

# MINORS OF TREE DISTANCE MATRICES

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ABSTRACT. We prove a formula for the determinant of a principal minor of the distance matrix of a weighted tree. This generalizes a result of Graham and Pollak.

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## 1. INTRODUCTION

Suppose  $G = (V, E)$  is a tree with  $n$  vertices. Let  $D$  denote the distance matrix of  $G$ . In [\[4\]](#), [Graham-Pollak](#) proved that

$$\text{eq:full-det} \quad (1) \quad \det D = (-1)^{n-1} 2^{n-2} (n-1).$$

This identity is remarkable in that the result does not depend on the tree structure, beyond the number of vertices. This identity was motivated by a problem in data communication, and inspired much further research on distance matrices.

The main result of this paper is to generalize [\(1\)](#) by replacing  $D$  with any principal submatrix. For  $S \subset V(G)$ , let  $D[S]$  denote the principal minor consisting of the  $S$ -indexed rows and columns.

**thm:main** **Theorem 1.** *Suppose  $G$  is a tree with  $n$  vertices, and distance matrix  $D$ . Let  $S \subset V(G)$  be a subset of vertices. Then*

$$\text{eq:main} \quad (2) \quad \det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left( (n-1) \kappa(G/S) - \sum_{\mathcal{F}_2(G/S)} k(F_*)^2 \right).$$

where  $G/S$  denotes the quotient graph that identifies together vertices in  $S$ ,  $\kappa(G/S)$  is the number of  $S$ -rooted spanning forests of  $G$ ,  $\mathcal{F}_2(G/S)$  is the set of  $(S, *)$ -rooted spanning forests of  $G$ ,  $F_*$  denotes the  $*$ -component of  $F$ , and

$$k(F_*) = 2 - \deg^o(F, *).$$

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Note: the quantities  $c(F, *)$ ,  $k(F_*)$  satisfy

$$1 \leq c(F, *) \leq |S|, \quad 2 - |S| \leq k(F_*) \leq 1.$$

When  $S = V$  is the full vertex set, the quotient graph  $G/V$  consists of a single vertex with  $n - 1$  loop edges, so  $\kappa(G/V) = 1$  and  $\mathcal{F}_2(G/V) = \emptyset$ .

Weighted version: A weighted version of (II) was proved by Bapat–Kirkland–Neumann [1]. eq:full-det

eq:w-full-det

$$(3) \quad \det D_\alpha = (-1)^{n-1} 2^{n-2} \prod_{e \in E} \alpha_e \sum_{e \in E} \alpha_e.$$

thm:w-main

**Theorem 2.** Suppose  $G$  is a finite, weighted tree, and  $A \subset V(G)$  is a subset of vertices. Then

eq:w-main

$$(4) \quad \det D[A] = (-1)^{|S|-1} 2^{|S|-2} \left( \sum_{E(G)} \ell(e) \sum_{\mathcal{T}(G/S)} w(T) - \sum_{\mathcal{F}_2(G/S)} w(F) k(F_*)^2 \right).$$

where  $\mathcal{T}(G/A)$  denotes the set of  $A$ -rooted spanning forests of  $G$ ,  $\mathcal{F}_2$  varies over all  $(A, *)$ -rooted spanning forests of  $G$ ,  $F_*$  denotes the  $*$ -component of  $F$ .

It is worth observing that the distances appearing in  $D[S]$  may ignore a large part of the ambient tree  $G$ . We could instead replace  $G$  by the subtree consisting of paths between vertices in  $S$ , which we call  $\text{conv}(S, G)$ , the *convex hull* of  $S \subset G$ . To apply formula (2) “efficiently”, we should replace  $G$  with this convex hull  $\text{conv}(S, G)$ . However, the formula as stated is true even without this replacement due to cancellation of terms. eq:main

**Corollary 3.**

$$(5) \quad \frac{\det D[S]}{\text{cof } D[S]} = \frac{1}{2} \left( \sum_{E(G)} \ell(e) - \frac{\sum_{\mathcal{F}_2(G/S)} w(F) k(F_*)^2}{\sum_{\mathcal{T}(G/S)} w(T)} \right).$$

**Theorem 4** (Monotonicity of principal minor ratios). Suppose  $G = (V, E)$  is a finite, weighted tree with distance matrix  $D$ .

