Geodesics and Special Points on Grassmannians

March 17, 2025

Broad Outline for today

- Review of material from last week

- Two ways to build geodesics on the Grassmannian
- Families of Random Polygons in $\mathbb{R}^2$

- Special Points on the Grassmannian
- Special Projections of *n*-simplices
- Daubechies Points on Gr(n,2n)

Projection of a $n+1$-dimensional convex shape to $\mathbb{R}^n$

Let C be a bounded convex shape in $\mathbb{R}^{n+1}$ with boundary ∂C.
Let $P(C)$ denote a random projection of C to $\mathbb{R}^n$.

Theorem : $\mathbb{E}[Vol(P(C))] = \dfrac{Vol(B^n)}{Vol(S^n)} Vol(\partial C)$

$$n = 2, 3 \quad \mathbb{E}[Vol(P(C))] = \frac{1}{4} Vol(\partial C) \quad \text{and} \quad \frac{2}{3\pi} Vol(\partial C)$$

$$n = 1, 2, 3, 4, 5, 6 \qquad \frac{Vol(B^n)}{Vol(S^n)} = \frac{1}{\pi}, \ \frac{1}{4}, \ \frac{2}{3\pi}, \ \frac{3}{16}, \ \frac{8}{15\pi}, \ \frac{5}{32}$$

Number of k-faces of an n-cube

Coefficients of $(x+2)^n$ give the number of k-faces of an n-cube.

$$(x+2)^2 = x^2 + 4x + 4$$

$$(x+2)^3 = x^3 + 6x^2 + 12x + 8$$

$$(x+2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$$

$$(x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

Expected volumes of n-shadows of $(n+1)$-cube for $n = 2, 3, 4, 5, 6$:

$$4 \cdot 2 \cdot \frac{1}{\pi}, \quad 6 \cdot 2^2 \cdot \frac{1}{4}, \quad 8 \cdot 2^3 \cdot \frac{2}{3\pi}, \quad 10 \cdot 2^4 \cdot \frac{3}{16}, \quad 12 \cdot 2^5 \cdot \frac{8}{15\pi}$$

A theorem from integral geometry

If A and B are great circles on S^2 then $\mathbb{E}[\ |U.A \cap B|\ ] = 2$.

If A and B are random curves of length $\ell(A)$ and $\ell(B)$ on S^2 then

Theorem: $\mathbb{E}[\ |U.A \cap B|\ ] = \frac{1}{2\pi^2}\ \ell(A)\ \ell(B) = 2\frac{Vol(A)}{Vol(S^1)}\frac{Vol(B)}{Vol(S^1)}$

Let A, B be submanifolds of S^n of dimensions a, b with $a + b \geq n$.
The expected dimension of $U.A \cap B$ is $d = a + b - n$ and

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Theorem: $$\frac{\mathbb{E}[\ Vol(U.A \cap B)\ ]}{Vol(S^d)} = \frac{Vol(A)}{Vol(S^a)}\ \frac{Vol(B)}{Vol(S^b)}$$

Expected number of real roots for a random polynomial

Suppose P is a random function that can be written as $\mathbf{a} \cdot \mathbf{v}(x)$ with

components $a_i \sim N(0,1)$ and $v_i(x)$ piecewise smooth.

Normalizing $\mathbf{v}(t)$ gives a curve $\gamma(t)$ on S^n.

Let $\ell(\gamma) = \int_a^b \, ||\gamma'(t)|| \, \mathrm{d}t$ denote the length of γ with $t \in [a, b]$

Theorem:
The expected number of real roots of P lying in $[a, b]$ is $\frac{1}{\pi} \, \ell(\gamma)$.

Finding the length of γ

If $v(t) = [f_1(t) \quad f_2(t) \quad \ldots \quad f_f(t)]$ and $\gamma(t) = \frac{1}{||v(t)||}\, v(t)$

$$\text{then} \quad ||\gamma'(t)||^2 = \frac{\partial^2}{\partial x \partial y} \log[v(x) \cdot v(y)]\Big|_{x=y=t}$$

Kostlan Polynomials

A Kostlan polynomial of degree d is one of the form:

$$P(x) = a_0\sqrt{\binom{d}{0}} + a_1\sqrt{\binom{d}{1}}\, x + a_2\sqrt{\binom{d}{2}}\, x^2 + \cdots + a_d\sqrt{\binom{d}{d}}\, x^d.$$

The expected number of real roots of a Kostlan Polynomial is $\sqrt{d}$.

