Dilated floor functions and their commutators

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joint w/ Jeff Lagarias and Takumi Murayama

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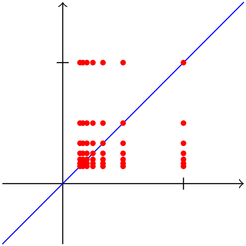

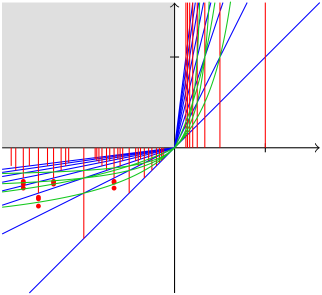

Floor functions

The **floor function** rounds a real number down to the nearest integer

$$\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$$

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$\lfloor x \rfloor$ is also known as the "greatest integer function":

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \le x\}$$

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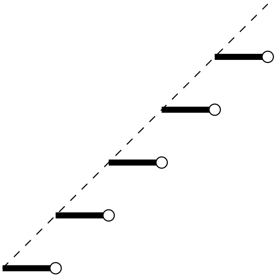

Figure: Graph of $f(x) = \lfloor x \rfloor$

Floor functions

A **dilated floor function** rounds down after rescaling by parameter α

$$f_{\alpha}(x) := \lfloor \alpha x \rfloor : \mathbb{R} \to \mathbb{Z}$$

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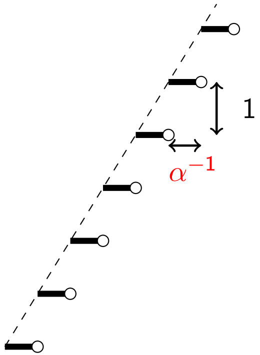

Figure: Graph of $f_\varphi(x) = \lfloor \varphi x \rfloor$, where $\varphi = \frac{1+\sqrt{5}}{2}$

Floor functions

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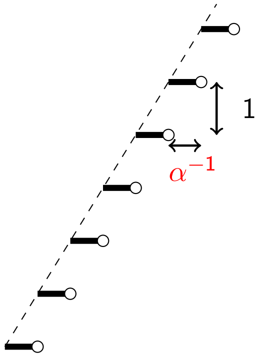
$\rightsquigarrow$ f_α discretizes $\mathbb{R}$
"at length scale α^{-1}"



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Dilated floor functions: Why care?

Elementary number theory:

$$\mathrm{val}_p(n!) = \left\lfloor \frac{1}{p} n \right\rfloor + \left\lfloor \frac{1}{p^2} n \right\rfloor + \left\lfloor \frac{1}{p^3} n \right\rfloor + \cdots$$

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Note: $\mathrm{val}_p(M)$ = largest exponent e such that p^e divides M

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Riemann zeta function:

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} \qquad \text{satisfies} \qquad \int_0^\infty \lfloor x \rfloor \, x^{-s} \frac{dx}{x} = \frac{1}{s} \zeta(s)$$

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Algebraic geometry: measuring singularities, minimal model program...

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$$\stackrel{?}{=} \left\lfloor \frac{1}{p} n \right\rfloor + \left\lfloor \frac{1}{p} \left\lfloor \frac{1}{p} n \right\rfloor \right\rfloor + \left\lfloor \frac{1}{p} \left\lfloor \frac{1}{p} \left\lfloor \frac{1}{p} n \right\rfloor \right\rfloor \right\rfloor + \cdots$$

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Algebraic geometry: measuring singularities, minimal model program...

Dilated floor functions: Why care?

Economics: currency exchange

$$\begin{aligned} \text{one US dollar} &= 0.80 \text{ UK pound (approximately)} \\ &= 1.33 \text{ Can. dollar} \\ &= 0.88 \text{ Eur. euro} \end{aligned}$$

Dilated floor functions: Why care?