(1) If  $S \subset V(G)$  is nonempty,

$$0 \leq \frac{\det D[S]}{\text{cof } D[S]} \leq \frac{1}{2} \sum_{E(G)} \ell(e).$$

(2) If  $\text{conv}(S, G)$  denotes the subtree of  $G$  consisting of all paths between points of  $S \subset V(G)$ ,

$$\frac{\det D[S]}{\text{cof } D[S]} \leq \frac{1}{2} \sum_{E(\text{conv}(S, G))} \ell(e).$$

(3) If  $A, B \subset V(G)$  are nonempty subsets with  $A \subset B$ , then

$$\frac{\det D[A]}{\text{cof } D[A]} \leq \frac{\det D[B]}{\text{cof } D[B]}.$$

**Theorem 5** (Nonsingular minors). Let  $G$  be a finite, weighted tree with distance matrix  $D$ , and let  $S \subset V(G)$  be a subset of vertices. If  $|S| \geq 2$  then  $\det D[S] \neq 0$ .

1.1. **Previous work.** A formula for the inverse matrix  $D^{-1}$  was found by Graham and Lovász in [3]. graham-lovasz

**1.2. Notation.**  $G$  a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$  edge set of  $G$

$V(G)$  vertex set of  $G$

## 2. BACKGROUND

For background on enumeration problems for graphs and trees, see Moon <sup>moon</sup>[5].

The following theorem is due to Kirchhoff. For any graph  $G$ , let  $\kappa(G)$  denote the number of spanning trees of  $G$ .

**Theorem 6** (All-minors matrix tree theorem). *Let  $G = (V, E)$  be a finite graph. Let  $L$  denote the Laplacian matrix of  $G$ . Then for any nonempty vertex set  $S \subset V(G)$ ,*

$$(6) \quad \det L[V \setminus S] = \kappa(G/S).$$

Note that  $\kappa(G/S)$  is also the number of  $S$ -rooted spanning forests of  $G$ .

**Theorem 7** (<sup>papat-sivasubramanian</sup>[2]). *Let  $T$  be a tree with  $m + 1$  vertices and  $m$  edges. Let  $D$  be the distance matrix of  $T$ , and  $L$  the Laplacian matrix. Let  $S \subset V(T)$  be a subset of vertices of  $T$ . Then*

$$\text{cof } D[S] = (-2)^{|S|-1} \det L[V \setminus S].$$

### 2.1. Trees and forests.

## 3. PROOFS

Outline of proof: given subset  $S$  and distance matrix minor  $D[S]$ , we will

- (1) Find vector  $\mathbf{m} \in \mathbb{R}^S$  such that  $D[S]\mathbf{m} = \lambda \mathbf{1}$ .
- (2) Compute the sum of entries of  $\mathbf{m}$ , i.e.  $\mathbf{1}^T \mathbf{m}$ .
- (3) Note the identity

$$\mathbf{1}^T \mathbf{m} = \lambda (\mathbf{1}^T D[S]^{-1} \mathbf{1}) = \lambda \frac{\text{cof } D[S]}{\det D[S]}.$$

where  $\text{cof } D[S]$  is the sum of cofactors of  $D[S]$ .

- (4) Use known expression for  $\text{cof } D[S]$  to compute

$$\det D[S] = \lambda (\text{cof } D[S]) (\mathbf{1}^T \mathbf{m})^{-1}.$$

The interesting part of this expression will be located in the constant  $\lambda$ .

**Example 8.** Suppose  $G$  is a tree consisting of three paths joined at a central vertex. Let  $S$  consist of the central vertex, and the three endpoints of the paths. The corresponding minor of the distance matrix is

$$D[S] = \begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix}.$$

Following the steps outlined above:

- (1) The vector  $\mathbf{m} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  satisfies  $D[S]\mathbf{m} = (a+b+c)\mathbf{1}$
- (2) The sum of entries of  $\mathbf{m}$  is  $\mathbf{1}^T \mathbf{m} = 2$ .

(3) We have

$$2 = \mathbf{1}^T \mathbf{m} = \lambda(\mathbf{1}^T D[S] \mathbf{1}) = \lambda \frac{\text{cof } D[S]}{\det D[S]}.$$

(4) The cofactor sum is  $\text{cof } D[S] = -8abc$ , so the determinant is

$$\det D[S] = \lambda \frac{\text{cof } A}{\mathbf{1}^T \mathbf{m}} = (a + b + c)(-8abc) \frac{1}{2} = -4(a + b + c)abc.$$

**Proposition 9.** *Let  $T = (V, E)$  a tree, and consider the vector  $\mathbf{m} \in \mathbb{R}^V$  defined by*

$$\mathbf{m}(v) = 2 - \deg v,$$

*where  $\deg v$  denotes the degree of  $v$  in  $T$ . Then  $\mathbf{1}^T \mathbf{m} = \sum_{v \in V} (2 - \deg v) = 2$ .*

Let  $\mathbf{1}$  denote the all-ones vector. When we choose a subset  $S \subset V(G)$ , we no longer have a single “obvious” replacement for  $\mathbf{m}$  inside  $\mathbb{R}^S$ . Instead, we can take an average over  $S$ -rooted spanning forests.

