First Applications using Grassmann Manifolds

March 3, 2025

Broad Outline for today

- Quick Recap of First Lecture
- Uniform Sampling of S^n, $O(n)$, and $Gr(k,n)$
- Random Polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Projections of n-cubes and n-simplices

The goal for the course is to gain insights in the interplay between

geometry
dimension
volume
randomness

through theory, computation, experiments, and applications.

A Quick Revisit to Real Projective Space

Real Projective space, $\mathbb{RP}^n$, is a geometric space whose points parameterize lines through the origin in $\mathbb{R}^{n+1}$.

Each line through the origin in $\mathbb{R}^{n+1}$ hits the sphere S^n in a pair of antipodal points. There is a 1-1 correspondence between unordered pairs of antipodal points and lines through the origin.

We can think of $\mathbb{RP}^n$ as the collection of lines through the origin in $\mathbb{R}^{n+1}$ or as S^n modulo antipodal points.

$\mathbb{RP}^n$ is an n-dimensional real manifold.

The volume of $\mathbb{RP}^n$ is half the volume of S^n.

A Quick Revisit to the Real Grassmannian $Gr(k, n)$

Grassmann manifolds are a generalization of Projective space.

$Gr(k, n)$ is a geometric space whose points parameterize k dimensional subspaces of $\mathbb{R}^n$.

$Gr(1, n+1) = \mathbb{RP}^n$

$Gr(k, n)$ is a $k(n-k)$-dimensional real manifold.

The volume of $Gr(k, n)$ can be expressed in terms of the volume of spheres.

The Orthogonal group $O(n)$

Let $O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I\}$

Let $SO(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I \text{ and } det(A) = 1\}$

As a manifold, $O(n)$ consists of 2 disconnected components.

$O(n)$ and $SO(n)$ are real manifolds of dimension $\frac{n(n-1)}{2}$.

The volume of $O(n)$ can be expressed in terms of the volume of spheres.

The volume of $SO(n)$ is half the volume of $O(n)$.

Grassmannian as a Homogenous Space

We can express $Gr(k,n)$ as a homogeneous space

$$Gr(k,n) = O(n)/O(k) \times O(n-k).$$

Let $Q \in O(n)$. When we write $Gr(k,n) = O(n)/O(k) \times O(n-k)$, we identify points in $Gr(k,n)$ with equivalence classes $[Q]$ where

$$[Q] = \left\{ Q \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \mid M_1 \in O(k),\ M_2 \in O(n-k) \right\}.$$

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$$[Q] = \left\{ Q \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \mid M_1 \in O(k),\ M_2 \in O(n-k) \right\}.$$

We can also express $Gr(k,n)$ in terms of $SO(n)$:

$$Gr(k,n) = SO(n)/S(O(k) \times O(n-k)).$$

Oriented Grassmannian

The Grassmannian is

$$\begin{aligned} Gr(k,n) &= O(n)/O(k) \times O(n-k) \\ &= SO(n)/S(O(k) \times O(n-k)) \end{aligned}$$

The oriented Grassmannian is

$$Gr(k,n)^\circ = SO(n)/SO(k) \times SO(n-k).$$

$Gr(k,n)^\circ$ is a $2:1$ cover of $Gr(k,n)$.

$$Gr(k,n) = SO(n)/S(O(k)\times O(n-k))$$

$$Gr(k,n)^\circ = SO(n)/SO(k)\times SO(n-k)$$

$Gr(k,n)^\circ$ is a $2:1$ cover of $Gr(k,n)$.

$$\mathbb{RP}^{n-1} = Gr(1,n) = SO(n)/S(O(1)\times O(n-1))$$

$$S^{n-1} = Gr(1,n)^\circ = SO(n)/SO(1)\times SO(n-1)$$

The sphere is a $2:1$ cover of projective space.

The volume of $Gr(k,n)^\circ$ is twice the volume of $Gr(k,n)$.

Dimension and Volume of a Grassmann manifold

From the model $Gr(k,n) = O(n)/(O(k) \times O(n-k))$, we can write the dimension and volume of $Gr(k,n)$ in terms of the dimension and volume of $O(n)$.

The dimension and volume of $O(n), O(k), O(n-k), Gr(k,n)$ can be written in terms of the dimension and volume of spheres, S^r .

$Vol(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$

$\dim(O(n)) = \sum_{i=0}^{n-1} \dim(S^i)$

$vol(O(n)) = \prod_{i=0}^{n-1} vol(S^i)$

SVD method for principal angles between subspaces

Let A and B be $n \times k$ matrices with orthonormal columns.

$[A]$ and $[B]$ correspond to points on $Gr(k, n)$.

Let θ_i denote the i^{th} principal angle between $[A]$ and $[B]$.