Economics: currency exchange

$$\begin{aligned} \text{one US dollar} &= \tfrac{4}{5} \text{ UK pound (approximately)} \\ &= \tfrac{4}{3} \text{ Can. dollar} \\ &= \tfrac{8}{9} \text{ Eur. euro} \end{aligned}$$

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To exchange money from US to UK

$$x \mapsto \left\lfloor \frac{4}{5} x \right\rfloor$$

(bank does not give back change)

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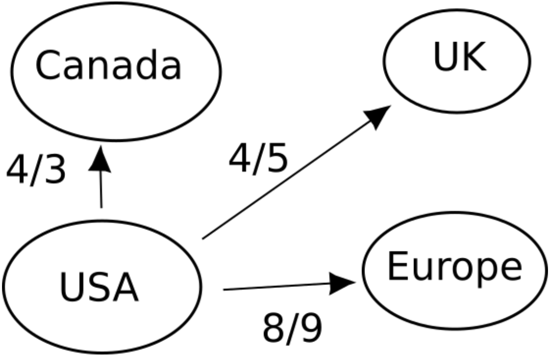

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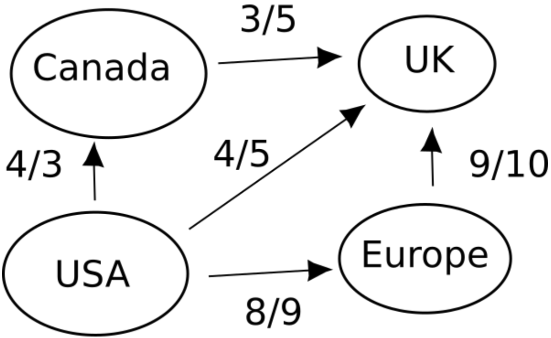

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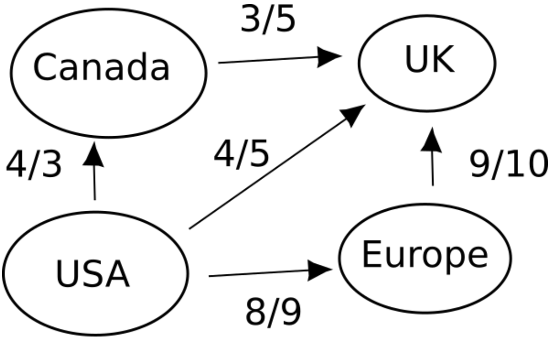

Problem: Would you rather exchange money through Canada or Europe?

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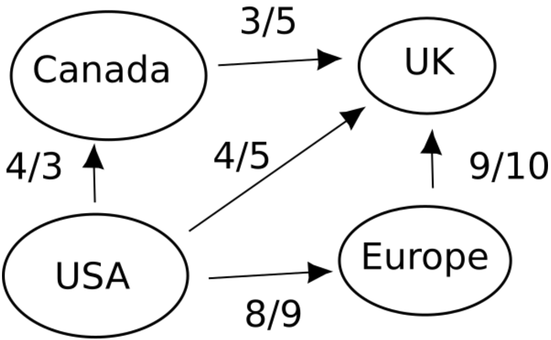

Problem: Would you rather exchange money through Canada or Europe?

$$\left\lfloor \frac{3}{5} \left\lfloor \frac{4}{3} x \right\rfloor \right\rfloor \quad \text{vs.} \quad \left\lfloor \frac{9}{10} \left\lfloor \frac{8}{9} x \right\rfloor \right\rfloor ?$$

Composing floor functions

Vague Question

What happens when we compose f_α and f_β?

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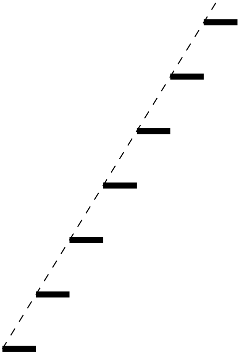

Figure: Graph of $f_1 \circ f_\varphi = \lfloor \lfloor \varphi x \rfloor \rfloor$ where $\varphi = \frac{1+\sqrt{5}}{2}$

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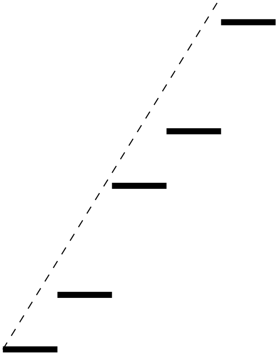