In the outline above, our first goal is to find a “special” vector  $\mathbf{m} \in \mathbb{R}^S$  satisfying  $D[S]\mathbf{m} = \lambda \mathbf{1}$ . We can approach this first goal as follows: consider  $\mathbb{R}^S$  inside the larger vector space  $\mathbb{R}^V = \mathbb{R}^S \oplus \mathbb{R}^{V \setminus S}$ , and we wish to find vectors  $\mathbf{n}_i \in \mathbb{R}^V$  satisfying  $\pi_S(D\mathbf{n}_i) = \lambda_i \mathbf{1}$ . By finding sufficiently many such vectors  $\mathbf{n}_i$ , we can hope to find a linear combination that lies inside  $\mathbb{R}^S \oplus \{0\}$ .

**Proposition 10.** *Suppose  $v \in V \setminus S$ . For each  $s \in S$ , let  $\mu(v, s) =$  current flowing to  $s$  when unit current enters  $G$  at  $v$  and  $G$  is grounded at  $S$ . Explicitly,*

$$\begin{aligned} \mu(v, s) &= \frac{\# \text{ of } S\text{-rooted spanning forests of } G \text{ whose } s\text{-component contains } v}{\# \text{ of } S\text{-rooted spanning forests of } G} \\ &= \frac{\sum_{T \in \mathcal{T}(G/S)} \mathbf{1}(v \in T(s))}{\kappa(G/S)} \end{aligned}$$

*Consider the vector  $\mathbf{n} \in \mathbb{R}^V$  defined by*

$$\mathbf{n}(v) = 1, \quad \mathbf{n}(s) = -\mu(v, s) \text{ if } s \in S, \quad \mathbf{n}(w) = 0 \text{ if } w \notin S \cup v$$

*Then  $\pi_S(D\mathbf{n}) = \lambda \mathbf{1}$  for some  $\lambda$ .*

*Proof sketch.* For any  $s, s' \in S$ , consider tracking value of  $D\mathbf{n}$  along path from  $s$  to  $s'$ . The value of  $D\mathbf{n}$  changes according to current flow in the corresponding system, i.e.  $D\mathbf{n}$  records electrical potential. By assumption  $S$  is grounded, so  $D\mathbf{n}$  takes the same value at  $s$  and  $s'$ .  $\square$

Note that we can express the tree distance  $d(v, w)$  as a sum over edges

$$d(v, w) = \sum_{e \in E(G)} \delta(e; v, w) \quad \text{where } \delta(e; v, w) = \begin{cases} 1 & \text{if } e \text{ lies on } v \sim w \text{ path,} \\ 0 & \text{otherwise.} \end{cases}$$

- If we fix  $v$  and  $w$ , then  $\delta(-; v, w) : E(G) \rightarrow \{0, 1\}$  is the indicator function for the unique  $v \sim w$  path in  $G$ .
- On the other hand if we fix  $e$  and  $v$ , then the deletion  $G \setminus e$  has two connected components, and  $\delta(e; v, -) : V(G) \rightarrow \{0, 1\}$  is the indicator function for the component of  $G \setminus e$  not containing  $v$ .

**Theorem 11.** *Let  $G$  be a tree,  $S$  a subset of vertices, and  $D[S]$  the corresponding minor of the distance matrix. Suppose  $\mathbf{m}_S \in \mathbb{R}^S$  is defined by*

$$\mathbf{m}_S(v) = \sum_{T \in \mathcal{T}(G/S)} \sum_{w \in T_v} (2 - \deg w) = \sum_{T \in \mathcal{T}(G/S)} 2 - c(T, v).$$

*Then  $D[S]\mathbf{m} = \lambda \mathbf{1}$  for some constant  $\lambda$ .*

*Proof.* Note that

$$\mathbf{m} = \kappa(G/S) \left( \sum + \sum_{v \in V \setminus S} (\deg v - 2) \mathbf{n}_v = \right)$$

□

**Theorem 12.** *With  $\mathbf{m}$  defined as above,  $D[S]\mathbf{m} = \lambda \mathbf{1}$  for*

$$(7) \quad \lambda = \sum_{T \in \mathcal{T}(G/S)} w(T) \sum_{E(G)} \ell(e) - \sum_{\mathcal{F}_2(G/S)} w(F) k(F, *)^2.$$

where  $c(T, w)$  is the “cut index” of the  $w$ -component of  $T$  (as a spanning forest).

