MINORS OF TREE DISTANCE MATRICES

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Abstract. We prove a formula for the determinant of a principal minor of the distance matrix of a weighted tree.

Contents

1. Introduction

Suppose T is a tree with n vertices and $m = n-1$ edges. let D denote the distance matrix of T. In graham-pollak [3], Graham and Pollak proved that

$$\det(D) = (-1)^{n-1}2^{n-2}(n-1). \tag{1}$$

thm:main **Theorem 1.** *Suppose G is a tree with n vertices, and $S \subset V(G)$ is a subset of vertices. Let D denote the distance matrix of G, and $D[S]$ the principal minor that includes the S-indexed rows and columns. Then*

eq:main
$$\det D[S] = (-1)^{|S|-1}2^{|S|-2}\left((n-1)\kappa(G/S) - \sum_{\mathcal{F}_2(G/S)} k(F_*)^2\right). \tag{2}$$

where G/S denotes the quotient graph that identifies together vertices in S, $\mathcal{F}_2$ is the set of two-component spanning forests, F_* denotes the $*$-component of F, and

$$k(F_*) = \sum_{x\in V(F_*)} 2 - \deg(x).$$

Weighted version:

thm:w-max-capacity **Theorem 2.** *Suppose G is a finite, weighted tree, and $A \subset V(G)$ is a subset of vertices. Then*

eq:w-max-capacity
$$\det D[A] = \frac{1}{4}\ell(\Gamma)\left(\sum_{\mathcal{T}} w(T)\right) - \frac{1}{4}\left(\sum_{\mathcal{F}^*} \mu_{can}(F_{2,*})^2 w(F_2)\right). \tag{3}$$

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where $\mathcal{T}(G/A)$ denotes the set of A-rooted spanning forests of G, F_2 varies over all $(A,*)$-rooted spanning forests of G, $F_{2,*}$ denotes the $*$-component of F_2, and

$$\mu_{can}(F_{2,*}) = \sum_{x \in V(F_{2,*})} 2 - \deg(x). \tag{4}$$

Theorem 3 (Monotonicity of principal minors)**.** *Suppose $G = (V,E)$ is a finite, weighted tree with distance matrix D. If $A, B \subset V(G)$ are vertex subsets with $A \subset B$, then*

$$\left| \frac{\det D[A]}{\operatorname{cof} D[A]} \right| \leq \left| \frac{\det D[B]}{\operatorname{cof} D[B]} \right|.$$

1.1. Previous work.

The following theorem is due to Kirchhoff.

Theorem 4 (All-minors matrix tree theorem)**.** *Let $G = (V,E)$ be a finite graph. Let L denote the Laplacian matrix of G. Then for any vertex subset $S \subset V(G)$,*

$$\det L[V \setminus S] = \kappa(G/S). \tag{5}$$

Theorem 5 (bapat-sivasubramanian [1])**.** *Let T be a tree with $m+1$ vertices and m edges. Let D be the distance matrix of T, and L the Laplacian matrix. Let $S \subset V(T)$ be a subset of vertices of T. Then*

$$\operatorname{cof} D[S] = (-2)^{|S|-1} \det L[V \setminus S].$$

1.2. Notation.

Γ a compact metric graph

G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G

(G, ℓ) a combinatorial model for a metric graph, where

$\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on edges of G

$\mathcal{C}(\Gamma)$ continuous $\mathbb{R}$-valued functions on Γ

$Meas(\Gamma)$ signed Borel measures on Γ

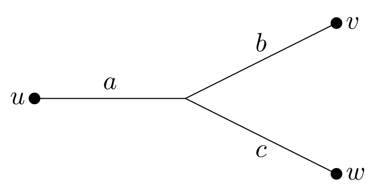
$Meas_{\geq 0}(\Gamma)$ positive Borel measures on Γ

2. Background

3. Proofs

4. Examples

Example 6. Suppose Γ is a tripod with lengths a, b, c and corresponding leaf vertices u, v, w.



Let $A = \{u, v, w\}$. Then

$$D[A] = \begin{bmatrix} 0 & a+b & a+c \\ a+b & 0 & b+c \\ a+c & b+c & 0 \end{bmatrix}.$$

and

$$\det D[A] = 2(a+b)(a+c)(b+c) = 2\left((a+b+c)(ab+ac+bc) - abc\right).$$

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