RESEARCH STATEMENT

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My research falls into two separate areas. In one area, I use *tropical geometry* to study algebraic curves, metric graphs, and their Jacobians. In another area, I study the behavior of *rounding functions* in the context of elementary algebra and number theory.

0.1. **Background: Tropical curves and Jacobians.** Algebraic geometry is the study of solutions to polynomial equations such as $x^4 + y^4 = 10$. Over the complex numbers, the set of solutions is known as a Riemann surface. Tropical geometry allows us to turn a Riemann surface into a graph, as shown in Figure 1. This can be achieved via *degenerating* a smooth algebraic curve to

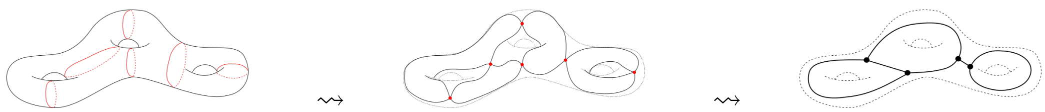

FIGURE 1. Tropicalizing a Riemann surface (left) to a graph (right).

a curve with nodal singularities, along a one-parameter family, then taking the dual graph of the nodal curve. This degeneration process turns meromorphic (i.e. rational) functions on the Riemann surface (i.e. complex algebraic curve) to piecewise linear functions on the dual graph. These tools were developed by Baker–Norine [BN07] and others [MZ, GK08].

I have studied how theorems from algebraic geometry concerning special discrete subsets of algebraic curves adapt to the tropical setting. These concern

(1) Weierstrass points on a metric graph, and
(2) torsion points of the Jacobian on a metric graph.

From one perspective this produces objects of inherent combinatorial interest. From another perspective, a goal of my research is to use the novel perspective of graph theory and combinatorics to prove results on algebraic curves and their discrete subsets that are currently beyond reach, e.g. uniform bounds on the number of torsion points on an algebraic curve.

Tropical geometry has been used to solve difficult problems in number theory and algebraic geometry. For instance, it is believed that there are *uniform bounds on rational points* on curves of genus $g \geq 2$, strengthening Faltings' theorem, but no such bound is currently known. Recent work of Katz, Rabinoff, and Zureick-Brown [KRZB16] made progress toward such a bound, for curves which satisfy an additional assumption on the Mordell–Weil rank. Tropical geometry was a fundamental ingredient in their proof.

0.2. **Background: Rounding functions.** The floor function $\lfloor x \rfloor$ takes a real number x and rounds it down to the nearest integer. This function is a basic operation of discretization, taking a continuous input to a discrete output. Discretization is fundamental in many applications, in areas such as computer imaging, digital communication, and finance. Mathematically, rounding can cause simple algebra rules to break down, but can leave others preserved in unexpected ways. In joint work with Jeffrey Lagarias and Takumi Murayama, we consider the effect of the floor function on commutativity of linear functions under composition.

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1. Tropical Weierstrass points

A projective embedding of an algebraic curve is naturally associated with a family of divisors on the curve, by intersecting the embedded curve with hyperplanes. The *Weierstrass points* of a divisor are the points in the corresponding embedding in $\mathbb{P}^r$ where the curve intersects some hyperplane with multiplicity at least $r+1$. On a genus one curve, this condition for a degree n divisor gives a set of n-torsion points; thus Weierstrass points are a higher genus analogue of torsion points [Mum77]. For a genus g curve and a generic divisor of degree $n \geq g$, the number of Weierstrass points is $g(n-g+1)^2$. Here we assume our base field is algebraically closed.

1.1. **Results.** In [R1], I study a natural analogue of Weierstrass points for a tropical curve.

