

MINORS OF TREE DISTANCE MATRICES

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ABSTRACT. We prove a formula for the determinant of a principal minor of the distance matrix of a weighted tree.

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1. INTRODUCTION

Suppose T is a tree with n vertices and $m = n - 1$ edges. let D denote the distance matrix of T . In [\[3\]](#), [Graham-Pollak](#) and Pollak proved that

$$(1) \quad \det(D) = (-1)^{n-1} 2^{n-2} (n-1).$$

[Graham-Hoffman-Hosoya](#)
[\[2\]](#),

Special case: Γ is a metric tree.

thm:max-capacity

Theorem 1. *Suppose G is a finite (weighted?) tree, and $A \subset V(G)$ is a subset of vertices. Then*

eq:max-capacity

$$(2) \quad \det D[A] = \frac{1}{4} \ell(\Gamma) \left(\sum_{\mathcal{T}} w(T) \right) - \frac{1}{4} \left(\sum_{\mathcal{F}^*} \mu_{can}(F_{2,*})^2 w(F_2) \right).$$

where $\mathcal{T}(G/A)$ denotes the set of A -rooted spanning forests of G , F_2 varies over all $(A, *)$ -rooted spanning forests of G , $F_{2,*}$ denotes the $*$ -component of F_2 , and

$$(3) \quad \mu_{can}(F_{2,*}) = \sum_{x \in V(F_{2,*})} 2 - \deg(x).$$

In other words, μ_{can} denotes the canonical measure on Γ .

Theorem 2 (Monotonicity of principal minors). *Suppose $G = (V, E)$ is a finite, weighted tree with distance matrix D . If $A, B \subset V$ are vertex subsets with $A \subset B$, then*

$$\left| \frac{\det D[A]}{\text{cof } D[A]} \right| \leq \left| \frac{\det D[B]}{\text{cof } D[B]} \right|.$$

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1.1. Previous work.

Theorem 3 ([\[1\]](#), [bapat-sivasubramanian](#)). *Let T be a tree with $m + 1$ vertices and m edges. Let D be the distance matrix of T , and L the Laplacian matrix. Let $S \subset V(T)$ be a subset of vertices of T . Then*

$$\text{cof } D[S] = (-2)^{|S|-1} \det L[V \setminus S].$$

1.2. Notation.

Γ a compact metric graph

G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G

(G, ℓ) a combinatorial model for a metric graph, where

$\ell : E(G) \rightarrow \mathbb{R}_{>0}$ is a length function on edges of G

$\mathcal{C}(\Gamma)$ continuous \mathbb{R} -valued functions on Γ

$\text{Meas}(\Gamma)$ signed Borel measures on Γ

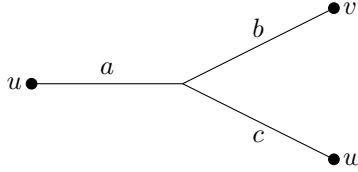
$\text{Meas}_{\geq 0}(\Gamma)$ positive Borel measures on Γ

2. BACKGROUND

3. PROOFS

4. EXAMPLES

Example 4. Suppose Γ is a tripod with lengths a, b, c and corresponding leaf vertices u, v, w .



Let $A = \{u, v, w\}$. Then

$$D[A] = \begin{bmatrix} 0 & a+b & a+c \\ a+b & 0 & b+c \\ a+c & b+c & 0 \end{bmatrix}.$$

and

$$\det D[A] = 2(a+b)(a+c)(b+c) = 2((a+b+c)(ab+ac+bc) - abc).$$

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REFERENCES

- [1] R. B. Bapat and S. Sivasubramanian. Identities for minors of the Laplacian, resistance and distance matrices. *Linear Algebra Appl.*, 435(6):1479–1489, 2011.
- [2] R. L. Graham, A. J. Hoffman, and H. Hosoya. On the distance matrix of a directed graph. *J. Graph Theory*, 1(1):85–88, 1977.
- [3] R. L. Graham and H. O. Pollak. On the addressing problem for loop switching. *Bell System Tech. J.*, 50:2495–2519, 1971.

bapat-sivasubramanian

graham-hoffman-hosoya

graham-pollak