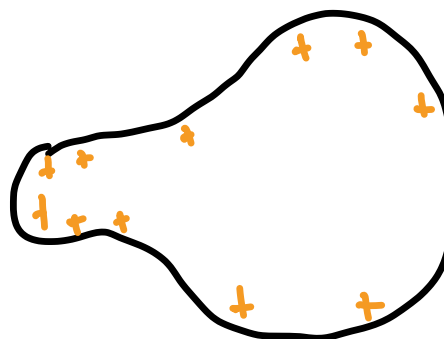
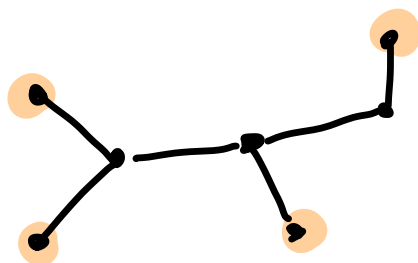


Tree distance matrices & their minors



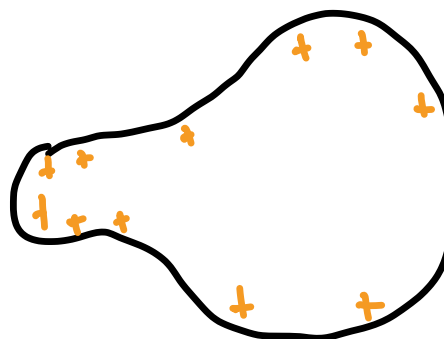
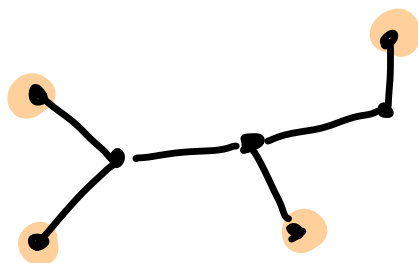
Harry Richman

NCTS, Taipei

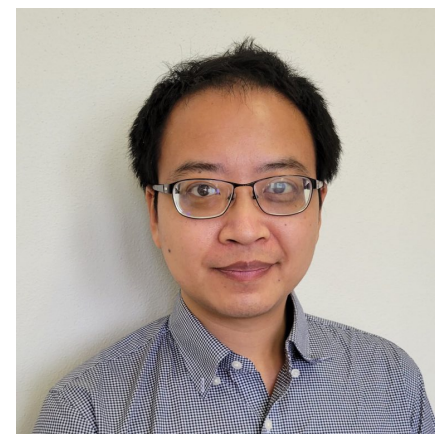
24 June 2025

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Tree distance matrices & their minors



joint work w/
Farbod Shokrieh,
U. Washington

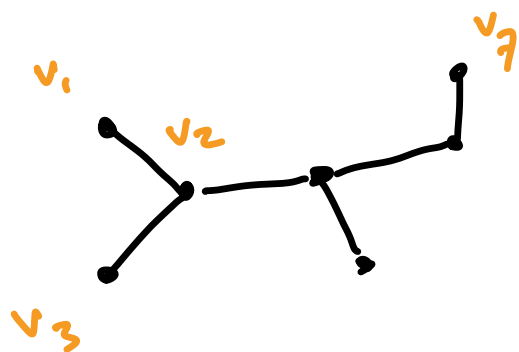


Chenxi Wu
U. Wisconsin

Distance matrices

Problem What does determinant of

a distance matrix tell us "combinatorially"?

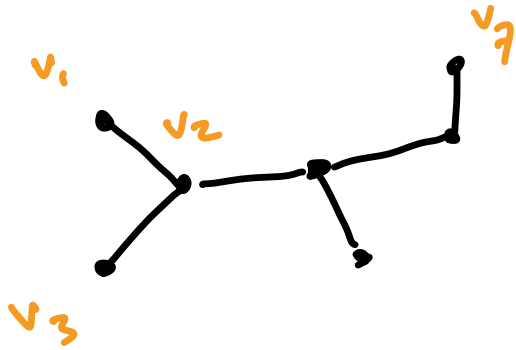


$$D = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & \dots & v_7 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_7 \end{matrix} & \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix} \end{matrix}$$

all nodes,
internal included

Distance matrices

Problem What does the determinant tell us ?