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Example: $f_1 \circ f_\varphi$ vs $f_\varphi \circ f_1$

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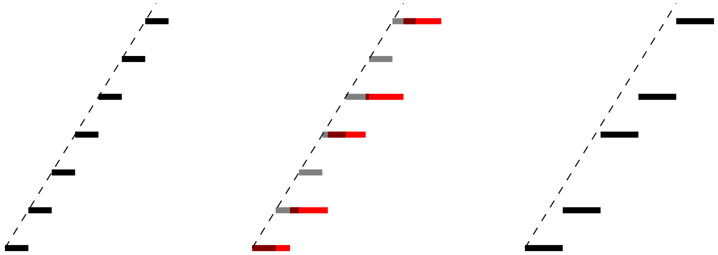

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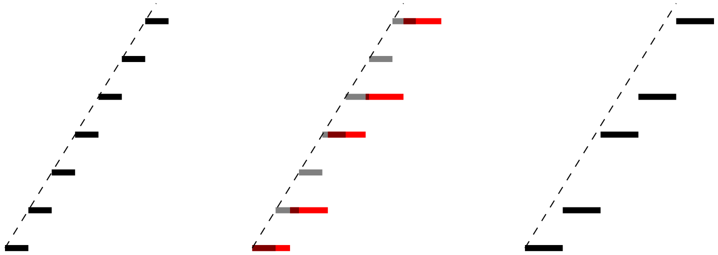

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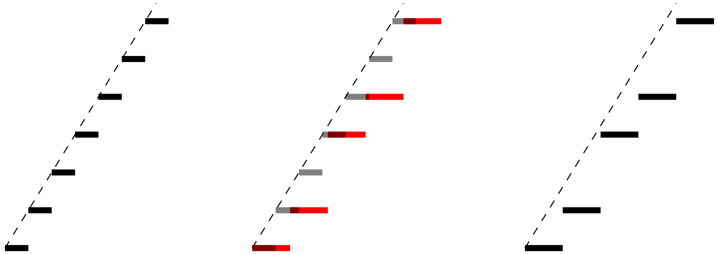

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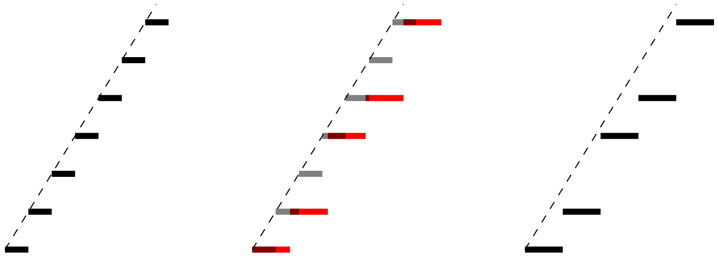

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Problem A

For which (α, β) do we have

$$f_\alpha \circ f_\beta = f_\beta \circ f_\alpha?$$

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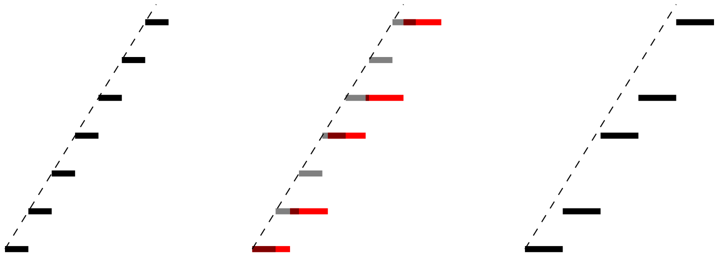

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For which (α, β) do we have

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Problem B

For which (α, β) do we have

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Composing floor functions: results

Problem A

For which (α, β) do we have $\quad f_\alpha \circ f_\beta = f_\beta \circ f_\alpha$?

Composing floor functions: results

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For which (α, β) do we have $\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor$?

