

MINORS OF TREE DISTANCE MATRICES

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ABSTRACT. We prove a combinatorial formula for the principal minors of the distance matrix of a weighted tree. This generalizes a result of Graham and Pollak. The identity involves counts of rooted spanning forests of the underlying tree.

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1. INTRODUCTION

Suppose $G = (V, E)$ is a tree with n vertices. Let D denote the distance matrix of G . In [\[4\]](#), [Graham and Pollak](#) proved that

$$\text{eq:full-det} \quad (1) \quad \det D = (-1)^{n-1} 2^{n-2} (n-1).$$

This identity is remarkable in that the result does not depend on the tree structure, beyond the number of vertices. The identity [\(I\)](#) was motivated by a problem in data communication, and inspired much further research on distance matrices.

The main result of this paper is to generalize [\(I\)](#) by replacing $\det D$ with any of its principal minors. For a subset $S \subset V(G)$, let $D[S]$ denote the submatrix consisting of the S -indexed rows and columns of D .

thm:main **Theorem 1.** *Suppose G is a tree with n vertices, and distance matrix D . Let $S \subset V(G)$ be a nonempty subset of vertices. Then*

$$\text{eq:main} \quad (2) \quad \det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left((n-1) \kappa(G; S) - \sum_{\mathcal{F}_2(G; S)} (2 - \deg^o(F, *))^2 \right),$$

where $\kappa(G; S)$ is the number of S -rooted spanning forests of G , $\mathcal{F}_2(G; S)$ is the set of $(S, *)$ -rooted spanning forests of G , and $\deg^o(F, *)$ denotes the outdegree of the $*$ -component of F .

For definitions of $(S, *)$ -rooted spanning forests and other terminology, see Section [7.7](#). [sec:graphs-matrices](#) Note that the quantity $\deg^o(F, *)$ satisfies the bounds

$$1 \leq \deg^o(F, *) \leq |S|.$$

When $S = V$ is the full vertex set, the set of V -rooted spanning forests is a singleton, consisting of the subgraph with no edges, so $\kappa(G; V) = 1$; and moreover the set $\mathcal{F}_2(G; V)$ of $(V, *)$ -rooted spanning forests is empty. Thus (2) recovers the Graham–Pollak identity (1) when $S = V$.

1.1. Weighted trees. If $\{\alpha_e : e \in E\}$ is a collection of positive edge weights, the α -distance matrix D_α takes the distance along edge e to be α_e . Namely, the (v, w) -entry of D_α is the sum of the weights α_e along the unique path from v to w . Then

$$(3) \quad \det D_\alpha = (-1)^{n-1} 2^{n-2} \sum_{e \in E} \alpha_e \prod_{e \in E} \alpha_e.$$

This weighted version of (1) was proved by Bapat–Kirkland–Neumann [1]. The weighted identity (1.1) reduces to (1) when taking all unit weights, $\alpha_e = 1$. We also prove the following weighted version of our main theorem.

thm:w-main

Theorem 2. Suppose $G = (V, E)$ is a finite, weighted tree with edge weights $\{\alpha_e : e \in E\}$, and corresponding weighted distance matrix D_α . For any nonempty subset $S \subset V$, we have

eq:w-main

$$(4) \quad \det D_\alpha[S] = (-1)^{|S|-1} 2^{|S|-2} \left(\sum_{T \in \mathcal{F}_1(G; S)} \alpha_e \sum_{F \in \mathcal{F}_1(G; S)} w(T) - \sum_{F \in \mathcal{F}_2(G; S)} (2 - \deg^o(F, *))^2 w(F) \right).$$

where $\mathcal{F}_1(G; S)$ is the set of S -rooted spanning forests of G , $\mathcal{F}_2(G; S)$ is the set of $(S, *)$ -rooted spanning forests of G , $w(T)$ and $w(F)$ denote the α -weights of the forests T and F , and $\deg^o(F, *)$ is the outdegree of the $*$ -component of F , as above.

Theorem 2 also reduces to Theorem 1 when taking all unit weights, $\alpha_e = 1$.

It is worth observing that depending on the chosen subset $S \subset V$, the distances appearing in the submatrix $D[S]$ may ignore a large part of the ambient tree G . We could instead replace G by the subtree $\text{conv}(S, G)$ consisting of the union of all paths between vertices in S , which we call the *convex hull* of $S \subset G$. To apply formula (2) or (2) “efficiently,” we should replace G with this subtree $\text{conv}(S, G)$. However, the formulas as stated are true even without this replacement due to cancellation of terms.

1.2. Applications of the main theorem. Given a matrix A , let $\text{cof } A$ denote the *sum of cofactors* of A , i.e.

$$\text{cof } A = \sum_{i=1}^{|S|} \sum_{j=1}^{|S|} (-1)^{i+j} \det A_{ij}$$

If A is invertible, then $\text{cof } A$ is related to the sum of entries of the matrix inverse A^{-1} by a factor of $\det A$, i.e. $\text{cof } A = (\det A)(\mathbf{1}^\top A^{-1} \mathbf{1})$. In [2], Bapat and Sivasubramanian showed the following identity for the sum of cofactors of $D[S]$.

eq:cof-trees

$$\text{cof } D[S] = (-2)^{|S|-1} \sum_{T \in \mathcal{F}_1(G; S)} w(T).$$

Using (1.2), an immediate corollary to Theorem 2 is the following identity.

Theorem 3. Suppose $G = (V, E)$ is a finite, weighted tree with edge weights $\{\alpha_e : e \in E\}$. Let $D = D_\alpha$ denote the weighted distance matrix of G . For any nonempty subset $S \subset V$, we have

$$\frac{\det D[S]}{\text{cof } D[S]} = \frac{1}{2} \left(\sum_{T \in \mathcal{F}_1(G; S)} \alpha_e - \frac{\sum_{F \in \mathcal{F}_2(G; S)} w(F) k(F, *)^2}{\sum_{T \in \mathcal{F}_1(G; S)} w(T)} \right)$$

where $k(F, *) = 2 - \deg^o(F, *)$.

