Weierstrass points}) = \infty$

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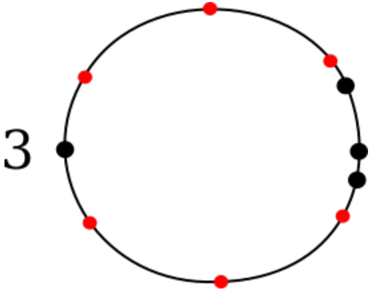


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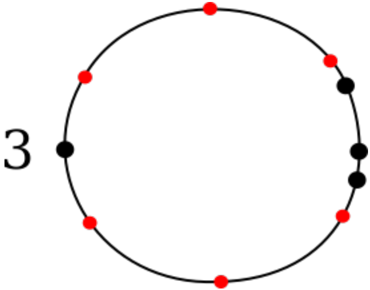


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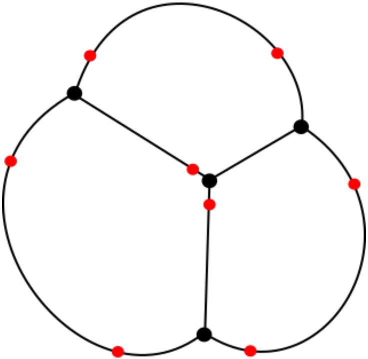

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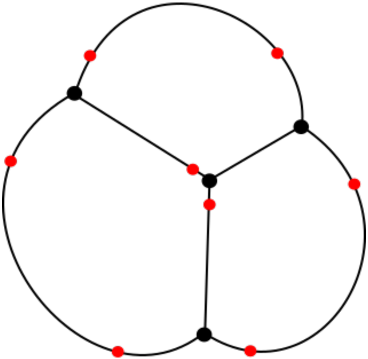

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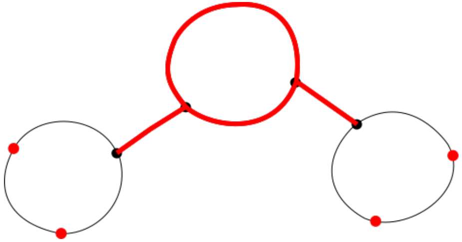

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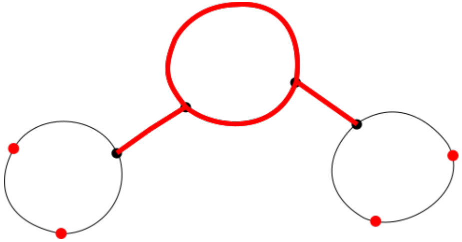

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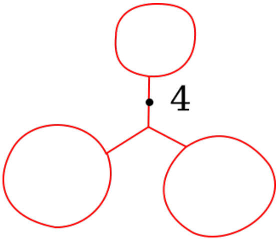


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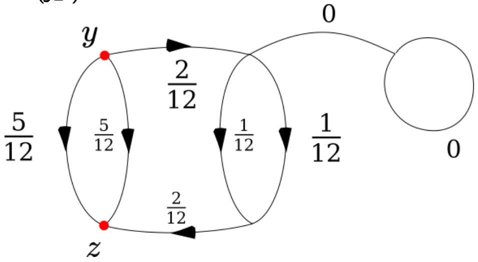

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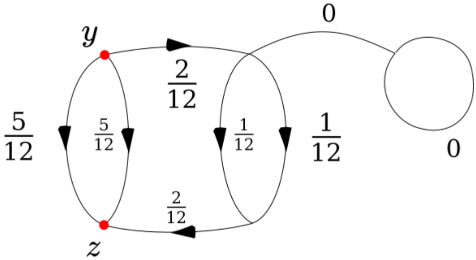
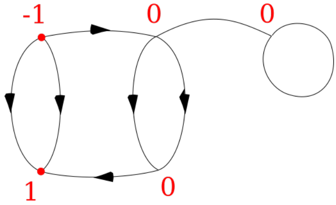



satisfies Laplacian $\Delta(j_z^y) = z - y$



# Electrical networks: Canonical measure

$\Gamma$ = metric graph

**Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)**

Zhang's **canonical measure** $\mu$ on an edge is the "current defect"

$$\begin{aligned} \mu(e) &= \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+ \\ &= 1 - (\text{current through } e \text{ when } \ldots\ ) \end{aligned}$$

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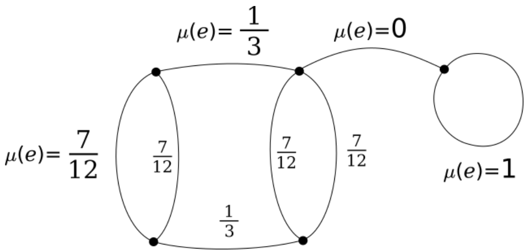

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Generally:

- $0 \leq \mu(e) \leq 1$
- $\mu(e) = 0 \Leftrightarrow e$ a bridge
- $\mu(e) = 1 \Leftrightarrow e$ a loop

Foster's Theorem: $\mu(\Gamma) = \sum_{e \in E} \mu(e) = g$

# Tropical Weierstrass distribution: proof idea

## Theorem (R)

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*Namely, for any edge $e$*

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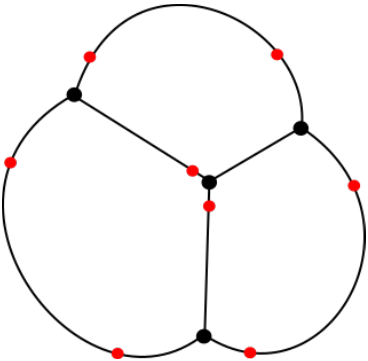
# Weierstrass points on a tropical curve



Thank you!