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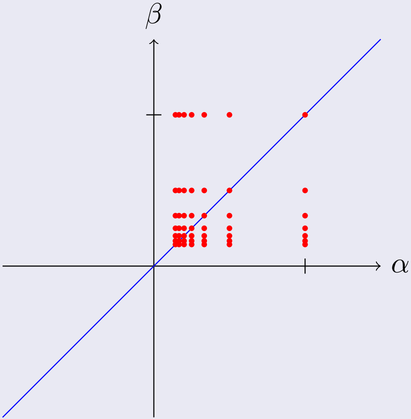
Theorem (Lagarias–Murayama–R)

All solutions to (A) are:



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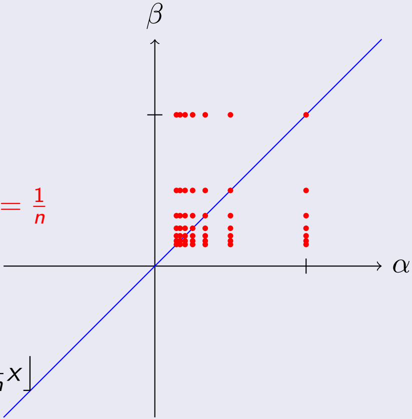
$\alpha = \frac{1}{m}$, $\beta = \frac{1}{n}$



In fact: $\left\lfloor \frac{1}{m} \left\lfloor \frac{1}{n} x \right\rfloor \right\rfloor = \left\lfloor \frac{1}{mn} x \right\rfloor$

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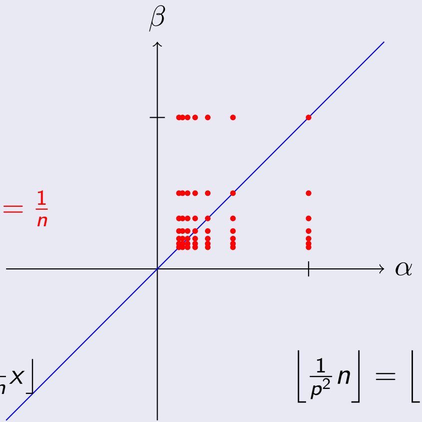

In fact: $\left\lfloor \frac{1}{m} \left\lfloor \frac{1}{n} x \right\rfloor \right\rfloor = \left\lfloor \frac{1}{mn} x \right\rfloor$

$\left\lfloor \frac{1}{p^2} n \right\rfloor = \left\lfloor \frac{1}{p} \left\lfloor \frac{1}{p} n \right\rfloor \right\rfloor$!

Composing floor functions: results

Problem B

For which (α, β) do we have $f_\alpha \circ f_\beta \geq f_\beta \circ f_\alpha$?

Composing floor functions: results

Problem B

For which (α, β) do we have $\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$?

Composing floor functions: results

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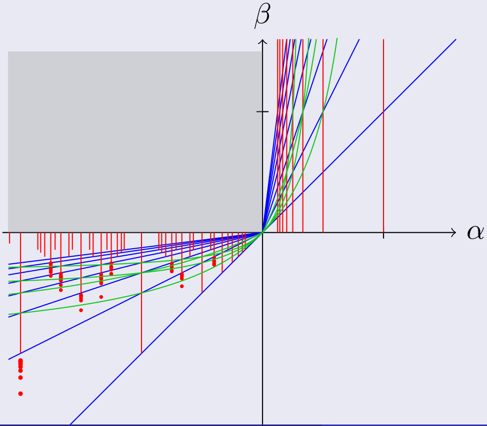
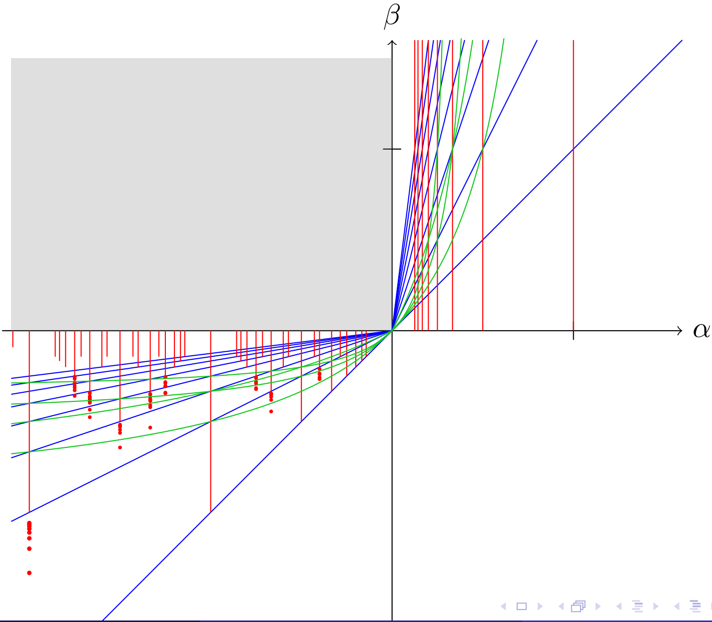
Theorem (Lagarias–R)