◇ add remark / theorem that \det/cof is achieved as result of optimization problem ◇

We remark that the calculation of $\det D[S]$ is related to the following quadratic optimization problem: for all vectors $\mathbf{m} \in \mathbb{R}^S$,

$$\begin{aligned} &\text{optimize objective function: } \mathbf{m}^\top D[S] \mathbf{m} \\ &\text{with constraint: } \mathbf{1}^\top \mathbf{m} = 1. \end{aligned}$$

The gradient of the objective function is $2D[S]\mathbf{m}$, and the gradient of the constraint is $\mathbf{1}$. By the theory of Lagrange multipliers, the optimal solution \mathbf{m}^* is a vector satisfying

$$D[S]\mathbf{m}^* = \lambda \mathbf{1} \quad \text{for some } \lambda \in \mathbb{R}.$$

It follows that the optimal objective value is

$$(\mathbf{m}^*)^\top D[S] \mathbf{m}^* = (D[S]\mathbf{m}^*)^\top \mathbf{m}^* = \lambda(\mathbf{1}^\top \mathbf{m}^*) = \lambda.$$

(The above computation uses the fact that $D[S]$ is a symmetric matrix, and the given constraint $\mathbf{1}^\top \mathbf{m} = 1$.) On the other hand, assuming $D[S]$ is invertible we have $\mathbf{m}^* = \lambda(D[S]^{-1}\mathbf{1})$, so that

$$1 = \mathbf{1}^\top \mathbf{m}^* = \lambda(\mathbf{1}^\top D[S]^{-1}\mathbf{1}) = \lambda \frac{\text{cof } D[S]}{\det D[S]}, \quad \text{so} \quad \lambda = \frac{\det D[S]}{\text{cof } D[S]}.$$

Proposition 4. *If $D[S]$ is any matrix with rows and columns indexed by S , then any critical value of the optimization problem ◇ cite above ◇ is equal to $\frac{\det D[S]}{\text{cof } D[S]}$, i.e.*

$$\frac{\det D[S]}{\text{cof } D[S]} = \max\{\mathbf{m}^\top D[S] \mathbf{m} : \mathbf{m} \in \mathbb{R}^S, \mathbf{1}^\top \mathbf{m} = 1\}$$

where $\text{cof } D[S]$ denotes the sum of cofactors,

$$\text{cof } A = \sum_{i=1}^{|S|} \sum_{j=1}^{|S|} (-1)^{i+j} \det A_{ij}$$

Theorem 5 (Monotonicity of principal minor ratios). *If $A, B \subset V(G)$ are nonempty subsets with $A \subset B$, then*

$$\frac{\det D[A]}{\text{cof } D[A]} \leq \frac{\det D[B]}{\text{cof } D[B]}.$$

Theorem 6 (Bounds on principal minor ratios). *Suppose $G = (V, E)$ is a finite, weighted tree with distance matrix D .*

(1) *If $S \subset V(G)$ is nonempty,*

$$0 \leq \frac{\det D[S]}{\text{cof } D[S]} \leq \frac{1}{2} \sum_{E(G)} \alpha_e.$$

(2) *If $\text{conv}(S, G)$ denotes the subtree of G consisting of all paths between points of $S \subset V(G)$,*

$$\frac{\det D[S]}{\text{cof } D[S]} \leq \frac{1}{2} \sum_{E(\text{conv}(S, G))} \alpha_e.$$

(3) *If γ is a simple path between vertices $s_0, s_1 \in S$, then*

$$\frac{1}{2} \sum_{e \in \gamma} \alpha_e \leq \frac{\det D[S]}{\text{cof } D[S]}$$

Theorem 7 (Nonsingular minors). *Let G be a finite tree with (weighted) distance matrix D , and let $S \subset V(G)$ be a subset of vertices. If $|S| \geq 2$ then $\det D[S] \neq 0$.*

1.3. Previous work. A formula for the inverse matrix D^{-1} was found by Graham and Lovász in [3].

1.4. **Notation.** G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G

$\kappa(G)$ number of spanning trees of G

$\mathcal{F}_1(G)$ the set of spanning trees of G

$\mathcal{F}_2(G)$ the set of 2-component forests of G

$\mathcal{F}_1(G; S)$ the set of S -rooted spanning forests of G

$\mathcal{F}_2(G; S)$ the set of $(S, *)$ -rooted spanning forests of G

2. GRAPHS AND MATRICES

For background on enumeration problems for graphs and trees, see Moon ^{moon}[5].

Given a graph $G = (V, E)$ with edge weights $\{\alpha_e : e \in E\}$, for any edge subset $A \subset E$ we define the *weight* of A as

$$w(A) = \prod_{e \in A} \alpha_e.$$

We define the *co-weight* of A as

$$w(\overline{A}) = \prod_{e \notin A} \alpha_e.$$

By abuse of notation, if H is a subgraph of G , we use $w(\overline{H})$ to denote $w(\overline{E(H)})$.

2.1. Spanning trees and forests. A *spanning tree* of a graph G is a subgraph which is connected, has no cycles, and contains all vertices of G . A *spanning forest* of a graph G is a subgraph which has no cycles and contains all vertices of G .

Given a set of vertices $S = \{v_1, v_2, \dots, v_k\}$, an *S -rooted spanning forest* of G is a spanning forest which has exactly one vertex v_i in each connected component. An *$(S, *)$ -rooted spanning forest* of G is a spanning forest which has $|S| + 1$ components, where $|S|$ components each contain one vertex of S , and the additional component is disjoint from S . We call the component disjoint from S the “floating component.”

Let

$$\kappa_k(v_1|v_2|\dots|v_k)$$

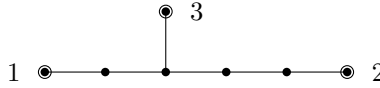
denote the number of k -component spanning trees which have a vertex v_i in each component. If $S = \{v_1, \dots, v_k\}$, then $\kappa_k(v_1|\dots|v_k) = \kappa(G/S)$.

If u, v, w are vertices, then let

$$\kappa_2(uv|w)$$

denote the number of two-forests which have u, v in one component and w in the other component.

Example 8. Suppose G is the tree with unit edge lengths shown below.



Let S be the set of three leaf vertices. Then $\mathcal{F}_1(G; S)$ contains 11 forests, while $\mathcal{F}_2(G; S)$ contains 19 forests. These are shown in Figures 1 and 2, respectively.

