Geodesics on Grassmannians

Let $A, B \in \mathbb{R}^{n \times k}$ be matrices satisfying $A^T A = B^T B = I_k$.

1) Find orthogonal matrices $U, V \in O(k)$ such that $R = AU$ and $S = BV$ have the property that $R^T S$ is a diagonal matrix whose diagonal entries are monotone decreasing and non-negative.

2) Goal: Show that if A is skew-symmetric then $exp(A)$ is an orthogonal matrix.

a) Let A, B be commuting matrices (i.e. $AB = BA$). Show that $exp(A + B) = exp(A)exp(B)$.

b) Find non-commuting matrices A, B with $exp(A + B) \neq exp(A)exp(B)$.

c) Show that if A is skew-symmetric then A commutes with A^T.

d) Let 0 denote a square matrix of zeros. Show that $exp(0)$ is an identity matrix.

e) Show that $exp(A^T) = exp(A)^T$.

f) Recall that a square matrix A is orthogonal if and only if $A^T A = AA^T = I$. Show that if A is skew-symmetric then $exp(A)$ is an orthogonal matrix.

3) Suppose we know an expression $A = exp(S)$ where S is a skew symmetric matrix. From the above problems, we know that A is an orthogonal matrix.

a) How can we use this to parameterize a curve on $O(n)$ from I to A?

b) Show that the determinant of an orthogonal matrix is either -1 or 1.

c) Show that if S is skew symmetric then $exp(S)$ is in $SO(n)$ (i.e. has determinant 1)

d) If B is an orthogonal matrix, how can we parameterize a path from B to BA?

4) In Maple, you can exponentiate the square matrix, E, with the command $MatrixExponential(E)$ (from the package "LinearAlgebra"). Recall that A is unitary if $A^* A = I$ where A^* is the conjugate transpose of A.

a) Let $E = \begin{bmatrix} 0 & .23 \\ -.23 & 0 \end{bmatrix}$. Find a diagonal matrix D and a unitary matrix U such that $E = UDU^*$

b) Compute $exp(E)$ and describe how the entries of E and of $exp(E)$ are related.

5) Below is Maple code for finding a geodesic between the subspaces spanned by the columns of A and the columns of B.

a) Look at the Maple code below and understand the command syntax and meaning.

b) Use the code to find the subspace halfway between the column spaces of A and B.

c) Read `https://nhigham.com/2020/10/27/what-is-the-cs-decomposition/`

```
restart;
with(LinearAlgebra);
with(ArrayTools);
Digits := 30;

R := evalf(RandomMatrix(5, 5));
S := evalf(RandomMatrix(5, 5));
QR := QRDecomposition(R)[1];
QS := QRDecomposition(S)[1];

A := SubMatrix(QR, [1 .. 5], [1, 2]);
AT := SubMatrix(QR, [1 .. 5], [3, 4, 5]);
B := SubMatrix(QS, [1 .. 5], [1, 2]);
BT := SubMatrix(QS, [1 .. 5], [3, 4, 5]);

SVD1 := SingularValues((Transpose(A)) . B, output = ['U', 'S', 'Vt']);
AA := A . (SVD1[1]);
BB := B . (Transpose(SVD1[3]));

(Transpose(AA)) . BB;

SVD2 := SingularValues((Transpose(AT)) . BT, output = ['U', 'S', 'Vt']);
AAT := AT . (SVD2[1]);
BBT := BT . (Transpose(SVD2[3]));

NA := <AA | AAT>;
NB := <BB | BBT>;

C := (Transpose(NA)) . NB;

Digits := 20;
CC := evalf(10^(-25)*round~(10^25*C));

SP1 := Diagonal(CC);
SP2 := CC - DiagonalMatrix(SP1);

X := DiagonalMatrix(cos~(arccos~(SP1)*t)) + sin~(arcsin~(SP2)*t);

eval(NA . X, {t = 0});
```