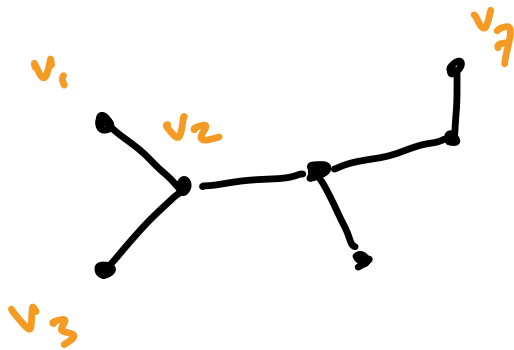


$$D = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & \dots & v_7 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_7 \end{matrix} & \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$\Rightarrow \det D = \underline{192}$$

Distance matrices

Problem What does the determinant tell us ?



$$D = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & \dots & v_7 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_7 \end{matrix} & \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix} \end{matrix}$$

Theorem (Graham-Pollak, 1971)

For any tree on n vertices,

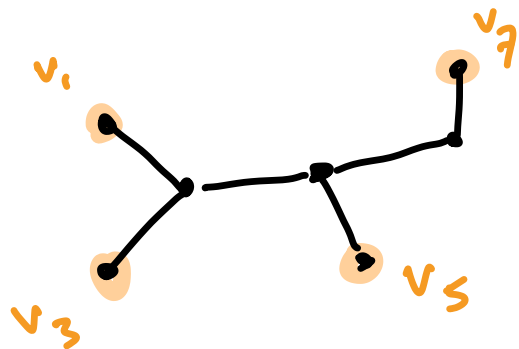
$$\det D = (-1)^{n-1} 2^{n-2} (n-1)$$

no
combinatorics ;)

Distance matrices

Problem What does determinant of
a distance submatrix tell us "combinatorially"?

$$D = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix}$$



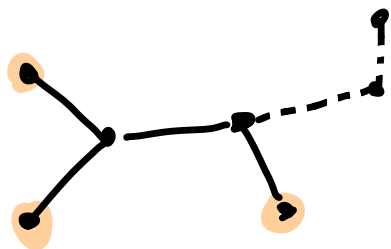
$$D[S] = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{bmatrix}$$

e.g.

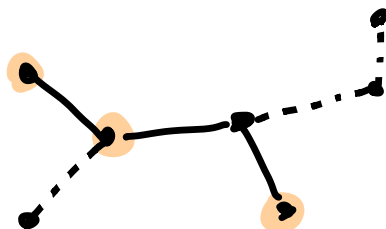
$$S = \partial G$$

Distance matrices

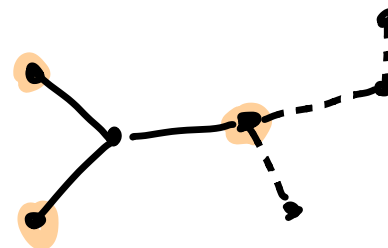
Problem What does determinant of
a distance submatrix tell us "combinatorially"?



$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$



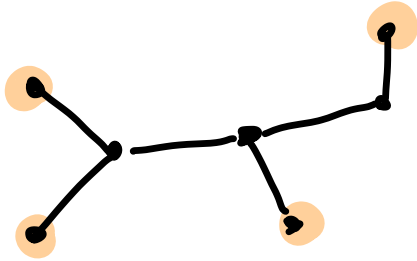
$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

Distance matrices

Problem What does determinant tell us ?



$$D[s] = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{bmatrix}$$

$$\Rightarrow \det D[s] = -252 \quad ??$$



No previously known combinatorial interpretation

Distance matrices

Thm (Graham - Pollak)

$$\det D = (-1)^{n-1} 2^{n-2} (n-1)$$

Theorem (R - Shokrieh - Wu)

Given a tree $G = (V, E)$ and vertex subset $S \subset V$,

$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left((n-1) K_1(G; S) - \sum_{F \in \mathcal{F}_2(G; S)} (\deg^o(F, *) - 2)^2 \right)$$

where $n = \#$ vertices

$K_1(G; S) = \#$ S -rooted spanning forests

$\mathcal{F}_2(G; S) = (S, *)$ -rooted spanning forests

$\deg^o(F, *) =$ out-degree of floating component

Distance matrices

Theorem (R - Shokrieh - Wu)

Given a tree $G = (V, E)$ and vertex subset $S \subset V$,

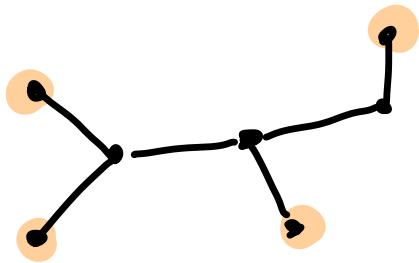
$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left((n-1) \kappa_1(G; S) - \sum_{F \in \mathcal{F}_2(G; S)} (\deg^o(F, *) - 2)^2 \right)$$