2.2. Laplacian matrix. Given a graph $G = (V, E)$, let $L \in \mathbb{R}^{V \times V}$ denote the *Laplacian matrix* of G . If G is a weighted graph with edge weights $\alpha_e \in \mathbb{R}_{>0}$ for $e \in E$, let L denote the weighted Laplacian matrix of G .

Given $S \subset V$, let $L[\overline{S}]$ denote the matrix obtained from L by removing the rows and columns indexed by S .

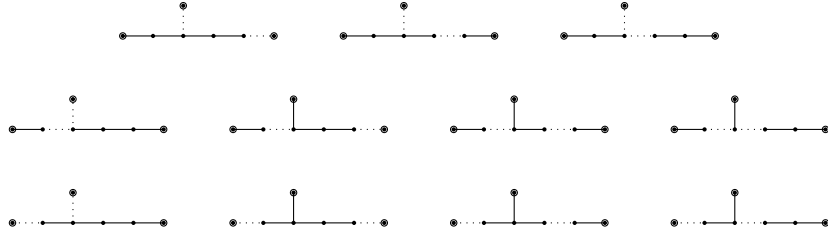

 FIGURE 1. Forests in $\mathcal{F}_1(G; S)$.

fig:1-forests

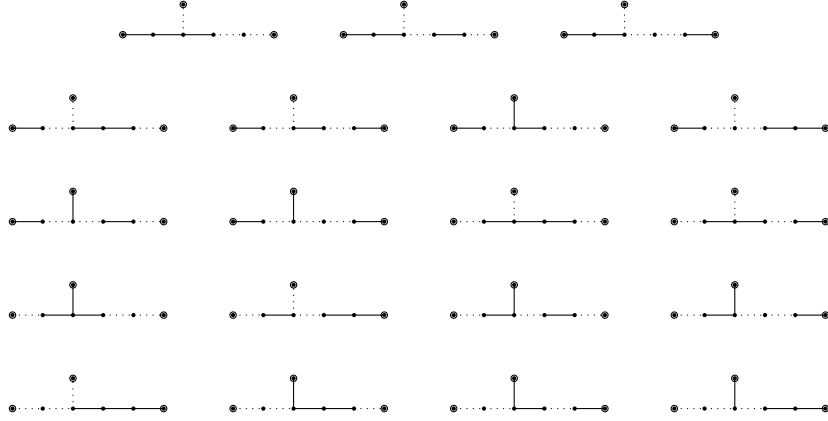

 FIGURE 2. Forests in $\mathcal{F}_2(G; S)$.

fig:2-forests

Definition 9 (Weighted Laplacian matrix). Given a graph $G = (V, E)$ and edge weights $\{\alpha_e : e \in E\}$, the *weighted Laplacian matrix* $L_\alpha \in \mathbb{R}^{V \times V}$ is defined by

$$(L_\alpha)_{v,w} = \begin{cases} 0 & \text{if } v \neq w \text{ and } (v,w) \notin E \\ \alpha_e^{-1} & \text{if } v \neq w \text{ and } (v,w) = e \in E \\ -\sum_{e \in N(v)} \alpha_e^{-1} & \text{if } v = w. \end{cases}$$

For any graph G , let $\kappa(G)$ denote the number of spanning trees of G . The following theorem is due to Kirchhoff. For any graph G , let $\kappa(G)$ denote the number of spanning trees of G .

Theorem 10 (All-minors matrix tree theorem). *Let $G = (V, E)$ be a finite graph, and let L denote the Laplacian matrix of G . Then for any nonempty vertex set $S \subset V$,*

$$\det L[\bar{S}] = \kappa(G; S).$$

Note that $\kappa(G; S)$ is also the number of spanning trees of the quotient graph G/S , which “glues together” all vertices in S as a single vertex.

The following result is due to Bapat–Sivasubramanian.

Theorem 11 (Distance matrix cofactor sums ^{bapat-sivasubramanian [2]}). *Given a tree G , let D be the distance matrix of G , and L the Laplacian matrix. Let $S \subset V(G)$ be a nonempty subset of vertices of G . Then*

$$\text{cof } D[S] = (-2)^{|S|-1} \det L[\bar{S}].$$

2.3. Tree distance. For $e \in E$ and $v, w \in V$, let

$$\delta(e; v, w) = \begin{cases} 1 & \text{if } e \text{ separates } v \text{ from } w, \\ 0 & \text{otherwise.} \end{cases}$$

thm:matrix-tree

ec:tree-distance

(Note that $\delta(e; v, v) = 0$ for any e and v .) Given an edge $e \in E$ and vertices $v, w \in V$, let

$$\delta(e; v, w) = \begin{cases} 1 & \text{if } e \text{ lies on } v \sim w \text{ path,} \\ 0 & \text{otherwise.} \end{cases}$$

We can express the tree distance $d(v, w)$ as a sum over edges

$$d(v, w) = \sum_{e \in E(G)} \delta(e; v, w) \quad \text{where } \delta(e; v, w) = \begin{cases} 1 & \text{if } e \text{ lies on } v \sim w \text{ path,} \\ 0 & \text{otherwise.} \end{cases}$$

We have the following perspectives on the function $\delta(e; v, w)$:

- If we fix v and w , then $\delta(-; v, w) : E(G) \rightarrow \{0, 1\}$ is the indicator function for the unique $v \sim w$ path in G .
- On the other hand if we fix e and v , then the deletion $G \setminus e$ has two connected components, and $\delta(e; v, -) : V(G) \rightarrow \{0, 1\}$ is the indicator function for the component of $G \setminus e$ not containing v .

prop:distance-sum

Proposition 12 (Weighted tree distance). *For a tree $G = (V, E)$ with weights $\{\alpha_e : e \in E\}$, the weighted distance function satisfies*

$$d_\alpha(v, w) = \sum_{e \in E} \alpha_e \delta(e; v, w).$$

2.4. Principal cuts. Given a tree $G = (V, E)$ and an edge $e \in E$, the edge deletion $G \setminus e$ contains two connected components. Using the implicit orientation on $e = (e^+, e^-)$, we let $(G \setminus e)^+$ denote the component that contains endpoint e^+ , respectively $(G \setminus e)^-$ and endpoint e^- .

For any $e \in E$ and $v \in V$, we let $(G \setminus e)^v$ denote the component of $G \setminus e$ containing v , respectively $(G \setminus e)^{\bar{v}}$ \diamond or $(G \setminus e)^{-v}$ \diamond for the component not containing v .