All solutions to (B) are:



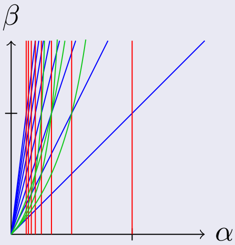
Composing floor functions: results



$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$: positive-dilation results

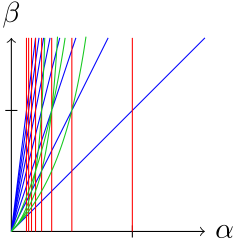
Theorem (Lagarias–R)

All positive solutions to (B) are:



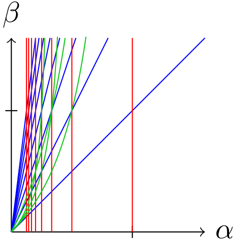
$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$: positive-dilation results

Coordinate change:

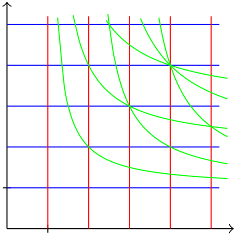


$\lfloor\alpha\lfloor\beta x\rfloor\rfloor \geq \lfloor\beta\lfloor\alpha x\rfloor\rfloor$: positive-dilation results

Coordinate change: $\mu = \frac{1}{\alpha}$, $\nu = \frac{\beta}{\alpha}$



$\rightsquigarrow$



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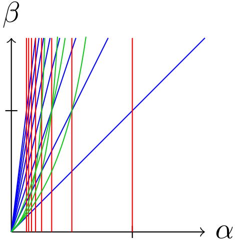

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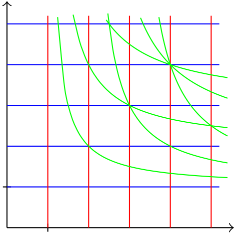
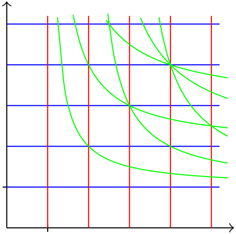


Symmetries:



$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$: positive-dilation results

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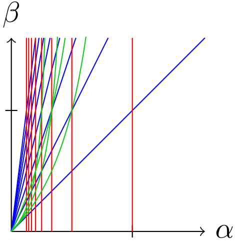

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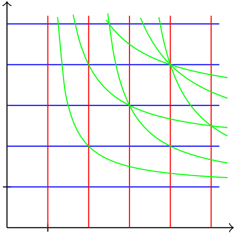

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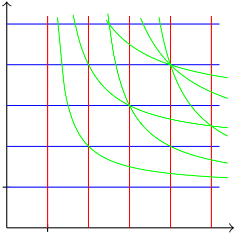

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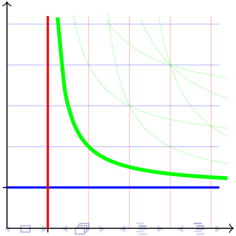
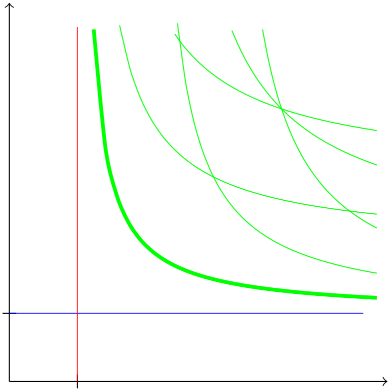


$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$: positive-dilation results

Where do **green** solution curves come from?