Ex.

where

$n = 7$

$$\kappa_1(G; S) = \# \left\{ \begin{array}{c} \text{[Diagram 1: Tree with 7 vertices, 6 edges, 3 leaves]} \\ \text{[Diagram 2: Tree with 7 vertices, 6 edges, 3 leaves]} \\ \dots \end{array} \right\}$$



$$\mathcal{F}_2(G; S) = \left\{ \begin{array}{c} \text{[Diagram 1: Tree with 7 vertices, 6 edges, 3 leaves, one edge highlighted in blue]} \\ \text{[Diagram 2: Tree with 7 vertices, 6 edges, 3 leaves, one edge highlighted in blue]} \\ \dots \end{array} \right\}$$

$$\deg^o(F, *) = \begin{array}{c} \text{[Diagram 1: Tree with 7 vertices, 6 edges, 3 leaves, one edge highlighted in blue]} \\ \text{[Diagram 2: Tree with 7 vertices, 6 edges, 3 leaves, one edge highlighted in blue]} \end{array}$$

$\deg^o = 3$ $\deg^o = 4$

Distance matrices

Theorem (R - Shokrieh - Wu)

$$\det D[S] = (-1)^{|S|-1} z^{|S|-2} \left((n-1) \kappa_1(G; S) - \sum_{F \in \mathcal{F}_2(G; S)} (\deg^0(F, *) - 2)^2 \right)$$

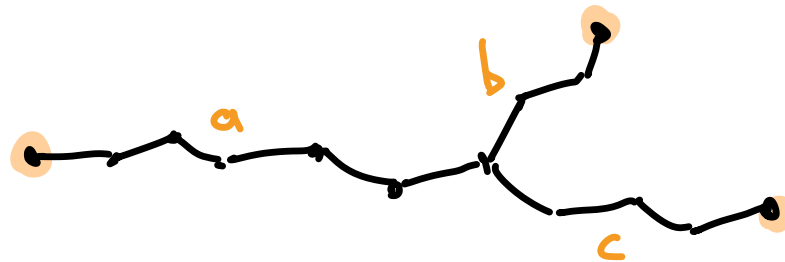
How to prove?



Potential theory on trees

Distance matrices

Ex. Tripod graph



$$D[S] = \begin{bmatrix} 0 & a+b & a+c \\ a+b & 0 & b+c \\ a+c & b+c & 0 \end{bmatrix}$$

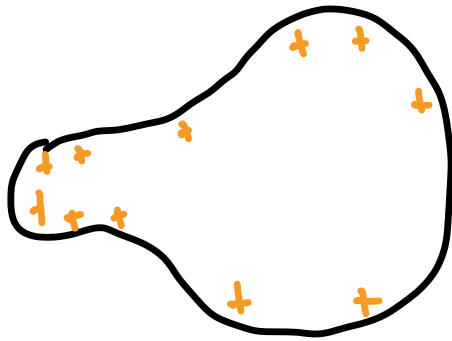
$$\det D[S] = 2(a+b)(a+c)(b+c)$$

By Theorem,

$$\det D[S] = 2 \left(\underbrace{(a+b+c)}_{n-1} \underbrace{(ab+bc+ac)}_{\kappa_1(G;S)} - \underbrace{abc}_{\sum i^2, F_2(G;S)} \right)$$

Potential theory

Problem How do particles "distribute" within
a region, given repulsive potential $U(x,y)$?



2-dim region



1-dim tree

Potential theory: tree case

Problem How do particles distribute ?



1-dim tree

- Minimize

$$\Sigma(\vec{\mu}) = -\frac{1}{2} \vec{\mu}^T D[s] \vec{\mu}$$

→ self-repulsion energy

- Constraint

$$\vec{1} \cdot \vec{\mu} = 1$$

→ conservation of mass

Potential theory



Problem Find

$$\min \left\{ -\frac{1}{2} \vec{\mu}^T D[S] \vec{\mu} : \vec{\mu} \in \mathbb{R}^S, \mathbf{1} \cdot \vec{\mu} = 1 \right\}$$

$$\nabla(\text{objective}) = -D[S] \vec{\mu}$$

$$\nabla(\text{constraint}) = \mathbf{1}$$

Proposition (Bapat)

a) Minimum occurs at $D[S] \vec{\mu}^* = \lambda \mathbf{1}$

b) $\min_{\mathbf{1} \cdot \vec{\mu} = 1} \left\{ -\frac{1}{2} \vec{\mu}^T D[S] \vec{\mu} \right\} = -\frac{1}{2} \frac{\det D[S]}{\text{cof } D[S]}$