2.5. Outdegree of rooted forest. Given a rooted forest F in $\mathcal{F}(G; S)$ and $s \in S$, let $F(s)$ denote the s -component of F . We define the *outdegree* $\deg^o(F, s)$ by

eq:outdeg

$$(5) \quad \deg^o(F, s) = \#\{e = (a, b) \in E : a \in F(s), b \notin F(s)\}.$$

In words, $\deg^o(F, s)$ is the number of edges which connect the s -component of F to a different component.

$$\#\{e \in E : e \text{ connects the } s\text{-component of } F \text{ to a different component}\}$$

If F is a forest in $\mathcal{F}_2(G; S)$, let $\deg^o(F, *)$ denote the outdegree of the floating component.

lem:outdeg-sum

Lemma 13. *Suppose G is a tree and $H \subset G$ is a (nonempty) connected subgraph. Then*

$$\sum_{v \in V(H)} (2 - \deg(v)) = 2 - \deg^o(H).$$

Proof. This is straightforward to check by induction on $|V(H)|$, with base case $|V(H)| = 1$: if $H = \{v\}$ consists of a single vertex, then $\deg^o(H) = \deg(v)$. \square

3. PROOFS

In this section we prove Theorem [2](#). ^{thm:w-main}

Outline of proof: given a subset $S \subset V$ and distance submatrix $D[S]$, we will

- Find vector $\mathbf{m} \in \mathbb{R}^S$ such that $D[S]\mathbf{m} = \lambda \mathbf{1} \in \mathbb{R}^S$.
- Compute the sum of entries of \mathbf{m} , i.e. $\mathbf{1}^\top \mathbf{m}$.
- Using (i), relate the sum $\mathbf{1}^\top \mathbf{m}$ to the sum of entries of the inverse matrix $D[S]^{-1}$:

$$\mathbf{1}^\top \mathbf{m} = \lambda (\mathbf{1}^\top D[S]^{-1} \mathbf{1}) = \lambda \frac{\text{cof } D[S]}{\det D[S]}.$$

where $\text{cof } D[S]$ is the sum of cofactors of $D[S]$.

(iv) Use known expression for $\text{cof } D[S]$ to compute

$$\det D[S] = \lambda(\text{cof } D[S]) (\mathbf{1}^\top \mathbf{m})^{-1}.$$

The interesting part of this expression will turn out to be in the constant λ .

Example 14. Suppose G is a tree consisting of three paths joined at a central vertex. Let S consist of the central vertex, and the three endpoints of the paths. The corresponding submatrix of the distance matrix is

$$D[S] = \begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix}.$$

Following the steps outlined above:

(i) The vector $\mathbf{m} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ satisfies $D[S]\mathbf{m} = (a+b+c)\mathbf{1}$

(ii) The sum of entries of \mathbf{m} is $\mathbf{1}^\top \mathbf{m} = 2$.

(iii) We have

$$2 = \mathbf{1}^\top \mathbf{m} = \lambda(\mathbf{1}^\top D[S]^{-1} \mathbf{1}) = \lambda \frac{\text{cof } D[S]}{\det D[S]}.$$

(iv) The cofactor sum $\text{cof } D[S]$ is $-8abc$, so the determinant is

$$\det D[S] = \lambda \frac{\text{cof } A}{\mathbf{1}^\top \mathbf{m}} = (a+b+c)(-8abc) \frac{1}{2} = -4(a+b+c)abc.$$

3.1. **Warmup case:** $S = V$.

Proposition 15. Let $G = (V, E)$ a tree, and consider the vector $\mathbf{m} \in \mathbb{R}^V$ defined by

$$\mathbf{m}_v = 2 - \deg v \quad \text{for each } v \in V.$$

Then $\mathbf{1}^\top \mathbf{m} = \sum_{v \in V} (2 - \deg v) = 2$.

Proof. For any graph, $\sum_{v \in V} \deg v = 2|E|$. Since G is a tree, $|E| = |V| - 1$. □

Proposition 16. Let \mathbf{m} be the vector defined above, and let D be the distance matrix of G . Then $D\mathbf{m} = \lambda \mathbf{1}$ for some constant λ .

Proof. It suffices to show that for each edge e , with endpoints (e^+, e^-) , we have

$$(D\mathbf{m})_{(e^+)} = (D\mathbf{m})_{(e^-)}.$$

We compute

$$\begin{aligned} (D\mathbf{m})_{(e^+)} - (D\mathbf{m})_{(e^-)} &= \sum_{v \in V} (d(v, e^+) - d(v, e^-))(2 - \deg v) \\ &= \sum_{v \in (G \setminus e)^-} \alpha_e (2 - \deg v) - \sum_{v \in (G \setminus e)^+} \alpha_e (2 - \deg v) \end{aligned} \tag{6}$$

since

$$d(v, e^+) - d(v, e^-) = \begin{cases} \alpha_e & \text{if } v \text{ is closer to } e^- \text{ than } e^+, \\ -\alpha_e & \text{if } v \text{ is closer to } e^+ \text{ than } e^-. \end{cases}$$

◇ **TO DO: define notation** $(G \setminus e)^\pm$ ◇ For each sum in (5.1), we apply Proposition ◇ **cite** ◇ to obtain

$$\alpha_e \sum_{v \in (G \setminus e)^-} (2 - \deg v) = \alpha_e (2 - \deg^o((G \setminus e)^-)) = \alpha_e.$$

The same identity applies to the sum over $(G \setminus e)^+$, so $(D\mathbf{m})_{(e^+)} = (D\mathbf{m})_{(e^-)}$ as desired. □

-distance-warmup

eq:12-1

eq:12-1

3.2. **General case:** $S \subset V$. Fix a tree $G = (V, E)$ and a subset $S \subset V$.

dfn:m-vector

Definition 17. Let $\mathbf{m} = \mathbf{m}(G; S) \in \mathbb{R}^S$ be defined by

eq:m-vector

$$(7) \quad \mathbf{m}_v = \sum_{T \in \mathcal{F}_1(G; S)} (2 - \deg^o(T, v)) w(T) \quad \text{for each } v \in S.$$

where $\deg^o(T, v)$ is the outdegree of the v -component of T , ([eq:outdeg](#), (2.4)).

Let $\mathbf{1}$ denote the all-ones vector.

