Random Geometry - First Steps

March 10, 2025

Broad Outline for today

- Comments on Random Polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Comments on Projections of n-cubes and n-simplices

- Random geometry on the sphere
- Real roots of random polynomials
- Random real algebraic curves

A comment on random n-gons

Let C_k denote a k-chord of a random polygon in $\mathbb{R}^3$.

Surprisingly, $\mathbb{E}[\ ||C_k||^2\ ]$ is a rational number.

See: "Probability Theory of Random Polygons from the Quaternionic Viewpoint" by Cantarella, Deguchi, Shonkwiler

A very readable paper that you might enjoy is:

"Random Triangles and Polygons in the Plane"
by Cantarella, Needham, Shonkwiler, Stewart

"Polygon Spaces and Grassmannians" by Hausmann and Knutson

Projection of a 3-dimensional convex shape to $\mathbb{R}^2$

Let C be a bounded convex shape in $\mathbb{R}^3$.

Let ∂C denote the boundary of C (i.e. its surface).

Let $P(C)$ denote a random projection of C to $\mathbb{R}^2$.

$$\mathbb{E}[Vol(P(C))] = \frac{Vol(B^2)}{Vol(S^2)} Vol(\partial C) = \frac{\pi}{4\pi} Vol(\partial C) = \frac{1}{4} Vol(\partial C)$$

"A tale of two problem solvers — Average cube shadow area"

3Blue1Brown YouTube video

Projection of cube, simplex, cross-polytope

Cube surface area is 24 so the expected shadow area is 6.

Simplex surface area is $2\sqrt{3}$ so the expected shadow area is $\frac{\sqrt{3}}{2}$.

Octahedron (cross-polytope in $\mathbb{R}^3$) surface area is $4\sqrt{3}$ so the expected shadow area is $\sqrt{3}$.

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The argument can be extended to find expected volumes of shadows of bounded convex shapes in $\mathbb{R}^{n+1}$ projected to $\mathbb{R}^n$.

Projection of a $n+1$-dimensional convex shape to $\mathbb{R}^n$

Let C be a bounded convex shape in $\mathbb{R}^{n+1}$ with boundary ∂C.

Let $P(C)$ denote a random projection of C to $\mathbb{R}^n$.

$$\mathbb{E}[Vol(P(C))] = \frac{Vol(B^n)}{Vol(S^n)} Vol(\partial C)$$

when $n = 3$, we have

$$\mathbb{E}[Vol(P(C))] = \frac{Vol(B^3)}{Vol(S^3)} Vol(\partial C) = \frac{2}{3\pi} Vol(\partial C)$$

The 3-volume of the boundary of a 4-cube is 64 so the expected volume of its shadow in $\mathbb{R}^3$ is $\frac{128}{3\pi}$ which is about 13.58.

A comment about hypercubes

How is an $n+1$ cube built from an n cube?

Let $C_n(k)$ denote the number of k faces of an n cube.

We have $C_{n+1}(k) = 2C_n(k) + C_n(k-1)$

This leads to the observation that the coefficients of $(x+2)^n$ give the number of k-faces of an n-cube.

Example: $(x+2)^3 = x^3 + 6x^2 + 12x + 8$.

An Experimental Observation

Shadows of random projections of an n-cube to $\mathbb{R}^2$ seem to have $2n$ vertices.

The expected volume seems to be $2n$ as well.

Can you find a justification for these observations?

Switching to a different topic

Random intersections, integral geometry, and random algebraic geometry.

Intersection of curves on a sphere

Let A and B be curves lying on the unit sphere.

Let $U \in O(3)$.

There is a natural action of U on each curve.

Let $|A \cap B|$ denote the number of intersection points of A and B.

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Problem: Find $\mathbb{E}[\ |U.A \cap B|\ ]$ for a random U.

A theorem from integral geometry

If A and B are great circles on S^2 then $\mathbb{E}[\ |U.A \cap B|\ ] = 2$.

What if A is only half a great circle?

What if A is a random curve of length π?

What if A and B are random curves of length $\ell(A)$ and $\ell(B)$?

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Theorem: $\mathbb{E}[\ |U.A \cap B|\ ] = \frac{1}{2\pi^2}\ \ell(A)\ \ell(B)$

Expected number of real roots for a random polynomial

Let $P \in \mathbb{C}[x]$ be a random degree 2 polynomial.