⌋
sum of cofactors

$$\sum_{i,j} (-1)^{i+j} \det A_{i,j}$$

Potential theory

Proposition (Bapat)



$$\xi = -\frac{1}{2} \vec{\mu}^T D[S] \vec{\mu}$$

a) Minimum $\xi(\vec{\mu})$ occurs at $D[S] \vec{\mu}^* = \lambda \mathbb{1}$

$$b) \min_{\mathbb{1} \cdot \vec{\mu} = 1} \left\{ \xi(\vec{\mu}) \right\} = -\frac{1}{2} \frac{\det D[S]}{\text{cof } D[S]}$$

Sum of cofactors $\sum_{i,j} (-1)^{i+j} \det A_{i,j}$

Aside

Theorem (RSW + Bapat - Sivasubramanian 2011)

$$\frac{\det D[S]}{\text{cof } D[S]} = \frac{1}{2} \left((n-1) - \frac{\sum_{F \in \mathcal{F}_2(G; S)} (\deg^0(F, *) - 2)^2}{\kappa_1(G; S)} \right)$$

Potential theory

Proposition (Bapat)



$$\mathcal{E} = -\frac{1}{2} \vec{\mu}^T D[S] \vec{\mu}$$

a) Minimum $\mathcal{E}(\vec{\mu})$ occurs at $D[S] \vec{\mu}^* = \lambda \mathbb{1}$

$$b) \min_{\mathbb{1} \cdot \vec{\mu} = 1} \left\{ \mathcal{E}(\vec{\mu}) \right\} = -\frac{1}{2} \frac{\det D[S]}{\text{cof } D[S]}$$

Sum of cofactors $\sum_{i,j} (-1)^{i+j} \det A_{i,j}$

Aside

Theorem (RSW, also Devriendt 2022)

$$\text{If } A \subset B \subset V(G), \text{ then } \frac{\det D[A]}{\text{cof } D[A]} \leq \frac{\det D[B]}{\text{cof } D[B]}$$

Potential theory

Summary:

How to find
 $\det D[S]$?



How to find
 $\min \mathcal{E}(\vec{\mu})$?



How to solve
 $D[S]\vec{\mu} = \lambda \mathbb{1}$?

Theorem (Bapat - Sivasubramanian 2011, et al.?)

Equilibrium vector is

$$\mu_i^* = \frac{1}{2 \kappa_i(G; S)} \sum_{T \in \mathcal{F}_i(G; S)} (2 - \deg^o(T, i))$$

$$\begin{cases} D[S]\vec{\mu}^* = \lambda \mathbb{1} \\ \mathbb{1} \cdot \vec{\mu}^* = 1 \end{cases}$$

Potential theory

Theorem (Bapat - Sivasubramanian 2011)

$$\mu_i^* = \frac{1}{2 \kappa_i(G; S)} \sum_{T \in \mathcal{F}_i(G; S)} (2 - \deg^o(T, i))$$

Idea:

combinatorics
of \mathcal{D}

\longleftrightarrow

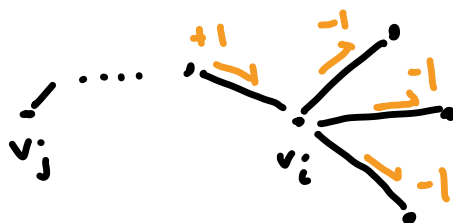
combinatorics of
Laplacian

• Laplacian $(L \vec{x})_i = \sum_{j \sim i} (x_i - x_j)$

• L times distance matrix

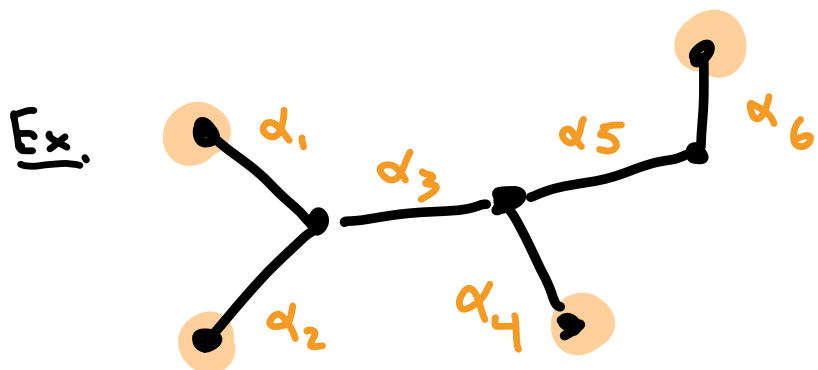
$$(LD)_{i,j} = \left(L \begin{bmatrix} d_{1,j} \\ \vdots \\ d_{i,j} \end{bmatrix} \right)_i = 1 - (\deg_i - 1) = 2 - \deg_i$$