Proposition 18. For \mathbf{m} defined above, $\mathbf{1}^\top \mathbf{m} = 2 \sum_{T \in \mathcal{F}_1(G; S)} w(T)$.

Proof. We have

$$\begin{aligned} \mathbf{1}^\top \mathbf{m} &= \sum_{s \in S} \mathbf{m}_s = \sum_{s \in S} \left(\sum_{T \in \mathcal{F}_1(G; S)} (2 - \deg^o(T, s)) w(T) \right) \\ &= \sum_{T \in \mathcal{F}_1(G; S)} w(T) \left(\sum_{s \in S} \sum_{v \in T(s)} 2 - \deg(v) \right) \\ &= \sum_{T \in \mathcal{F}_1} w(T) \left(\sum_{v \in V} 2 - \deg(v) \right) = \sum_{T \in \mathcal{F}_1} w(T) \cdot 2. \end{aligned}$$

In the second line we apply Lemma [9](#) and exchange the outer summations. To obtain the third line, we observe that the vertex sets of $T(s)$ for $s \in S$ form a partition of V , since T is an S -rooted spanning forest. Finally we again apply Lemma [9](#) for the last equality, as $\deg^o(G) = 0$. \square

Corollary 19. If G is a graph with unit edge weights $\alpha_e = 1$, then the vector \mathbf{m} defined in ([eq:m-vector](#), (14)) satisfies $\mathbf{1}^\top \mathbf{m} = 2\kappa(G; S)$.

Theorem 20. With $\mathbf{m} = \mathbf{m}(G; S)$ defined as in ([eq:m-vector](#), (14)), $D[S]\mathbf{m} = \lambda \mathbf{1}$ for the constant

$$\lambda = \sum_{E(G)} \alpha_e \sum_{\mathcal{F}_1(G; S)} w(T) - \sum_{\mathcal{F}_2(G; S)} (2 - \deg^o(F, *))^2 w(F)$$

where $\deg^o(F, w)$ is the out-degree of the w -component of F (as a spanning forest).

Proof. For $e \in E$ and $v, w \in V$, let $\delta(e; v, w)$ denote the function defined in Section [1.1](#) [cite earlier](#).

\diamond For any $v \in S$, we have

eq:14-1

$$\begin{aligned} (D[S]\mathbf{m})_v &= \sum_{s \in S} d(v, s) \mathbf{m}_s \\ &= \sum_{s \in S} \left(\sum_{e \in E(G)} \alpha_e \delta(e; v, s) \right) \left(\sum_{T \in \mathcal{F}_1(G; S)} (2 - \deg^o(T, s)) w(T) \right) \\ &= \sum_{T \in \mathcal{F}_1} w(T) \sum_{e \in E} \alpha_e \left(\sum_{s \in S} \delta(e; v, s) (2 - \deg^o(T, s)) \right) \\ &= \sum_{T \in \mathcal{F}_1} w(T) \sum_{e \in E} \alpha_e \left(\sum_{s \in S} \delta(e; v, s) \sum_{u \in T(s)} (2 - \deg(u)) \right). \end{aligned} \quad (8)$$

Each S -rooted spanning tree T is naturally associated with a surjection $\pi_T : V \rightarrow S$, defined by

$$\pi_T(u) = s \quad \text{if and only if} \quad u \in T(s).$$

Using this notation,

$$(D[S]\mathbf{m})_v = \sum_{T \in \mathcal{F}_1} w(T) \sum_{e \in E} \alpha_e \left(\sum_{u \in V} (2 - \deg(u)) \delta(e; v, \pi_T(u)) \right)$$

$$(D[S]\mathbf{m})_v - \sum_{T \in \mathcal{F}_1} w(T) \sum_{e \in E} \alpha_e = \sum_{T \in \mathcal{F}_1} w(T) \sum_{e \in E} \alpha_e \sum_{u \in V} (2 - \deg(u)) (\delta(e; v, \pi_T(u)) - \delta(e; v, u))$$

We have

$$\delta(e; v, \pi_T(u)) - \delta(e; v, u) = \begin{cases} 0 & \text{if } \delta(e; \pi_T(u), u) = 0 \\ 1 & \text{if } \delta(e; \pi_T(u), u) = 1 \text{ and } \delta(e; v, \pi_T(u)) = 1 \\ -1 & \text{if } \delta(e; \pi_T(u), u) = 1 \text{ and } \delta(e; v, \pi_T(u)) = 0 \end{cases}$$

Summing the terms $(2 - \deg(u))$ against the above over all vertices u , we have the following cases:

Case 1: if $e \notin T$, then u and $\pi_T(u)$ are on the same side of the principal cut $G \setminus e$, for every vertex u .

Case 2: if $e \in T(s_0)$ and s_0 is separated from v by e , then $\delta(e; v, \pi_T(\cdot)) - \delta(e; v, \cdot)$ is the indicator function for the floating component of $T \setminus e$.

Case 3: if $e \in T(s_0)$ and s_0 is on the same component as v from e , then $\delta(e; v, \pi_T(\cdot)) - \delta(e; v, \cdot)$ is the negative of the indicator function for the floating component of $T \setminus e$.



FIGURE 3. Edge $e \in T(s_0)$ with $\delta(e; v, s_0) = 1$ (left) and $\delta(e; v, s_0) = 0$ (right).

Thus we obtain

$$\sum_{u \in V} (2 - \deg(u)) (\delta(e; v, \pi_T(u)) - \delta(e; v, u)) = \begin{cases} 0 & \text{if } e \notin T, \\ 2 - \deg^o(T \setminus e, *) & \text{if } e \in T(s_0) \text{ and } \delta(e; v, s_0) = 1, \\ -(2 - \deg^o(T \setminus e, *)) & \text{if } e \in T(s_0) \text{ and } \delta(e; v, s_0) = 0. \end{cases}$$

Thus

$$\begin{aligned} \text{eq:1} \quad (D[S]\mathbf{m})(v) - \sum_{T \in \mathcal{F}_1} w(T) \sum_{e \in E} \alpha_e \\ = \sum_{T \in \mathcal{F}_1} w(T) \sum_{s_0 \in S} \left(\sum_{\substack{e \in T(s_0) \\ \delta(e; v, s_0) = 1}} \alpha_e (2 - \deg^o(T \setminus e, *)) - \sum_{\substack{e \in T(s_0) \\ \delta(e; v, s_0) = 0}} \alpha_e (2 - \deg^o(T \setminus e, *)) \right). \end{aligned}$$