With probability one, P has 2 distinct roots over $\mathbb{C}$ and 0 real roots.

Let $P \in \mathbb{R}[x]$ be a random degree 2 polynomial.

With probability one, P has 2 distinct roots over $\mathbb{C}$.

With positive probability, P has 0 real roots and with positive probability P has 2 real roots.

Question: How many real roots does P have on average?

Polynomials and orthogonality

Consider a degree 2 polynomial $P(x) = a + bx + cx^2$.

A root of this polynomial is a number t such that $a + bt + ct^2 = 0$.

This can be rewritten as $[a \quad b \quad c] \cdot [1 \quad t \quad t^2] = 0$.

Being a root is a statement about orthogonality!

Sampling from the uniform distribution

If $a, b, c \sim U[-1, 1]$ then the expected number of real roots can be found using calculus.

The equation $b^2 - 4ac = 0$ chops the cube into two pieces. We get

$$\mathbb{E}[\# real\ roots] = \frac{41}{36} + \frac{ln(2)}{6} \approx 1.2544$$

For degree 10 polynomials experimentally about 2.1 real roots.

For degree 100 polynomials experimentally about 3.5 real roots.

Orthgonality and lengths of curves on spheres

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The expected number of real roots for $a, b, c \sim N(0,1)$ is $\frac{1}{\pi} \; \ell(\gamma)$.

Finding the length of γ

We have $\ell(\gamma) = \int_{-\infty}^{\infty} ||\gamma'(t)|| \, \mathrm{d}t$.

The expected number of real roots is $\frac{1}{\pi} \, \ell(\gamma)$.

Let $v(t) = [f_1(t) \;\; f_2(t) \;\; \dots \;\; f_f(t)]$ and $\gamma(t) = \frac{1}{||v(t)||} \, v(t)$.

It can be shown that

$$||\gamma'(t)||^2 = \frac{\partial^2}{\partial x \partial y} \log[v(x) \cdot v(y)] \Big|_{x=y=t}$$

We have $v(t) = [1 \quad t \quad t^2]$ and $\gamma(t) = \frac{1}{||v(t)||} v(t)$.

From the formula

$$||\gamma'(t)||^2 = \frac{\partial^2}{\partial x \partial y} \log[v(x) \cdot v(y)]\Big|_{x=y=t}$$

we compute that

$$||\gamma'(t)||^2 = \frac{t^4 + 4t^2 + 1}{(t^4 + t^2 + 1)^2}$$

so

$$\frac{1}{\pi} \int_{-\infty}^{\infty} ||\gamma'(t)|| \, \mathrm{d}t = \frac{1}{\pi} \int_{-\infty}^{\infty} \sqrt{\frac{t^4 + 4t^2 + 1}{(t^4 + t^2 + 1)^2}} \, \mathrm{d}t = 1.297\ldots$$

Expected number of real roots of a degree d polynomial

Let $P(x) = a_0 + a_1 x + \cdots + a_d x^d$ with $a_i \sim N(0,1)$.

If E_d denotes the expected number of real roots of $P(x)$, then

$$E_d \sim \frac{2}{\pi}\log(d) + .62575... + \frac{2}{\pi d} + O(\frac{1}{d^2}).$$

This is nice, in a way, but sort of messy.

Large degree polynomials have fewer real roots than might be expected. For instance $E_{1000} \approx 5.024$.

A different way to sample polynomials

- t is a root of $a + bx + cx^2 \iff [a \;\; b \;\; c] \cdot [1 \;\; t \;\; t^2] = 0.$
- $v(t) = [1 \;\; t \;\; t^2]$ and $\gamma(t) = \frac{1}{||v(t)||} v(t).$
- $\mathbb{E}[\#realroots] = \frac{1}{\pi}\, \ell(\gamma) = \frac{1}{\pi} \int ||\gamma'(t)|| \mathrm{d}t.$
- $||\gamma'(t)||^2 = \frac{\partial^2}{\partial x \partial y} \log[v(x) \cdot v(y)] \Big|_{x=y=t}$

If $v(t) = [1 \;\; \sqrt{2}t \;\; t^2]$ then $v(x) \cdot v(y) = (1 + xy)^2.$

The integral becomes easy and we get $\frac{1}{\pi}\, \ell(\gamma) = \sqrt{2}.$

Note that $[a \;\; b \;\; c] \cdot [1 \;\; \sqrt{2}t \;\; t^2] = [a \;\; \sqrt{2}b \;\; c] \cdot [1 \;\; t \;\; t^2]$.