Theorem 1 (Tropical Weierstrass point count [R1])**.** *On a metric graph of genus g, a generic divisor of degree $n \geq g$ has $g(n-g+1)$ Weierstrass points.*

Theorem 2 (Tropical Weierstrass point distribution [R1])**.** *Let Γ be a metric graph of genus $g \geq 2$, and let δ_n be the unit discrete measure supported on the Weierstrass locus of a generic divisor of degree n. Then the sequence of normalized measures $\frac{1}{gn}\delta_n$ converges weakly to Zhang's canonical measure on Γ.*

The distribution result mirrors parallel results of Neeman and Amini for algebraic curves over the complex numbers $\mathbb{C}$, respectively, over a field with non-Archimedean valuation.

Theorem 3 (Neeman [Nee84])**.** *On a compact Riemann surface X of genus $g \geq 2$, let δ_n be the unit discrete measure supported on the Weierstrass locus of a divisor of degree n. Then the sequence of normalized meausres $\frac{1}{gn^2}\delta_n$ converges weakly to the Bergman measure on X.*

Theorem 4 (Amini [Ami])**.** *On a smooth proper Berkovich curve X^{an} of genus $g \geq 2$, let δ_n be the unit discrete measure supported on the Weierstrass locus of a divisor of degree n. Then the sequence of normalized measures $\frac{1}{\delta_n(X^{an})}\delta_n$ converges weakly to Zhang's canonical measure on X^{an}.*

The Berkovich analytification of a curve [Ber90] contains a tropical curve as its *skeleton*, and divisor theory behaves well with respect to retraction to the skeleton [Bak08]. Amini's result suggested that the distribution of Weierstrass points could be a purely tropical phenomenon. In constrast to [Ami], the proof of Theorem 2 does not rely on any non-Archimedean or analytic geometry.

1.2. **Future research objectives.** To address the distribution of tropical Weierstrass points, a fundamental tool I use is the theory of electrical current flow; it turns out that the Weierstrass point condition can be viewed as a problem in the flow of "discrete" electrical current. To be precise, the voltage function j_z^y which arises from sending one unit of current from point y to point z is approximated, up to a rescaling, by the piecewise linear function whose zeros and poles are *reduced divisors* at y and z, respectively [R1, Section 3]. Zhang's measure [Zha93] originated in work unrelated to Weierstrass points, but Chinburg and Rumely found an equivalent formulation in terms of electrical current flow through a resistor network [CR93, BF06]. These current flows are closely related to random spanning trees and random walks on a graph, connections which go back to work of Kirchhoff [Kir47, Tet91].

These techniques look very different than the ones used by Neeman and Amini to study the distribution of Weierstrass points on an algebraic curve over $\mathbb{C}$ or over a non-Archimedean field. In future work, I plan to investigate whether there is a common strategy for proving Theorems 2, 3 and 4, unifying the contexts of complex, non-Archimedean, and tropical geometry. The current proofs do not seem to extend beyond one of the three contexts. One possible strategy is to utilize the *hybrid valuation* on the complex numbers developed by Berkovich and Jonsson [Ber90, Jon15].

Problem 1. Is there a proof of the convergence of Weierstrass points to a limiting distribution that works for complex curves, Berkovich curves, and tropical curves?

1.2.1. **Special divisors.** Another goal is to address the behavior of Weierstrass points of low-degree non-generic divisors, such as the *canonical divisor* of a metric graph. The methods of [R1] give a good understanding of Weierstrass points of a divisor which is *Riemann–Roch nonspecial*, meaning that its rank $r(D) = \deg(D) - g$. The canonical divisor K has degree $2g-2$ and rank $g-1$. For many metric graphs, the Weierstrass locus of K contains components of dimension 1, which is not possible on an algebraic curve. It would be desireable to define a "stable" Weierstrass locus which does not suffer from excess dimensionality.

Problem 2. Is there a stable Weierstrass locus for the canonical divisor on a metric graph?

To address this problem, I plan to build on work of Ulirsch and Zakharov [UZ] on moduli spaces of finite ramified coverings of metric trees with prescribed ramification data. On an algebraic curve, the existence of a ramified cover of $\mathbb{P}^1$ by a curve X with prescribed fibers over $0, \infty \in \mathbb{P}^1$ is equivalent to a linear equivalence between the prescribed fibers. In the tropical case this equivalence breaks down, and families of linearly equivalent divisors will generally have "excess dimension." In constrast, [UZ] shows that the space of finite effective ramified covers does have the dimension predicted from the analogous space for algebraic curves.

