Ricci flow on graphs from effective resistance

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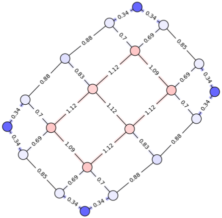
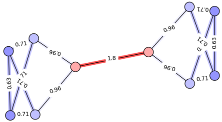
AMS MRC and Fred Hutch Cancer Center

JMM: Ricci curvatures on graphs and applications
4 January 2024





Motivation

Problem: How to understand "geometry" of a graph?

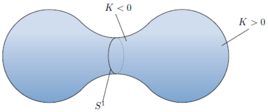
- Real world: max flow / min cut, community detection
- Arithmetic geometry: bounding number of rational points
- Combinatorics: Laplacian eigenvalues, Kemeny's constant, ...

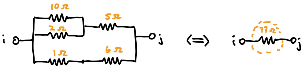
Differential geometry

curvature, Ricci flow

Combinatorics

effective resistance

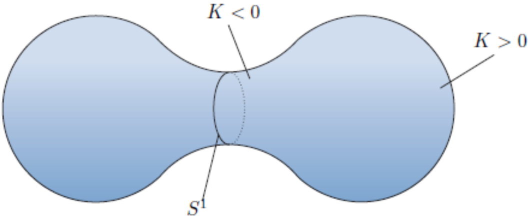




Why Ricci flow?

Related Problem: How to understand "geometry" of a manifold?

- Poincare Conjecture: what conditions suffice for $\mathcal{M}^n \cong S^n$?



(image from Topping 2006)

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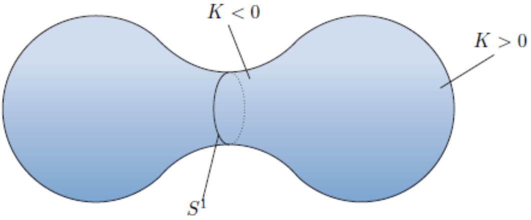

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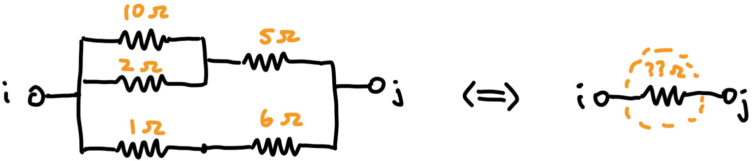
Apply Ricci flow:

- positive curvature $\longrightarrow$ shrink metric
- negative curvature $\longrightarrow$ expand metric

Why effective resistance?



Close connections to:

- simple random walk on G
- uniformly random spanning trees on G

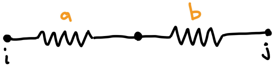
Recent breakthrough applications:

- graph sparsification (Spielman–Srivastava, 2009)
- traveling salesman problem (Anari–Oveis-Gharan, 2015)

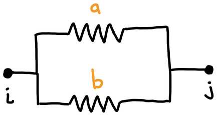
Effective resistance

Setting: graph $G = (V, E)$, each edge e has a positive resistance ℓ_e

How to compute the **effective resistance** ω_{ij} for vertices $i, j \in V$?

- series rule: 

$\omega_{ij} = a + b$

- parallel rule: 

$\omega_{ij} = \dfrac{ab}{a+b}$

- general case (??): combine series and parallel rules

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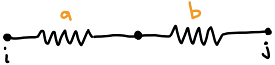
- series rule: 

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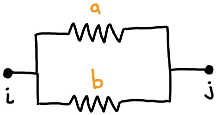
- parallel rule: 

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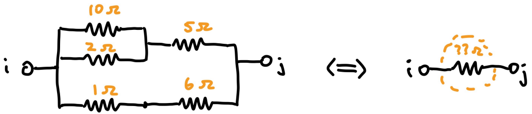

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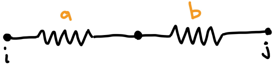

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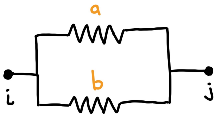
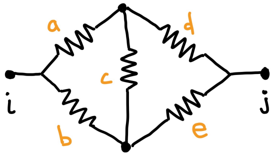


$$\omega_{ij} = \frac{ab}{a+b}$$

- general case (??): Wheatstone bridge



$$\omega_{ij} = ?$$

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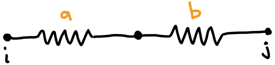
- series rule: 