*Proof.* We have

$$\begin{aligned} (D[S]\mathbf{m})(v) &= \sum_{s \in S} d(v, s) \mathbf{m}(s) \\ &= \sum_{s \in S} \left( \sum_{e \in E(G)} \ell(e) \delta(e; v, s) \right) \left( \sum_{T \in \mathcal{T}(G/S)} w(T) (2 - \deg^o(T, s)) \right) \\ &= \sum_T \sum_e \ell(e) w(T) \sum_{s \in S} (2 - \deg^o(T, s)) \delta(e; v, s) \\ &= \sum_T \sum_e \ell(e) w(T) \sum_{s \in S^*(e, v)} (2 - \deg^o(T, s)). \end{aligned}$$

where

$$S^*(e, v) = \{s \in S : e \text{ lies on path from } v \text{ to } s\}.$$

If  $e \in \text{conv}(G, S)$ , then  $S^*(e, v)$  is nonempty and we have

$$\sum_{s \in S^*(e, v)} (2 - c(T, s)) = \begin{cases} 1 & \text{if } e \notin T, \\ 1 - (2 - c(T \setminus e, *)) & \text{if } e \in T(s'), s' \in S^*(e, v), \\ 1 + (2 - c(T \setminus e, *)) & \text{if } e \in T(s'), s' \notin S^*(e, v) \end{cases}$$

Here  $c(T \setminus e, *)$  refers to the cut index of the “free” component of the spanning forest  $T \setminus e$ .

If  $e \notin \text{conv}(G, S)$  on the other hand,  $S^*(e, v)$  is empty and we have

$$\sum_{s \in S^*(e, v)} 2 - c(T, s) = 0 = 1 - (2 - \deg^o(T \setminus e, *))^2,$$

since  $\deg^o(T \setminus e, *) = 1$  in this case. From <sup>above</sup> (7.7) we have

$$(D[S]\mathbf{m})(v) = \sum_e \sum_T \ell(e) w(T) (1 - f(v, e, T))$$

where

$$f(v, e, T) = \begin{cases} 0 & \text{if } e \notin T \\ 2 - \deg^o(T \setminus e, *) & \text{if } e \in T(s') \text{ for some } s' \notin S^*(e, v) \\ -(2 - \deg^o(T \setminus e, *)) & \text{if } e \in T(s') \text{ for some } s' \in S^*(e, v) \end{cases}$$

$$\begin{aligned} (D[S]\mathbf{m})(v) - \sum_e \sum_T \ell(e)w(T) &= - \sum_T \sum_{e \in T \setminus T(S^*)} \ell(e)w(T)(2 - \deg^o(T \setminus e, *)) \\ &\quad + \sum_T \sum_{e \in T(S^*)} \ell(e)w(T)(2 - \deg^o(T \setminus e, *)) \end{aligned}$$

The deletion  $T \setminus e$  is an  $(S, *)$ -rooted spanning forest of  $G$ , so we may rewrite the above expression in terms of  $\mathcal{F}_2(G/S)$ .

$$\begin{aligned} (1) &= - \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \sum_T \sum_{e \in T \setminus T(S^*)} \mathbb{1}(F = T \setminus e) \\ &\quad + \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \sum_T \sum_{e \in T(S^*)} \mathbb{1}(F = T \setminus e) \end{aligned}$$

Next, we note that  $F = T \setminus e$  is equivalent to  $T = F \cup e$ , and in particular this only occurs when we choose  $e \in \partial F(*)$ :

$$\begin{aligned} (1) &= - \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \sum_{e \in \partial F(*)} \mathbb{1}(\delta(e; v, F(*)) = 1) \\ &\quad + \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *)) \sum_{e \in \partial F(*)} \mathbb{1}(\delta(e; v, F(*)) = 0) \end{aligned}$$

Finally, we observe that for any forest  $F$ ,

$$\#\{e \in \partial F(*) : \delta(e; v, F(*)) = 0\} = \deg^o(F, *) - 1$$

and

$$\#\{e \in \partial F(*) : \delta(e; v, F(*)) = 1\} = 1$$

Thus

$$\begin{aligned} (1) &= - \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *))(1) \\ &\quad + \sum_{F \in \mathcal{F}_2} w(F)(2 - \deg^o(F, *))( \deg^o(F, *) - 1) \\ &= - \sum_{F \in \mathcal{F}_2(G/S)} w(F)(2 - \deg^o(F, *))^2. \end{aligned}$$