Conditions on canonical Weierstrass points would naturally cut out subsets in the moduli space of pointed tropical curves.

Problem 3. How is the moduli space of pointed tropical curves stratified according to conditions on Weierstrass points?

Similar stratifications for the moduli space of algebraic curves are studied by Eisenbud and Harris [EH87], and by Pflueger [Pfl18]. The moduli space of algebraic curves is compatible under tropicalization with a natural moduli space of tropical curves [ACP15].

1.2.2. **The universal break divisor.** To understand the Weierstrass points of a nonspecial divisor, an important step in [R1] is to define a *stable Weierstrass locus*, which relies on the fundamental tropical notion of break divisor [MZ, ABKS14]. A divisor class $[D] \in \mathrm{Pic}^g(\Gamma)$ of degree g has a canonical *break divisor representative* $\mathrm{br}[D]$, meaning roughly that the complement $\Gamma \setminus \mathrm{br}[D]$ is a metric tree.

We may consider the space $\widetilde{\mathrm{Br}}^g(\Gamma) \to \mathrm{Pic}^g(\Gamma)$ whose fiber over a divisor class $[D]$ is defined to be the break divisor $\mathrm{br}[D]$; we call $\widetilde{\mathrm{Br}}^g(\Gamma)$ the *universal break divisor* of Γ. Following [ABKS14], $\widetilde{\mathrm{Br}}^g(\Gamma)$ has the structure of a g-dimensional cell complex. When Γ is trivalent, the space $\widetilde{\mathrm{Br}}^g(\Gamma)$ is in fact a manifold. It would be interesting to have a better understanding of the topology of these spaces.

Problem 4. What are the homology groups of the universal break divisor $\widetilde{\mathrm{Br}}^g(\Gamma)$? What are its homotopy groups?

The universal break divisor is equipped with a natural projection $\widetilde{\mathrm{Br}}^g(\Gamma) \to \mathrm{Pic}^{g-1}(\Gamma)$ sending $(x, [D]) \mapsto [D - x]$. A general fiber of this projection is a circle; there are also singular fibers which are cyclic subgraphs of Γ of higher genus. This suggests that one could compute the homology of $\widetilde{\mathrm{Br}}^g(\Gamma)$ efficiently with an application of the Leray spectral sequence.

2. Tropical Manin–Mumford conjecture

Given an algebraic curve with fixed basepoint x_0, we say that x is a *torsion point* if the divisor $n(x - x_0)$ is linearly equivalent to 0 for some positive n. Equivalently, x is a torsion point if the Abel–Jacobi embedding (with respect to x_0) sends x to the torsion subgroup $\mathrm{Jac}(X)_{tors}$ of the Jacobian. By analogy with Mordell's conjecture on finiteness of rational points, Manin and Mumford conjectured that an algebraic curve of genus 2 or more has finitely many torsion points. This conjecture was proved by Raynaud [Ray83].

Theorem 5 (Raynaud; formerly the Manin–Mumford Conjecture)**.** *For any smooth curve X of genus $g \geq 2$, the set of torsion points $X \cap \mathrm{Jac}(X)_{tors}$ is finite.*

The following stronger result remains open, though it is suspected to be true [BP01]. (Baker and Poonen use the equivalent language of *torsion packets* on curves.)

Problem 5 (Uniform bound on torsion points)**.** Is there a constant $N(g)$ such that any algebraic curve X of genus $g \geq 2$ has $\#(X \cap \mathrm{Jac}(X)_{tors}) \leq N(g)$?

2.1. **Results.** For a metric graph Γ, there is an analogous Jacobian [MZ] which is compatible with the Jacobian of an algebraic curve under taking the skeletons of non-Archimedean varieties [BR15]. In [R2], I investigate the tropical analogue of the Manin–Mumford condition.