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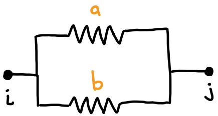
- parallel rule: 

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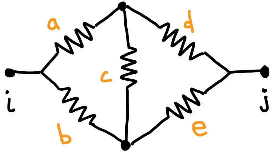


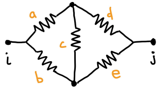
$$\omega_{ij} = \frac{abd+abe+ade+bde+acd+ace+bcd+bce}{ad+ae+bd+be+ac+bc+cd+ce}$$

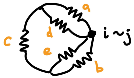
⚠ Series and parallel rules not sufficient to find effective resistance

- General case: use weighted sums of spanning trees

Theorem (Kirchhoff)

$$\omega_{ij} = \frac{\sum_{\mathcal{T}(G/ij)} \prod_{e \notin T} \ell_e}{\sum_{\mathcal{T}(G)} \prod_{e \notin T} \ell_e}$$

Example: $G =$ 

, $G/ij =$ 

$$\omega_{ij} = \frac{abd + abe + ade + bde + acd + ace + bcd + bce}{ad + ae + bd + be + ac + bc + cd + ce}$$

Effective resistance

Theorem (Rayleigh's law)

For any edge e and vertices i, j we have

$$\frac{\partial}{\partial \ell_e} \omega_{ij} \geq 0.$$

- physically "obvious"
- mathematically ...

$$\frac{\partial}{\partial c} \omega_e = \frac{\partial}{\partial c} \left(\frac{abd + abe + ade + bde + acd + ace + bcd + bce}{ad + ae + bd + be + ac + bc + cd + ce} \right) =?$$

Effective resistance

Theorem (Rayleigh's law)

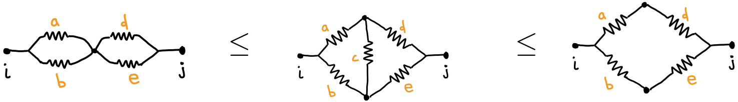
For any edge e and vertices i, j we have

$$\frac{\partial}{\partial \ell_e} \omega_{ij} \geq 0.$$

- delete edge $\quad \leftrightarrow \quad \ell_e = +\infty$
- contract edge $\quad \leftrightarrow \quad \ell_e = 0$

Corollary (usual Rayleigh's law)

$$\omega_{ij}(G/e) \leq \omega_{ij}(G) \leq \omega_{ij}(G \setminus e)$$



Resistance curvature on nodes

(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

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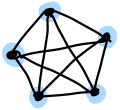
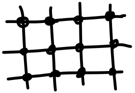
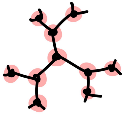
- A finite, vertex-transitive graph has (constant) positive node curvature.
- An infinite regular lattice is flat (zero curvature).
- An infinite tree has negative node curvature everywhere.







Ricci curvature on edges

(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

Can we make edge curvature "more local", in the sense that

$$p_i = \sum_{e \ni i} \mathrm{K}_{\vec{e}} \qquad \text{for edge curvatures } \mathrm{K}_{\vec{e}}?$$

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Yes! Define

$$\text{oriented edge curvature} \qquad \mathrm{K}_{\vec{e}} = \frac{1}{\deg_i} - \frac{1}{2} \frac{\omega_e}{\ell_e}$$

$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

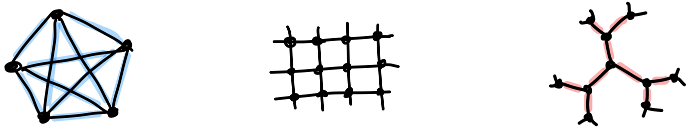
Resistance curvature on edges

Definition

On weighted graph (G, ℓ), the *Foster–Ricci curvature* on edge e is

$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

- Constant-curvature graphs:



- Edge curvature gives more information than node curvature:

Ricci flow from resistance

Definition

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$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

Consider resulting Ricci flow

$$\frac{d}{dt}\ell_e(t) = -\mathrm{K}_e(t)$$

where $\mathrm{K}_e(t) = \mathrm{K}_e(\ell(t))$.

