MINORS OF TREE DISTANCE MATRICES

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Abstract. We prove a formula for the determinant of a principal minor of the distance matrix of a weighted tree.pr

Contents

1. Introduction

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Special case: Γ is a metric tree.

thm:max-measure **Theorem 1.** *Suppose $\Gamma = (G, \ell)$ is a metric tree, and $A \subset V(G)$ a finite subset of vertices. The measure in $Meas^1(A)$ which maximizes distance energy is given by*

eq:max-measure
$$\mu_A = \frac{1}{\sum\limits_{\mathcal{F}} w(F)} \sum_{\mathcal{F}} w(F) \sum_{x \in A} \mu_{can}(F_x)\, \delta_x = \sum_{x \in A} \left(\frac{\sum_{\mathcal{F}} w(F) \mu_{can}(F_x)}{\sum_{\mathcal{F}} w(F)} \right) \delta_x \tag{1}$$

where δ_x is the Dirac measure at x, $\mathcal{F} = \mathcal{F}(A, G)$ denotes the set of A-rooted spanning forests of G, $w(F) = \prod_{e \notin F} \ell(e)$ is the weight of F, F_x denotes the x-component of the forest F and

$$\mu_{can}(F_x) = \sum_{y \in V(F_x)} \frac{2 - \mathrm{val}(y)}{2}.$$

Equivalently, the optimal measure is

$$\mu_A = \sum_{x \in A} \delta_x \left(\frac{2 - \mathrm{val}(x)}{2} + \sum_{y \in \Gamma \setminus A} \frac{2 - \mathrm{val}(y)}{2} c(y, x; \Gamma) \right)$$

where $c(y, x; \Gamma)$ denotes the current to $x \in A$ when unit current flows from y to A. In particular,

$$c(y, x; \Gamma) = \frac{\sum_F \epsilon(y, x, F) w(F)}{\sum_F w(F)} \tag{2}$$

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where both sums are taken over A-rooted spanning forests F of Γ, and

$$\epsilon(y, x, F) = \begin{cases} 1 & \text{if } x\text{-component of } F \text{ contains } y \\ 0 & \text{otherwise.} \end{cases}$$

thm:max-capacity **Theorem 2.** *Suppose $\Gamma = (G, \ell)$ is a metric tree, $A \subset V(G)$ is a finite subset, and μ_A is the measure* (eq:max-measure 1). *Then*

eq:max-capacity (3)
$$\mathcal{E}(\mu_A) = \frac{1}{4}\ell(\Gamma) - \left(\frac{\sum_{\mathcal{F}^*} \mu_{can}(F_{2,*})^2 w(F_2)}{\sum_{\mathcal{F}} w(F_1)} \right).$$

where F_1 varies over A-rooted spanning forests of G, F_2 varies over all $(A, *)$-rooted spanning forests of G, $F_{2,*}$ denotes the $*$-component of F_2, and

(4)
$$\mu_{can}(F_{2,*}) = \frac{1}{2} \sum_{x \in V(F_{2,*})} 2 - \mathrm{val}_\Gamma(x).$$

In other words, μ_{can} denotes the canonical measure on Γ.

1.1. **Previous work.**

1.2. **Notation.** Γ a compact metric graph

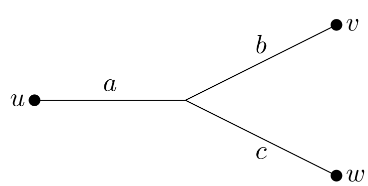
G a finite graph, loops and parallel edges allowed, possibly disconnected
$E(G)$ edge set of G
$V(G)$ vertex set of G
(G, ℓ) a combinatorial model for a metric graph, where
$\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on edges of G
$\mathcal{C}(\Gamma)$ continuous $\mathbb{R}$-valued functions on Γ
$Meas(\Gamma)$ signed Borel measures on Γ
$Meas_{\geq 0}(\Gamma)$ positive Borel measures on Γ

2. Background

3. Proofs

4. Examples

Example 3. Suppose Γ is a tripod with lengths a, b, c and corresponding leaf vertices u, v, w.



Let $A = \{u, v, w\}$. Then

$$\begin{aligned} \mu_A &= \frac{ab(\frac{1}{2}\delta_u + \frac{1}{2}\delta_v) + ac(\frac{1}{2}\delta_u + \frac{1}{2}\delta_w) + bc(\frac{1}{2}\delta_v + \frac{1}{2}\delta_w)}{ab + ac + bc} \\ &= \frac{1}{2}\left(\frac{ab + ac}{ab + ac + bc}\delta_u + \frac{ab + bc}{ab + ac + bc}\delta_v + \frac{ac + bc}{ab + ac + bc}\delta_w \right). \end{aligned}$$

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References

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[1] R. L. Graham, A. J. Hoffman, and H. Hosoya. On the distance matrix of a directed graph. *J. Graph Theory*, 1(1):85–88, 1977.