Theorem 6 (Tropical uniform bound, conditioned on finiteness [R2])**.** *Let Γ be a metric graph of genus $g \geq 2$. If $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{tors})$ is finite, then $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{tors}) \leq 3g - 3$.*

The bound of $3g-3$ in Theorem 6 answers a tropical analogue of Problem 5. However, the Manin–Mumford conjecture fails for tropical curves: for any metric graph with integer edge lengths, there are infinitely many torsion points with respect to any basepoint, no matter how large the genus is. On the other hand, I show that a metric graph of genus 2 or more does satisfy the Manin–Mumford condition if we impose certain additional constraints.

Theorem 7 (Tropical uniform bound, general edge lengths [R2])**.** *Let G be a biconnected graph of genus $g \geq 2$. For a very general choice of edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, the metric graph $\Gamma = (G, \ell)$ has $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{tors}) \leq g + 1$.*

A graph is *biconnected* if it cannot be separated into two parts by cutting out one vertex; a *very general* choice of edge lengths means we exclude countably many families of positive codimension in the parameter space of edge-lengths.

2.2. **Future research objectives.** There are natural higher-dimensional analogues of the Manin–Mumford condition, where we embed the d-th symmetric power of a curve, or metric graph, into its Jacobian. If D_0 is a fixed divisor of degree d, we define the d-dimensional Abel–Jacobi map

$$AJ^{(d)}_{D_0} : \mathrm{Sym}^d(\Gamma) \to \mathrm{Jac}(\Gamma) \qquad \text{by} \qquad x_1 + \cdots + x_d \mapsto [x_1 + \cdots + x_d - D_0].$$

Problem 6. Identify $\mathrm{Sym}^d(\Gamma)$ with its image under $AJ^{(d)}_{D_0}$. When is $\mathrm{Sym}^d(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}$ finite?

In progress on this problem, I have found that the *girth* of a graph gives an upper bound on the dimension d for which the higher Manin–Mumford condition is satisfied. For very general edge lengths, we conjecture that $\mathrm{Sym}^d(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}$ is finite if and only if

$$d \leq \min_{C \subset E(G)} \{\mathrm{rank}_{\mathcal{M}^\perp(G)}(C)\},$$

where the minimum is taken over all (simple) cycles of G and $\mathcal{M}^\perp(G)$ denotes the cographic matroid of G. We call this number the *independent girth* of G; note that the girth is $\min_{C \subset E(G)}\{\#C\}$.

3. Rounding functions

Multiplication of real numbers is commutative:

$$10.1 \times 20.3 = 20.3 \times 10.1.$$

However, if we are in a world where the output of any computation is rounded down, then commutativity may fail:

$$\lfloor \lfloor 10.1 \rfloor \times 20.3 \rfloor = 203 \neq 202 = \lfloor \lfloor 20.3 \rfloor \times 10.1 \rfloor.$$

3.1. **Results.** In joint work with Jeffrey Lagarias and Takumi Murayama, we studied this failure of commutativity when composing two floor functions with dilation factors. We completely classify the pairs of dilation factors for which changing the order of composition of dilated floor functions does not change the resulting function [LMR16], and for which changing the order of composition makes the resulting function larger [LR1, LR2].

Theorem 8 ([LMR16])**.** *For $\alpha, \beta > 0$, the equality $\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ holds for all real x if and only if $\alpha = \beta$, or $\alpha = 1/m$ and $\beta = 1/n$ for positive integers m, n.*

Theorem 9 ([LR1])**.** *For $\alpha, \beta > 0$, the inequality $\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$ holds for all real x if and only if $m\alpha + n\alpha/\beta = 1$ for positive integers m, n.*

The question of composing dilated floor functions in different orders was initially motivated by work of Cardinal, who constructed families of matrices whose entries depend on compositions of floor functions dilated by ratios of whole numbers [Car10]. In Theorem 8, the dilation pairs α, β are precisely those that show up in the construction of Cardinal's matrices.