Proof ingredient: Beatty sequences

Parameter $\mu \geq 1$,

$$\mathcal{B}(\mu) = \{\lfloor \mu \rfloor, \lfloor 2\mu \rfloor, \lfloor 3\mu \rfloor, \ldots\} \subset \mathbb{N}$$

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Theorem ("Beatty's Theorem," Ostrowski, Hyslop, Aitken, ..)

If μ and ν are irrational and satsify $\frac{1}{\mu} + \frac{1}{\nu} = 1$, then

$$\mathcal{B}(\mu) \cap \mathcal{B}(\nu) = \emptyset \qquad \textit{and} \qquad \mathcal{B}(\mu) \cup \mathcal{B}(\nu) = \mathbb{N}$$

i.e. their Beatty sequences **partition** $\mathbb{N}$.

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Theorem ("Beatty's Theorem" 1926)

Beatty sequences $\mathcal{B}(\mu)$, $\mathcal{B}(\nu)$ partition $\mathbb{N}$, i.e.

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For parameters $(\alpha, \beta) > 0$,

$$f_\alpha \circ f_\beta \geq f_\beta \circ f_\alpha \qquad \textbf{iff} \qquad \mathcal{B}(\mu) \cap \mathcal{B}^{<}(\nu) = \emptyset$$

where $\mu = \frac{1}{\alpha}$ and $\nu = \frac{\beta}{\alpha}$.

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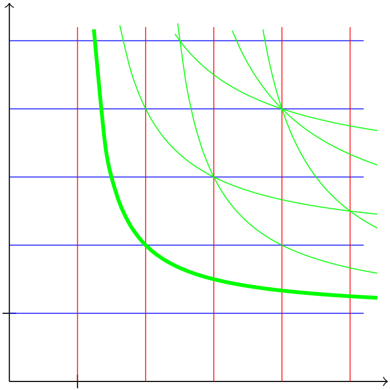
Proof ingredient: Beatty sequences

⇝ **Green** solution curves come from Beatty sequences



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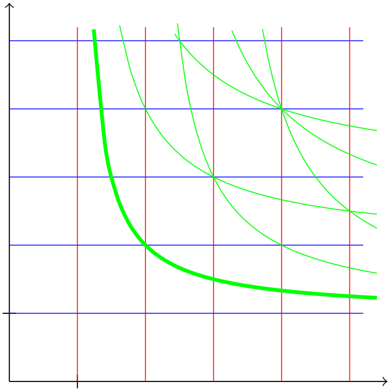

How do we know there are **no more** solutions?

Proof ingredient: Torus subgroups

Torus surface $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$
A point $(\sigma, \tau) \in \mathbb{T}$ generates a **cyclic subgroup** of $\mathbb{T}$

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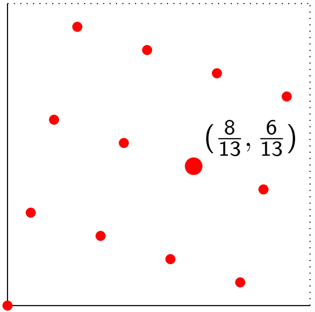

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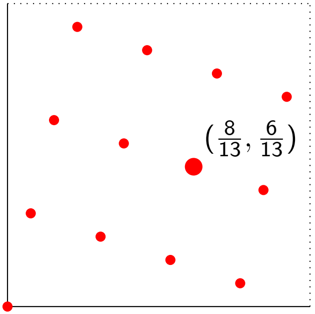

Vague Question

When is $(\sigma, \tau) \in \mathbb{T}$ a "minimal" generator for its subgroup?