We now rewrite the above expression in terms of $\mathcal{F}_2(G; S)$, observing that the deletion $T \setminus e$ is an $(S, *)$ -rooted spanning forest of G , if $e \in T$, and that the corresponding weights satisfy

$$w(F) = \alpha_e \cdot w(T) \quad \text{if} \quad F = T \setminus e.$$

Thus

$$\begin{aligned}
(\text{eq.1}) &= \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \sum_{T \in \mathcal{F}_1} \sum_{s_0 \in S} \left(\sum_{\substack{e \in T(s_0) \\ \delta(e; v, s_0) = 1}} \mathbb{1}(F = T \setminus e) - \sum_{\substack{e \in T(s_0) \\ \delta(e; v, s_0) = 0}} \mathbb{1}(F = T \setminus e) \right) \\
&= \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \left(\#\{T \in \mathcal{F}_1 : F = T \setminus e \text{ for some } e \in T(s_0), \delta(e; v, s_0) = 1\} \right. \\
&\quad \left. - \#\{T \in \mathcal{F}_1 : F = T \setminus e \text{ for some } e \in T(s_0), \delta(e; v, s_0) = 0\} \right)
\end{aligned}$$

Next, we note that $F = T \setminus e$ is equivalent to $T = F \cup e$, and in particular this only occurs when we choose the edge e to be in the floating boundary $\partial F(*)$:



FIGURE 4. Edge $e \in \partial F(*)$ with $\delta(e; v, F(*)) = 0$ (left) and $\delta(e; v, F(*)) = 1$ (right). The floating component $F(*)$ is highlighted.

$$\begin{aligned}
(\text{eq.1}) &= \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \sum_{e \in \partial F(*)} \left(\sum_{\substack{s_0 \in S \\ \delta(e; v, s_0) = 0}} \sum_{\substack{T \in \mathcal{F}_1 \\ e \in T(s_0)}} \mathbb{1}(T = F \cup e) \right. \\
&\quad \left. - \sum_{\substack{s_0 \in S \\ \delta(e; v, s_0) = 1}} \sum_{\substack{T \in \mathcal{F}_1 \\ e \in T(s_0)}} \mathbb{1}(T = F \cup e) \right). \\
(\text{eq.1}) &= \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \left(\#\{e \in \partial F(*) : \delta(e; v, F(*)) = 0\} \right. \\
&\quad \left. - \#\{e \in \partial F(*) : \delta(e; v, F(*)) = 1\} \right).
\end{aligned}$$

Here we let $\delta(e; v, F(*)) = \delta(e; v, x)$ for any $x \in F(*)$, i.e.

$$\delta(e; v, F(*)) = \begin{cases} 1 & \text{if } e \text{ lies on path from } v \text{ to } F(*), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we observe that for any forest F in $\mathcal{F}_2(G; S)$, there is exactly one edge e in the boundary $\partial F(*)$ of the floating component which satisfies $\delta(e; v, F(*)) = 1$, namely the unique boundary edge on the path from the floating component $F(*)$ to v . Thus

$$\#\{e \in \partial F(*) : \delta(e; v, F(*)) = 1\} = 1$$

and

$$\#\{e \in \partial(F, *) : \delta(e; v, F(*)) = 0\} = \deg^o(F, *) - 1.$$

Thus

$$\begin{aligned} \stackrel{\text{eq:1}}{(\dagger\dagger)} &= \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \left((\deg^o(F, *) - 1) - (1) \right) \\ &= - \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *))^2. \end{aligned}$$

as desired. \square

Old proof. Regarding the vertex sets that appear in the last summation, we have

$$\bigcup_{\substack{s \in S \\ \delta(e; v, s) = 1}} \{u \in T(s)\} = \begin{cases} \{u \in (G \setminus e)^{\bar{v}}\} & \text{if } e \notin T, \\ \{u \in (G \setminus e)^{\bar{v}}\} \cup (T \setminus e)(*) & \text{if } e \in T(s_0), \delta(e; v, s_0) = 0 \\ \{u \in (G \setminus e)^{\bar{v}}\} \setminus (T \setminus e)(*) & \text{if } e \in T(s_0), \delta(e; v, s_0) = 1. \end{cases}$$

alternative:

$$\bigcup_{\substack{s \in S \\ \delta(e; v, s) = 1}} \{w \in T(s)\} = \begin{cases} \{w \in V : \delta(e; v, w) = 1\} & \text{if } e \notin T, \\ \{w \in (G \setminus e)^{\bar{v}}\} \cup (T \setminus e)(*) & \text{if } e \in T, s_0(e, T) \in (G \setminus e)^v \\ \{w \in (G \setminus e)^{\bar{v}}\} \setminus (T \setminus e)(*) & \text{if } e \in T, s_0(e, T) \in (G \setminus e)^{\bar{v}}. \end{cases}$$

Here $T \setminus e$ is a $(S, *)$ -rooted forest, i.e. $T \setminus e \in \mathcal{F}_2(G; S)$, and $\deg^o(T \setminus e, *)$ refers to the out-degree of the floating component of $T \setminus e$. These cases correspond to

$$\sum_{s \in S \cap (G \setminus e)^{\bar{v}}} (2 - \deg^o(T, s)) = \begin{cases} 1 & \text{if } e \notin T, \\ 1 - (2 - \deg^o(T \setminus e, *)) & \text{if } s_0(e, T) \in (G \setminus e)^{\bar{v}}, \\ 1 + (2 - \deg^o(T \setminus e, *)) & \text{if } s_0(e, T) \in (G \setminus e)^v \end{cases}$$

From $\stackrel{\text{eq:14-1}}{(3.2)}$ we have

$$\begin{aligned} (D[S]\mathbf{m})_v &= \sum_{e \in E} \alpha_e \sum_{T \in \mathcal{F}_1} w(T)(1 - f(v, e, T)) \\ (D[S]\mathbf{m})_v &= \sum_{T \in \mathcal{F}_1} w(T) \left(\sum_{e \notin T} \alpha_e + \sum_{s_0 \in S} \sum_{e \in T(s_0)} \alpha_e (1 - f(T, v, e, s_0)) \right) \end{aligned}$$

where

$$f(T, v, e, s_0) = \begin{cases} 0 & \text{if } e \notin T \\ 2 - \deg^o(T \setminus e, *) & \text{if } \delta(e; v, s_0) = 0 \\ -(2 - \deg^o(T \setminus e, *)) & \text{if } e \in T(s_0) \text{ for some } s_0 \in S \text{ with } \delta(e; v, s_0) = 1 \end{cases}$$

\square

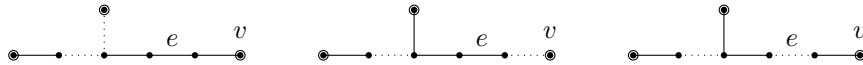


FIGURE 5. Components rooted in $S(G \setminus e)^{\bar{v}}$.