If r is a root, consider the points on a circle $\pm(\frac{r}{\sqrt{r^2+1}}, \frac{1}{\sqrt{r^2+1}})$.

Orthogonal invariance of the Kostlan distribution $\implies$ the roots sample the circle uniformly.

Homogeneous Multivariate Real Kostlan polynomials

Kostlan polynomials are also defined in the multivariable case and are typically considered in homogeneous form.

They arise by contracting an array with $N(0,1)$ entries, with a vector of indeterminates, in every possible direction.

Theorem: The expected number of real intersection points, in $\mathbb{RP}^n$, of n Kostlan hypersurfaces of degrees $d_1, \ldots d_n$ is $\sqrt{\prod_i d_i}$

The topology of Kostlan curves and surfaces

Kostlan curves are, topologically, a smooth 1-manifold without boundary: a disjoint union of circles.

Kostlan surfaces are, topologically, a smooth 2-manifold without boundary: a disjoint union of surfaces.

An odd degree Kostlan surface in $\mathbb{RP}^3$ has exactly 1 non-orientable component.

An even degree Kostlan surface in $\mathbb{RP}^3$ has 0 non-orientable components.

The topology of smooth cubic surfaces in $\mathbb{RP}^3$

There are 5 topological types that arise.

Let W_r denote the connected sum of r copies of $\mathbb{RP}^2$.

The space of real points of a smooth cubic surface is diffeomorphic to $W_7,\ W_5,\ W_3,\ W_1$, or $W_1 \coprod S^2$.

Each occur with positive probability.

The number of real lines on these surfaces is $27, 15, 7, 3, 3$.

The expected number of real lines on a Kostlan cubic is $6\sqrt{2} - 3$.

Geodesics on Grassmannians

We will go over two ways to produce geodesics on a Grassmannian:

1) Exponentiation of a skew-symmetric matrix.

2) CS decomposition.

Skew Symmetric Matrices - Geodesics on $O(n)$

Let A be an $n \times n$ skew symmetric matrix (so $A^T = -A$).

Theorem: $A = UDU^T$ with $U \in O(n)$, D a purely imaginary diagonal matrix.

From this we have $Exp(At) = U\ Exp(Dt)\ U^T$.

Recall: $Exp(i\theta) = \cos(\theta) + i\sin(\theta)$ and $Exp(A)$ is orthogonal.

$Exp(At)$ parameterizes a geodesic on $O(n)$.

Block Skew Symmetric Matrices - Geodesics on $Gr(k,n)$

Let 0_k denote a square $k \times k$ matrix of zeros.

$$\text{If} \quad A = \begin{bmatrix} 0_k & -B^T \\ B & 0_{n-k} \end{bmatrix} \quad \text{then} \quad Exp(At) \;\; \text{is a geodesic on } Gr(k,n).$$

The length of this geodesic is $\sqrt{\frac{1}{2} Trace(A^T A)}$ (for $t \in [0,1]$).

Note that $\sqrt{\frac{1}{2} Trace(A^T A)} = \sqrt{Trace(B^T B)}$.

CS Decomposition

Let R, T be $n \times n$ orthogonal matrices.

Let A, B be the first k columns of R, T.

Find $A^T B = U \Sigma V^T$. Define $A_1 = AU, B_1 = BV$.

Let C, D be the last $n - k$ columns of R, T.

Find $C^T D = U \Sigma V^T$. Define $A_2 = CU, B_2 = DV$.

Let $\mathbf{A} =< A_1 | A_2 >$ and $\mathbf{B} =< B_1 | B_2 >$

We will call $\mathbf{A^T B}$ the Grassmann CS decomposition.

Parameterizing the CS decomposition

$$\mathbf{A}^{\mathbf{T}}\mathbf{B} = \begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix}. \text{ Let } Q_t = \begin{bmatrix} C_t & 0 & -S_t \\ 0 & I & 0 \\ S_t & 0 & C_t \end{bmatrix}$$

$$\text{where} \quad C_t = Diag(Cos(\Theta t)) \quad S_t = Diag(Sin(\Theta t))$$

then $\mathbf{A}Q_t$ parametrizes a family of orthogonal matrices that start at $\mathbf{A}$ when $t = 0$ and end at $\mathbf{B}$ when $t = 1$.

Q_t is a geodesic path on $SO(n)$. The first k columns of $\mathbf{A}Q_t$ gives a geodesic path between points on $Gr(k, n)$.

Connection to Skew Symmetric Matrices

$$\begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix} = Exp(\begin{bmatrix} 0 & 0 & \arcsin(-S) \\ 0 & 0 & 0 \\ \arcsin(S) & 0 & 0 \end{bmatrix})$$

Time to compute!

