Random Polygons

Random Polygons in $\mathbb{R}^2$

Let a, b be a pair of orthonormal vectors in $\mathbb{R}^n$.

Consider $c = a + bi \in \mathbb{C}^n$.

Apply the map $z \mapsto z^2$ to each entry of c to get a vector $d \in \mathbb{C}^n$.

Write $d = e + fi$ with $e, f \in \mathbb{R}^n$ then form the matrix U whose columns are e and f.

The rows of U are an ordered collection of vectors in $\mathbb{R}^2$ whose total length is 2 and whose sum is the zero vector.

This ordered set of vectors can be used to build a polygon in $\mathbb{R}^2$.

Problem 1: Show that if (a, b) is a row of your starting matrix then $(a^2 - b^2, 2ab)$ is the corresponding row of your ending matrix.

Problem 2: a) Show that the sum of the rows of C is equal to $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

b) Show that the sum of the lengths of the rows of C is equal to 2.

Set $S_0 = (0, 0)$ and $S_{i+1} = S_i + R_i$ where R_i denotes the i^{th} row of C. You can build a polygon with n-sides whose k^{th} side starts at S_{k-1} and ends at S_k.

Problem 3: Write a program to produce random n-gons.

Problem 4: If we order the rows of C by the angle the associated vector makes with the x-axis then we can use this ordered set of vectors to make a convex n-gon. Write a program to produce random convex n-gons.

Problem 5: One can experimentally compute $\mathbb{E}(||R_i||)$ and $\mathbb{E}(||R_i||^2)$. It should be unsurprising that $\mathbb{E}(||R_i||) = 2/n$. Can you guess a value for $\mathbb{E}(||R_i||^2)$?

Problem 6: Experimentally find the expected distance from vertex i to the origin in a random n-gon and a random convex n-gon.

Quaternions:
The algebra of quaternions is denoted $\mathbb{H}$.
Elements are of the form $a + bi + cj + dk$.
If $q = a + bi + cj + dk$ then $\bar{q} = a - bi - cj - dk$.
Multiplication of quaternions use the rules:
$ij = k, jk = i, ki = j$ and $ji = -k, kj = -i, ik = ij$ and $i^2 = j^2 = k^2 = -1$.

See the wikipedia page on "Quaternions" for more details.

Random Polygons in $\mathbb{R}^3$

Let a, b be a pair of orthonormal vectors in $\mathbb{C}^n$.

Consider $c = a + bj \in \mathbb{H}^n$.

Apply the map $q \mapsto \bar{q}iq$ to each entry of c to get a vector $d \in \mathbb{H}^n$.

Write $d = ei + fj + gk$ with $e, f, g \in \mathbb{R}^n$ then form the matrix U whose columns are e, f and g.

The rows of U are a collection of vectors in $\mathbb{R}^3$ whose total length is 2 and whose sum is the zero vector.

This ordered set of vectors can be used to build a polygon in $\mathbb{R}^3$.

Problem 6: Show that if $(a+bi, c+di)$ is a row of your starting matrix then $(a^2+b^2-c^2-d^2, 2(bc-ad), 2(ac+bd))$ is the corresponding row of your ending matrix.

Problem 7): a) Show that the sum of the rows of G is equal to $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

b) Show that the sum of the lengths of the rows of G is equal to 2.

Set $S_0 = (0,0,0)$ and $S_{i+1} = S_i + R_i$ where R_i denotes the i^{th} row of G. You can build a polygon with n-sides whose k^{th} side starts at S_{k-1} and ends at S_k.

Problem 8: Write a program to produce random n-gons in $\mathbb{R}^3$.

Problem 9: One can experimentally compute $\mathbb{E}(||R_i||)$ and $\mathbb{E}(||R_i||^2)$. It should be unsurprising that $\mathbb{E}(||R_i||) = 2/n$. Can you guess a value for $\mathbb{E}(||R_i||^2)$?

Problem 10: Experimentally find the square expected distance from vertex i to the origin in a random n-gon.

Problem 11: Can you come up with some possible extensions of Problem 3 for random "convex" polygons in $\mathbb{R}^3$?

https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.21480
Probability Theory of Random Polygons from the Quaternionic Viewpoint

https://www.tandfonline.com/doi/full/10.1080/00029890.2019.1535735
Random Triangles and Polygons in the Plane

https://arxiv.org/abs/dg-ga/9602012
Polygon Spaces and Grassmannians