◇ MOVE TO REMARK? If $e \in \text{conv}(G, S)$, then $S(G \setminus e)^{\bar{v}}$ is nonempty and ◇

Remark 21. The set $\mathcal{F}_2(G; S)$ of $(S, *)$ -rooted spanning forests of G can be partitioned into two types: “active” and “inactive”.

$$\mathcal{F}_2(G/S) = \mathcal{F}_2^{\text{in}}(G; S) \sqcup \mathcal{F}_2^{\text{out}}(G; S),$$

where

$$\begin{aligned} \mathcal{F}_2^{\text{in}}(G; S) &= \{F \in \mathcal{F}_2(G; S) \text{ such that } \deg^o(*, F) \geq 2\}, \\ \mathcal{F}_2^{\text{out}}(G; S) &= \{F \in \mathcal{F}_2(G; S) \text{ such that } \deg^o(*, F) = 1\}. \end{aligned}$$

Remark 22. A key step in the above proof is the use of the map

$$E(G) \times \mathcal{F}_1(G; S) \mapsto S \sqcup \{\text{error}\}$$

defined by

$$(e, T) \mapsto \begin{cases} s & \text{if } e \in T(s), \\ \text{error} & \text{if } e \notin T. \end{cases}$$

Remark 23. For a given spanning forest $F \in \mathcal{F}_2(G; S)$, there are exactly $\deg^o(F, *)$ -many choices of pairs $(T, e) \in \mathcal{F}_1(G; S) \times E(G)$ such that $F = T \setminus e$. Consider the “deletion” map

$$E(G) \times \mathcal{F}_1(G; S) \rightarrow \mathcal{F}_2(G; S) \sqcup \mathcal{F}_1(G; S)$$

defined by ...

$$(e, T) \mapsto \begin{cases} T \setminus e & \text{if } e \in T, \\ T & \text{if } e \notin T. \end{cases}$$

For a forest F in $\mathcal{F}_2(G; S)$, the preimage under this map has $\deg^o(F, *)$ elements.

There is an associated “union” map

$$E(G) \times \mathcal{F}_2(G; S) \longrightarrow \mathcal{F}_1(G; S) \sqcup \mathcal{F}_2(G; S)$$

defined by

$$(e, F) \mapsto \begin{cases} F \cup e & \text{if } e \notin F, \\ F & \text{if } e \in F \end{cases}$$

Proposition 24. Let $G = (V, E)$ be a tree, and $S \subset V$. Suppose we label $S = \{s_1, \dots, s_r\}$ and $V \setminus S = \{t_1, \dots, t_{n-r}\}$. For each $t_i \in V \setminus S$, consider $\mathbf{f}_i \in \mathbb{R}^V$ defined by

Example 25. If

$$D[S \cup t] = \begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix}$$

then

$$\begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix} \begin{bmatrix} ab+ac+bc \\ -bc \\ -ac \\ -ab \end{bmatrix} = \begin{bmatrix} -3abc \\ -abc \\ -abc \\ -abc \end{bmatrix}$$

4. PHYSICAL INTERPRETATION

If we consider G as a network of wires with each edge containing a unit resistor, which is grounded at all nodes in S , then \mathbf{m}_S records the currents flowing to S when current is added on $V \setminus S$ in the amount $2 - \deg v$ for each $v \notin S$.

4.1. Alternate proof. Let $\mathbf{1}$ denote the all-ones vector. When we choose a subset $S \subset V(G)$, we no longer have a single “obvious” replacement for \mathbf{m} inside \mathbb{R}^S . Instead, we can take an average over S -rooted spanning forests.

In the outline above, our first goal is to find a “special” vector $\mathbf{m} \in \mathbb{R}^S$ satisfying $D[S]\mathbf{m} = \lambda\mathbf{1}$. We can approach this first goal as follows: consider \mathbb{R}^S inside the larger vector space $\mathbb{R}^V = \mathbb{R}^S \oplus \mathbb{R}^{V \setminus S}$, and we wish to find vectors $\mathbf{n}_i \in \mathbb{R}^V$ satisfying $\pi_S(D\mathbf{n}_i) = \lambda_i\mathbf{1}$. By finding sufficiently many such vectors \mathbf{n}_i , we can hope to find a linear combination that lies inside $\mathbb{R}^S \oplus \{0\}$.

prop:n-vector

Proposition 26. Suppose $v \in V \setminus S$. For each $s_j \in S$, let $\mu(v, s_j) =$ current flowing to s_j when unit current enters G at v and G is grounded at S . Explicitly,

$$\begin{aligned} \mu(v, s) &= \frac{\# \text{ of } S\text{-rooted spanning forests of } G \text{ whose } s_j\text{-component contains } v}{\# \text{ of } S\text{-rooted spanning forests of } G} \\ &= \frac{\sum_{T \in \mathcal{F}_1(G/S)} \mathbf{1}(v \in T(s))}{\kappa(G/S)} \\ &= \frac{\kappa_r(s_1 | \cdots | s_j v | \cdots | s_r)}{\kappa_r(s_1 | \cdots | s_r)} \end{aligned}$$

Consider the vector $\mathbf{n} = \mathbf{n}(G; S, v) \in \mathbb{R}^V$ defined by

$$\mathbf{n}_v = 1, \quad \mathbf{n}_s = -\mu(v, s) \text{ if } s \in S, \quad \mathbf{n}_w = 0 \text{ if } w \notin S \cup v$$

Then $D\mathbf{n}$ is constant on S , i.e. $\pi_S(D\mathbf{n}) = \lambda\mathbf{1}$ for some λ .

