Random Polygons

Random Polygons: This document explores a particular model for random polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$ that is closely related to Grassmannians.

Random Polygons in $\mathbb{R}^2$: Let A be a random $n \times 2$ real matrix with i.i.d. entries drawn from $N(0,1)$. Let B be the associated $n \times 2$ real matrix obtained from A via QR factorization (or Gram-Schmidt orthogonalization). Note that the columns space of A is equal to the columns space of B. By construction, B consists of a pair of orthogonal columns, each of length 1. If we multiply B by the vector $\begin{bmatrix} 1 \\ I \end{bmatrix}$ then we get an $n \times 1$ matrix with complex entries. Now replace each entry with its square to get a new $n \times 1$ matrix C. Build a new $n \times 2$ real matrix from C whose first column is the real part of C and whose second column is the imaginary part of C. We will consider the rows of C to be an ordered set of vectors in $\mathbb{R}^2$ and will use these vectors to build our random n-gon.

Problem 1): a) Show that the sum of the rows of C is equal to $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

b) Show that the sum of the lengths of the rows of C is equal to 2.

Set $S_0 = (0,0)$ and $S_{i+1} = S_i + R_i$ where R_i denotes the i^{th} row of C. You can build a polygon with n-sides whose k^{th} side starts at S_{k-1} and ends at S_k.

Problem 2: Write a program to produce random n-gons.

Problem 3: If we order the rows of C by the angle the associated vector makes with the x-axis then we can use this ordered set of vectors to make a convex n-gon. Write a program to produce random convex n-gons.

Problem 4: One can experimentally compute $\mathbb{E}(||R_i||)$ and $\mathbb{E}(||R_i||^2)$. It should be unsurprising that $\mathbb{E}(||R_i||) = 2/n$. Can you guess a value for $\mathbb{E}(||R_i||^2)$?

Problem 5: Experimentally find the expected distance from vertex i to the origin in a random n-gon and a random convex n-gon.

Random Polygons in $\mathbb{R}^3$: Let $A1, A2$ be random $n \times 2$ real matrices with i.i.d. entries drawn from $N(0,1)$. Let $B1, B2$ be complex $n \times 1$ matrices obtained from $A1, A2$ by multiplying by the vector $\begin{bmatrix} 1 \\ I \end{bmatrix}$. Let C be the $n \times 2$ complex matrix whose columns are $B1$ and $B2$. Let D be the associated $n \times 2$ complex matrix obtained from C via QR factorization. By construction, D consists of a pair of orthogonal columns, each of length 1. Now form an $n \times 1$ quaternionic matrix, E, by multiplying D with the vector $\begin{bmatrix} 1 \\ J \end{bmatrix}$. If we consider the entrywise map on E given by $q \mapsto \bar{q} I q$ then we end up with an $n \times 1$ quaternionic matrix, F, whose entries have no real component. Now form the $n \times 3$ real matrix, G, whose columns store the I, J and K components of each entry of F. We will consider the rows of G to be an ordered set of vectors in $\mathbb{R}^3$ and will use these vectors to build our random n-gon in $\mathbb{R}^3$.

Problem 6): a) Show that the sum of the rows of G is equal to $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

b) Show that the sum of the lengths of the rows of G is equal to 2.

Set $S_0 = (0,0,0)$ and $S_{i+1} = S_i + R_i$ where R_i denotes the i^{th} row of G. You can build a polygon with n-sides whose k^{th} side starts at S_{k-1} and ends at S_k.

Problem 7: Write a program to produce random n-gons in $\mathbb{R}^3$.

Problem 8: One can experimentally compute $\mathbb{E}(||R_i||)$ and $\mathbb{E}(||R_i||^2)$. It should be unsurprising that $\mathbb{E}(||R_i||) = 2/n$. Can you guess a value for $\mathbb{E}(||R_i||^2)$?

Problem 9: Experimentally find the square expected distance from vertex i to the origin in a random n-gon.

Problem 10: What are some possible extensions of Problem 3 to random polygons in $\mathbb{R}^3$?

https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.21480
Probability Theory of Random Polygons from the Quaternionic Viewpoint

https://www.tandfonline.com/doi/full/10.1080/00029890.2019.1535735
Random Triangles and Polygons in the Plane