□

Remark:

The set  $\mathcal{F}_2(G/S)$  of  $(S, *)$ -rooted spanning forests of  $G$  can be partitioned into two types: “active” and “inactive”.

$$\mathcal{F}_2(G/S) = \mathcal{F}_2^{in}(G/S) \sqcup \mathcal{F}_2^{out}(G/S),$$

where

$$\begin{aligned} \mathcal{F}_2^{in}(G/S) &= \{F \in \mathcal{F}_2(G/S) \text{ such that } \deg^o(*, F) \geq 2\}, \\ \mathcal{F}_2^{out}(G/S) &= \{F \in \mathcal{F}_2(G/S) \text{ such that } \deg^o(*, F) = 1\}. \end{aligned}$$

Remark:

Moreover, for a given spanning forest  $F \in \mathcal{F}_2(G/S, *)$ , there are exactly  $c(F, *)$  choices of pairs  $(T, e) \in \mathcal{T}(G/S) \times E(G)$  such that  $F = T \setminus e$ . Consider the map

$$E(G) \times \mathcal{T}(G/S) \rightarrow \mathcal{F}_2(G/S)$$

defined by ...

$$(e, T) \mapsto T \setminus e.$$

For a forest  $F$  in  $\mathcal{F}_2(G/S)$ , the preimage under this map has  $c(F, *)$  elements.

**Proposition 13.** *Let  $G = (V, E)$  be a tree, and  $S \subset V$ . Suppose we label  $S = \{s_1, \dots, s_r\}$  and  $V \setminus S = \{t_1, \dots, t_{n-r}\}$ . For each  $t_i \in V \setminus S$ , consider  $\mathbf{f}_i \in \mathbb{R}^V$  defined by*

**Example 14.** If

$$D[S \cup t] = \begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix}$$

then

$$\begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix} \begin{bmatrix} ab+ac+bc \\ -bc \\ -ac \\ -ab \end{bmatrix} = \begin{bmatrix} -3abc \\ -abc \\ -abc \\ -abc \end{bmatrix}$$

#### 4. PHYSICAL INTERPRETATION

If we consider  $G$  as a network of wires with each edge containing a unit resistor, which is grounded at all nodes in  $S$ , then  $\mathbf{m}_S$  records the currents flowing to  $S$  when current is added on  $V \setminus S$  in the amount  $2 - \deg v$  for each  $v \notin S$ .

#### 5. EXAMPLES

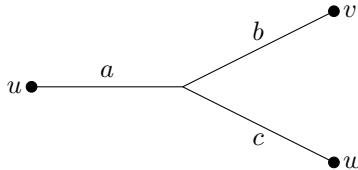
**Example 15.** Suppose  $G$  is a tree consisting of three paths joined at a central vertex. Let  $S$  consist of the central vertex, and the three endpoints of the paths. The corresponding minor of the distance matrix is

$$D[S] = \begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & a & b & c \\ a & -a & a & a \\ b & b & -b & b \\ c & c & c & -c \end{bmatrix} \sim \begin{bmatrix} 0 & a & b & c \\ a & -2a & 0 & 0 \\ b & 0 & -2b & 0 \\ c & 0 & 0 & -2c \end{bmatrix}.$$

The determinant is

$$\det D[S] = -4(a+b+c)abc.$$

**Example 16.** Suppose  $\Gamma$  is a tripod with lengths  $a, b, c$  and corresponding leaf vertices  $u, v, w$ .



Let  $B = \{u, v, w\}$ . Then

$$D[B] = \begin{bmatrix} 0 & a+b & a+c \\ a+b & 0 & b+c \\ a+c & b+c & 0 \end{bmatrix}.$$

and

$$\det D[B] = 2(a+b)(a+c)(b+c) = 2((a+b+c)(ab+ac+bc) - abc).$$

The “special vector” in this example is  $\mathbf{m}^T = [a(b+c) \quad b(a+c) \quad c(a+b)]$ .

## 6. FURTHER WORK

See [richman-shokrieh-wu](#)  
[6].

## ACKNOWLEDGEMENTS

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[bapat-kirkland-neumann](#)

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[graham-lovasz](#)

[graham-pollak](#)

[moon](#)

[richman-shokrieh-wu](#)