Proof sketch. For any $s, s' \in S$, consider tracking value of $D\mathbf{n}$ along path from s to s' . The value of $D\mathbf{n}$ changes according to current flow in the corresponding network, i.e. $D\mathbf{n}$ records electrical potential. By assumption S is grounded, so $D\mathbf{n}$ takes the same value at s and s' . \square

Theorem 27. Let G be a tree, S a nonempty subset of vertices, and $D[S]$ the corresponding submatrix of the distance matrix. Suppose $\mathbf{m} = \mathbf{m}(G; S) \in \mathbb{R}^S$ is defined by (14);

$$\mathbf{m}(G; S)_v = \sum_{T \in \mathcal{F}_1(G; S)} \sum_{w \in T_v} (2 - \deg w) = \sum_{T \in \mathcal{F}_1(G; S)} 2 - \deg^o(T, v).$$

Then $D[S]\mathbf{m} = \lambda\mathbf{1}$ for some constant λ .

Proof. Note that

$$\begin{aligned} \mathbf{m}(G; S) &= \kappa(G/S) \left(\sum_{v \in V} (2 - \deg v) \delta(v) - \sum_{v \in V \setminus S} (2 - \deg v) \mathbf{n}(G; S, v) \right) \\ &= \kappa(G; S) \left(\mathbf{m}(G; V) - \sum_{v \in V \setminus S} (2 - \deg v) \mathbf{n}(G; S, v) \right) \end{aligned}$$

◇ **TODO: elaborate on this equation** ◇ From Proposition 13 we know that $D\mathbf{m}(G; V)$ is constant on V , and from Proposition 21 we know that $D\mathbf{n}(G; S, v)$ is constant on S . Hence by linearity, $D\mathbf{m}(G; S)$ is constant on S . \square

5. EXAMPLES

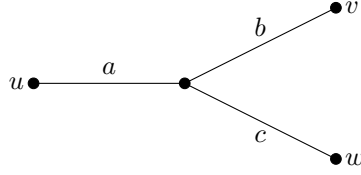
Example 28. Suppose G is a tree consisting of three paths joined at a central vertex. Let S consist of the central vertex, and the three endpoints of the paths. The corresponding minor of the distance matrix is

$$D[S] = \begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & a & b & c \\ a & -a & a & a \\ b & b & -b & b \\ c & c & c & -c \end{bmatrix} \sim \begin{bmatrix} 0 & a & b & c \\ a & -2a & 0 & 0 \\ b & 0 & -2b & 0 \\ c & 0 & 0 & -2c \end{bmatrix}.$$

The determinant is

$$\det D[S] = -4(a + b + c)abc.$$

Example 29. Suppose Γ is a tripod with lengths a, b, c and corresponding leaf vertices u, v, w .



Let $S = \{u, v, w\}$. Then

$$D[S] = \begin{bmatrix} 0 & a+b & a+c \\ a+b & 0 & b+c \\ a+c & b+c & 0 \end{bmatrix}.$$

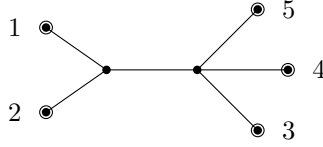
and

$$\det D[S] = 2(a+b)(a+c)(b+c) = 2((a+b+c)(ab+ac+bc) - abc).$$

The “special vector” that satisfies $D[S]\mathbf{m} = \lambda \mathbf{1}$ in this example is

$$\mathbf{m}^\top = [a(b+c) \quad b(a+c) \quad c(a+b)].$$

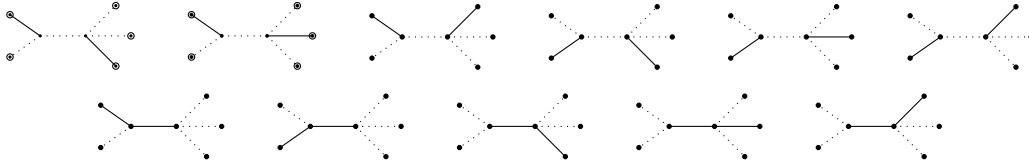
Example 30. Suppose Γ is the tree with unit edge lengths shown below, with five leaf vertices.



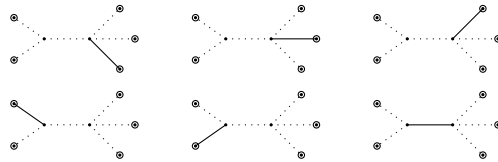
Let S denote the set of five leaf vertices. Then

$$D[S] = \begin{bmatrix} 0 & 2 & 3 & 3 & 3 \\ 2 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 2 & 2 \\ 3 & 3 & 2 & 0 & 2 \\ 3 & 3 & 2 & 2 & 0 \end{bmatrix}.$$

Forests in $\mathcal{F}_1(G; S)$:



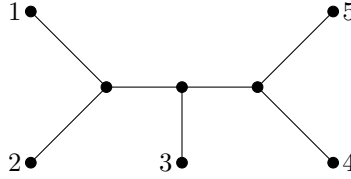
Forests in $\mathcal{F}_2(G; S)$:



and

$$\det D[S] = 368 = (-1)^{4 \cdot 2^3} (6 \cdot 11 - (3 \cdot 1^2 + 2 \cdot 2^2 + 1 \cdot 3^2))$$

Example 31. Suppose Γ is the tree with unit edge lengths shown below, with five leaf vertices and three internal vertices.



Let S denote the set of five leaf vertices. Then

$$D[S] = \begin{bmatrix} 0 & 2 & 3 & 4 & 4 \\ 2 & 0 & 3 & 4 & 4 \\ 3 & 3 & 0 & 3 & 3 \\ 4 & 4 & 3 & 0 & 2 \\ 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

and

$$\det D[S] = 864 = (-1)^4 2^3 (7 \cdot 21 - (14 \cdot 1^2 + 4 \cdot 2^2 + 1 \cdot 3^2))$$

6. FURTHER WORK

See [richman-shokrieh-wu](#)
[6].

6.1. Symanzik polynomials.

ACKNOWLEDGEMENTS

The authors would like to thank Ravindra Bapat for helpful discussion.

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kirkland-neumann

-sivasubramanian

graham-lovasz

graham-pollak

moon

hman-shokrieh-wu