MINORS OF TREE DISTANCE MATRICES

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ABSTRACT. We prove a formula for the determinant of a principal minor of the distance matrix of a weighted tree. This generalizes a result of Graham and Pollak.

CONTENTS

1. INTRODUCTION

Suppose T is a tree with n vertices and $m = n - 1$ edges. Let D denote the distance matrix of T. In graham-pollak [3], Graham and Pollak proved that

$$\det(D) = (-1)^{n-1} 2^{n-2} (n-1). \tag{1}$$

A weighted version was proved by Bapat–Kirkland–Neumann bapat-kirkland-neumann [1].

thm:main **Theorem 1.** *Suppose G is a tree with n vertices, and $S \subset V(G)$ is a subset of vertices. Let D denote the distance matrix of G, and $D[S]$ the principal minor that includes the S-indexed rows and columns. Then*

eq:main
$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left((n-1)\, \kappa(G/S) - \sum_{\mathcal{F}_2(G/S)} k(F_*)^2 \right). \tag{2}$$

where G/S denotes the quotient graph that identifies together vertices in S, $\mathcal{F}_2$ is the set of two-component spanning forests, F_* denotes the $*$-component of F, and

$$k(F_*) = \sum_{x \in V(F_*)} 2 - \deg(x) = 2 - c(F, *).$$

Weighted version:

thm:w-max-capacity **Theorem 2.** *Suppose G is a finite, weighted tree, and $A \subset V(G)$ is a subset of vertices. Then*

eq:w-max-capacity
$$\det D[A] = (-1)^{|S|-1} 2^{|S|-2} \left(\sum_e \ell(e) \sum_{\mathcal{T}} w(T) - \sum_{\mathcal{F}^*} k(F_{2,*})^2 w(F_2) \right). \tag{3}$$

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where $\mathcal{T}(G/A)$ denotes the set of A-rooted spanning forests of G, F_2 varies over all $(A,*)$-rooted spanning forests of G, $F_{2,*}$ denotes the $*$-component of F_2.

Theorem 3 (Monotonicity of principal minors)**.** *Suppose $G=(V,E)$ is a finite, weighted tree with distance matrix D. If $A,B\subset V(G)$ are vertex subsets with $A\subset B$, then*

$$\left|\frac{\det D[A]}{\operatorname{cof} D[A]}\right| \le \left|\frac{\det D[B]}{\operatorname{cof} D[B]}\right|.$$

Theorem 4 (Nonsingular minors)**.** *Let G be a finite, weighted tree with distance matrix D, and let $S\subset V(G)$ be a subset of vertices. If $|S|\ge 2$ then $\det D[S]\neq 0$.*

1.1. **Previous work.** The following theorem is due to Kirchhoff. For any graph G, let $\kappa(G)$ denote the number of spanning trees of G.

Theorem 5 (All-minors matrix tree theorem)**.** *Let $G=(V,E)$ be a finite graph. Let L denote the Laplacian matrix of G. Then for any nonempty vertex set $S\subset V(G)$,*

$$\det L[V\setminus S]=\kappa(G/S). \tag{4}$$

Note that $\kappa(G/S)$ is also the number of S-rooted spanning forests of G.

Theorem 6 ([2] bapat-sivasubramanian)**.** *Let T be a tree with $m+1$ vertices and m edges. Let D be the distance matrix of T, and L the Laplacian matrix. Let $S\subset V(T)$ be a subset of vertices of T. Then*

$$\operatorname{cof} D[S]=(-2)^{|S|-1}\det L[V\setminus S].$$

1.2. **Notation.** G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G

2. Background

3. Proofs

Outline of proof: given subset S and distance matrix minor $D[S]$, we will

(1) Find vector $\mathbf{m}$ such that $D[S]\mathbf{m}=\lambda\mathbf{1}$.
(2) Compute the sum of entries of $\mathbf{m}$, i.e. $\mathbf{1}^T\mathbf{m}$.
(3) Note the identity
$$\mathbf{1}^T\mathbf{m}=\lambda(\mathbf{1}^T D[S]^{-1}\mathbf{1})=\lambda\frac{\operatorname{cof} D[S]}{\det D[S]}.$$
where $\operatorname{cof} D[S]$ is the sum of cofactors of $D[S]$.
(4) Use known expression for $\operatorname{cof} D[S]$ to compute
$$\det D[S]=\lambda(\operatorname{cof} D[S])\left(\mathbf{1}^T\mathbf{m}\right)^{-1}.$$

The interesting part will be hidden in the constant λ.

Example 7. Suppose G is a tree consisting of three paths joined at a central vertex. Let S consist of the central vertex, and the three endpoints of the paths. The corresponding minor of the distance matrix is

$$D[S]=\begin{bmatrix}0 & a & b & c\\ a & 0 & a+b & a+c\\ b & a+b & 0 & b+c\\ c & a+c & b+c & 0\end{bmatrix}.$$

Following the steps outlined above:

(1) The vector $\mathbf{m} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ satisfies $D[S]\mathbf{m} = (a+b+c)\mathbf{1}$

(2) The sum of entries of $\mathbf{m}$ is $\mathbf{1}^T\mathbf{m} = 2$.

