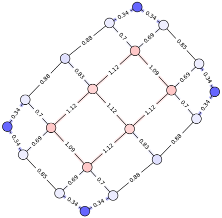
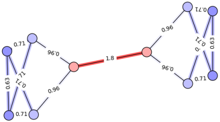
Ricci flow on graphs from effective resistance

Harry Richman, joint with Aleyah Dawkins, Vishal Gupta, Mark Kempton, William Linz, Jeremy Quail, Zachary Stier

AMS MRC and Fred Hutch Cancer Center

JMM: Ricci curvatures on graphs and applications
4 January 2024





Motivation

Problem: How to understand “geometry” of a graph?

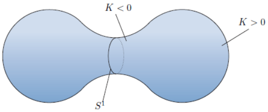
- Real world: max flow / min cut, community detection
- Arithmetic geometry: bounding number of rational points
- Combinatorics: Laplacian eigenvalues, Kemeny’s constant, ...

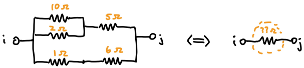
Differential geometry

curvature, Ricci flow

Combinatorics

effective resistance

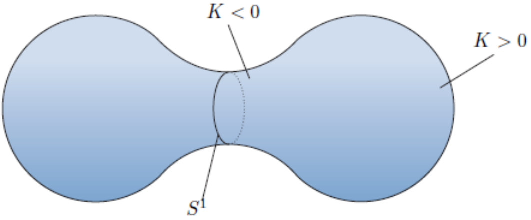




Why Ricci flow?

Related Problem: How to understand “geometry” of a manifold?

- Poincare Conjecture: what conditions suffice for $\mathcal{M}^n \cong S^n$?

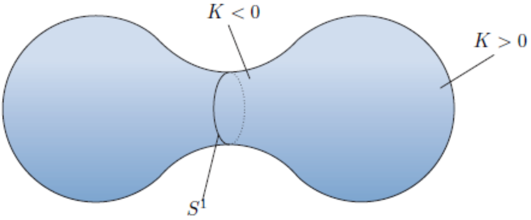


(image from Topping 2006)

Why Ricci flow?

Related Problem: How to understand "geometry" of a manifold?

- Poincare Conjecture: what conditions suffice for $\mathcal{M}^n \cong S^n$?

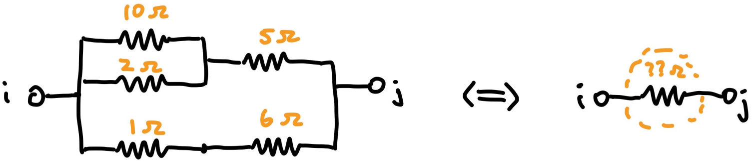


(image from Topping 2006)

Apply Ricci flow:

- positive curvature $\longrightarrow$ shrink metric
- negative curvature $\longrightarrow$ expand metric

Why effective resistance?



Close connections to:

- simple random walk on G
- uniformly random spanning trees on G

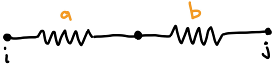
Recent breakthrough applications:

- graph sparsification (Spielman–Srivastava, 2009)
- traveling salesman problem (Anari–Oveis-Gharan, 2015)

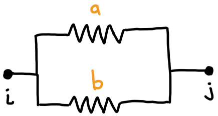
Effective resistance

Setting: graph $G = (V, E)$, each edge e has a positive resistance ℓ_e

How to compute the **effective resistance** ω_{ij} for vertices $i, j \in V$?

- series rule: 

$\omega_{ij} = a + b$

- parallel rule: 

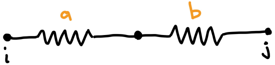
$\omega_{ij} = \dfrac{ab}{a+b}$

- general case (??): combine series and parallel rules

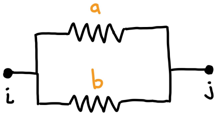
Effective resistance

Setting: graph $G = (V, E)$, each edge e has a positive resistance ℓ_e

How to compute the **effective resistance** ω_{ij} for vertices $i, j \in V$?

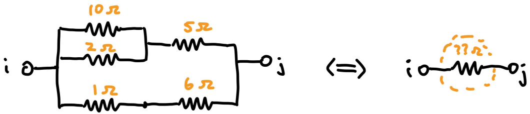
- series rule: 

$\omega_{ij} = a + b$

- parallel rule: 

$$\omega_{ij} = \frac{ab}{a+b}$$

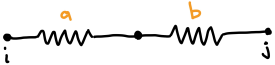
- general case (??): combine series and parallel rules



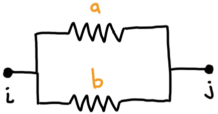
Effective resistance

Setting: graph $G = (V, E)$, each edge e has a positive resistance ℓ_e

How to compute the **effective resistance** ω_{ij} for vertices $i, j \in V$?