3.2. **Future research objectives.** The investigation of the compositional commutator

$$f_{\alpha,\beta}(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor$$

fits in the larger context of *generalized polynomials* of Bergelson and Liebman [BL07], which are functions composed from addition, multiplication, their inverses, and the floor function. In [BL07], it is shown that all such functions have nice ergodic properties.

A consequence of Theorem 9 is that the parameter space of all α, β satisfying $f_{\alpha,\beta} \geq 0$ has a dense set of rational points. It is not clear a priori why this should occur. In general, the question of when a real algebraic variety has a dense subset of rational point is quite subtle [HB02, Maz92] and not completely understood. We are interested in studying the density of rational points in spaces cut out by generalized polynomials with rational coefficients.

3.2.1. **Almost divisors.** In multiplicative number theory, one studies the prime factorization of numbers by considering the numbers as a partially ordered set (poset) under divison. From this poset, we have the Möbius function $\mu(n)$, which depends on the prime factorization structure of n. It is a classical result that the Riemann Hypothesis is equivalent to the bound

$$\sum_{n \leq x} \mu(n) = O(x^{1/2+\epsilon}) \quad \text{for any } \epsilon > 0.$$

We may use the floor function to define the *almost divisors* of n as the numbers of the form $\{d : d = \lfloor n/k \rfloor \text{ for } k = 1, 2, \ldots, n\}$. Surprisingly, this definition gives the structure of a poset on the whole numbers. This observation was implicit in [Car10]. In Figure 2, we show the almost divisors of $n = 16$ and their poset structure.

In [LR3], we investigate the structure of the poset of almost divisors, answering questions in parallel with usual results in multiplicative number theory. We obtain polynomial bounds on the Möbius function of the almost-divisor poset. (The results in [Car10] suggest that $\mu_{AD}(n)$ should be considered an analogue of the sum $\sum_{n/2<i\leq n} \mu(i)$ of the usual Möbius function, rather than $\mu(n)$.)

Theorem 10 (Lagarias–R [LR3])**.** *Let $\mu_{AD}(n)$ denote the Möbius function of the almost-divisor poset interval from 1 to n. There is some constant $C > 0$ such that*

$$|\mu_{AD}(n)| < Cn^{1.729} \qquad \textit{for all } n.$$

We believe the bound in Theorem 10 is not optimal. We hope that further investigation will yield improved bounds on the Möbius function.

Problem 7. For any $\epsilon > 0$, is there some constant $C(\epsilon)$ such that

$$|\mu_{AD}(n)| < C(\epsilon) n^{1/2+\epsilon} \qquad \text{for all } n?$$

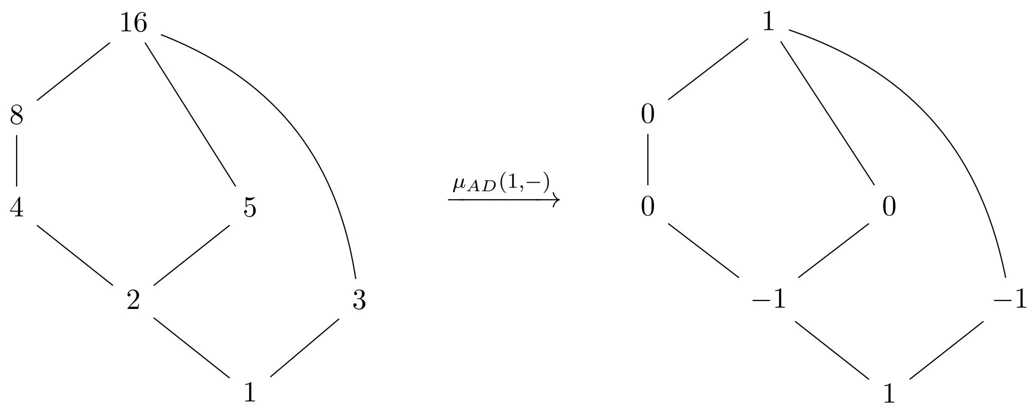

FIGURE 2. Poset of almost divisors of 16, with Möbius values $\mu_{AD}(1, d)$ on right.

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