(3) We have
$$2 = \mathbf{1}^T\mathbf{m} = \lambda(\mathbf{1}^T D[S]\mathbf{1}) = \lambda\frac{\operatorname{cof} D[S]}{\det D[S]}.$$

(4) The cofactor sum is $\operatorname{cof} D[S] = -8abc$, so the determinant is
$$\det D[S] = \lambda\frac{\operatorname{cof} A}{\mathbf{1}^T\mathbf{m}} = (a+b+c)(-8abc)\frac{1}{2} = -4(a+b+c)abc.$$

Proposition 8. *Let $T = (V, E)$ a tree, and consider the vector $\mathbf{m} \in \mathbb{R}^V$ defined by*
$$\mathbf{m}(v) = 2 - \deg v,$$
where $\deg v$ denote the degree of v in T. Then $\mathbf{1}^T\mathbf{m} = \sum_{v\in V}(2 - \deg v) = 2$.

Let $\mathbf{1}$ denote the all-ones vector.

Theorem 9. *Let T be a tree, S a subset of vertices, and $D[S]$ the corresponding minor of the distance matrix. Suppose $\mathbf{m} \in \mathbb{R}^S$ is defined by*
$$\mathbf{m}(v) = \sum_{T\in\mathcal{T}(T/S)}\sum_{w\in T_v}(2 - \deg w)$$
Then $D[S]\mathbf{m} = \lambda\mathbf{1}$ for some constant λ.

Proof. For any $u, v \in S$, we must show that $(D[S]\mathbf{m})(u) = (D[S]\mathbf{m})(v)$. We have
$$\begin{aligned}(D[S]\mathbf{m})(v) &= \sum_{w\in S} d(v,w)\mathbf{m}(w) \\ &= \sum_{w\in S} d(v,w)\sum_{T\in\mathcal{T}(G/S)}\sum_{z\in T(w)}(2-\deg z) \\ &= \sum_{T\in\mathcal{T}(G/S)}\sum_{w\in S}\sum_{z\in T(w)}(2-\deg z)d(v,w) \\ &= \sum_{T\in\mathcal{T}(G/S)}\sum_{w\in S}(2 - c(T,w))d(v,w).\end{aligned}$$
where $c(T, w)$ is the "cut index" of the w-component of T (as a spanning forest).

Note that we can express the tree distance $d(v, w)$ as a sum over edges
$$d(v,w) = \sum_{e\in E(G)}\delta(e; v, w) \qquad \text{where } \delta(e; v, w) = \begin{cases} 1 & \text{if } e \text{ lies on } v \sim w \text{ path,} \\ 0 & \text{otherwise.}\end{cases}$$
Thus
$$\begin{aligned}(D[S]\mathbf{m})(v) &= \sum_{T\in\mathcal{T}(G/S)}\sum_{w\in S}(2-c(T,w))\sum_{e\in E(G)}\delta(e;v,w) \\ &= \sum_{T\in\mathcal{T}(G/S)}\sum_{e\in E(G)}\left(\sum_{w\in S}(2-c(T,w))\delta(e;v,w)\right)\end{aligned}$$
$\square$

4. Examples

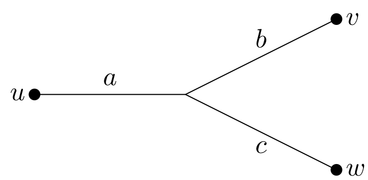
Example 10. Suppose G is a tree consisting of three paths joined at a central vertex. Let S consist of the central vertex, and the three endpoints of the paths. The corresponding minor of the distance matrix is

$$D[S] = \begin{bmatrix} 0 & a & b & c \\ a & 0 & a+b & a+c \\ b & a+b & 0 & b+c \\ c & a+c & b+c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & a & b & c \\ a & -a & a & a \\ b & b & -b & b \\ c & c & c & -c \end{bmatrix} . \sim \begin{bmatrix} 0 & a & b & c \\ a & -2a & 0 & 0 \\ b & 0 & -2b & 0 \\ c & 0 & 0 & -2c \end{bmatrix} .$$

The determinant is

$$\det D[S] = -4(a+b+c)abc.$$

Example 11. Suppose Γ is a tripod with lengths a, b, c and corresponding leaf vertices u, v, w.



Let $A = \{u, v, w\}$. Then

$$D[A] = \begin{bmatrix} 0 & a+b & a+c \\ a+b & 0 & b+c \\ a+c & b+c & 0 \end{bmatrix} .$$

and

$$\det D[A] = 2(a+b)(a+c)(b+c) = 2\left((a+b+c)(ab+ac+bc) - abc\right).$$

The "special vector" in this example is $\mathbf{m}^T = \begin{bmatrix} a(b+c) & b(a+c) & c(a+b) \end{bmatrix}$.

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References

bapat-kirkland-neumann [1] R. Bapat, S. J. Kirkland, and M. Neumann. On distance matrices and Laplacians. *Linear Algebra Appl.*, 401:193–209, 2005.

bapat-sivasubramanian [2] R. B. Bapat and S. Sivasubramanian. Identities for minors of the Laplacian, resistance and distance matrices. *Linear Algebra Appl.*, 435(6):1479–1489, 2011.

graham-pollak [3] R. L. Graham and H. O. Pollak. On the addressing problem for loop switching. *Bell System Tech. J.*, 50:2495–2519, 1971.