Consider the singular value decomposition $A^T B = U \Sigma V^T$.

We have $[AU] = [A], \;\; [BV] = [B], \;\; (AU)^T (BV) = \Sigma$.

Theorem: $\Sigma_{i,i} = \cos(\theta_i)$ and $u_i, v_i = i^{th}$ column of AU, BV

Orthogonally invariant distances on Grassmann manifolds

Orthogonally invariant distance measure between subspaces can be expressed in terms of principal angles between the subspaces.

Let $\Theta(U, V) = (\theta_1, \ldots, \theta_k)$.

- $d(U, V) = ||\Theta(U, V)||_2$ is the geodesic distance
- $d(U, V) = ||\sin(\Theta(U, V))||_2$ is the projection distance
- $d(U, V) = \cos^{-1}(\prod_i \cos(\theta_i))$ is the Fubini-Study distance

Uniform sampling of points on S^n

Let $N(\mu, \sigma^2)$ denote the Gaussian distribution with mean μ, variance σ^2.

Given a Gaussian random variable X, let $\mathbb{E}(X)$ denote its expected value and let $V(X)$ denote its variance.

If X, Y are independent Gaussians then

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) \quad \text{and} \quad V(kX) = k^2 V(X).$$

Let $X = [X_1, \ldots, X_n]$ with $X_i \sim N(0, 1)$ and let $U \in O(n)$.

$\frac{X}{||X||}$ is a random point on S^n.

XU has the same distribution as X.

Uniform sampling of points on $O(n)$ and $Gr(k,n)$

To sample $O(n)$, build an $n \times n$ matrix with entries from $N(0,1)$ then find Q in its QR factorization.

To sample $Gr(k,n)$, build an $n \times k$ matrix with entries from $N(0,1)$ then consider its column space.

Summary: $N(0,1)$ leads to orthogonally invariant distributions on

S^n, $O(n)$, $Gr(k,n)$.

Time for Computer Experiments!

Today's Experiments

- Random polygons in $\mathbb{R}^2$
- Random polygons in $\mathbb{R}^3$
- Projections of hypercubes
- Projections of simplices

Experiment 1: Random Polygons in $\mathbb{R}^2$

Let a, b be a pair of orthonormal vectors in $\mathbb{R}^n$.

Consider $c = a + bi \in \mathbb{C}^n$.

Apply the map $z \mapsto z^2$ to each entry of c to get a vector $d \in \mathbb{C}^n$.

Write $d = e + fi$ with $e, f \in \mathbb{R}^n$ then form the matrix U whose columns are e and f.

The rows of U are an ordered collection of vectors in $\mathbb{R}^2$ whose total length is 2 and whose sum is the zero vector.

This ordered set of vectors can be used to build a polygon in $\mathbb{R}^2$.

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$(a, b) \to (a^2 - b^2, 2ab)$

Experiment 2: Random Polygons in $\mathbb{R}^3$

Let a, b be a pair of orthonormal vectors in $\mathbb{C}^n$.

Consider $c = a + bj \in \mathbb{H}^n$.

Apply the map $q \mapsto \bar{q}iq$ to each entry of c to get a vector $d \in \mathbb{H}^n$.

Write $d = ei + fj + gk$ with $e, f, g \in \mathbb{R}^n$ then form the matrix U whose columns are e, f and g.

The rows of U are a collection of vectors in $\mathbb{R}^3$ whose total length is 2 and whose sum is the zero vector.

This ordered set of vectors can be used to build a polygon in $\mathbb{R}^3$.

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$$(a + bi, c + di) \to (a^2 + b^2 - c^2 - d^2, 2(bc - ad), 2(ac + bd))$$

Experiment 3: Zonotopes - Projections of a Hypercube

An *n*-cube is the convex hull of the points with coordinates $(\pm 1, \ldots, \pm 1)$

A zonotope is the projection of an *n*-cube.

What does a "typical" zonotope in $\mathbb{R}^2$ and $\mathbb{R}^3$ look like?

How many vertices and faces does one expect?

What is the expected volume?

What about the projection of an *n* cube for *n* very large?

Experiment 4: Polytopes - Projections of an n-Simplex

The standard n-simplex is the convex hull of the points given by the rows of an $(n+1) \times (n+1)$ identity matrix.

Subtracting $(1/n, \ldots, 1/n)$ from each of these points centers the standard n-simplex about the origin.

A polytope is the convex hull of a set of points in $\mathbb{R}^k$.

Every non-degenerate polytope is the projection of a scaled/translated regular simplex.

What does a "typical" projection to $\mathbb{R}^2$ and $\mathbb{R}^3$ look like?

How many vertices and faces does one expect? What is the expected volume?