If $X \;\sim\; N(0,1)$ then $\sqrt{2}\,X \;\sim\; N(0,2)$.

More generally, a degree d polynomial with $a_i \sim N(0, \binom{d}{i})$

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d$$

is the same as the polynomial with $a_i \sim N(0,1)$

$$P(x) = a_0\sqrt{\binom{d}{0}} + a_1\sqrt{\binom{d}{1}}\;x + a_2\sqrt{\binom{d}{2}}\;x^2 + \cdots + a_d\sqrt{\binom{d}{d}}\;x^d.$$

Random degree d polynomials from the Kostlan distribution

Let $a = [a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_d]$ with $a_i \sim N(0,1)$ and let

$$v(t) = [\sqrt{\binom{d}{0}} \quad \sqrt{\binom{d}{1}}\ t \quad \sqrt{\binom{d}{2}}\ t^2 \quad \ldots \quad \sqrt{\binom{d}{d}}\ t^d]$$

Then $P(x) = a \cdot v(x)$ is a random polynomial from the Kostlan distribution.

Similar to before, we have $v(x) \cdot v(y) = (1 + xy)^d$.

The integral for the length of γ becomes easy and we get:

The expected number of real roots of $P(x)$ is $\frac{1}{\pi}\ \ell(\gamma) = \sqrt{d}$.

The Kostlan distribution is orthogonally invariant

If $F(x)$ is a polynomial then we can homogenize to get $G(x, y)$.

If $G(x, y)$ is a homogenized "Kostlan" polynomial then an orthogonal change of variables gives a polynomial from the same distribution.

If $F(r) = 0$ then $G(cr, c) = 0$ for any c.

The multiples of $(r, 1)$ is a line through the origin in $\mathbb{R}^2$. This line will intersect $x^2 + y^2 = 1$ in two points: $\pm(\frac{r}{\sqrt{r^2+1}}, \frac{1}{\sqrt{r^2+1}})$

Orthogonal invariance of the Kostlan distribution means that the roots of Kostlan polynomials sample the circle uniformly.

The Kostlan distribution as a contraction of an array

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [ax^2 + bxy + cyx + dy^2]$$

which we can write as $[ax^2 + (b+c)xy + dy^2]$.

If $a, b, c, d \sim N(0,1)$ then $ax^2 + (b+c)xy + dy^2$ is a sample from the Kostlan distribution.

In general, a $2 \times 2 \times \cdots \times 2$ d-way array can be d-contracted with $[x, y]$ to produce a degree d polynomial, $P(x, y)$.

If the array has $N(0,1)$ entries then $P(x, y)$ will be Kostlan distributed.

Real Kostlan polynomials in 3 variables

A $3 \times 3 \times \cdots \times 3$ d-way array can be contracted with $[x, y, z]$ to produce a degree d polynomial, $P(x, y, z)$.

If $N(0, 1)$ entries then $P(x, y, z)$ will be Kostlan distributed.

$P(x, y, z) = 0$ intersects the sphere S^2 in a curve.

This curve is closed under the antipodal map.

The expected number of intersection points of 2 homogeneous Kostlan polynomials, F, G, of degree d, e is:

$2\sqrt{de}$ on the sphere

$\sqrt{de}$ if we mod out by the antipodal map.

The topology of Kostlan curves and surfaces

Question: What do the zeros of a homogeneous polynomial in 3 variables look like when it intersects the sphere S^2?

How about after we mod out by antipodal points?

Question: What do the zeros of a homogeneous polynomial in 4 variables look like when it intersects the sphere S^3?

How about after we mod out by antipodal points?

The topology of smooth cubic surfaces in $\mathbb{RP}^3$

It turns out there are 5 topological types that arise.

Let W_r denote the connected sum of r copies of $\mathbb{RP}^2$.

The space of real points of a smooth cubic surface is diffeomorphic to W_7, W_5, W_3, W_1 or the disjoint union of W_1 and S^2.

Each of these cases occur with positive probability.

The number of real lines on each of these surfaces is $27, 15, 7, 3, 3$.

The expected number of real lines is $6\sqrt{2} - 3$.

Time to compute!