⚠ no j -index



except when
 $i = j$

Further extensions: edge weights



$$D = \begin{bmatrix} 0 & \alpha_1 & \alpha_1 + \alpha_2 & \dots \\ \alpha_1 & 0 & \alpha_2 & \\ \alpha_1 + \alpha_2 & \alpha_2 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

Theorem (R - Shokrieh - Wu)

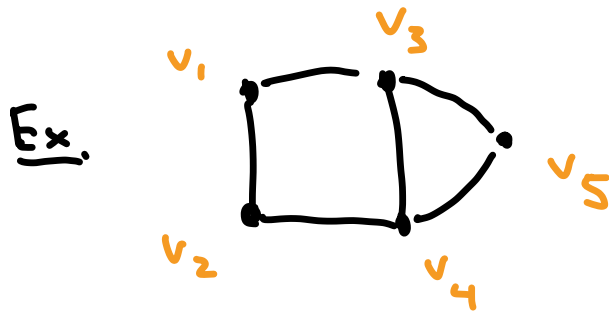
$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left(\sum_E \alpha_e \sum_{T \in \mathcal{F}_1} w(T) - \sum_{F \in \mathcal{F}_2} (\deg^0(F, *) - 2)^2 w(F) \right)$$

edge weights

(Bapat - Kirkland - Neumann, 2005 when $S = V$)

Further extensions:

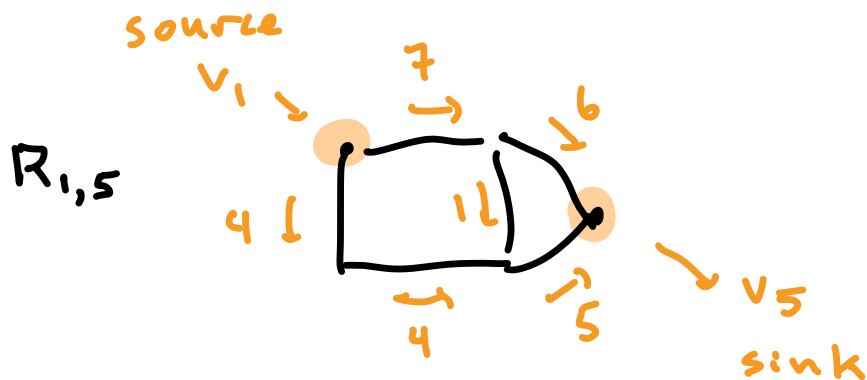
graphs w/ cycles



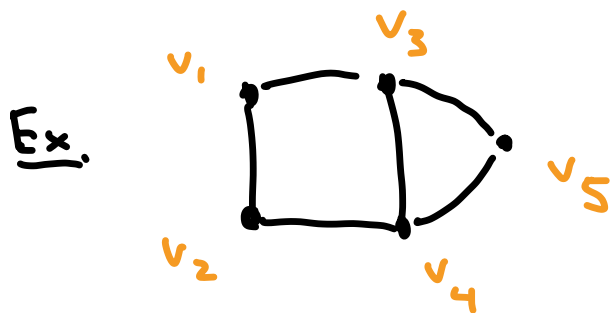
$$D = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 1 & 2 \\ 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{bmatrix}$$

effective resistance
matrix

$$R = \begin{bmatrix} 0 & 8/11 & 8/11 & 10/11 & 13/11 \\ 8/11 & 0 & 10/11 & 8/11 & 13/11 \\ 8/11 & 10/11 & 0 & 6/11 & 7/11 \\ 10/11 & 8/11 & 6/11 & 0 & 7/11 \\ 13/11 & 13/11 & 7/11 & 7/11 & 0 \end{bmatrix}$$



Further extensions: graphs w/ cycles



$$R = \begin{bmatrix} 0 & 8/11 & 8/11 & 10/11 & 13/11 \\ 8/11 & 0 & 10/11 & 8/11 & 13/11 \\ 8/11 & 10/11 & 0 & 6/11 & 7/11 \\ 10/11 & 8/11 & 6/11 & 0 & 7/11 \\ 13/11 & 13/11 & 7/11 & 7/11 & 0 \end{bmatrix}$$