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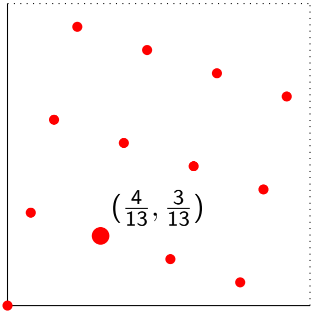

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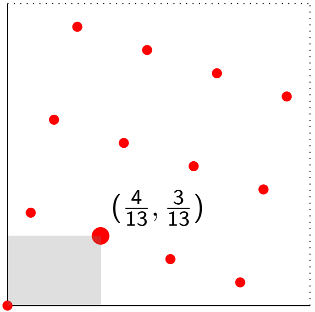

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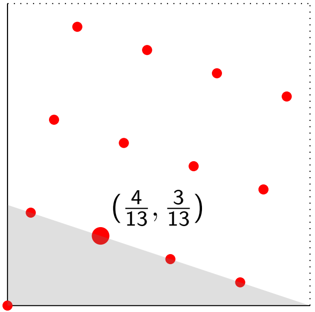

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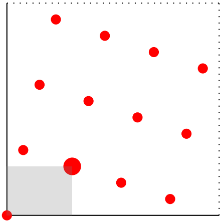

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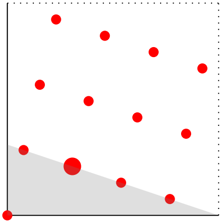

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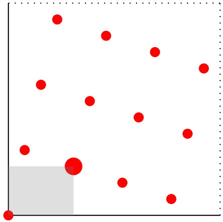

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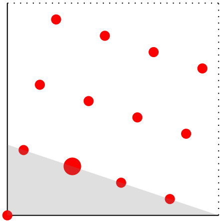

Proposition 3 (Lagarias–R)

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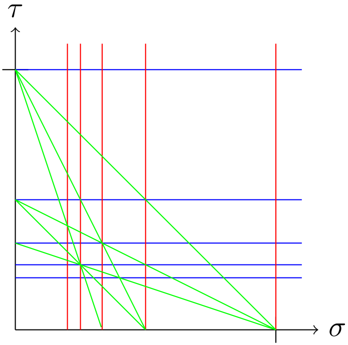
All minimal generators of cyclic subgroups, in $\mathbb{T}$:



Composing floor functions: Why **really** care?

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(Recall: P.N.T. says $\pi(x) = \#\{\text{prime } p \leq x\} \approx \frac{x}{\log x}$)

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Jean-Paul Cardinal (2010) defined a "2-dimensional analogue" of the Mertens function

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Let $\{d_i\} = \{n, \left\lfloor \frac{1}{2}n \right\rfloor, \left\lfloor \frac{1}{3}n \right\rfloor, \left\lfloor \frac{1}{4}n \right\rfloor, \ldots, 1\}$ be the "almost divisors" of n.

In Cardinal's matrix $\mathcal{M}_n$, the entry in position i, j is

$$\mathcal{M}_n(i,j) = M\left(\left\lfloor \frac{1}{d_i d_j} n \right\rfloor\right) = M\left(\left\lfloor \frac{1}{d_i} \left\lfloor \frac{1}{d_j} n \right\rfloor \right\rfloor\right) = M\left(\left\lfloor \frac{1}{d_j} \left\lfloor \frac{1}{d_i} n \right\rfloor \right\rfloor\right)$$

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Riemann hypothesis is equivalent to

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(Note: "almost divisors of almost divisors are almost divisors"!)

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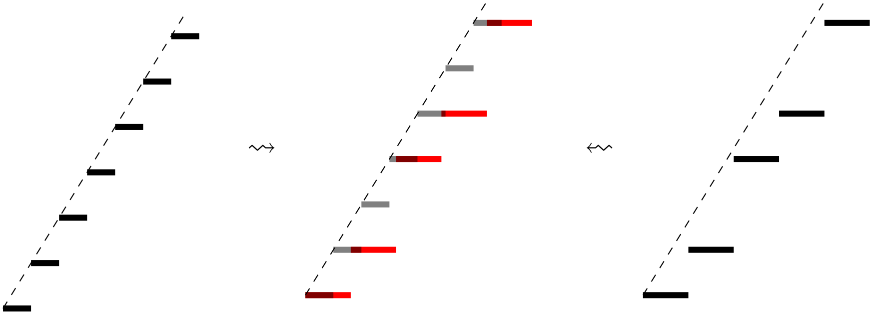
Dilated floor function commutators



Thank you!