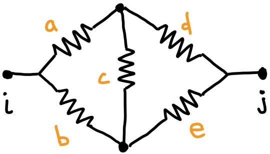
- series rule: 

$\omega_{ij} = a + b$

- parallel rule: 

$\omega_{ij} = \dfrac{ab}{a+b}$

- general case (??): Wheatstone bridge

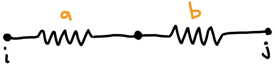


$\omega_{ij} = ?$

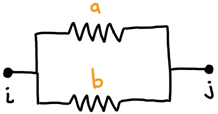
Effective resistance

Setting: graph $G = (V, E)$, each edge e has a positive resistance ℓ_e

How to compute the **effective resistance** ω_{ij} for vertices $i, j \in V$?

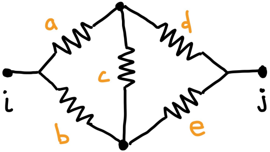
- series rule: 

$\omega_{ij} = a + b$

- parallel rule: 

$$\omega_{ij} = \frac{ab}{a+b}$$

- general case (??): Wheatstone bridge



$$\omega_{ij} = \frac{abd+abe+ade+bde+acd+ace+bcd+bce}{ad+ae+bd+be+ac+bc+cd+ce}$$

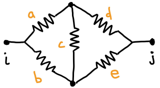
Effective resistance

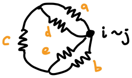
⚠ Series and parallel rules not sufficient to find effective resistance

- General case: use weighted sums of spanning trees

Theorem (Kirchhoff)

$$\omega_{ij} = \frac{\sum_{\mathcal{T}(G/ij)} \prod_{e \notin T} \ell_e}{\sum_{\mathcal{T}(G)} \prod_{e \notin T} \ell_e}$$

Example: $G =$ 

, $G/ij =$ 

$$\omega_{ij} = \frac{abd + abe + ade + bde + acd + ace + bcd + bce}{ad + ae + bd + be + ac + bc + cd + ce}$$

Effective resistance

Theorem (Rayleigh's law)

For any edge e and vertices i, j we have

$$\frac{\partial}{\partial \ell_e} \omega_{ij} \geq 0.$$

- physically "obvious"
- mathematically ...

$$\frac{\partial}{\partial c} \omega_e = \frac{\partial}{\partial c} \left(\frac{abd + abe + ade + bde + acd + ace + bcd + bce}{ad + ae + bd + be + ac + bc + cd + ce} \right) =?$$

Effective resistance

Theorem (Rayleigh's law)

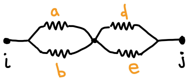
For any edge e and vertices i, j we have

$$\frac{\partial}{\partial \ell_e} \omega_{ij} \geq 0.$$

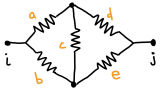
- delete edge $\quad \leftrightarrow \quad \ell_e = +\infty$
- contract edge $\quad \leftrightarrow \quad \ell_e = 0$

Corollary (usual Rayleigh's law)

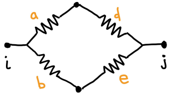
$$\omega_{ij}(G/e) \leq \omega_{ij}(G) \leq \omega_{ij}(G \setminus e)$$



$\leq$



$\leq$



Resistance curvature on nodes

(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

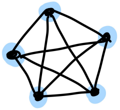
$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

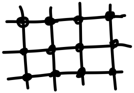
Resistance curvature on nodes

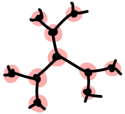
(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

- A finite, vertex-transitive graph has (constant) positive node curvature.
- An infinite regular lattice is flat (zero curvature).
- An infinite tree has negative node curvature everywhere.







Ricci curvature on edges

(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

Can we make edge curvature "more local", in the sense that

$$p_i = \sum_{e \ni i} \mathrm{K}_{\vec{e}} \qquad \text{for edge curvatures } \mathrm{K}_{\vec{e}}?$$

Ricci curvature on edges

(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

Can we make edge curvature "more local", in the sense that

$$p_i = \sum_{e \ni i} \mathrm{K}_{\vec{e}} \qquad \text{for edge curvatures } \mathrm{K}_{\vec{e}}?$$

Yes! Define

$$\text{oriented edge curvature} \qquad \mathrm{K}_{\vec{e}} = \frac{1}{\deg_i} - \frac{1}{2}\frac{\omega_e}{\ell_e}$$

$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

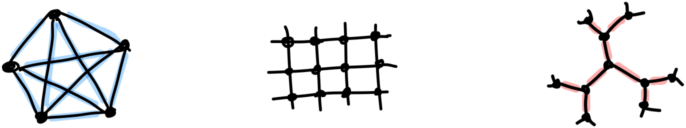
Resistance curvature on edges

Definition

On weighted graph (G, ℓ), the *Foster–Ricci curvature* on edge e is

$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

- Constant-curvature graphs:



- Edge curvature gives more information than node curvature:

Ricci flow from resistance

Definition

On weighted graph (G, ℓ), the *Foster–Ricci curvature* on edge e is

$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

Consider resulting Ricci flow

$$\frac{d}{dt}\ell_e(t) = -\mathrm{K}_e(t)$$

where $\mathrm{K}_e(t) = \mathrm{K}_e(\ell(t))$.

What does Ricci flow look like?

