

# Singular Value Decomposition

1) Find and compare the SVD of  $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \\ 4 & 7 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 4 & 7 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix}$ .

2) Find the closest rank 1 matrix and rank 2 matrix to  $A$  from problem 1.

3) Find the closest matrix  $E$ , satisfying  $E^T E = I$ , to the matrix  $A$  from problem 1.

Let  $\Theta(U, V) = [\theta_1, \dots, \theta_k]$  be the vector of principal angles between  $U, V$ . Let  $P(U)$  denote the projection matrix for the subspace  $U$ .

- The geodesic distance is  $d_g(U, V) = \|\Theta(U, V)\|_2 = \sqrt{\theta_1^2 + \dots + \theta_k^2}$
- The Fubini-Study distance is  $d_F(U, V) = \cos^{-1}(\cos(\theta_1) \cdots \cos(\theta_k))$
- The chordal distance is  $d_c(U, V) = \|P(U) - P(V)\|_F = \|\sin \Theta(U, V)\|_2$

4) Let  $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ .

a) Find the principal angles between the column spaces of  $A$  and  $B$ ?

b) How far apart are the column spaces of  $A$  and  $B$  on  $\text{Gr}(2,4)$  with respect to the distance measures above?

c) Check that the formula for the chordal distance is correct.

d)  $\wedge^2(A)$  denotes the matrix of all  $2 \times 2$  minors of  $A$ . How far apart are the column spaces of  $\wedge^2(A)$  and  $\wedge^2(B)$  on  $\text{Gr}(1,6)$  with respect to the distance measures above?