Theorem (RSW)

$$\frac{\det R}{\text{cof } R} = \frac{2}{3} \frac{\kappa_2(G)}{\kappa(G)} - \frac{1}{6} \sum_{e \in E} \left(\frac{\kappa(G/e)}{\kappa(G)} \right)^2$$

↗ # Z-forests

↘ # trees

Further extensions

Still unresolved:

- q -distance matrices, e.g.

$$D_q[S] = \frac{1}{(1-q)^4} \begin{bmatrix} 0 & 1-q^2 & \dots \\ 1-q^2 & 0 & \\ 1-q^3 & 1-q^3 & \\ 1-q^4 & 1-q^4 & \dots \end{bmatrix}$$

Bapat - Lai - Pati 2006

Choudhury - Khare 2024

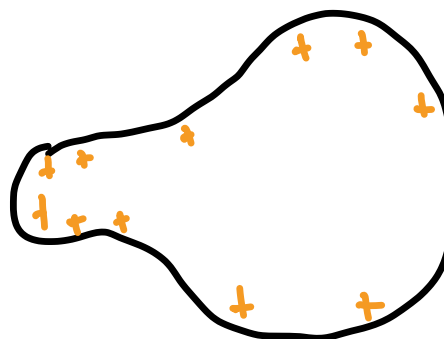
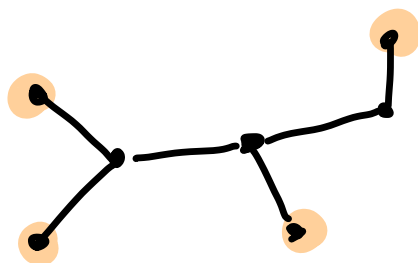
- Steiner distance hyperdeterminants, k -subsets

Cooper - Tauschek 2024.

- Combinatorial proof via sign-reversing involution

Briand - Esquivias - Gutiérrez - Lillo - Rosas 2024.

Thank you!




Harry Richman


NCTS, Taipei


24 June 2025


ILAS Kaohsiung, NSYSU

Aside: Transitions between $\mathcal{F}_1(G; S)$ and $\mathcal{F}_2(G; S)$
 form interesting dynamical system

$$\mathcal{F}_1(G; S) = \left\{ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\}$$


delete
edge $e \in T$ 

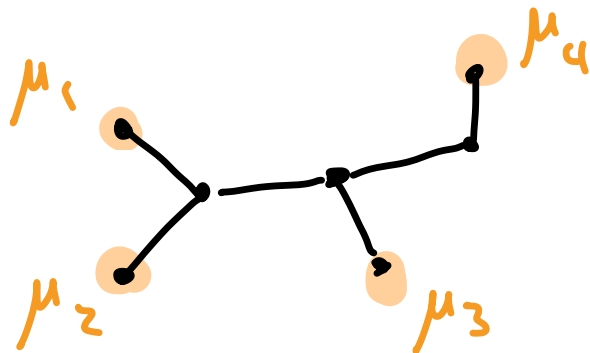
 add edge
 $e \in \partial F(\star)$

$$\mathcal{F}_2(G; S) = \left\{ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\}$$


see: Amini et al., Branden-Huh, Vinzant et al.

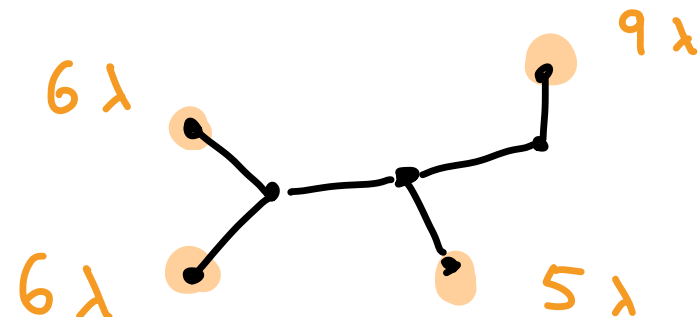
Potential theory

\mathbb{E}_x .



\rightsquigarrow

Equilibrium



equilibrium
ratio
 \downarrow

$$D[s] \vec{\mu}^* = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 63 \\ 63 \\ 63 \\ 63 \end{bmatrix}$$

Recall:

$$\det D[s] = -252$$

$$= -4(63)$$