Geodesics parametrizing a sequence of polygons in $\mathbb{R}^2$

Geodesics parametrizing a sequence of projections of n-cubes and n-simplices to $\mathbb{R}^k$

https://chrispetersonmath.github.io/Taiwan2025.html

Maple Code: "CS Decomposition" and "Family of polygons in R2"

Special points on the Grassmannian

Below are three special points on $Gr(2, n)$ that produce extremal effects for n-gons in $\mathbb{R}^2$ and for projections of n-simplices to $\mathbb{R}^2$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & 0 & \dots & 0 \end{bmatrix}^T$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & -1 & -1 & 0 & \dots & 0 \end{bmatrix}^T$$

$$\begin{bmatrix} \sin(\frac{2\pi 0}{n}) & \sin(\frac{2\pi 1}{n}) & \sin(\frac{2\pi 2}{n}) & \sin(\frac{2\pi 3}{n}) & \dots & \sin(\frac{2\pi(n-1)}{n}) \\ \cos(\frac{2\pi 0}{n}) & \cos(\frac{2\pi 1}{n}) & \cos(\frac{2\pi 2}{n}) & \cos(\frac{2\pi 3}{n}) & \dots & \cos(\frac{2\pi(n-1)}{n}) \end{bmatrix}^T$$

Effect of Projection on Simplices and constructing n-gons

What will be the shadows of an n-simplex in each case?

What kind of n-gons do you think these points make?

What if you make a geodesic between the points?

Projection Matrices

Suppose A satisfies $A^T A = I$. There is a corresponding projection matrix $P = AA^T$.

$P^2 = P$ and the eigenvalues are all 0's and 1's. The number of 1's is the rank.

$I - P$ is the projection matrix for the orthogonal complement.

If $B = AU$ where $U \in O(k)$ then $B^T B = I$ and $BB^T = P$.

If $v \in \mathbb{R}^n$ then $Pv \in \mathbb{R}^n$ and Pv is the point in $\mathbb{R}^n$ closest to v that lies on the column space of A.

Daubechies Points

Daubechies wavelets have a continuous form and a discrete form.

The discrete Daubechies wavelets are called $D_2, D_4, D_6, D_8, \ldots$.

These can be used to build a sequence of points on $Gr(n, 2n)$.

Let $P_2, P_4, P_6, \ldots$ be the corresponding projection matrices.

If v is a vector of sampled points from a relatively smooth curve then $P_a v$ and v are fairly close.

If A is a matrix of sampled points from a relatively smooth surface then $P_a A P_b$ and A are fairly close.

Constructing Daubechies points

The discrete Daubechies wavelets correspond to special n-dimensional subspaces of $\mathbb{R}^{2n}$.

They are generated by a basic vector v by padding it with zeros to make a vector of length $2n$ then cyclically permitting this vector with a skip of two and considering their span.

The basic vector for D_2 is $\begin{bmatrix} .7071 \\ .7071 \end{bmatrix}$ and for D_4 is $\begin{bmatrix} .4829 \\ .8365 \\ .2241 \\ -.1294 \end{bmatrix}$.

Daubechies Examples

In $Gr(4,8)$, D_2 is $\begin{bmatrix} .7071 & 0 & 0 & 0 \\ .7071 & 0 & 0 & 0 \\ 0 & .7071 & 0 & 0 \\ 0 & .7071 & 0 & 0 \\ 0 & 0 & .7071 & 0 \\ 0 & 0 & .7071 & 0 \\ 0 & 0 & 0 & .7071 \\ 0 & .0 & 0 & .7071 \end{bmatrix}$

Daubechies Examples Continued

In $Gr(4,8)$, D_4 is
$$\begin{bmatrix} .4829 & 0 & 0 & .2241 \\ .8365 & 0 & 0 & -.1294 \\ .2241 & .4829 & 0 & 0 \\ -.1294 & .8365 & 0 & 0 \\ 0 & .2241 & .4829 & 0 \\ 0 & -.1294 & .8365 & 0 \\ 0 & 0 & .2241 & .4829 \\ 0 & .0 & -.1294 & .8365 \end{bmatrix}$$

The columns are orthonormal for Discrete Daubechies Wavelets.

Time to compute again!

Special Projections of an n-simplex

Special Polygons

Daubechies points and effect on curves and surfaces

https://chrispetersonmath.github.io/Taiwan2025.html

Maple Code: "Daubechies Points on Grassmannians and effect on curves and surfaces"