Problem Set 1: Problems related to tensor operations and SVD

Let V_1, V_2 be vector spaces. Let $\mathbf{a}$ denote the basis $a_1, a_2, \ldots, a_n$ for V_1. Let $\mathbf{b}$ denote the basis $b_1, b_2, \ldots, b_m$ for V_2.

- A basis for $S^2(V)$ is $s(\mathbf{a}, 2) = \{a_i \otimes a_j \mid i \leq j\}$.
- A basis for $\wedge^2(V)$ is $\wedge(\mathbf{a}, 2) = \{a_i \wedge a_j \mid i < j\}$.
- A basis for $V_1 \otimes V_2$ is $t(\mathbf{a}, \mathbf{b}) = \{a_i \otimes b_j \mid i \leq n, j \leq m\}$.
- In a similar manner, we have bases for $S^k(V), \wedge^k(V), V_1 \otimes \cdots \otimes V_k$.
- A basis can be turned into an ordered basis by choosing an ordering of the basis elements. In the problems below, matrix representations of induced linear transformations may vary depending on the ordering of the basis elements that you choose. However, the basic ideas relating tensors to the SVD will be consistent.
- if $L : V_1 \to V_2$ is a linear transformation, then define $S^2(L)(a_i \otimes a_j) = L(a_i) \otimes L(a_j)$ and $\wedge^2(L)(a_i \wedge a_j) = L(a_i) \wedge L(a_j)$. With this definition, $S^2(L)$ and $\wedge^2(L)$ are linear transformations. Note that $a_i \wedge a_j = -a_j \wedge a_i$ and $a_i \otimes a_j = a_j \otimes a_i$.

1) Let $\mathbf{a}$ denote the ordered basis a_1, a_2, a_3 for $\mathbb{R}^3$ and let $\mathbf{b}$ denote the ordered basis b_1, b_2 for $\mathbb{R}^2$. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation and let $[L]_{\mathbf{b}}^{\mathbf{a}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ be the matrix representing L with respect to $\mathbf{a}$ and $\mathbf{b}$.

a) Write down the matrix $[\wedge^2 L]_{\wedge(\mathbf{b},2)}^{\wedge(\mathbf{a},2)}$ representing $\wedge^2 L$ with respect to $\wedge(\mathbf{a}, 2)$ and $\wedge(\mathbf{b}, 2)$.

b) Write down the matrix $[S^2 L]_{s(\mathbf{b},2)}^{s(\mathbf{a},2)}$ representing $S^2 L$ with respect to $s(\mathbf{a}, 2)$ and $s(\mathbf{b}, 2)$.

c) What are the singular values of $[L]_b^a, [\wedge^2 L]_{\wedge(\mathbf{b},2)}^{\wedge(\mathbf{a},2)}, [S^2 L]_{s(\mathbf{b},2)}^{s(\mathbf{a},2)}$ and how are they related?

d) Can you relate the Singular Value Decompositions of $[L]_b^a, [\wedge^2 L]_{\wedge(\mathbf{b},2)}^{\wedge(\mathbf{a},2)}, [S^2 L]_{s(\mathbf{b},2)}^{s(\mathbf{a},2)}$?

2) Let $[L_1]_b^a = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ represent a linear transformation $L_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and let $[L_2]_d^c = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ represent a linear transformation $L_2 : \mathbb{R}^1 \to \mathbb{R}^2$.

a) Write down the matrix, $[L_1 \otimes L_2]_{t(\mathbf{b},\mathbf{d})}^{t(\mathbf{a},\mathbf{c})}$, representing the linear transformation $L_1 \otimes L_2 : \mathbb{R}^3 \otimes \mathbb{R}^1 \to \mathbb{R}^2 \otimes \mathbb{R}^2$ with respect to the ordered bases $t(\mathbf{a}, \mathbf{c})$ and $t(\mathbf{b}, \mathbf{d})$.

b) What are the singular values of $[L_1]_b^a, [L_2]_d^c, [L_1 \otimes L_2]_{b \otimes d}^{a \otimes c}$ and how are they related?

c) Can you relate the Singular Value Decompositions of $[L_1]_b^a, [L_2]_d^c, [L_1 \otimes L_2]_{b \otimes d}^{a \otimes c}$?

3) Let A, B be matrices. How are the singular value decompositions of A, B, AB related?

4) Pick ordered bases $\mathbf{a}, \mathbf{b}, \mathbf{c}$ for vector spaces U, V, W. Let $L_1 : U \to V$ and $L_2 : V \to W$ be linear transformations.

a) Is it true that $[S^2(L_2 \circ L_1)]^{s(\mathbf{a},2)}_{s(\mathbf{c},2)} = [S^2(L_2)]^{s(\mathbf{a},2)}_{s(\mathbf{b},2)} \; [S^2(L_1)]^{s(\mathbf{b},2)}_{s(\mathbf{c},2)}$?

b) Is it true that $[\wedge^2(L_2 \circ L_1)]^{\wedge(\mathbf{a},2)}_{\wedge(\mathbf{c},2)} = [\wedge^2(L_2)]^{\wedge(\mathbf{a},2)}_{\wedge(\mathbf{b},2)} \; [\wedge^2(L_1)]^{\wedge(\mathbf{b},2)}_{\wedge(\mathbf{c},2)}$?

5) Suppose $[L]^{\mathbf{a}}_{\mathbf{a}}$ and $[L']^{\mathbf{b}}_{\mathbf{b}}$ are unitary.

a) Is $[\wedge^2 L]^{\wedge(\mathbf{a},2)}_{\wedge(\mathbf{a},2)}$ unitary?

b) Is $[S^2 L]^{s(\mathbf{a},2)}_{s(\mathbf{a},2)}$ unitary?

c) Is $[L \otimes L']^{t(\mathbf{a},\mathbf{b})}_{t(\mathbf{a},\mathbf{b})}$ unitary?

Let $\Theta(U, V) = [\theta_1, \ldots, \theta_k]$ be the vector of principal angles between U, V (with $\theta_1 \leq \cdots \leq \theta_k$). Let $P(U)$ denote the projection matrix for the subspace U.

- The geodesic distance is $d_g(U, V) = ||\Theta(U, V)||_2 = \sqrt{\theta_1^2 + \ldots \theta_k^2}$
- The Fubini-Study distance is $d_F(U, V) = cos^{-1}(cos(\theta_1) \cdots cos(\theta_k))$
- The chordal distance is $d_c(U, V) = ||P(U) - P(V)||_F = ||sin \; \Theta(U, V)||_2$
- The Asimov distance is $d_A(U, V) = \theta_k$

6) Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 1 \\ 7 & 1 \end{bmatrix}$.

a) Find the principal angles between $[A]$ and $[B]$?

b) How far apart are $[A]$ and $[B]$ on Gr(2,4) with respect to the distance measures above?

c) How far apart are $[\wedge^2(A)]$ and $[\wedge^2(B)]$ on Gr(1,6) with respect to the distance measures above?

d) How far apart are $[S^2(A)]$ and $[S^2(B)]$ on Gr(3,10) with respect to the distance measures above?