What does Ricci flow look like?

Ricci flow from resistance

Theorem (Ricci flow existence, DGKLQRS)

For any edge-weighted graph (G, ℓ_0), where $\ell_0 = \{\ell_{0,e} > 0 : e \in E(G)\}$, there exists $T > 0$ such that there exists a unique solution to Ricci flow for $t \in [0, T)$.

Proof sketch:

- On any finite box in positive orthant, curvature function $\{\ell_e : e \in E\} \mapsto \{\mathrm{K}_e(\ell) : e \in E\}$ is differentiable
- Differentiable function on compact domain is Lipschitz
- Apply Picard–Lindelöf theorem

Ricci flow on positively curved graphs

Conjecture

Ricci flow preserves positively curved graphs.

Chain rule: $$\frac{d}{dt}\mathrm{K}_e(t) = \sum_{f \in E} \frac{\partial \mathrm{K}_e}{\partial \ell_f} \cdot \frac{d\ell_f}{dt}$$

Lemma

1. For any edge e,
$$\frac{\partial}{\partial \ell_e}\mathrm{K}_e \geq 0;$$
2. For any edges $e \neq j$,
$$\frac{\partial}{\partial \ell_e}\mathrm{K}_f \leq 0.$$

Proof sketch: apply Rayleigh's law.

Discussion

Previous work:

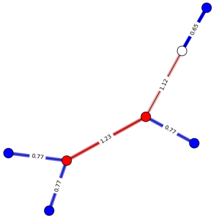
1. Bai–Lin–Lu–Wang–Yau (2021) show existence of Ricci flow for Ollivier–Ricci curvature
2. Devriendt–Lambiotte (2022) study Ricci flow for a different resistance-based edge curvature

Further questions: many notions of Ricci curvature on graphs exist.

1. For which curvatures is it true that

$$\frac{\partial}{\partial \ell_e} \mathrm{K}_e \geq 0, \qquad \frac{\partial}{\partial \ell_e} \mathrm{K}_f \leq 0?$$

2. For which curvatures is it true that Ricci flow preserves positively-curved graphs?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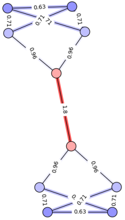
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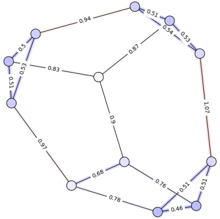
Thank you!


0.94
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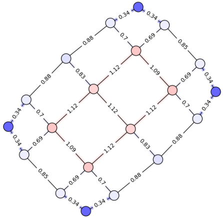
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0.7
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Alternative Ricci flow from resistance

Recall that Devriendt–Lambiotte define

$$\text{node curvature} \qquad p_i = 1 - \frac{1}{2}\sum_{j\sim i} \omega_{ij},$$

$$* \text{ edge curvature} \qquad \kappa_{ij} = \frac{2}{\omega_{ij}}(p_i + p_j)$$

Devriendt–Lambiotte consider *Ricci flow* defined by differential equation

$$\frac{d}{dt}\omega_{ij}(t) = -\kappa_{ij}(t)\,\omega_{ij}(t) \qquad \text{where } \kappa_{ij} = \kappa_{ij}(G(\omega(t)))$$

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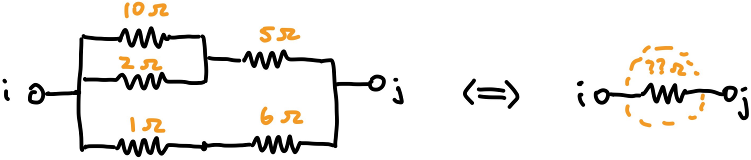
Features:

- in a path, leaf-edges shrink to zero-resistance, “edge contraction”

Downsides:

- in trees with higher-degree vertices, leaf-edges don’t always shrink
- positive values of ω_{ij} may be “invalid”



Answer: $\omega_{ij} = \dfrac{140}{41}$