Ricci flow from resistance

Theorem (Ricci flow existence, DGKLQRS)

For any edge-weighted graph (G, ℓ_0), where $\ell_0 = \{\ell_{0,e} > 0 : e \in E(G)\}$, there exists $T > 0$ such that there exists a unique solution to Ricci flow for $t \in [0, T)$.

Proof sketch:

- On any finite box in positive orthant, curvature function $\{\ell_e : e \in E\} \mapsto \{\mathrm{K}_e(\ell) : e \in E\}$ is differentiable
- Differentiable function on compact domain in Lipschitz
- Apply Picard–Lindelöf theorem

Ricci flow on positively curved graphs

Conjecture

Ricci flow preserves positively curved graphs.

Chain rule: $$\frac{d}{dt}\mathrm{K}_e(t) = \sum_{f\in E} \frac{\partial \mathrm{K}_e}{\partial \ell_f} \cdot \frac{d\ell_f}{dt}$$

Lemma

1. For any edge e,
$$\frac{\partial}{\partial \ell_e}\mathrm{K}_e \geq 0;$$
2. For any edges $e \neq j$,
$$\frac{\partial}{\partial \ell_e}\mathrm{K}_f \leq 0.$$

Proof sketch: apply Rayleigh's law.

Discussion

Previous work:

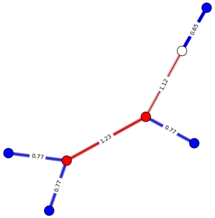
1. Bai–Lin–Lu–Wang–Yau (2021) show existence of Ricci flow for Ollivier–Ricci curvature
2. Devriendt–Lambiotte (2022) study Ricci flow for a different resistance-based edge curvature

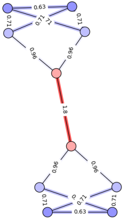
Further questions: many notions of Ricci curvature on graphs exist.

1. For which curvatures is it true that

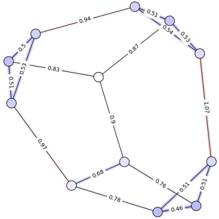
$$\frac{\partial}{\partial \ell_e} \mathrm{K}_e \geq 0, \qquad \frac{\partial}{\partial \ell_e} \mathrm{K}_f \leq 0?$$

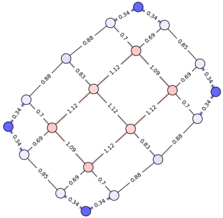
2. For which curvatures is it true that Ricci flow preserves positively-curved graphs?


0.65
1.12
0.77
1.23
0.77
0.77


0.63
0.71
0.71
0.71
0.71
0.96
0.96
1.8
0.96
0.96
0.71
0.71
0.71
0.63

Thank you!


0.94
0.51
0.54
0.53
0.5
0.53
0.83
0.87
0.51
1.07
0.9
0.97
0.68
0.76
0.51
0.51
0.78
0.46


0.34
0.34
0.88
0.7
0.69
0.85
0.88
0.83
1.12
1.09
0.69
0.34
0.34
0.7
1.12
1.12
1.12
0.7
0.34
0.69
1.09
1.12
0.83
0.88
0.85
0.69
0.7
0.88
0.34
0.34

Alternative Ricci flow from resistance

Recall that Devriendt–Lambiotte define

$$\text{node curvature} \qquad p_i = 1 - \frac{1}{2}\sum_{j\sim i} \omega_{ij},$$

$$* \text{ edge curvature} \qquad \kappa_{ij} = \frac{2}{\omega_{ij}}(p_i + p_j)$$

Devriendt–Lambiotte consider *Ricci flow* defined by differential equation

$$\frac{d}{dt}\omega_{ij}(t) = -\kappa_{ij}(t)\,\omega_{ij}(t) \qquad \text{where } \kappa_{ij} = \kappa_{ij}(G(\omega(t))$$

Alternative Ricci flow from resistance

Recall that Devriendt–Lambiotte define

$$\text{node curvature} \qquad p_i = 1 - \frac{1}{2}\sum_{j\sim i} \omega_{ij},$$

$$* \text{ edge curvature} \qquad \kappa_{ij} = \frac{2}{\omega_{ij}}(p_i + p_j)$$

Devriendt–Lambiotte consider *Ricci flow* defined by differential equation

$$\frac{d}{dt}\omega_{ij}(t) = -\kappa_{ij}(t)\,\omega_{ij}(t) \qquad \text{where } \kappa_{ij} = \kappa_{ij}(G(\omega(t)))$$

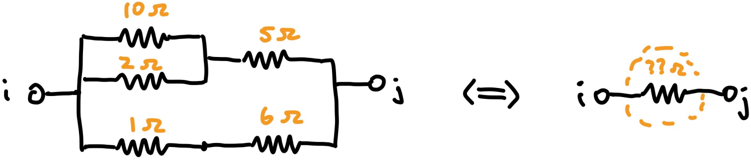
Features:

- in a path, leaf-edges shrink to zero-resistance, "edge contraction"

Downsides:

- in trees with higher-degree vertices, leaf-edges don't always shrink
- positive values of ω_{ij} may be "invalid"

Effective resistance: quiz answer



Answer: $\omega_{ij} = \dfrac{140}{41}$